Microcanonical entropy of a black hole

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(Received 10 December 2003; published 19 May 2004)

It has been suggested recently that the microcanonical entropy of a system may be accurately reproduced by including a logarithmic correction to the canonical entropy. In this paper we test this claim both analytically and numerically by considering three simple thermodynamic models whose energy spectrum may be defined in terms of one quantum number only, as in a non-rotating black hole. The first two pertain to collections of noninteracting bosons, with logarithmic and power-law spectra. The last is an area ensemble for a black hole with equi-spaced area spectrum. In this case, the many-body degeneracy factor can be obtained analytically in a closed form. We also show that in this model, the leading term in the entropy is proportional to the horizon area A, and the next term is $\ln A$ with a negative coefficient.

DOI: 10.1103/PhysRevD.69.104018 PACS number(s): 04.70.Dy, 04.60.−m, 05.30.−d

I. INTRODUCTION

The entropy of a macroscopic black hole is known to be proportional to the area of its horizon [1], in units of the Planck length squared. It has also been shown by several authors using a variety of approaches that the leading order correction to this is proportional to the logarithm of the area [2] (also see the references in [3]). Recently, a universal form for the (negative) coefficient of this logarithmic term has been obtained in Ref. [4] by assuming a power-law dependence of the area on the mass of the (non-rotating) black hole. For an isolated black hole, it is of course appropriate to consider the microcanonical entropy. For a quantum system, the microcanonical entropy may be defined uniquely in terms of the degeneracy of the state at a given energy, and it has no fluctuation in energy. The many-body degeneracy factor, for any nontrivial system, however, is exceedingly difficult to calculate. For this reason, it is desirable to approximate the microcanonical entropy by the canonical entropy (the leading term), minus a logarithmic term due to fluctuations in the canonical ensemble-averaged energy from the equilibrium value. The main objective of this paper is to test this formula quantitatively in three solvable models, where the microcanonical entropy can also be calculated exactly. The first two of these are systems of noninteracting bosons, and *not* related to black holes. These many-body systems, however, have eigenenergies that depend on a single quantum number, similar to a non-rotating quantum black hole. When the logarithmic correction to the canonical entropy are included in the canonical expression, its agreement with the microcanonical entropy improves markedly. Next, we consider a canonical area ensemble with equi-spaced spectrum and distinguishable area components as a model for a non-rotating black hole. The area of the surface of the event horizon plays a role analogous to the energy [5]. This equi-spaced spectrum was first proposed in the early seventies and later confirmed by several authors using different techniques [6]. The model has been studied earlier by Alekseev *et al.* [7] using the grand canonical ensemble. In this paper, we calculate the exact microcanonical entropy for this spectrum in a closed form, and obtain the area law for the entropy. We also show that the next term in the entropy is proportional to $\ln A$ with a negative coefficient. This is also verified using the canonical ensemble when the area fluctuation is subtracted out.

The origin of the black hole entropy, in theories of quantum gravity, is believed to arise from the microstates that are generated by a quantum mechanical operator such as area. In standard statistical mechanics, when many particles in the mean-field model are trapped in a potential well, the microcanonical entropy of the many-body system at a (quantized) energy E_n is obtained by taking the logarithm of the number of distinct microstates that all give the same energy E_n. To be more explicit, $E_n=\Sigma_{\{N_i\}_n}N_i\epsilon_i$, where N_i is the number of particles with single-particle energy ϵ_i. The set $\{N_i\}_n$ denotes a given occupancy configuration of single-particle levels that make up a microstate with total energy E_n. There may be $\Omega(E_n)$ such distinct microstates for an energy E_n, each denoted by a set $\{N_i\}$, and the microcanonical entropy is then uniquely defined as

$$\mathcal{S}(E_n)=k_B\ \ln[\Omega(E_n)], \tag{1}$$

where the Boltzmann constant k_B will henceforth be set to unity. Similarly, in models of quantum gravity, the (macroscopic) area eigenvalue A_n is taken to be coming from the elementary components a_i, such that $A_n=\Sigma_{\{N_i\}_n}N_i a_i$, where N_i is the number of elementary components with area a_i [7]. Each microstate is specified by a distinct set $\{N_i\}$, and there may be $\Omega(A)$ such microstates for a given area A. The microcanonical entropy is then $\mathcal{S}(A_n)=\ln[\Omega(A_n)]$.

II. CORRECTION TO THE CANONICAL ENTROPY

Consider a many-body quantum system with eigenenergies E_n that are completely specified by a single quantum number n,

$$E_n=f(n), \quad n=0,1,2,3, \ldots, \tag{2}$$

where we assume $f(n)$ to be an arbitrary monotonic function with a differentiable inverse, $f^{-1}(x)=F(x)$, such that $n=F(E_n)$. The degeneracy of the states at energy E_n is given by $\Omega(E_n)$, a function characterizing the quantum spectrum. At this point, we need not assume that this many-body system is described in the mean-field picture. The quantum density of the system is defined as

$$\rho(E)=\sum_n \Omega(E_n)\delta(E-E_n). \tag{3}$$

Using general properties of the delta function, we write

$$\delta(E-E_n)=\delta(E-f(n))=\delta(n-F(E))|F'(E)|, \tag{4}$$

where the prime denotes differentiation with respect to the continuous variable E. Using the Poisson sum formula, we thus obtain [8]

$$\rho(E)=\Omega(E)|F'(E)|\left(1+2\sum_{k=1}^{\infty}\cos[2\pi kF(E)]\right). \tag{5}$$

We assume the function $\Omega(E)$ of the continuous variable E to be smooth. The first term on the right-hand side (RHS) of the above relation is then the smoothly varying part of the density of states, while the second part consists of the oscillating components coming from the discreteness of the energy levels. For a macroscopic system with large E, the oscillating part may be neglected. We then obtain the important relation

$$\tilde{\rho}(E)=\Omega(E)|F'(E)|, \tag{6}$$

where $\tilde{\rho}(E)$ denotes the averaged smooth density of states. Now we specialize to a system of N noninteracting particles (or in a mean field) constituting the many-body system. Then the degeneracy $\Omega(E_n)$ is just the number of distinct microstates that all have the same energy E_n, as described earlier. The microcanonical entropy is given by Eq. (1), which, using Eq. (6), may now be expressed as

$$\mathcal{S}(E)\simeq\ln[\tilde{\rho}(E)|F'(E)|^{-1}], \tag{7}$$

where the oscillating part has been dropped.

Our next task is to calculate $\tilde{\rho}(E)$ for a many-body system. This may be obtained by considering the canonical partition function of the N-particle system, and taking its inverse Laplace transform using the saddle-point method [3]. The canonical N-particle partition function is given by

$$Z(\beta)=\sum_n \Omega(E_n)\exp(-\beta E_n)=\int_0^{\infty}\rho(E)\exp(-\beta E)dE, \tag{8}$$

where $\rho(E)$ was defined earlier. Note that we are using the canonical ensemble only as a tool to obtain $\tilde{\rho}(E)$ by Laplace inversion with respect to β, which is just a variable of integration along the imaginary axis. The saddle-point approximation simply gives the smooth part $\tilde{\rho}(E)$. The well-known result [9] is

$$\tilde{\rho}(E)=\frac{\exp[S_C(\beta_0)]}{\sqrt{2\pi S_C''(\beta_0)}}, \tag{9}$$

where

$$S_C(\beta_0)=\beta_0 E+\ln Z(\beta_0)$$

is the canonical entropy evaluated at the stationary point β_0, and the prime denotes differentiation with respect to β. The energy E is related to the saddle point β_0 via the condition that $S_C'(\beta_0)=0$. Using Eqs. (7) and (9), we obtain

$$\mathcal{S}(E)\simeq S_C(\beta_0)-\frac{1}{2}\ln[2\pi S_C''(\beta_0)]-\ln(|F'(E)|). \tag{10}$$

The formula that was originally suggested in [3] missed the last term on the RHS. This was also pointed out earlier in Ref. [4]. It turns out that for two of the models that we consider in this paper, $F'(E)=1$, and the last term in Eq. (10) does not contribute. On the other hand, in the model with a logarithmic energy spectrum, this term plays a crucial role. Note that within the canonical formalism, $S_C''(\beta_0)=(\langle E^2\rangle-\langle E\rangle^2)$ is the fluctuation squared of the energy [9]. When this energy fluctuation is subtracted out from the canonical entropy, as in Eq. (10), we obtain an estimate for the microcanonical entropy (for quantum gravitational fluctuations, see e.g. [10]).

The approximation (10) for the microcanonical entropy $S(E)$ is very useful, since it is prohibitively difficult to calculate it directly from Eq. (1). Generally, in a mean-field model, one is given the single-particle quantum spectrum. The direct computation of the many-body degeneracy factor $\Omega(E)$ from this starting point is very time consuming. Instead, it is much simpler to obtain the canonical N-body partition function by well-known recursion relations (depending on quantum statistics) [11], and then compute the canonical entropy $S_c(\beta_0)$. Going one step further, one may calculate the canonical energy fluctuation, and use Eq. (10) to obtain $\mathcal{S}$. By following this canonical route, no computation of $\Omega(E)$ is necessary. The approximate formula (10), relevant to black hole physics, has not been yet explicitly tested. The main objective of this paper is to test this formula quantitatively in some model systems where the microcanonical entropy $\mathcal{S}(E)$ can be calculated exactly.

A. The logarithmic energy spectrum

In the first idealized example, we consider N noninteracting bosons ($N\to\infty$) occupying a set of single-particle energy levels (and also the ground state, which is at zero energy) $\epsilon_p=\ln p$, where p runs over all the prime numbers 2,3,5, ... As recently pointed out in Ref. [12], the many-body microcanonical entropy $\mathcal{S}$ of this system is *exactly* zero. This follows from the fundamental theorem of arithmetic, known from the time of Euclid. It states that every positive integer n can be expressed in only one way as a product of the prime number powers:

$$n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r},$$

where the p_r's are distinct primes, and n_r's are positive integers, including zero, and need not be distinct. It immediately follows that the eigenenergies of the many-body system are given by $E_n = \ln n = \Sigma_r n_r \ln p_r$, and that each eigenstate is *non-degenerate*. This means that for every macro-state of the many-body system, there is exactly one microstate, and $\Omega(E_n) = 1$. This implies, by Eq. (1), that the microcanonical entropy $\mathcal{S}(E_n) = 0$. We would now like to check if this can be verified from Eq. (10). For the above $\ln n$ many-body spectrum, note that the inverse function $F(E) = \exp(E)$, and we immediately obtain the density of states using Eq. (5)

$$\rho(E) = e^E \left(1 + 2 \sum_{k=1}^{\infty} \cos(2\pi k e^E) \right). \tag{11}$$

The second term on the RHS is the intrinsic quantum fluctuation, due to E_n's taking only discrete values. The smooth part of the density of states is $\tilde{\rho}(E) = e^E$. To obtain the canonical entropy $S_c(\beta_0)$, we need to calculate the canonical partition function $Z(\beta)$. The exact $Z(\beta)$ for $N \to \infty$ in this case is the Riemann zeta function $\zeta(\beta) = \Sigma_{n=1}^{\infty} n^{-\beta}$, which includes the quantum fluctuations. We pick up the smooth part of $Z(\beta)$ by evaluating it using Eq. (8) with $\tilde{\rho}(E) = e^E$, obtaining $Z(\beta) = (\beta - 1)^{-1}$, for $\beta > 1$. From this, we get $S_c(\beta) = -\ln(\beta - 1) + \beta E$, so the saddle point is given by $\beta_0 = (1/E + 1)$. Thus the equilibrium canonical entropy is $S_c(\beta_0) = E + \ln E + 1$, that contains both a linear and a logarithmic term. Evaluation of the fluctuation term is elementary, and the microcanonical entropy using Eq. (10) is

$$\mathcal{S}(E) = E + \ln E + 1 - \frac{1}{2}\ln(2\pi E^2) - E = 1 - \frac{1}{2}\ln(2\pi). \tag{12}$$

We see that the E-dependent terms in the canonical entropy are entirely canceled by the fluctuation term; the small residual constant is due to the use of the saddle-point method. This example is atypical, because the canonical term contains both the linear and the logarithmic terms, and still Eq. (10) yields (almost) the correct microcanonical estimate.

B. The power-law single-particle spectrum

For our second example, we consider N noninteracting bosons confined in a mean field with a single particle spectrum given by $\epsilon_m = m^s$, where the integer $m \geqslant 0$, and $s > 0$. The energy is measured in dimensionless units. This model is considered here because the canonical partition function (for $N \to \infty$) is exactly known [13]:

$$Z(\beta) = \prod_{m=1}^{\infty} [1 - \exp(-\beta m^s)]^{-1}. \tag{13}$$

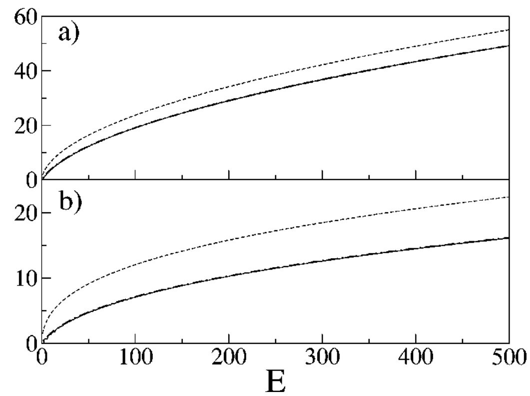

FIG. 1. (a) Comparison of the exact microcanonical entropy $\mathcal{S}(E)$ (solid line) and the canonical entropy $S_C(E)$ (dashed line) for the $\epsilon = m^s$ spectrum, where $s = 1$. The particles are taken to be N non-interacting bosons, where $N \to \infty$. The dot-dashed curve, given by Eq. (10), overlaps with the exact solid curve. (b) Same as (a), except $s = 2$.

This expression is well known in number theory. It is the generating function for $\Omega(E)$, which pertains to the number of ways that an integer E can be partitioned into a sum of powers. This is illustrated in the Appendix by taking a quadratic single-particle spectrum ($s = 2$), and showing that the exact combinatorial result for $\Omega(E)$ of the many-body system can be reproduced by expanding the canonical partition function above. It is important to note that even though the single-particle energies may not be equi-spaced (for $s > 1$), the many-body $E_n = 1,2,3, \ldots$ is. This has the consequence that $F'(E) = 1$, and $\Omega(E) = \tilde{\rho}(E)$ when the quantum oscillations are dropped. The numerical calculations for a large number of particles, using different power-law single-particle spectra, were done in a different context in Ref. [14], where the details may be found. We test the accuracy of Eq. (10) by comparing it with the exact $\mathcal{S}(E)$ from Eq. (1) for $s = 1,2$. The quantum oscillations have been included in the exact microcanonical calculations, but are difficult to see in this scale. In Figs. 1(a) and 1(b), the dashed curve denotes the canonical entropy $S_C(E)$ without the correction, and the continuous curve the exact microcanonical entropy $\mathcal{S}(E)$ for the two power laws. We see from these curves that the two differ substantially as a function of the excitation energy E, specially for $s = 2$. Inclusion of the logarithmic correction to the canonical entropy using Eq. (10) results, however, in almost perfect agreement, as shown by the dot-dashed curves in these figures.

III. MODEL FOR AN AREA ENSEMBLE OF A BLACK HOLE

The above bosonic model with the power law spectrum $\epsilon_n = n^s$ is not directly applicable to the black hole problem since the leading term in the entropy varies as $E^{1/(1+s)}$ [13,14]. Following [7], we consider instead *distinguishable* elementary components, and consider the situation where the elementary area components are equi-spaced, $a_j = j$ with j taking values $0,1,2, \ldots$, etc., with a degeneracy of $g(j) = (j+1)$. This model has been considered by the authors in

Ref. [7], but the results were derived in the grand-canonical ensemble. For a macroscopic black hole with a horizon area A, we assume that the number of independent components is

$$N=\eta\frac{A}{l_p^2}, \tag{14}$$

where η is a positive constant, and l_p the Planck length, which is set to unity. Both N and A are fixed quantities in the microcanonical picture. As will be shown shortly, the advantage of this model is that the expression for the multiplicity $\Omega(A,N)$ may be explicitly found, and therefore the exact microcanonical entropy may be calculated directly. By expanding the microcanonical entropy for large A, we find that the leading term is proportional to the area A of the horizon, and the next term varies as $\ln A$. This expression is exactly reproduced in the canonical ensemble calculation, when the ensemble averaged $\langle A\rangle$ is identified with A, and the fluctuation in $\langle A\rangle$ is subtracted out.

The one-body partition function is

$$Z_1=\sum_{j=0}^{\infty}(j+1)\exp(-\alpha j)=(1-e^{-\alpha})^{-2}=(1-x)^{-2}, \tag{15}$$

where α is a variable canonical to the area, and $x=\exp(-\alpha)$. The canonical N-particle partition function for distinguishable elementary components is

$$\begin{aligned} Z_N&=(Z_1)^N=(1-x)^{-2N},\\ &=1+2Nx+\frac{2N(2N+1)}{2}x^2\\ &\quad+\frac{2N(2N+1)(2N+2)}{3!}x^3+\dots\\ &=\sum_{A=1}^{\infty}\frac{\prod_{i=0}^{A-1}(2N+i)}{A!}x^A,\\ &=\sum_A\Omega(A,N)e^{-\alpha A}, \end{aligned} \tag{16}$$

which is analogous to Eq. (8). The multiplicity of states of area is therefore

$$\Omega(A,N)=\frac{\prod_{i=0}^{A-1}(2N+i)}{A!}. \tag{17}$$

This is the microcanonical partition function. It is not difficult to check by combinatorics that $\Omega(A,N)$ for a given A and N is indeed given by Eq. (17). Note that unlike the case of bose statistics where the multiplicity $\Omega(E)$ was found only by expanding the partition function [for example, see Eq. (A1)] or by exact counting, here due to the distinguishability property of the system it is given by an explicit formula. Thus an analytical expression for the microcanonical $\mathcal{S}(A)$ may be found directly from Eq. (17): $\mathcal{S}(A)=\ln\Omega(A,N)$. Using Stirling's series and the Euler-Maclaurin summation formula, we get

$$\begin{aligned}\mathcal{S}(A)\simeq A\ln\left(1+\frac{2N}{A}\right)+2N\ln\left(1+\frac{A}{2N}\right)\\ -\frac{1}{2}\ln\left[2\pi\left(A+\frac{A^2}{2N}\right)\right].\end{aligned} \tag{18}$$

We now calculate the canonical entropy and show that inclusion of the logarithmic correction term [Eq. (10)] gives a formula that agrees with Eq. (18) for the microcanonical entropy. The canonical calculations are performed for a fixed N, and the ensemble averaged area is given by $\langle A\rangle=-\partial\ln Z_N/\partial\alpha$ at the equilibrium α_0. For large A, we identify $\langle A\rangle=A$ and later correct for the fluctuation. The canonical entropy is $S_C(\alpha)=\alpha A+\ln Z_N=\alpha A-2N\ln(1-e^{-\alpha})$. The saddle point is obtained from the condition that $S_C'(\alpha_0)=0$, and gives $\alpha_0=\ln(2N/A+1)$. For this α_0, $S_C(\alpha_0)$ and $S_C''(\alpha_0)$ can easily be evaluated. Inserting these into Eq. (10) immediately yields Eq. (18), which was obtained from the asymptotic expansion of the microcanonical entropy. Finally, we use the relation (14) to eliminate N from the above equation. We obtain

$$\mathcal{S}(A)\simeq\xi A-\frac{1}{2}\ln A-\frac{1}{2}\ln[2\pi(1+1/2\eta)], \tag{19}$$

where $\xi=\ln[(1+2\eta)(1+1/2\eta)^{2\eta}]$. Note that the leading term, proportional to A, comes from the canonical entropy S_c, and the correction $\ln A$ arises from the fluctuation in $\langle A\rangle$. If we had included the zero-point energy in the a_j spectrum by taking $a_j=(j+1)$, as in [7], we again obtain the same form of Eq. (19), but the expression for ξ, as well as the coefficient of $\ln A$ are different (the latter is still negative).

We have shown that for a class of statistical mechanical systems, the log-corrected entropy formula (10) accurately reproduces the microcanonical entropy of these systems. Our results are applicable to a large class of black holes, with uniformly spaced area spectrum [6]. Although the area spectrum in loop quantum gravity is not strictly uniform [15], it is effectively equi-spaced for large areas. Moreover, as shown in [7] and [16], an exact equi-spaced spectrum may emerge in loop gravity as well. It would be interesting to extend our analysis to charged and rotating black holes, described by more than one quantum number.

ACKNOWLEDGMENTS

We are grateful to M.V.N. Murthy for some of the results reported in this paper and to P. Majumdar for his comments and criticism of this manuscript. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by funding from The University of Lethbridge.

APPENDIX

We give an example of the calculation of the exact entropy for a system of N non-interacting bosons with a single-particle spectrum given by $\epsilon_m = m^2$, $m = 0,1,2,\ldots$. Initially, at $T=0$ (or $E=0$), the particles all reside in the ground state, where $m=0$. Denote by N_{ex} the number of particles in the excited states. An excitation energy E may be shared by N_{ex} out of N particles such that $E=\Sigma_i N_i \epsilon_i$, $N_{ex}=\Sigma_i N_i$. The multiplicity $\Omega(E,N)$ is the number of ways of doing this. For example, take $E=8$ and $N=8$; then there are three distinct configurations. First, $N_{ex}=2$ particles can be excited in which case each takes $2^2=4$ quanta and goes to the second level above the ground state. In this case, $N_2=2$ and $N_i=0$, $i\neq 2$. Second, $N_{ex}=5$ particles can be excited, four of which each takes one quantum to the first level ($N_1=4$) above the ground state and the other takes $2^2=4$ quanta to the second level ($N_2=1$). Finally, all 8 particles can be excited; each takes one excitation quantum to the first level ($N_1=8$, and $N_i=0$, $i\neq 1$). The energy in all three cases is the same:

$$E=8=N_1\epsilon_1+N_2\epsilon_2+\ldots=0+2\times 2^2+\ldots+0,$$

$$\text{or } =4\times 1^2+1\times 2^2+\ldots+0,$$

$$\text{or } =8\times 1^2+\ldots+0.$$

Hence, $\Omega(8,8)=3$. We see that this problem is identical to counting the number of ways that an integer E can be partitioned into a sum of squares. Note that had we taken $5\leqslant N<8$ in this example, then there would be only 2 configurations instead of 3, since the last case in which each particle takes one quantum is eliminated. In number theory this is known as restricted partitioning as opposed to the unrestricted case considered above. Clearly, as long as $N\geqslant E$, the number of accessible microstates may be enumerated as if $N=\infty$. In Table I we enumerate the multiplicity for several values of E, assuming unrestricted partitioning.

A few remarks are in order. First, as mentioned before, as long as $E\leqslant N$, the enumeration of the multiplicity is N-independent. Second, as illustrated above, E is given by a set of consecutive integers, $E_n=n$, even though the single-particle energy spectrum is not equi-spaced. Each many-body energy level E_n has a degeneracy $\Omega(E_n,N)$. This is in fact general for any power-law single-particle spectrum and non-interacting particles. Note that the multiplicity $\Omega(E,N)$ enumerated in the table is the same as the expansion coefficient of the partition function, i.e.

$$Z(x)=\prod_{m=1}^{\infty}\frac{1}{[1-x^{m^2}]}$$

$$=1+x+x^2+x^3+2x^4+2x^5+2x^6+2x^7+3x^8+4x^9+\ldots, \tag{A1}$$

where $x=e^{-\beta}$, and the power of x corresponds to the many-body energy E_n.

TABLE I. Calculation of the multiplicity $\Omega(E,N)$ for N bosons at an excitation energy E. The single-particle energy spectrum is given by $\epsilon_m=0,1,4,9,\ldots,m^2$. For a given integer E, the partitions of E are tabulated in column 1, and the corresponding number N_{ex} in column 2. The microstate $\omega(E,N_{ex},N)$, enumerated in column 3, is defined as the number of ways of exciting *exactly* N_{ex} particles. The last column gives the multiplicity $\Omega(E,N)=\Sigma_{N_{ex}=1}^{N}\omega(E,N_{ex},N)$. It is to be noted that for a large excitation energy E (not considered in the table), $\omega(E,N_{ex},N)$ may take on values larger than unity.

E	N_{ex}	$\omega(E,N_{ex},N)$	$\Omega(E,N)$
$1=1^2$	1	1	1
$2=1^2+1^2$	2	1	1
$3=1^2+1^2+1^2$	3	1	1
$4=2^2$	1	1	
$=1^2+1^2+1^2+1^2$	4	1	2
$5=1^2+2^2$	2	1	
$=1^2+1^2+1^2+1^2+1^2$	5	1	2
$6=1^2+1^2+2^2$	3	1	
$=1^2+\ldots+1^2$	6	1	2
$7=1^2+1^2+2^2$	3	1	
$=1^2+\ldots+1^2$	7	1	2
$8=2^2+2^2$	2	1	
$=1^2+1^2+1^2+1^2+2^2$	5	1	
$=1^2+\ldots+1^2$	8	1	3

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