

Hamiltonian treatment of the gravitational collapse of thin shells

Juan Crisóstomo*

Instituto de Física, Facultad de Ciencias Básicas y Matemáticas, Pontificia Universidad Católica de Valparaíso, Avenida Brasil 2950, Valparaíso, Chile

and Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile

Rodrigo Olea†

Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile

and Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

(Received 6 November 2003; published 24 May 2004)

A Hamiltonian treatment of the gravitational collapse of thin shells is presented. The direct integration of the canonical constraints reproduces the standard shell dynamics for a number of known cases. The formalism is applied in detail to three-dimensional spacetime and the properties of the $(2+1)$ -dimensional charged black hole collapse are further elucidated. The procedure is also extended to deal with rotating solutions in three dimensions. The general form of the equations providing the shell dynamics implies the stability of black holes, as they cannot be converted into naked singularities by any shell collapse process.

DOI: 10.1103/PhysRevD.69.104023

PACS number(s): 04.20.Dw, 04.20.Fy, 04.40.-b, 04.70.Bw

I. INTRODUCTION

The gravitational collapse of thin shells was beautifully discussed in the classic paper of Israel [1]. The generalization to include electric charge was given by Kuchař [2], and an interesting further development and applications were given by Ipser and Sikivie [3]. In all these treatments the analysis is based on the discontinuities in the intrinsic and extrinsic curvatures of the world tube of the collapsing matter as it is regarded as embedded in either its exterior or its interior.

In this paper we introduce, in addition to the intrinsic and extrinsic geometry of the world tube, another structure, namely, a foliation of spacetime by constant time surfaces which intersect the tube. The reason for doing this is that the charges of the black hole which results from the collapse of the shell are conserved quantities given by surface integrals at spacelike infinity, which are naturally treated in terms of the Hamiltonian formalism [4]. Furthermore, the local properties of the horizon, such as its area, are also economically treated in Hamiltonian terms. The formalism which emerges from combining the Israel treatment with the Hamiltonian formalism is quite compact and permits one to economically analyze a number of situations of interest.

The plan of the paper is as follows. Sections II and III briefly review the Israel method for thin shell collapse and the Hamiltonian formalism, respectively. Section IV applies the canonical formalism to the gravitational collapse of a spherically symmetric shell in a spacetime of arbitrary dimension and recovers and further clarifies results previously found in the literature. Section V studies the radial gravitational collapse in three dimensional spacetimes, including the electrically charged case. Section VI extends the treatment to deal with angular momentum in three-dimensional

spacetimes. Finally, Sec. VII is devoted to brief concluding remarks.

II. THIN SHELL FORMALISM RECONSIDERED

The standard treatment for dealing with thin shells in general relativity, arising from the seminal work of Israel [1], has provided a useful tool to tackle a large variety of cases, ranging from lower dimensional static black hole formation [5] to interesting recent applications in the analysis of the junction conditions for extended objects in Gauss-Bonnet extended gravity (see, e.g., [6]).

The standard procedure considers a timelike hypersurface Σ_ξ , generated by the time evolution of the shell. This hypersurface divides the spacetime into two regions, namely, V_+ and V_- . Let ξ^μ be the outer pointing unit normal to the *world tube*, which is spacelike, and h_{ab} the induced metric on the tube. Here, the indices $a, b = 1, \dots, (d-1)$ label the tangent directions along the hypersurface. The coordinate set $\{x^\mu\}$ describes the spacetime with a metric $g_{\mu\nu}$, and another set $\{\sigma^a\}$ represents the intrinsic coordinates of the induced geometry, related to each other by a transformation matrix $e_a^\mu = \partial x^\mu / \partial \sigma^a$. Any point on the spacetime shell trajectory can be endowed with a local basis $\{\xi^\mu, e_a^\mu\}$. In this way, the standard definition of intrinsic metric over the hypersurface $h_{ab} = e_a^\mu e_b^\nu g_{\mu\nu}$ is recovered in terms of the spacetime metric.

The surface stress tensor $S_{\mu\nu}$ can be obtained from the volume tensor $T_{\mu\nu}$ as the limit process in the shell thickness,

$$S_{\mu\nu} = \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{+\varepsilon} T_{\mu\nu} d\xi. \quad (1)$$

The projections of Einstein tensor $G_{\mu\nu}$ along the normal coordinate ξ and the remaining directions over the hypersurface Σ_ξ lead to a set of relations

$$G_{\xi a} = K_{|a} - K_{a|b}^b, \quad (2)$$

*Electronic address: jcrisost@gacg.cl

†Electronic address: rolea@fis.puc.cl

$$2G_{\xi\xi} = (d-1)R(h) - (K^2 - K_{ab}K^{ab}), \quad (3)$$

$$G_{ab} = (d-1)G_{ab} + \partial_\xi(K_{ab} - h_{ab}K) - KK_{ab} + \frac{1}{2}h_{ab}(K^2 + K_{cd}K^{cd}). \quad (4)$$

Here, $(d-1)R(h)$ stands for the Ricci scalar of h_{ab} and K_{ab} is the extrinsic curvature of Σ_ξ .

Integrating Eq. (4) across the shell, the Lanczos equation is obtained,

$$\gamma_{ab} - h_{ab}\gamma = 8\pi\tilde{G}S_{ab}, \quad (5)$$

relating the discontinuity of the extrinsic curvature $\gamma_{ab} = [K_{ab}] = K_{ab}^+ - K_{ab}^-$ and its trace γ , with the projected surface stress tensor S_{ab} .

From Eq. (2) we see that the jump across the shell leads to the continuity equation for S_{ab}

$$S_{a|b}^b = -[T_{\mu\nu}e_a^\mu e_b^\nu] = -[T_{a\xi}]. \quad (6)$$

For many cases of physical interest, we consider a perfect fluid with a bulk stress tensor

$$T_{\mu\nu} = [\sigma u_\mu u_\nu - \tau(h_{\mu\nu} + u_\mu u_\nu)]\delta(X), \quad (7)$$

where u^μ is the shell d -dimensional velocity, and σ and τ stand for the surface energy density and tension, respectively. The delta function represents a matter distribution localized at the boundary of Σ_ξ .

Even though the Israel treatment for thin shells has proceeded along a line of increasing success on the understanding of gravitational collapse, the complexity brought about, for instance, by adding angular momentum can make this method hard to use in practice.

On the other hand, some authors have proposed alternative approaches, based on the canonical formalism, to rederive the thin shell dynamics obtained by the Israel method in a number of cases [7–12]. Following this line, we present a simple method to reproduce the equations of motion for the radial collapse of thin shells, but that can also be extended to deal with rotating solutions in three-dimensional spacetimes.

In the next section, we show that the direct integration of Hamiltonian constraints provides a complete set of equations equivalent to the ones obtained from the standard thin shell method.

III. HAMILTONIAN TREATMENT OF THIN SHELL COLLAPSE

The Einstein-Hilbert action with cosmological constant in d dimensions is written as

$$I = -\kappa \int d^d x \sqrt{-^{(d)}g} (R - 2\Lambda), \quad (8)$$

with the constant in front of the gravitational action given as $\kappa = 1/2(d-2)\Omega_{d-2}G$ [14]. The general approach presented

here is equally valid for any value of the constant Λ . For later purposes, Λ is chosen as $\Lambda = -(d-1)(d-2)/2l^2$ in terms of the AdS radius l .

Taking a timelike Arnowitt-Deser-Misner (ADM) foliation for the spacetime [13], we write the line element as

$$ds^2 = -(N^\perp)^2 dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j), \quad (9)$$

where g_{ij} is the spatial metric and the functions N^\perp and N^i represent the time lapse and the spatial shift, respectively. The quantities N^\perp and N^i play the role of Lagrange multipliers of the constraints $\mathcal{H}_\perp \approx 0$ and $\mathcal{H}_i \approx 0$, so that the gravitational action can be cast in Hamiltonian form

$$I = \int dt d^{d-1}x (\pi^{ij}\dot{g}_{ij} - N^\perp\mathcal{H}_\perp - N^i\mathcal{H}_i), \quad (10)$$

where \mathcal{H}_\perp and \mathcal{H}_i are given by the formulas

$$\mathcal{H}_\perp = -\frac{1}{\kappa\sqrt{g}} \left(\pi_{ij}\pi^{ij} - \frac{1}{(d-2)}(\pi_i^i)^2 \right) + \kappa\sqrt{g}({}^{(d-1)}R(g) - 2\Lambda) + \sqrt{g}T_{\perp\perp}, \quad (11)$$

$$\mathcal{H}_i = -2\pi_{ij}^i + \sqrt{g}T_{\perp i} \quad (12)$$

in the presence of matter fields. Here ${}^{(d-1)}R(g)$ stands for the Ricci scalar of the spatial metric g_{ij} and π^{ij} are the conjugate momenta.

IV. NONROTATING CASE

By radial collapse, it is possible to add mass and electric charge to an already existing (un)charged black hole, or produce the black hole itself over a vacuum state.

In a similar procedure to the ones developed in Refs. [9,11] for massive shells, here we show that the integration of the Hamiltonian constraints along an infinitesimal radial distance on a constant-time slice reproduce the results of the standard formalism. It will be shown that this treatment also implies the stability of the event horizon in a generic case.

A spherically symmetric collapsing shell has static interior and exterior geometries described by Schwarzschild-like coordinates

$$ds_\pm^2 = -N_\pm^2(r)f_\pm^2(r)dt_\pm^2 + f_\pm^{-2}(r)dr^2 + r^2d\Omega_{d-2}^2, \quad (13)$$

where the matching condition for the time is given by the choice $N_\pm = 1$. The radial coordinate r is continuous across the shell, because it measures the (intrinsic) area of the shell, which is the same as looked at from the inside and the outside.

The induced metric of the world tube is simply the one of a $(d-2)$ -sphere,

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega_{d-2}^2. \quad (14)$$

For spherical symmetry, the Hamiltonian generator \mathcal{H}_\perp becomes [15]

$$\mathcal{H}_\perp = -\frac{\sqrt{g}}{2\Omega_{d-2}G} \left[\frac{(d-3)}{r^2} (1-f^2) - \frac{(f^2)'}{r} + \frac{(d-1)}{l^2} \right] + \sqrt{g} T_{\perp\perp}. \quad (15)$$

We are going to integrate out the constraint $\mathcal{H}_\perp = 0$ across a radial infinitesimal length centered at the shell position $r = R(\tau)$ at a constant time, to express the discontinuities in this component of the Hamiltonian in terms of $T_{\perp\perp}$. It is straightforward to prove that all the terms—but the radial derivative—contribute with a finite value jump proportional to ε , and they can indeed be ruled out in the limit $\varepsilon \rightarrow 0$. Thus, the only nonvanishing contribution comes from the second term

$$-\int_{-\varepsilon}^{+\varepsilon} \frac{(f^2)'}{r} dr = -\frac{\Delta(f^2)}{R} = 2\Omega_{d-2}G \int_{-\varepsilon}^{+\varepsilon} T_{\perp\perp} dr. \quad (16)$$

In the right-hand side (RHS) of the above equation, $T_{\perp\perp}$ is given by $T_{\perp\perp} = T^{\mu\nu} n_\mu n_\nu$, the contraction with the time-like normal vector in the ADM foliation $n_\mu = (-N^\perp, 0, \vec{0})$, that generates the sequence of constant-time surfaces Σ_t .

On the other hand, adapting another frame to the hypersurface Σ_ξ , we have a set of coordinates $\{T, X\}$. The tangential axis T that runs along the velocity u^μ and the direction X goes along the spacelike normal ξ^μ , at whose origin the delta function is located. In this way,

$$T_{\perp\perp} = T^{\mu\nu} n_\mu n_\nu = \{\sigma u_\perp u_\perp - \tau(h_{\perp\perp} + u_\perp u_\perp)\} \delta(X). \quad (17)$$

Without loss of generality, we take a Schwarzschild-like coordinate set $x^\mu = \{t, r, \phi^i\}$ for the outer description of a shell collapse. Then we can compute $u^\mu = \{f^{-2} \dot{R}, \dot{R}, \vec{0}\}$ and $\xi^\mu = \{f^{-2} \dot{R}, \alpha, \vec{0}\}$, where the function α is given by $\alpha = \sqrt{f^2 + \dot{R}^2}$. Thus, we obtain an expression for Eq. (17) as seen from the $\{T, X\}$ frame

$$T_{\perp\perp} = \sigma \frac{\alpha^2}{f^2} \delta(X). \quad (18)$$

However, to carry out the integration over r , we need to rewrite the delta function in the spacetime coordinate system $\{t, r\}$.

From the Fig. 1, any point on the shell is described by both coordinate systems as

$$dt = u^t dT + \xi^t dX, \quad (19)$$

$$dr = u^r dT + \xi^r dX. \quad (20)$$

Integrating along dr , on a time-constant ADM slice Σ_t ($dt = 0$), we get

$$\frac{dr}{dX} = \xi^r - \frac{u^r}{u^t} \xi^t = \frac{f^2}{\alpha}, \quad (21)$$

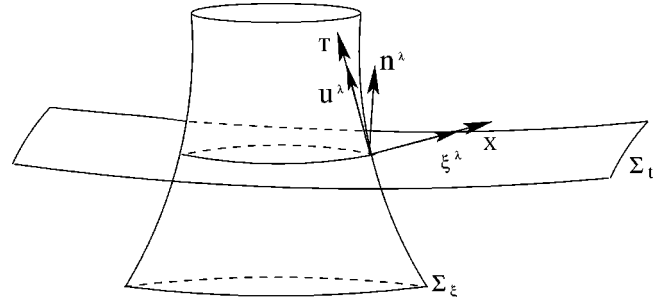


FIG. 1. The hypersurfaces Σ_t and Σ_ξ are defined by the normal vectors n^μ and ξ^μ . The intersection between Σ_t and Σ_ξ is the shell itself at the time t .

and the delta function transforms in such a way that it gets an additional “relativistic” factor $\delta(X) = (f^2/\alpha) \delta(r-R)$ and the final form for the stress tensor (18) is

$$T_{\perp\perp} = \alpha \sigma \delta(r-R). \quad (22)$$

Integrating the above relation, the RHS represents the mean value of the function α as seen embedded in both inside and outside spacetimes

$$2\Omega_{d-2}G \int_{-\varepsilon}^{+\varepsilon} T_{\perp\perp} dr = \Omega_{d-2} \sigma (\alpha_+ + \alpha_-) G. \quad (23)$$

Notice that the tension value τ does not appear on the right hand side of Eq. (23).

With these simple arguments, this method recovers and extends the dynamics for radial collapse computed using the thin shell formalism [1–3]

$$-\Delta f^2(R) = (\Omega_{d-2} R \sigma) (\sqrt{f_+^2 + \dot{R}^2} + \sqrt{f_-^2 + \dot{R}^2}) G. \quad (24)$$

For three spacetime dimensions, the same formula has been obtained by Steif and Peleg [5] for the gravitational collapse of a dust thin shell.

Note, as is well known, that in order to regard Eq. (24) as a first integral of the equation of motion for $R(\tau)$, one needs to specify the density σ as a function of the tension τ . Replacing in the continuity equation (6) the expression for S_{ab} and taking the parallel components to the velocity u^a we have

$$(\sigma u^b)_{|b} - \tau u^b_{|b} = 0, \quad (25)$$

which is the relation that provides the conserved quantities in the system. For example, for coherent dust one has that $R^{d-2} \sigma$ is a constant, whereas for a domain wall σ is a constant. In the first case, the interpretation of Eq. (24) is quite intuitive. For Schwarzschild-AdS black holes, the function in the metric reads $f^2 = 1 - (2GM/r^{d-3}) + (r^2/l^2)$, and the term $m = \Omega_{d-2} R^{d-2} \sigma$ is the rest-frame mass, as seen by an intrinsic observer. Hence, Eq. (24) reduces to

$$\Delta M = \frac{1}{2} (\alpha_+ + \alpha_-) m, \quad (26)$$

which relates the inertial mass to the semisum of the gravitational mass at each side of the shell. For Minkowskian spacetime, the factor α becomes the special relativity γ factor, thanks to the useful identity $\gamma^2 = 1 + (dR/d\tau)^2 = [1 - (dR/dt)^2]^{-1}$.

To complete the present picture of radial collapse, it is necessary to analyze the consistency of the remaining non-vanishing components of the Hamiltonian.

The angular components of the constraint (12) are identically zero. In common cases, the condition of spherical symmetry is sufficient to ensure that the radial constraint $\mathcal{H}_r^{(g)}$ vanishes. However, here it is different from zero because $T_{\perp i}$ is proportional to the radial velocity. One can expect \mathcal{H}_r to be proportional to \mathcal{H}_{\perp} , since Eq. (11) already provides the equation of motion for $R(\tau)$. It is interesting to see explicitly that this indeed occurs. The proof also illustrates again how efficiently in this approach one obtains a feature already known in the Israel method.

Computing the extrinsic curvature by definition in terms of the Lie derivative, we get

$$K_{ij} = -\frac{1}{2}\mathcal{L}_n g_{ij} = -\frac{1}{2}\partial_{\perp} g_{ij}. \quad (27)$$

Here $\partial_{\perp} = n^{\mu}\partial_{\mu}$ defines the derivative along the ADM time-like normal n^{μ} . This requires the projection of the vector n^{μ} on the shell frame, which is decomposed on the basis $\{u^{\mu}, \xi^{\mu}\}$ as $n^{\mu} = au^{\mu} + b\xi^{\mu}$, on the intersection between the shell hypersurface Σ_{ξ} and the constant-time slice Σ_t . Projecting between the frames, we obtain the coefficients $a = f^{-1}\alpha$ and $b = -f^{-1}\dot{r}$, which allows us to express the normal derivative as

$$\partial_{\perp} = \frac{\alpha}{f}\frac{\partial}{\partial\tau} - \frac{\dot{r}}{f}\frac{\partial}{\partial X}. \quad (28)$$

Here, we have used the definitions $u^{\mu}\partial_{\mu} = \partial/\partial\tau$ and $\xi^{\mu}\partial_{\mu} = \partial/\partial X$.

The metric g_{ij} has no dependence on X , because the coordinate X can always be set to zero over Σ_{ξ} . Then the explicit form for the extrinsic curvature K_{ij} as a proper time derivative of g_{ij} is

$$K_{ij} = -\frac{\alpha}{2f}\frac{\partial g_{ij}}{\partial\tau}. \quad (29)$$

Imposing the constraint over the radial component of Eq. (12) leads to

$$-2\pi_{r|j} + \sqrt{g}T_{\perp r} = 0, \quad (30)$$

where π^{ij} are obtained by means of the above formula for K_{ij} calculated with the spatial metric g_{ij} of ADM foliation.

Computing the stress tensor in terms of velocity and intrinsic metric, and using the Jacobian of the basis change, produces

$$T_{\perp r} = \frac{f^2}{\alpha}S_{\perp r}\delta(r-R) = -\frac{\dot{r}}{f}\sigma\delta(r-R), \quad (31)$$

resulting in the equation

$$\frac{d}{dr}(\alpha\dot{r}) = -\Omega_{d-2}r\dot{r}\sigma\delta(r-R)G. \quad (32)$$

The integration of this relation gives the discontinuity in the function α across the shell

$$\alpha_+ - \alpha_- = -\Omega_{d-2}R\sigma G. \quad (33)$$

In the context of standard thin shell formalism, this equation comes from the discontinuity in the normal acceleration across the hypersurface Σ_{ξ} . However, this does not stand for an independent relation from the energy conservation law (24), since it can also be recovered by multiplying that equation by $(\alpha_+ - \alpha_-)$.

Equation (24) clearly has a limited range of validity in the Schwarzschild-like radial coordinate R , since $-(\Delta f^2)$ must be strictly positive. For instance, for radial collapse of a massive thin shell, the LHS of this relation is just the difference of the outer solution mass with respect to AdS spacetime, which is positive for all the solutions of physical interest. In a more general case, there might be a radial position where Δf^2 vanishes. However, the same formula, written in the form of Eq. (33), tells us that the shell must bounce back before this happens, because for $\dot{R} = 0$

$$f_+ - f_- = -\Omega_{d-2}R\sigma G. \quad (34)$$

As is well known, the analysis of the shell motion can be carried out until the point where $f_+^2 = 0$. The change in the signature of the metric on the outer side leads to an inconsistency in the matching conditions on the shell.

Whereas the previous discussion in general imposes a lower bound for R , the positive definiteness of the functions α_{\pm} makes the analysis break down beyond a critical radius, for instance, in the black hole formation from a domain wall collapse, discussed below.

V. RADIAL COLLAPSE IN THREE DIMENSIONAL SPACETIME

For simplicity, we will focus on the problem of black hole creation in 2+1 dimensions, setting the inner solution as AdS spacetime ($M = -1$).

A. Coherent dust shell collapse

For pressureless dust, $m = 2\pi R\sigma$ is a constant of motion. In this case, already studied in Ref. [5], we have that Eq. (24) takes the form

$$M + 1 = \frac{1}{2}(\alpha_+ + \alpha_-)m, \quad (35)$$

with $M + 1 \geq 0$. For a given value of m , the collapse comes from the radial speed expression

$$\dot{R}^2 = \left(\frac{a^2}{16m^2} - 1\right) - R^2, \quad (36)$$

with $a = m^2 + 4(M + 1)$, gravity constant $G = \frac{1}{2}$ and AdS radius $l = 1$.

Its analysis leads to a confined motion for the dust ring, because the dust ring cannot be located beyond the turning point $R_0^2 = (a^2 - 16m^2)/16m^2$. This distance turns out to be greater than the black hole horizon for any outer solution with $M > 0$.

Depending on initial velocity and position, either a naked singularity ($-1 < M < 0$) or a black hole ($M > 0$) can be formed from this radial gravitational collapse process, as stated by Peleg and Steif. For negative mass solutions, there exists a critical shell mass $m = 2\sqrt{|M|} + 1$ below which the motion is impossible in the whole space. Apart from this condition, the analysis of the effective potential does not constitute a physical impediment to prevent the creation of a naked singularity in the black hole mass gap $(-1, 0)$. However, as we shall see in Sec. VI, the introduction of an amount of rotation, however small, gets rid of the naked singularities.

B. Closed fundamental string collapse

The radial collapse of a fundamental string can also generate a black hole (or naked singularity) as the external configuration starting up from AdS spacetime as the interior solution, for certain initial conditions.

In this case, it is more useful to analyze the equation of the radial acceleration, rather than its first integral (24). One obtains, by differentiation of Eq. (24)

$$\ddot{R} = -R - \pi \frac{\alpha_+ \alpha_-}{R}, \quad (37)$$

which implies that $\ddot{R} < 0$ and therefore there is no bounce, because the functions α_{\pm} are always positive. Hence, gravitational collapse is unavoidable for any shell density σ and black hole mass M .

Another interesting feature is that, just as it happens in $3 + 1$ dimensions as pointed out in [3], due to the particular form of the functions f_{\pm}^2 present in the metric, the constraint (33) is violated if $R > R_{\max} = (\pi^2 \sigma^2 - 1)^{-1/2}$, with R_{\max} as the maximum value of R (the value for which $\dot{R} = 0$). The existence of this bound for the radial coordinate makes it impossible to treat by dynamical analysis the cases where this critical radius is located within the event horizon r_+ and, therefore, the system is already collapsed.

From this consideration, it follows that there exists only an allowed interval in the mass spectrum for the exterior solution with a given density σ of

$$M = 2\pi\sigma R_{\max} \sqrt{1 + R_{\max}^2} - (\pi\sigma)^2 R_{\max}^2 - 1. \quad (38)$$

In the same way as in the $(3 + 1)$ -dimensional counterpart, this process cannot create black hole solutions beyond that mass range, where the too large spherical walls are already collapsed inside their corresponding Schwarzschild radius [3].

C. General case

The existence of the equation of state determines the nature of the collapsing matter, ranging from coherent dust ($\tau = 0$) to a domain wall ($\tau = \sigma$). Interpolating between these cases, we can set a parameter α such that $\tau = \alpha\sigma$.

Choosing the commoving frame in Eq. (25) and introducing the value of the tension, we can see that the density at any time satisfies the relation

$$\sigma = C_0 R^{\alpha-1}, \quad (39)$$

where C_0 is a constant throughout the motion.

For an even more general dependence of the tension τ , we can always write the equation of motion as

$$\ddot{R} = -R - \pi \frac{\alpha_+ \alpha_-}{R} \frac{\tau}{\sigma}. \quad (40)$$

Provided $\tau \geq 0$, Eq. (40) tells us that \ddot{R} is always negative. As a consequence, the shell accelerates inward and it will always collapse to either a black hole or a naked singularity, depending on the initial conditions.

D. Electrically charged solutions

Electrically charged solutions are obtained by supplementing the Einstein-Hilbert action (8) with the Maxwell term

$$I_{\text{Maxwell}} = \frac{1}{4\epsilon\Omega_{d-2}} \int d^d x \sqrt{-^{(d)}g} F_{\mu\nu} F^{\mu\nu}, \quad (41)$$

in an arbitrary dimension d . The constant ϵ can be written in terms of the vacuum permeability as $\epsilon = \epsilon_0/\Omega_{d-2}$.

For an static, spherically symmetric ansatz, the Reissner-Nordström (RN)-AdS black hole metric appropriately describes the geometry of both inner and outer regions of spacetime,

$$f_{\pm}^2 = 1 + \frac{r^2}{l^2} - \left(\frac{2GM_{\pm}}{r^{d-3}} - \frac{\epsilon G}{d-3} \frac{Q_{\pm}^2}{r^{2(d-3)}} \right), \quad (42)$$

where the shell carries an electric charge $q = Q_+ - Q_-$.

The general form of Eq. (24) that governs the radial collapse in any dimension remains the same in this case because the electromagnetic stress tensor does not contribute to the Hamiltonian component \mathcal{H}_{\perp} . Therefore, the equation of motion becomes

$$\Delta M - \frac{\epsilon}{2(d-3)} \frac{\Delta Q^2}{R^{d-3}} = \frac{1}{2} (\Omega_{d-2} R^{d-2} \sigma) (\alpha_+ + \alpha_-), \quad (43)$$

which recovers the thin shell dynamics studied in [1,2] for the 4-dimensional case.

In $2 + 1$ dimensions, the solution corresponding to an electrically charged static black hole was first presented in

[17] as the three-dimensional counterpart of the RN black hole. The metric contains a logarithmic dependence on the radial coordinate,

$$f^2 = r^2 - M - \frac{1}{4} Q^2 \ln r^2, \quad (44)$$

with the constant ϵ and the cosmological length l set equal to unity.

From the analysis of this function, the condition for the existence of extremal black holes is

$$M = \frac{Q^2}{4} \left[1 - \ln \frac{Q^2}{4} \right], \quad (45)$$

which is the curve that separates black hole configurations from naked singularities in the plane (M, Q) . If the electric charge is large enough, there exist black hole solutions for arbitrarily negative values for the mass.

In order to study the creation of charged black holes over a vacuum state, we set the inner solution as AdS spacetime, with $f_-^2 = (1 + r^2)$. With a dust shell carrying a total mass $m = 2\pi R\sigma$, Eq. (24) becomes

$$M + 1 + \frac{Q^2}{4} \ln R^2 = \frac{m}{2} (\alpha_+ + \alpha_-), \quad (46)$$

and the exterior mass and charge are $M_+ = M$ and $Q_+ = Q$, respectively. The LHS of the above expression must be positive in order to ensure the validity of the treatment in this coordinate set, and therefore it imposes a lower bound for the radial coordinate $R^2 > e^{-4(M+1)/Q^2}$. It can be proved that this quantity is larger than the inner horizon r_- for any charged black hole and its existence is relevant only in the context of creation of naked singularities, discussed below.

The radial velocity for this case is obtained by quadrature and takes the form

$$\dot{R}^2 = -(R^2 + 1) + \frac{1}{16m^2} (a + b \ln R^2)^2, \quad (47)$$

with the constants a and b defined in terms of the parameters of the solution as $a = m^2 + 4(M + 1)$ and $b = Q^2$. A quick analysis of the function shows that there must necessarily be a turning point as we move toward infinity (in the most general case there could even be two more). To find the local maximal and minimal points \bar{R} for the effective potential one solves the transcendental equation

$$\frac{8m^2}{b} \bar{R}^2 = a + b \ln \bar{R}^2. \quad (48)$$

Keeping a and m to a fixed value, the limit of $b \rightarrow 0$ produces the result that both intersection points move to $\bar{R}_{1,2}^2 \rightarrow 0$. On the contrary, if the limit in the parameter $b \rightarrow \infty$ is taken, the extremal points are shifted to $\bar{R}_1^2 \rightarrow 1$ and $\bar{R}_2^2 \rightarrow \infty$.

An inflection point exists at the position $\bar{R}^2 = b^2/8m^2$ when the parameters satisfy $a = b[1 - \ln(b^2/8m^2)]$. The cor-

responding radial velocity at that point is always purely imaginary. This relation represents a critical value for a and b , which permits the existence of local extremal points in the curve for a over that value. For values of a below the one given by the equality, there is neither a local maximum nor a minimum, the curve is monotonically decreasing, and the only turning point is immersed in the zone where the equation of motion is no longer valid.

Another critical situation is represented by a static thin shell, where the dust ring has been put in a fixed radial position

$$R_*^2 = \frac{b}{8m^2} [b + \sqrt{16m^2 + b^2}], \quad (49)$$

and it is not able to collapse or expand. For this particular case, the solution parameters satisfy the relation

$$a = b \left\{ 1 - \ln \left(\frac{b}{8m^2} [b + \sqrt{16m^2 + b^2}] \right) \right\} + \sqrt{16m^2 + b^2}. \quad (50)$$

Note that if we take the total mass of the shell as $m^2 = b + 4$, the process creates an extremal black hole with a ring standing still at the event horizon $R_+^2 = b/4$.

However, it is important to stress that this situation represents just a critical case in the extremal black hole formation, as there are many different sets of initial conditions that also generate them. From this perspective, extremal black holes cannot be regarded as “fundamental” objects, because Eq. (47) allows their creation from the dynamic process depicted in this section.

Finally, from the analysis of the effective potential (47) we conclude that a charged spherically symmetric shell cannot collapse to form a naked singularity in three dimensions. It is worthwhile to stress that, in spite of the different form of the charged black hole metric and the extremality condition derived from it, this property is also found in the four-dimensional case [16].

VI. ROTATING BLACK HOLE SOLUTIONS IN THREE DIMENSIONS

A different case is represented by the rotating black hole in a $(2+1)$ -dimensional spacetime. This time, the line element possesses a shift along the angular direction, responsible for the existence of two horizons and an ergosphere [17,18], in an analogous way to the Kerr metric in $3+1$ dimensions,

$$ds^2 = -N^2 f^2 dt^2 + f^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \quad (51)$$

where

$$f^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad (52)$$

$$N^\phi = -\frac{J}{2r^2} + N^\phi(\infty), \quad (53)$$

$$N = N(\infty). \quad (54)$$

The residual arbitrariness constitutes the choice of N at infinity, which is usually set as $N(\infty) = 1$, and the angular shift N^ϕ . For this case, we will choose $N^\phi(R(\tau)) = 0$, which represents a null angular velocity on the shell at every time, and simply corresponds to a reparametrization in the angular variable. In this form it is possible to attain suitable matching conditions on the shell, for instance, for a static internal solution.

The rotating solution possesses the same isometries as the static one, the Killing vectors ∂_t and ∂_ϕ . This makes sensible the vector basis choice for both outside and inside spaces in a similar way as in the previous case. Therefore, the projection of the 3-velocity along the basis $\{n, \partial/\partial r, \partial/\partial\phi\}$ can be cast in the form

$$u^\mu = \frac{\alpha}{f} n^\mu + \dot{r} \left(\frac{\partial}{\partial r} \right)^\mu + u^\phi \left(\frac{\partial}{\partial\phi} \right)^\mu, \quad (55)$$

and the normal vector ξ^μ in terms of the same orthogonal set

$$\xi^\mu = \frac{\dot{r}}{f\gamma} n^\mu + \frac{\alpha}{\gamma} \left(\frac{\partial}{\partial r} \right)^\mu, \quad (56)$$

with the angular velocity defined as $u^\phi = d\phi/d\tau$.

The functions α and γ have the explicit expressions

$$\alpha^2 = f^2 + \dot{r}^2 + f^2 r^2 (u^\phi)^2 \quad (57)$$

and

$$\gamma^2 = 1 + r^2 (u^\phi)^2. \quad (58)$$

It is useful to define a new time coordinate

$$d\lambda = \sqrt{1 + r^2 \left(\frac{d\phi}{d\tau} \right)^2} d\tau, \quad (59)$$

which corresponds to the proper time measured by an observer in radial collapse. In this way, the angular velocity can be expressed as

$$\Omega = \frac{d\phi}{d\lambda} = \frac{\dot{\phi}}{\sqrt{1 + r^2 \dot{\phi}^2}}, \quad (60)$$

which, in turn, permits us to write down the time variable and the angular velocity as

$$d\lambda = \gamma d\tau, \quad (61)$$

$$\dot{\phi} = \gamma \Omega, \quad (62)$$

in an analogous way to special relativity, using the (dilation) relativistic factor

$$\gamma = \frac{1}{\sqrt{1 - r^2 \Omega^2}}. \quad (63)$$

Once more, the equation $T_{\mu\nu} = [\sigma u_\mu u_\nu - \tau (h_{\mu\nu} + u_\mu u_\nu)] \delta(X)$ provides the shell stress tensor, with a delta distribution located at the origin of the X axis, along the ξ^μ direction. Computing the relevant components and expressing them in terms of the normal time λ ,

$$T_{\perp\perp} = \frac{\alpha^2}{f^2} \{ \gamma^2 \sigma - \tau (\gamma^2 - 1) \} \delta(X), \quad (64)$$

$$T_{\perp\phi} = -\frac{\alpha}{f} \gamma^2 r^2 \Omega (\sigma - \tau) \delta(X). \quad (65)$$

In this case, the function α has been defined as

$$\alpha^2 = f^2 + \left(\frac{dr}{d\lambda} \right)^2. \quad (66)$$

Performing the required change of variable to integrate out in the radial coordinate r , the Jacobian $dr/dX = f^2/\alpha$ remains unchanged with the new time definition. It is clear that the whole procedure matches the radial collapse case when $\Omega = 0$.

Again, the discontinuity in \mathcal{H}_\perp is caused by only one term in ${}^{(2)}R$, because all other terms represent finite jumps in a null-measure interval. Thus, Eq. (24) undergoes a change, due to the different form of $T_{\perp\perp}$, and becomes

$$-\Delta(f^2) = \pi R (\alpha_+ + \alpha_-) \{ \gamma^2 \sigma - \tau (\gamma^2 - 1) \}. \quad (67)$$

The fact that γ has the same value at each side of the shell is a direct consequence of the junction conditions imposed on the shell position $r = R(\tau)$.

Direct integration of the angular component of Hamiltonian \mathcal{H}_ϕ is possible considering the only nonvanishing component of the gravitational momentum $\pi_\phi^r = p(r)/2\pi$. This contributes a difference in the angular momentum ΔJ , coming from Eq. (65), given by

$$-2\Delta p = \Delta J = 2\pi \gamma^2 R^3 \Omega (\sigma - \tau). \quad (68)$$

The equivalent of Eq. (33) can be obtained from Eq. (67) by repeating the same analysis depicted in Sec. IV,

$$\alpha_+ - \alpha_- = -\pi R \sigma \{ \gamma^2 \sigma - \tau (\gamma^2 - 1) \}. \quad (69)$$

This is a useful version of the equation of motion for the study of the dynamical interval in the radial coordinate.

These equations provide the starting point for the analysis of the collapse of a rotating shell. In the cases shown below, the extremal values for shell energy density and tension are explicitly developed. We will focus on the process of black hole formation on a ‘‘vacuum’’ inner solution (AdS space-time).

A. Domain wall ($\sigma = \tau$)

A rotating shell with a tension equal to the mass density represents a singular case of the equations of motion governing the collapse dynamics. From Eq. (68) we see that the contribution to the angular momentum is vanishing for a collapsing domain wall. This was geometrically expected due to the fact that for $\sigma = \tau$ this object can be obtained from the Nambu-Goto action for a fundamental string. The Poincaré symmetry defines an angular momentum tensor that is identically vanishing for a perfectly circular rotating string.

This result states the impossibility of generating rotating solutions with this “fundamental object.” Furthermore, the condition imposed on Eq. (67) reproduces the same expression (24) as for nonrotating domain wall collapse, for an observer falling radially with the shell.

B. Dust shell

The collapse of a pressureless shell represents a system of particles traveling inward with no mutual interaction. Thus, the path of every infinitesimal piece of matter is given by the geodesics in an external gravitational field, spinning around the radial potential because of the initial angular velocity.

For this case, Eqs. (67) and (68), which set the change in the parameters between AdS and the outer spacetime, take the form

$$M + 1 - \frac{J^2}{4R^2} = \pi\sigma R(\alpha_+ + \alpha_-)\gamma^2 \quad (70)$$

for the energy conservation, and

$$J = 2\pi\gamma^2 R^3 \sigma \Omega \quad (71)$$

for the angular momentum. The description here is from the frame of an observer falling radially with the shell (nonrotating), which measures a time λ . Eq. (25) gives the conservation of the total mass, enlarged in a γ factor with respect to the commoving (rest) frame

$$2\pi\gamma R\sigma = m. \quad (72)$$

Replacing the latter expression in Eq. (71) allows us to obtain the angular velocity

$$\Omega = \frac{\pm J}{R\sqrt{J^2 + m^2 R^2}}, \quad (73)$$

where the plus (minus) sign stands for the shell rotating (counter)clockwise; and the explicit form for the relativistic γ factor

$$\gamma = \frac{1}{\sqrt{1 + (J^2/m^2 R^2)}}. \quad (74)$$

Finally, inserting all these results in Eq. (70), the radial velocity $\dot{R} = dR/d\lambda$ as a function of the solution parameters and the radial coordinate is

$$\dot{R}^2 = \frac{a^2 R^2}{16(m^2 R^2 + J^2)} - (R^2 + 1), \quad (75)$$

with the constant a again defined as $a = m^2 + 4(M + 1)$.

The maximum value of the above function is found to be $R_{\max}^2 = J(a - 4J)/4m^2$. A quick analysis of the effective potential shows that the shell cannot reach the origin $R = 0$, nor infinity, confining the motion between two turning points. In order to ensure that these turning points do not coalesce—when they indeed exist—the maximum value for \dot{R} is

$$\dot{R}_{\max}^2 = \frac{(a - 4J)^2}{16m^2} - 1, \quad (76)$$

must be greater than zero. Therefore, for the motion to exist at all, the parameters must satisfy the condition $m/2 > \sqrt{J - M + 1}$ or $0 < m/2 < \sqrt{J - M} - 1$ for the possible creation of a naked singularity ($J > M$). However, as the shell does not disappear beyond an event horizon, necessarily the bounce is produced for any value of the initial conditions. Thus, the dust ring cannot generate the naked singularity at the origin. The presence of angular momentum provides a “centrifugal barrier” that is not infinite as in the Keplerian case, and whose effect is clear when we put Eq. (75) into the form

$$\dot{R}^2 = V_{\text{eff}}(J=0) - \frac{a^2 J^2}{16m^2(m^2 R^2 + J^2)}, \quad (77)$$

where $V_{\text{eff}}(J=0)$ corresponds to the RHS of Eq. (36).

In view of the above result, we can reinterpret the only case in three dimensions where it was possible to form naked singularities: the radial collapse of a massive shell onto AdS vacuum. Because of the existence of a mass gap between AdS and the $M = 0$ black hole, the outer solution can have a negative mass even for a shell with $\sigma > 0$. However, this case is somehow ill defined because the particles would need to free fall with infinite precision along the radial direction. Any angular perturbation in the initial condition would prevent the shell to reach the origin.

In turn, outer black hole solutions ($J < M$) are created for any value of the shell mass m , since the smallest turning point is always inside the horizon r_+ . The time evolution is completely determined once the initial conditions are set. In particular, for a collapsing shell starting from zero radial velocity at a distance $R = R_0$, we obtain the expression for the mass of the external solution

$$M = \frac{2\sqrt{(R_0^2 + 1)(m^2 R_0^2 + J^2)}}{R_0} - \left(\frac{m^2}{4} + 1\right). \quad (78)$$

For extremal black holes, there is no restriction on the total mass of the collapsing ring, either. The limit case is represented by the situation where both turning points coalesce. Because the shell mass must be $m = 2$, the ring is orbiting steadily at a fixed radius $R^2 = J/2$, the radius corresponding to an extremal black hole horizon. As a consequence, the shell dynamics sees no objection to the forma-

tion of extremal black holes from a collapse process with an appropriate set of initial conditions, in a similar way as in charged black hole creation.

VII. CONCLUSIONS

Apart from the relative ease with which this alternative treatment reproduces and extends the dynamics for collapsing thin shells obtained by the Israel method, this formalism presents a few additional interesting features, especially because of the general statements that can be derived from it.

The geometrical scheme applied in the derivation of the formula (24)—and the corresponding version in the rotating case—permits us to write them generically in terms of the change in the geometry through the shell Δf^2 and not explicitly in terms of any particular solution parameters (M, J, Q, Λ , etc.). Even more subtle, it is precisely this difference that generalizes the method, opening the possibility of dealing with a number of interesting cases: from black hole creation—as presented in this article—to thin shell collapse over an existing black hole, and also the possibility of extension to higher dimensional spacetimes.

In 2+1 dimensions, a direct consequence of Eqs. (24) and (67) is the well-known thermodynamical law stating that the horizon area always grows. This can be derived from the energy conservation law for both nonrotating and rotating cases as follows. Let us assume that a thin shell of physical matter—satisfying $\sigma > \tau > 0$ —is dropped over an already existing black hole configuration. Therefore, the LHS in Eqs. (24) and (67) is strictly positive for any value of R , that is, $f_+^2(R) < f_-^2(R)$. For the interior black hole, there exists an even horizon $R_+^{(in)}$ such that $f_-^2(R_+^{(in)}) = 0$, and the function f_+^2 must be negative for the same position. Hence, this last function should vanish at a larger distance than the inner horizon $R_+^{(in)}$. The conditions imposed mean that any mechanical perturbation would not move faster than the speed of light around the shell and it is equivalent to the usual *dominant* energy condition in cosmology (see, e.g., Ref. [19]). The former argument is also valid for radial collapse in higher dimensions.

Another general consequence, regarding naked singularity formation from the collapse of a thin shell over a black hole interior solution, can be made from the analysis of Eq. (24).

Let the set of parameters be such that the collapse will turn the inner black hole solution into a naked singularity, as seen by a distant external observer. For example, we can imagine a near-extremal electrically charged black hole and a

dust shell carrying more charge q than the proper mass m . For this system, we have an inner event horizon $R_+^{(in)}$, such that $f_-^2(R_+^{(in)}) = 0$, whereas the exterior function $f_+^2(R)$ is positive throughout the space. Roughly speaking, if the shell does not gather enough speed during the collapse, it will not become massive enough to prevent the formation of a naked singularity. Furthermore, from the shell dynamics we know that a matter sphere released with certain speed is equivalent to one dropped from rest at another distance. Then, in principle, it might always be possible to find a set of initial conditions to destroy the black hole configuration.

However, Eq. (24) expresses that by the time the shell has reached the black hole horizon $R_+^{(in)}$ the conservation of energy has already been violated. In addition, the shell must have bounced before, because for $\dot{R} = 0$ at the horizon

$$f_+(R_+^{(in)}) = -\Omega_{d-2} R \sigma G, \quad (79)$$

in open contradiction with the fact we have an external naked singularity. A similar argument can be developed for the rotating case in three dimensions, stating the impossibility of turning black holes into naked singularities by throwing thin shells of physical matter over them.

The previous reasoning cannot be repeated *verbatim* in the case of naked singularity formation over an empty space. Nevertheless, as we discussed in the corresponding sections, the absence of a horizon and the explicit form of the metric for the cases with angular momentum and electric charge, prevents the shell from reaching the origin.

Finally, the Hamiltonian formalism for the collapse of thin shells developed in this paper can be applied to create magnetic black holes in three spacetime dimensions [20]. It can also be extended to deal with the problem of gravitational collapse in gravity theories with higher powers in the curvature [21].

ACKNOWLEDGMENTS

We wish to thank Professor C. Teitelboim for having proposed the problem to us and for numerous enlightening discussions. Institutional support to Centro de Estudios Científicos (CECS) from Empresas CMPC is acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation. This work was partially funded by Ministerio de Educación through MECESUP grant FSM 9901 and the grant 3030029 from FONDECYT.

-
- [1] W. Israel, *Nuovo Cimento B* **44**, 1 (1966); **48**, 463(E) (1967).
 [2] K.V. Kuchař, *Czech. J. Phys., Sect. B* **B18**, 435 (1968).
 [3] J. Ipser and P. Sikivie, *Phys. Rev. D* **30**, 712 (1984).
 [4] T. Regge and C. Teitelboim, *Ann. Phys. (N.Y.)* **88**, 286 (1974).
 [5] Y. Peleg and A. Steif, *Phys. Rev. D* **51**, 3992 (1995).
 [6] S.C. Davis, *Phys. Rev. D* **67**, 024030 (2003).
 [7] P. Hájíček and J. Bičák, *Phys. Rev. D* **56**, 4706 (1997).
 [8] P. Hájíček and J. Kijowski, *Phys. Rev. D* **57**, 914 (1998).

- [9] P. Hájíček, *Phys. Rev. D* **57**, 936 (1998).
 [10] V.A. Berezin, N.G. Kozimirov, V.A. Kuzmin, and I.I. Tkachev, *Phys. Lett. B* **212**, 415 (1988).
 [11] A. Ansoldi, A. Aurilia, R. Balbinot, and E. Spalluci, *Class. Quantum Grav.* **14**, 2727 (1997).
 [12] P. Kraus and F. Wilczek, *Nucl. Phys.* **B433**, 403 (1995).
 [13] R. Arnowitt, S. Deser, and C.W. Misner, *Phys. Rev.* **117**, 1595 (1960); in *Gravitation: An Introduction to Current Research*,

- edited by L. Witten (Wiley, New York, 1962).
- [14] The choice of the factor in front of the action differs from the usual $-(16\pi\tilde{G})^{-1}$ for all spacetime dimensions, where \tilde{G} is the Newton constant in four dimensions. The use of this convention simplifies the form of the black hole metric in higher dimensions. The gravity constants are related by the formula $G=[8\pi/(d-2)\Omega_{d-2}]\tilde{G}$, where $\Omega_p=2\pi^{(p+1)/2}/\Gamma((p+1)/2)$ stands for the volume of S^p .
- [15] M. Henneaux and C. Teitelboim, *Commun. Math. Phys.* **98**, 391 (1985).
- [16] D.G. Boulware, *Phys. Rev. D* **8**, 2363 (1973).
- [17] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [18] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993).
- [19] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [20] R. Olea, "Charged Rotating Black Hole Formation from Thin Shell Collapse in Three Dimensions," hep-th/0401109.
- [21] J. Crisóstomo, S. del Campo, and J. Saavedra, "Hamiltonian Treatment of Collapsing Thin Shells in Lanczos-Lovelock's Theories," hep-th/0311259.