Spatial infinity in higher dimensional spacetimes

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Motivated by recent studies on the uniqueness or nonuniqueness of higher dimensional black hole spacetime, we investigate the asymptotic structure of spatial infinity in n-dimensional spacetimes ($n \geq 4$). It turns out that the geometry of spatial infinity does not have maximal symmetry due to the nontrivial Weyl tensor $^{(n-1)}C_{abcd}$ in general. We also address static spacetime and its multipole moments $P_{a_1 a_2 \cdots a_s}$. Contrasting with four dimensions, we stress that the local structure of spacetimes cannot be unique under fixed multipole moments in static vacuum spacetimes. For example, we consider the generalized Schwarzschild spacetimes which are deformed black hole spacetimes with the same multipole moments as spherical Schwarzschild black holes. To specify the local structure of the static vacuum solution we need some additional information, at least the Weyl tensor $^{(n-2)}C_{abcd}$ at spatial infinity.

I. INTRODUCTION

The fundamental study of higher dimensional black holes is gaining importance due to TeV gravity [1,2] and superstring theory. In four dimensions, the no-hair theorem [3] and uniqueness theorems [4] are the main results obtained during the golden age of study of black hole physics. Here we have a question about black holes in higher dimensions. What about the uniqueness theorem? Recently a static black hole has been proven to be unique in higher dimensional and asymptotically flat spacetimes [5–7]. However, we cannot show the uniqueness of stationary black holes. This is because there is a counterexample, that is, there are higher dimensional Kerr solutions [8] and black ring solutions [9] which have the same mass and angular momentum parameters. See also Ref. [10] for a related issue of supersymmetric black holes. Even if we concentrate on static spacetimes, the asymptotic boundary conditions are not unique [5]. Indeed, we could have a generalized Schwarzschild solution which is not asymptotically flat. There are also important issues about the final fate of the unstable black string or stable configuration of Kaluza-Klein black holes [11]. They are still under investigation.

In this paper, we focus on the fundamental issue of the asymptotic structure of spatial infinity, which is closely related to the asymptotic boundary condition in the uniqueness theorem and numerical study. See Ref. [12] for null infinity in higher dimensions, but with a different motivation. First we investigate the geometrical structure of spatial infinity in higher dimensions. Spatial infinity is essentially an $(n-1)$-dimensional manifold in general n-dimensional spacetimes. In four dimensions, it should be restricted to being a three-dimensional unit timelike hyperboloid with maximal symmetry [13]. In higher dimensions, as shown later, there are many varieties due to the nontrivial $(n-1)$-dimensional Weyl tensor. Next, we discuss the higher

multipole moments in static spacetimes. For four-dimensional spacetimes, the local structure of static and vacuum spacetime is uniquely determined by specifying all the multipole moments [14]. On the other hand, as we see later, higher dimensional static spacetimes cannot be fixed by multipole moments alone. We need some additional information to fix the spacetimes. One of them is the $(n-2)$-dimensional Weyl tensor on the surface normal to the radial direction.

The rest of this paper is organized as follows. In Sec. II, we define the spatial infinity following Ashtekar and Romano [13] and then discuss the leading structure of spatial infinity. In Sec. III, we concentrate on static spacetimes and again define spatial infinity on spacelike hypersurfaces. Then we define and discuss the multipole moments following Geroch [15]. Finally, we give a discussion and summary in Sec. IV.

II. STRUCTURE OF SPATIAL INFINITY

A. Definition

We begin with the definition of spatial infinity by Ashtekar and Romano [13]. If one is interested only in spatial infinity, their definition is useful.

Definition. Physical spacetime $(\hat{M}, \hat{g}_{ab})$ has a spatial infinity i_0 if there is a smooth function Ω satisfying the following features (i) and (ii) and the energy-momentum tensor satisfies the fall off condition (iii).

(i) $\Omega \hat{=} 0$ and $d\Omega| \hat{\neq} 0$.

(ii) The following quantities have smooth limits on i_0:

$$q_{ab} = \Omega^2(\hat{g}_{ab} - \Omega^{-4}F^{-1}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega) = \Omega^2\hat{q}_{ab}, \quad (1)$$

$$n^a := \Omega^{-4}\hat{g}^{ab}\hat{\nabla}_b\Omega, \quad (2)$$

where

$$F = \Omega^{-4} \hat{g}^{ab} \hat{\nabla}_a \Omega \hat{\nabla}_b \Omega = \pounds_n \Omega, \qquad (3)$$

and $\hat{=}$ denotes evaluation on i_0. q_{ab} has the signature $(-, +, +, \cdots, +)$.

(iii)　$\hat{T}_{\mu\nu} := \hat{T}_{ab} e^a_\mu e^b_\nu = O(\Omega^{2+m})$　near　i_0,　where $\{\hat{e}^a_\mu\}_{\mu=0,1,2,\cdots,n-1}$ is a quasiorthogonal basis of the metric $\hat{g}_{ab}$ and $m>0$. The definition is exactly the same as that in four dimensions.

We write the physical metric in terms of the quasiorthogonal basis

$$\hat{g}^{ab} = \hat{n}^a \hat{n}^b + \hat{e}^a_I \hat{e}^{bI}, \qquad (4)$$

where

$$\hat{n}^a = -\frac{n^a}{\sqrt{g(n,n)}} = -\Omega^2 F^{-1/2} n^a \qquad (5)$$

and

$$\hat{e}^a_I = e^a_I \Omega. \qquad (6)$$

e^a_I represents the parts of the quasiorthogonal basis of $\hat{q}_{ab}$.

B. Leading order structure

From the above the asymptotic behavior near i_0 is determined by the regular quantities q_{ab} and n^a. For example, the extrinsic curvature $\hat{K}_{ab}$ of $\Omega = \text{const}$ surfaces is written as

$$\hat{K}_{ab} = \frac{1}{2} \pounds_{\hat{n}} \hat{q}_{ab} = \Omega^{-1} F^{1/2} q_{ab} - \frac{1}{2} F^{-1/2} \pounds_n q_{ab}. \qquad (7)$$

Since it is not regular at $\Omega = 0$, we defined the regular tensor K_{ab} as

$$K_{ab} =: \Omega \hat{K}_{ab} = F^{1/2} q_{ab} - \frac{1}{2} \Omega F^{-1/2} \pounds_n q_{ab}. \qquad (8)$$

Then we see that

$$K_{ab} \hat{=} F^{1/2} q_{ab}. \qquad (9)$$

In the physical spacetime, the Codacci equation is

$$\hat{e}^a_I \hat{n}^b \hat{T}_{ab} = [\hat{D}_b \hat{K}^b_a - \hat{D}_a \hat{K}] \hat{e}^a_I. \qquad (10)$$

It is also expressed as

$$\Omega^{-2} \hat{e}^a_I \hat{n}^b \hat{T}_{ab} = D_b K^b_a - D_a K \qquad (11)$$

in terms of (q_{ab}, n^a). At i_0 it becomes

$$0 \hat{=} D_b K^b_a - D_a K. \qquad (12)$$

Substituting Eq. (9) into Eq. (12), we see that

$$D_a F \hat{=} 0 \qquad (13)$$

and then

$$F \hat{=} \text{const}. \qquad (14)$$

Since we can set $F \hat{=} 1$ without loss of generality,

$$K_{ab} \hat{=} q_{ab}. \qquad (15)$$

Here, we used the gauge freedom of the conformal factor $\Omega \to \omega \Omega$, that is, since under this transformation F transforms as

$$F \to F' \hat{=} \omega^{-2} F, \qquad (16)$$

we may choose ω to satisfy $\omega = F^{1/2}$. From the Gauss equation

$$\Omega^{-2} \hat{e}^a_I \hat{e}^b_J {}^{(n)} R_{ab} = [\, {}^{(n-1)} R_{ab} - K K_{ab} - F^{1/2} K_{ab} + 2 K_{ac} K^c_b \\ - F^{1/2} D_a D_b F^{-1/2} + \Omega F^{-1/2} \pounds_n K_{ab}] e^a_I e^b_J, \qquad (17)$$

we have

$$^{(n-1)} R_{ab} \hat{=} (n-2) q_{ab} \qquad (18)$$

and then

$$^{(n-1)} R_{abcd} \hat{=} {}^{(n-1)} C_{abcd} + 2 q_{a[c} q_{d]b}. \qquad (19)$$

This is simple but the main consequence in our paper. In four dimensions, due to the absence of the three-dimensional Weyl tensor $^{(3)} C_{abcd} = 0$,

$$^{(3)} R_{abcd} \hat{=} 2 q_{a[c} q_{d]b}. \qquad (20)$$

This implies that i_0 is a three-dimensional unit hyperboloid. In the case of $n \geq 5$, the situation is drastically changed because $^{(n-1)} C_{abcd} \neq 0$ in general. Indeed, we have an n-dimensional solution with nonzero Weyl tensor as shown in the next section. Such spacetimes are not included in the category of asymptotically flat spacetimes.

III. STATIC SPACETIMES

In this section, we focus on static spacetimes in higher dimensions. To investigate the asymptotic structure, it is better to adopt a definition separately.

In the static spacetime, the metric can be written as

$$ds^2 = -V^2 dt^2 + q_{ij} dx^i dx^j \qquad (21)$$

where $i,j = 1,2,\ldots,n-1$. The Einstein equation becomes

$$^{(n)} \hat{R}_{\hat{0}\hat{0}} = \frac{1}{V} D^2 V = \hat{T}_{\hat{0}\hat{0}} + \frac{1}{n-2} \hat{T} \qquad (22)$$

and

$$^{(n)} \hat{R}_{ij} = {}^{(n-1)} R_{ij} - \frac{1}{V} \hat{D}_i \hat{D}_j V = \hat{T}_{ij} - \frac{1}{n-2} g_{ij} \hat{T}. \qquad (23)$$

A. Structure of spatial infinity in static slices

Definition. A physical static slice $(\hat{\Sigma}, \hat{q}_{ab})$ has a spatial infinity $\tilde{i}_0$ if there is a smooth function Ω satisfying the following features (i),(ii) and an appropriate falloff condition for the energy-momentum tensor.

(i) $\Omega \hat{=} 0$ and $d\Omega| \hat{\neq} 0$.

(ii) The following quantities have smooth limits on $\tilde{i}_0$:

$$h_{ab} = \Omega^2(\hat{q}_{ab} - \Omega^{-4}F^{-1}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega) = \Omega^2\hat{h}_{ab}, \quad (24)$$

$$n^a := \Omega^{-4}\hat{g}^{ab}\hat{\nabla}_b\Omega, \quad (25)$$

where

$$F = \Omega^{-4}\hat{g}^{ab}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega = \pounds_n\Omega. \quad (26)$$

h_{ab} has the signature $(+,+,\cdots,+)$.

The extrinsic curvature defined by

$$\hat{k}_{ab} = \frac{1}{2}\pounds_{\hat{n}}\hat{h}_{ab} \quad (27)$$

is singular at $\Omega = 0$. In the same way as in the previous section, we define $k_{ab} = \Omega\hat{k}_{ab}$ and then we see that $k_{ab} \hat{=} h_{ab}$ from the Codacci equation. From the Gauss equation, $^{(n-2)}R_{ab} \hat{=} (n-3)h_{ab}$. Thus

$$^{(n-2)}R_{abcd} \hat{=} {}^{(n-2)}C_{abcd} + 2h_{a[c}h_{d]b}. \quad (28)$$

In five- or four-dimensional spacetimes,

$$^{(3,2)}R_{abcd} \hat{=} 2h_{a[c}h_{d]b}. \quad (29)$$

It represents a three- or two-sphere.

B. Multipole moments

In this subsection we define the multipole moments in a covariant way. To do so it is better to change the formalism and use the conformal completion defined by Geroch [15].

Definition. A physical static slice $(\hat{\Sigma}, \hat{q}_{ab})$ has a spatial infinity $\tilde{i}_0$ if there is a smooth function Ω such that

$$\Omega \hat{=} 0, \quad \tilde{\nabla}_a\Omega \hat{=} 0, \quad \tilde{\nabla}_a\tilde{\nabla}_b\Omega \hat{\neq} 0 \quad (30)$$

and

$$\tilde{q}_{ab} = \Omega^2\hat{q}_{ab} \quad (31)$$

has a smooth limit on $\tilde{i}_0$.

As an example, consider Euclid space. The metric is

$$d\ell^2 = dr^2 + r^2 d\Omega_{n-2}. \quad (32)$$

Ω is taken to be $\Omega = r^{-2}$. Then

$$\Omega^2 d\ell^2 = r^{-4}dr^2 + r^{-2}d\Omega_{n-2} = dR^2 + R^2 d\Omega_{n-2}, \quad (33)$$

where $R = r^{-1}$. Then $\tilde{i}_0$ is just the center in an unphysical slice. Moreover, $\tilde{\nabla}_a\Omega = 2R\tilde{\nabla}_aR \hat{=} 0$ and $\tilde{\nabla}_a\tilde{\nabla}_b\Omega \hat{=} 2\tilde{\nabla}_aR\tilde{\nabla}_bR \hat{\neq} 0$.

Following Geroch's argument, we might be able to identify the values of the following tensor at spatial infinity as multipole moments:

$$P = \frac{1}{2}(1-V)\Omega^{-(n-3)/2}$$

$$P_{a_1 a_2 \cdots a_{s+1}} = \mathcal{O}\left[\tilde{\nabla}_{a_1}P_{a_2 a_3 \cdots a_{s+1}} - \frac{s(2s+n-5)}{2(n-3)}{}^{(n-1)}\tilde{R}_{a_1 a_2}\right.$$

$$\left. \times P_{a_3 a_4 \cdots a_{s+1}}\right], \quad (34)$$

where $\mathcal{O}[T_{a_1 a_2 \cdots a_r}]$ denotes the totally symmetric, trace-free parts of $T_{a_1 a_2 \cdots a_r}$. This is recursive and a coordinate-free definition. The definition relies on the argument of the conformal rescaling $(\Omega' = \Omega\omega)$ [15,16]. (The multipole moments in a Newtonian system depend on the choice of the origin of the coordinates. This behavior of the multipole moments is reflected by the transformation of the multipole moments under a change of the conformal factor. The second term in the above definition reflects this in curved spacetimes.) Since the rescaling corresponds to a translational transformation, we wish the following transformation for $P_{a_1 a_2 \cdots a_{s+1}}$:

$$P'_{a_1 a_2 \cdots a_{s+1}} = P_{a_1 a_2 \cdots a_{s+1}} - \frac{(2s+n-3)(s+1)}{2}$$

$$\times \mathcal{O}[P_{a_1 \cdots a_s}\tilde{\nabla}_{a_{s+1}}\omega]. \quad (35)$$

We can check that it indeed holds for the definition of Eq. (34). Note that the definition does not contain the Weyl tensor $^{(n-1)}\tilde{C}_{abcd}$.

In four-dimensional asymptotically flat spacetimes, we can show that they become identical with the coordinate dependent multipole moments defined by Thorne [17,18]. The most important feature is that stationary and vacuum spacetimes having the same multipole moments are isometric with each other in four dimensions. That is, the local structure of the stationary and vacuum spacetimes is completely determined by the multipole moments [14]. In Newtonian gravity, this fact is trivial. However, in general relativity, it is not so. As demonstrated by an example below, the situation will be drastically changed in higher dimensions.

There are generalized Schwarzschild spacetimes and the metric is[1]

$$ds^2 = -f(r)^2dt^2 + h(r)^{4/(n-3)}[dr^2 + r^2\sigma_{AB}dx^Adx^B], \quad (36)$$

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[1]The form is similar to that in the isotropic coordinates of the Schwarzschild solution. If we use the coordinate $\rho = rh(r)^{2/(n-3)}$, the metric becomes the familiar form: $ds^2 = -F(\rho)dt^2 + F(\rho)^{-1}d\rho^2 + \rho^2\sigma_{AB}dx^Adx^B$, where $F(\rho) = 1 - 4(\mu/\rho)^{n-3}$.

where $A, B = 2, 3, \ldots, n-1$. $f(r)$ and $h(r)$ are given by

$$f(r) = \frac{1 - (\mu/r)^{n-3}}{1 + (\mu/r)^{n-3}} \qquad (37)$$

and

$$h(r) = 1 + \left(\frac{\mu}{r}\right)^{n-3}. \qquad (38)$$

σ_{AB} is the metric of the Einstein space, that is, it obeys

$$^{(n-2)}R_{AB}(\sigma) = (n-3)\sigma_{AB}, \qquad (39)$$

where $^{(n-2)}R_{AB}(\sigma)$ is the Ricci tensor of σ_{AB}. The metric σ_{AB} found by Bohm is given by [19]

$$\sigma_{AB}dx^A dx^B = d\theta^2 + a^2(\theta)d\Omega_p + b^2(\theta)d\Omega_{n-3-p}, \qquad (40)$$

where $5 \leq n-3 \leq 9$ with $p \geq 2$ and $q := n-3-p \geq 2$. See Refs. [20,21] for the stability of such spacetimes.

Taking $\Omega = [(1-V)/2]^{2/(n-3)}$, the unphysical metric $\tilde{q}$ becomes

$$\tilde{q} = \Omega^2 \hat{q} = \left(\frac{\mu}{r}\right)^4 [dr^2 + r^2 \sigma_{AB}dx^A dx^B]. \qquad (41)$$

Defining

$$R := \mu^2 r^{-1}, \qquad (42)$$

then

$$\tilde{q} = dR^2 + R^2 \sigma_{AB}dx^A dx^B. \qquad (43)$$

For this metric, the Ricci tensor becomes

$$^{(n-1)}\tilde{R}_{ij} = 0. \qquad (44)$$

Finally, we can see at spatial infinity

$$P = \frac{1}{2}(1-V)\Omega^{-(n-3)/2} = 1,$$

$$P_{a_1 a_2 \cdots a_s} = 0 \quad \text{for} \quad s \geq 1. \qquad (45)$$

Thus this spacetime has the same multipole moments as spherical Schwarzschild spacetimes. We cannot distinguish them from one another using only multipole moments. This problem comes from the absence of the Weyl tensor in the definition Eq. (34). Because of the total antisymmetricity of the Weyl tensor, there is no room for the Weyl tensor in the definition. We need the information related to the Weyl tensor independently. Hence, we might be able to expect that we can uniquely specify higher dimensional spacetimes by the multipole moments and Weyl tensor. The Bohm metric has the following nontrivial Weyl tensor $^{(n-2)}C_{abcd}$ [20]:

$$^{(n-2)}C_{\theta\hat{A}_1\theta\hat{A}_2} = c_1(\theta)\delta_{\hat{A}_1\hat{A}_2},$$

$$^{(n-2)}C_{\theta\hat{B}_1\theta\hat{B}_2} = c_2(\theta)\delta_{\hat{B}_1\hat{B}_2},$$

$$^{(n-2)}C_{\hat{A}_1\hat{B}_1\hat{A}_1\hat{B}_2} = c_3(\theta)\delta_{\hat{A}_1\hat{A}_2}\delta_{\hat{B}_1\hat{B}_2}, \qquad (46)$$

$$^{(n-2)}C_{\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4} = 2c_4(\theta)\delta_{\hat{A}_1[\hat{A}_3}\delta_{\hat{A}_4]\hat{A}_2},$$

$$^{(n-2)}C_{\hat{B}_1\hat{B}_2\hat{B}_3\hat{B}_4} = 2c_5(\theta)\delta_{\hat{B}_1[\hat{B}_3}\delta_{\hat{B}_4]\hat{B}_2},$$

where

$$c_1(\theta) = -1 - \frac{a''}{a}, \quad c_2(\theta) = -1 - \frac{b''}{b},$$

$$c_3(\theta) = -1 - \frac{a'}{a}\frac{b'}{b}, \qquad (47)$$

$$c_4(\theta) = \frac{1 - a'^2 - a^2}{a^2}, \quad c_5(\theta) = \frac{1 - b'^2 - b^2}{b^2}.$$

In the above we used the orthogonal basis $\{\hat{e}^{A_1}, \hat{e}^{A_2}, \ldots, \hat{e}^{A_p}\}$ and $\{\hat{e}^{B_1}, \hat{e}^{B_2}, \ldots, \hat{e}^{B_{n-3-p}}\}$, that is,

$$a^2(\theta)d\Omega_p = \delta_{A_1 A_2}\hat{e}^{A_1} \otimes \hat{e}^{A_2},$$

$$b^2(\theta)d\Omega_{n-3-p} = \delta_{B_1 B_2}\hat{e}^{B_1} \otimes \hat{e}^{B_2},$$

where $A_1, A_2, \ldots = 3, 4, \ldots, p+2$ and $B_1, B_2, \ldots = p+3, p+4, \ldots, n-1$.

IV. SUMMARY AND DISCUSSION

In this paper, we investigated the asymptotic structure at spatial infinity in higher dimensional spacetimes. One will realize that this is quite important when one tries to perform numerical computations or prove the uniqueness theorem. This is because one must impose asymptotic boundary conditions on them. In higher dimensions, it turned out that there are many varieties. That is, it is unlikely that the asymptotic symmetry is raised automatically due to the nontrivial Weyl tensor [see Eqs. (19) and (28)] at spatial infinity. If and only if we set the Weyl tensor to zero, the asymptotic flatness seems to be guaranteed. Since the definition of the multipole moments cannot include the Weyl tensor part, the static solutions are degenerate in terms of multipole moments. We must at least use the Weyl tensor if one wants to split these solutions. This is contrasted with the four-dimensional spacetimes where the local structure of static and vacuum solutions can be uniquely figured out from the higher moments. The point is just the dimension. From our study we must specify the multipole moments $P_{a_1 a_2 \cdots a_r}$ and Weyl tensor $^{(n-2)}C_{abcd}$ for each individual solution. This is a lesson for the boundary condition in a numerical study and gives us an insight into the argument of the uniqueness theorem. Therein we should carefully think of the Weyl tensor or something similar.

Now, we might be able to have the following conjecture: If the two static vacuum spacetimes defined here have the same multipoles and Weyl tensor $^{(n-2)}C_{abcd}$ at spatial infinity, they are isometric in a local sense.

There are many remaining issues; first of all, the details of the structure of spatial infinity. It is unlikely that there is asymptotic symmetry because of the lack of maximal symmetry. Even if this is so, it is important to ask why asymptotic symmetry cannot exist. The next issue is the proof of the statement that the static spacetimes can be uniquely specified by higher multipole moments and the Weyl tensor. We can also extend our argument on static spacetimes to stationary cases. Since the Weyl tensor appears in our conjecture, the relation with the peeling theorem associated with null infinity is also interesting (see Ref. [22] for the peeling theorem in four dimensions). Finally, in four dimensions, since the Weyl curvature on the static slices vanishes, it, of course, never contributes to the multipole moments. However, in more than four dimensions, the Weyl curvature on spacelike hypersurfaces in general does not vanish. Since the multipole moments imply deviation from spherical symmetry, they seem to contain the Weyl tensor. We may be able to extend Geroch's definition of multipole moments (34) to a refined form to contain the Weyl curvature in higher dimensional spacetime.

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