Second-order perturbations of the Friedmann world model

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We consider the instability of the Friedmann world model to second order in perturbations. We present the perturbed set of equations up to second order in the Friedmann background world model with a general spatial curvature and cosmological constant. We consider systems with completely general imperfect fluids, minimally coupled scalar fields, an electromagnetic field, and generalized gravity theories. We also present the case of null geodesic equations, and one based on the relativistic Boltzmann equation. In due time, a decomposition is made for scalar-, vector-, and tensor-type perturbations which couple with each other to second order. A gauge issue is resolved to each order. The basic equations are presented without imposing any gauge condition, and thus in a gauge-ready form so that we can take full advantage of having gauge freedom in analyzing the problems. As an application we show that to second order in perturbation the relativistic pressureless ideal fluid of the scalar type reproduces exactly the known Newtonian result. As another application we rederive the large-scale conserved quantities (of the pure scalar and tensor perturbations) to second order, first shown by Salopek and Bond, now from the exact equations. Several other applications are shown as well.

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I. INTRODUCTION

We consider instabilities of the spatially homogeneous and isotropic cosmological spacetime to second order in perturbations. The relativistic cosmological perturbation plays a fundamental role in the modern theory of large-scale cosmic structure formation. The original analysis of linear perturbations based on Einstein gravity with a hydrodynamic fluid was made by Lifshitz in 1946 [1] in an almost complete form. Because of the extremely low level anisotropies of the cosmic microwave background (CMB) radiation, the cosmological dynamics of the structures on the large scale and in the early Universe are generally believed to be small deviations from the homogeneous and isotropic background Friedmann world model [2]. The conventional relativistic cosmological perturbation analysis considers such deviations small enough so that one can treat them as *linear*. The linear perturbation theory works as the basic framework in handling the cosmological structure formation processes. Recent observations of the CMB anisotropies in the full sky by the Wilkinson Microwave Anisotropy Probe (WMAP) satellite and others [3], for example, assure the validity of the basic assumptions used in cosmological perturbation theory, i.e., the linearity of the relevant cosmic structures [4].

Still, as the observed relatively small-scale structures are apparently nonlinear, the gravitational instability based on the pure linear theory is not enough for a complete picture. It is agreed that such small-scale nonlinear structures could be handled by Newtonian gravity often based on numerical simulations. The current paradigm of large-scale structure generation and evolution processes is based on an underlying *assumption* that linear processes dominate until nonlinear processes take over on subhorizon scales in the Newtonian regime. Thus, it seems this paradigm of our understanding of the origin and evolution of the large-scale structure is rather satisfactory within linear theory concerning the regimes where relativistic gravity theory is needed. It is well known that in linear theory there can be no structure formation. In fact, this ‘‘no structure formation’’ in the scenario is precisely why we were successful in describing the structure generation and evolution processes in a simple manner as we describe below. In the standard scenario, the initial conditions (seed fluctuations) generated from quantum fluctuations are imprinted into ripples in the spacetime, and its spatial structures are preserved as the raw large-scale structures (i.e., are as yet unaffected by nonlinear processes). This is a trait which can be traced to the linearity assumption we adopt. It gives a simple but fictitious system. We could, perhaps, fairly describe the current situation as that the linear paradigm is *not inconsistent* with observations, especially with the low level of observed anisotropy in the CMB [3]. However, we should remember that the actual equations we are dealing with, in both gravity and the quantum regime, are highly nonlinear. It forms an intrinsically complex system.

We can decompose the perturbations into three different types: the scalar-type (associated with density condensation), the vector-type (rotation), and the tensor-type (gravitational wave) perturbations. To linear-order perturbation, due to the high symmetry of the background space, these three types of perturbation decouple from each other and evolve independently. Both (the linearity and the homogeneous-isotropic background) conditions are necessary to have natural descriptions of the three types of perturbations independently. We will see how couplings occur to the second-order perturbations; for the couplings in the simplest spatially homogeneous but anisotropic spacetime, see [5].

Another aspect of the simple nature of the linear processes in relativistic gravity theories is characterized by conservations (in expanding phase) of certain amplitudes on the superhorizon scale where we naively anticipate the independent evolution of causally disconnected regions. On superhorizon scales this conserved character is presented by an equation of the form

$$\Phi(\mathbf{x},t)=C(\mathbf{x}), \quad (1)$$

which applies for both the scalar- and tensor-type perturbations; for the vector-type perturbation the angular momentum is conserved on all scales. $C(\mathbf{x})$ is an integration constant of the integral form general solutions available in the large-scale limit; see Eqs. (311),(332),(333). The coefficient $C(\mathbf{x})$ contains information about the spatial structure which will eventually grow into the large-scale structure and the gravitational wave background. It can be considered as an initial condition for each perturbation variable which is preserved during the linear evolution. Whether a similar conserved variable can exist even in a nonlinear analysis is apparently an interesting question: for the presence of such variables to second order in perturbation see Secs. VII D–VII F. In analyses of the large-scale structures in the linear stage, the simple behavior of the conserved variables is practically important. In fact, if we know C, the behavior of all the other variables can be determined through linear algebra. Using the conserved quantity one can trivially relate the currently observable (or deducible) linear structure directly to the initial state of the structure in the early universe; probably just after the scale effectively becomes the large scale during the hypothetical early acceleration (inflation) stage. Of course, the underlying assumption for all of these results is the applicability of a linear analysis. As long as this assumption is valid, the initial condition is imprinted onto the large-scale structure and is preserved until the nonlinear effects become important. Although this is a big advantage, in a sense this is very consistent with the fact that no structure formation occurs in the linearized system.

The linear perturbation theory is currently well developed; see [6–11]. Although the observations do not particularly demand going beyond the linear theory, second-order perturbation theory is a natural next step in the theoretical investigations. The second-order perturbation theory, if well developed, will have important implications for our understanding of the large-scale structure formation processes. Not only are the structures we discover nonlinear, according to the gravitational instability there should occur (perhaps smooth) transitions from the linear to the nonlinear ones. Even from the theoretical point of view, in order to know the limit of linear perturbation theory we need the behavior of perturbations beyond the linear theory. It is not possible to know the limit of linear theory within the context of linear theory. It is yet unclear whether the second-order "perturbation theory" will provide an answer to such a question, but we expect it could provide a better perspective on the problem than simple linear theory. There will be more practical applications as well, like investigating the non-Gaussian signature in the inflation generated seed fluctuations which could have left a detectable signature in the CMB anisotropies and the large-scale structures. Other possible situations where translinear analyses might be useful are summarized in Sec. VIII.

Now we briefly discuss the gauge issue present in relativistic perturbation theory. Since the unperturbed background spacetime is spatially homogeneous and isotropic, to linear order the ambiguity caused by the spatial gauge (coordinate transformation) freedom does not play a role [7]. Thus, to linear order it is appropriate to write the perturbed set of equations in terms of natural combinations of variables which are invariant under spatial gauge transformations. However, the temporal gauge freedom can be conveniently used in analyzing various aspects of the perturbation problem in the Friedmann background. There are infinitely many different ways of taking the temporal gauge (choosing the spatial hypersurface) conditions, and we can identify several fundamental gauge choices [7,12]. Except for a widely used temporal synchronous gauge fixing condition ($\delta g_{00}\equiv 0$), each of the other gauge conditions completely fixes the temporal gauge freedom, thus each having its own corresponding gauge-invariant formulation. In our study of the linear theory we found that some particular gauge-invariant variables show the correct Newtonian behaviors. A perturbed density variable in the comoving gauge and a perturbed potential variable and a perturbed velocity variable in the zero-shear gauge most closely resemble the behaviors of the corresponding Newtonian variables [13]. Also, the scalar field perturbation in the uniform-curvature gauge most closely resembles the scalar field equation in the quantum field in curved space [14,15]. Since the gauge conditions mentioned allow no room for a remaining gauge mode, the variable in a given such gauge has a unique corresponding gauge-invariant combination of variables. Thus, the variable in such a gauge is equivalent to the corresponding gauge-invariant combination.

In the gauge theory it is well known that proper choice of the gauge condition is often necessary for proper handling of the problem. Either by fixing certain gauge conditions or by choosing certain gauge-invariant combinations in the early calculation stage, we are likely to lose possible advantages available in the other gauge conditions. In order to use the various gauge conditions as an advantage in handling cosmological perturbations, we have proposed a *gauge-ready* method which allows the flexible use of the various fundamental gauge conditions. The strategy is that, in order to use the various available temporal gauge conditions as an advantage, we had better present the basic equations without choosing any temporal gauge condition, and arrange the equations so that we could choose the fundamental gauge conditions conveniently. In this work we will further elaborate the gauge-ready approach to the second-order perturbations. Our gauge-ready strategy, together with our notation for indicating the gauge-invariant combinations, allows us to use the gauge freedom as an advantage in analyzing various given problems. We follow the wisdom suggested by Bardeen in 1988 "the moral is that one should work in the gauge that is mathematically most convenient for the problem at hand" [7].

As long as we are taking a perturbative approach the gauge issue in higher orders can be resolved similarly as in the linear theory. To second order we will identify two variables which can be used to fix the spatial gauge freedom.

One gauge condition completely removes the spatial gauge mode, whereas the other condition does not; i.e., in the latter case even after imposing the gauge condition there still remains a degree of freedom which is a gauge mode (a coordinate artifact). We call this incomplete gauge condition the B gauge ($\tilde{g}_{0\alpha} \equiv 0$ where α is a spatial index) whereas the other complete condition is called the C gauge. To second order we can identify the same several temporal gauge-fixing conditions. Again, except for the synchronous gauge each of the other gauge conditions completely removes the temporal gauge modes.

It is amusing to note that the classic study by Lifshitz [1] adopted the synchronous gauge condition, which is a combination of the temporal synchronous gauge condition and the spatial B gauge condition, thus failing to fix both the temporal and spatial gauge modes completely. This has caused some prevalent errors in the literature based on the synchronous gauge: see the Appendix of [13]. However, we note that these errors are simple algebraic ones probably caused by slightly more complicated algebra due to the presence of the gauge mode after the synchronous gauge fixing. We would like to emphasize that the gauge condition should be appropriately used according to the character of each problem at hand. We have this freedom because Einstein's gravity theory might be regarded as a gauge theory [16]. In this sense, although the temporal synchronous and the spatial B gauge conditions do not remove the gauge modes completely, often even these conditions could possibly turn out to be convenient in certain problems. Since physically measurable quantities should be gauge invariant we propose to use the gauge conditions in this pragmatic sense.

In a classic study of CMB anisotropy in 1967 Sachs and Wolfe mentioned that "the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible" [17]. In this work, we will present the basic formulation of the second-order perturbation of the Friedmann world model in detail. We will present the basic equations needed to investigate the second-order perturbation in a rather general context. We will consider the most general Friedmann background with K and Λ. We will consider the most general imperfect fluid situation. This includes multiple imperfect fluids with general interactions among them. We will also include minimally coupled scalar fields, a class of generalized gravity theories, the electromagnetic fields, the null geodesic, and the relativistic Boltzmann equation. In order to use the gauge-fixing conditions optimally we will present the complete sets of perturbed equations in the gauge-ready form. In this manner, as in the linear theory, we can easily apply the equations to any gauge conditions which make the mathematical analyses of given problems simplest. Our formulation will be suitable to handle nonlinear evolutions in the perturbative manner. Ours will be a useful complement to the other methods suggested in the literature to investigate the translinear regimes. In the Discussion we summarize the related studies, the different methods we could employ for further applications, and the cosmological situations where our formulation could be applied fruitfully. Although we will present some trivial applications in the later part, the main applications are left for future studies.

In Sec. II we summarize the basic equations of Einstein's gravity expressed in the Arnowitt-Deser-Misner (ADM) (3+1) formulation and in the covariant (1+3) formulation. In Sec. III we introduce our definition of the metric and the energy-momentum tensor to second order in the perturbation and present some useful quantities appearing in the ADM and the covariant formulations. In Sec. IV we present the complete sets of perturbed equations up to second order in the Friedmann world model. We consider the general spatial curvature and the cosmological constant in the background. We consider systems with completely general imperfect fluids. Such a general formulation can be reinterpreted to include the cases of minimally coupled scalar fields, the electromagnetic field, and even generalized gravity theories. We present the complete sets of equations for these systems as well. We also present the case of null geodesic equations, and the one based on the relativistic Boltzmann equation. In Sec. V we introduce decomposition of the perturbations to three different types and show how these couple to each other to second order. All equations up to this point are presented without introducing any gauge condition. Thus, the equations are presented in the most general forms, and any suitable gauge conditions can easily be deployed in these equations. In this sense, our set of equations is in a gauge-ready form. In Sec. VI we address the gauge issue, and show that the gauge issue can be resolved to each perturbation order, just as in the case of linear perturbation. We implement our gauge-ready strategy to second order in perturbations. In Sec. VII we make several applications. In Sec. VIII we summarize the main results and outline future applications of our work.

As a unit we set $c \equiv 1$.

II. BASIC EQUATIONS

A. ADM (3+1) equations

The ADM (Arnowitt-Deser-Misner) equations [18] are based on splitting the spacetime into spatial and the temporal parts using a normal vector field $\tilde{n}_a$. The metric is written as (we put a tilde on the covariant variables)

$$\tilde{g}_{00} \equiv -N^2 + N^\alpha N_\alpha, \quad \tilde{g}_{0\alpha} \equiv N_\alpha, \quad \tilde{g}_{\alpha\beta} \equiv h_{\alpha\beta},$$

$$\tilde{g}^{00} = -N^{-2}, \quad \tilde{g}^{0\alpha} = N^{-2} N^\alpha,$$

$$\tilde{g}^{\alpha\beta} = h^{\alpha\beta} - N^{-2} N^\alpha N^\beta, \tag{2}$$

where N_α is based on $h_{\alpha\beta}$ as the metric and $h^{\alpha\beta}$ is an inverse metric of $h_{\alpha\beta}$. Indices $a,b, \dots$ indicate the spacetime indices, and $\alpha,\beta, \dots$ indicate the spatial ones. The normal vector $\tilde{n}_a$ is introduced as

$$\tilde{n}_0 \equiv -N, \quad \tilde{n}_\alpha \equiv 0, \quad \tilde{n}^0 = N^{-1}, \quad \tilde{n}^\alpha = -N^{-1} N^\alpha. \tag{3}$$

The fluid quantities are defined as

$$E \equiv \tilde{n}_a \tilde{n}_b \tilde{T}^{ab}, \quad J_\alpha \equiv -\tilde{n}_b \tilde{T}^b_\alpha, \quad S_{\alpha\beta} \equiv \tilde{T}_{\alpha\beta},$$

$$S \equiv h^{\alpha\beta} S_{\alpha\beta}, \quad \overline{S}_{\alpha\beta} \equiv S_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} S, \tag{4}$$

where J_α and $S_{\alpha\beta}$ are based on $h_{\alpha\beta}$. The extrinsic curvature is introduced as

$$K_{\alpha\beta} \equiv \frac{1}{2N}(N_{\alpha:\beta} + N_{\beta:\alpha} - h_{\alpha\beta,0}), \quad K \equiv h^{\alpha\beta} K_{\alpha\beta},$$

$$\overline{K}_{\alpha\beta} \equiv K_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} K, \tag{5}$$

where $K_{\alpha\beta}$ is based on $h_{\alpha\beta}$. A colon denotes a covariant derivative based on $h_{\alpha\beta}$. The connections become

$$\tilde{\Gamma}^0_{00} = \frac{1}{N}(N_{,0} + N_{,\alpha} N^\alpha - K_{\alpha\beta} N^\alpha N^\beta),$$

$$\tilde{\Gamma}^0_{0\alpha} = \frac{1}{N}(N_{,\alpha} - K_{\alpha\beta} N^\beta), \quad \tilde{\Gamma}^0_{\alpha\beta} = -\frac{1}{N} K_{\alpha\beta},$$

$$\tilde{\Gamma}^\alpha_{00} = \frac{1}{N} N^\alpha (-N_{,0} - N_{,\beta} N^\beta + K_{\beta\gamma} N^\beta N^\gamma) + N N^{,\alpha} + N^\alpha_{\ ,0} - 2NK^{\alpha\beta} N_\beta + N^{\alpha:\beta} N_\beta,$$

$$\tilde{\Gamma}^\alpha_{0\beta} = -\frac{1}{N} N_{,\beta} N^\alpha - NK^\alpha_\beta + N^\alpha_{\ :\beta} + \frac{1}{N} N^\alpha N^\gamma K_{\beta\gamma},$$

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^{(h)\alpha}_{\ \ \ \ \beta\gamma} + \frac{1}{N} N^\alpha K_{\beta\gamma}, \tag{6}$$

where $\Gamma^{(h)\alpha}_{\beta\gamma}$ is the connection based on $h_{\alpha\beta}$ as the metric, $\Gamma^{(h)\alpha}_{\beta\gamma} \equiv \frac{1}{2} h^{\alpha\delta}(h_{\beta\delta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta})$. The intrinsic curvatures are based on $h_{\alpha\beta}$ as the metric:

$$R^{(h)\alpha}_{\ \ \ \ \beta\gamma\delta} \equiv \Gamma^{(h)\alpha}_{\ \ \ \ \beta\delta,\gamma} - \Gamma^{(h)\alpha}_{\ \ \ \ \beta\gamma,\delta} + \Gamma^{(h)\epsilon}_{\ \ \ \ \beta\delta} \Gamma^{(h)\alpha}_{\ \ \ \ \gamma\epsilon} - \Gamma^{(h)\epsilon}_{\ \ \ \ \beta\gamma} \Gamma^{(h)\alpha}_{\ \ \ \ \delta\epsilon},$$

$$R^{(h)}_{\alpha\beta} \equiv R^{(h)\gamma}_{\ \ \ \ \alpha\gamma\beta}, \quad R^{(h)} \equiv h^{\alpha\beta} R^{(h)}_{\alpha\beta},$$

$$\overline{R}^{(h)}_{\alpha\beta} \equiv R^{(h)}_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} R^{(h)}. \tag{7}$$

A complete set of ADM equations is the following [6].

Energy constraint equation

$$R^{(h)} = \overline{K}^{\alpha\beta} \overline{K}_{\alpha\beta} - \frac{2}{3} K^2 + 16\pi G E + 2\Lambda, \tag{8}$$

where Λ is the cosmological constant.

Momentum constraint equation

$$\overline{K}^\beta_{\alpha:\beta} - \frac{2}{3} K_{,\alpha} = 8\pi G J_\alpha. \tag{9}$$

Trace of ADM propagation equation

$$K_{,0} N^{-1} - K_{,\alpha} N^\alpha N^{-1} + N^{:\alpha}_{\ \ \ \alpha} N^{-1} - \overline{K}^{\alpha\beta} \overline{K}_{\alpha\beta} - \frac{1}{3} K^2 - 4\pi G(E+S) + \Lambda = 0. \tag{10}$$

Trace-free ADM propagation equation

$$\overline{K}^\alpha_{\beta,0} N^{-1} - \overline{K}^\alpha_{\beta:\gamma} N^\gamma N^{-1} + \overline{K}_{\beta\gamma} N^{\alpha:\gamma} N^{-1} - \overline{K}^\alpha_\gamma N^\gamma_{\ :\beta} N^{-1} = K\overline{K}^\alpha_\beta - \left(N^{:\alpha}_{\ \ \ \beta} - \frac{1}{3} \delta^\alpha_\beta N^{:\gamma}_{\ \ \ \gamma} \right) N^{-1} + \overline{R}^{(h)\alpha}_{\ \ \ \ \beta} - 8\pi G \overline{S}^\alpha_\beta. \tag{11}$$

Energy conservation equation

$$E_{,0} N^{-1} - E_{,\alpha} N^\alpha N^{-1} - K\left(E + \frac{1}{3} S \right) - \overline{S}^{\alpha\beta} \overline{K}_{\alpha\beta} + N^{-2} (N^2 J^\alpha)_{:\alpha} = 0. \tag{12}$$

Momentum conservation equation

$$J_{\alpha,0} N^{-1} - J_{\alpha:\beta} N^\beta N^{-1} - J_\beta N^\beta_{\ :\alpha} N^{-1} - KJ_\alpha + EN_{,\alpha} N^{-1} + S^\beta_{\alpha:\beta} + S^\beta_\alpha N_{,\beta} N^{-1} = 0. \tag{13}$$

B. Covariant (1+3) equations

The covariant formulation of Einstein gravity was investigated in [19,20]. The (1+3) covariant decomposition is based on the timelike normalized ($\tilde{u}^a \tilde{u}_a \equiv -1$) four-vector field $\tilde{u}_a$ introduced at all spacetime points. The expansion ($\tilde{\theta}$), the acceleration ($\tilde{a}_a$), the rotation ($\tilde{\omega}_{ab}$), and the shear ($\tilde{\sigma}_{ab}$) are kinematic quantities of the projected covariant derivative of the flow vector $\tilde{u}_a$ introduced as

$$\tilde{h}^c_a \tilde{h}^d_b \tilde{u}_{c;d} = \tilde{h}^c_{[a} \tilde{h}^d_{b]} \tilde{u}_{c;d} + \tilde{h}^c_{(a} \tilde{h}^d_{b)} \tilde{u}_{c;d} \equiv \tilde{\omega}_{ab} + \tilde{\theta}_{ab} = \tilde{u}_{a;b} + \tilde{a}_a \tilde{u}_b,$$

$$\tilde{\sigma}_{ab} \equiv \tilde{\theta}_{ab} - \frac{1}{3} \tilde{\theta} \tilde{h}_{ab}, \quad \tilde{\theta} \equiv \tilde{u}^a_{\ ;a}, \quad \tilde{a}_a \equiv \dot{\tilde{u}}_a \equiv \tilde{u}_{a;b} \tilde{u}^b, \tag{14}$$

where $\tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b$ is the projection tensor with $\tilde{h}_{ab} \tilde{u}^b = 0$ and $\tilde{h}^a_a = 3$. An overdot with tilde indicates a covariant derivative along $\tilde{u}^a$. We have

$$\tilde{u}_{a;b} = \tilde{\omega}_{ab} + \tilde{\sigma}_{ab} + \frac{1}{3} \tilde{\theta} \tilde{h}_{ab} - \tilde{a}_a \tilde{u}_b. \tag{15}$$

We introduce

$$\tilde{\omega}^a \equiv \frac{1}{2} \tilde{\eta}^{abcd} \tilde{u}_b \tilde{\omega}_{cd}, \quad \tilde{\omega}_{ab} = \tilde{\eta}_{abcd} \tilde{\omega}^c \tilde{u}^d,$$

$$\tilde{\omega}^2 \equiv \frac{1}{2} \tilde{\omega}^{ab} \tilde{\omega}_{ab} = \tilde{\omega}^a \tilde{\omega}_a, \quad \tilde{\sigma}^2 \equiv \frac{1}{2} \tilde{\sigma}^{ab} \tilde{\sigma}_{ab}, \tag{16}$$

where $\tilde{\omega}^a$ is a *vorticity vector* which has the same information as the vorticity tensor $\tilde{\omega}_{ab}$. We have $\tilde{\eta}^{abcd}=\tilde{\eta}^{[abcd]}$ with $\tilde{\eta}^{1234}=1/\sqrt{-\tilde{g}}$; indices surrounded by () and [] are the symmetrization and antisymmetrization symbols, respectively.

Our conventions of the Riemann curvature and Einstein's equation are

$$\tilde{u}_{a;bc}-\tilde{u}_{a;cb}=\tilde{u}_d\tilde{R}^d{}_{abc}, \tag{17}$$

$$\tilde{R}_{ab}-\frac{1}{2}\tilde{R}\tilde{g}_{ab}=8\pi G\tilde{T}_{ab}-\Lambda\tilde{g}_{ab}. \tag{18}$$

The Weyl (conformal) curvature is introduced as

$$\tilde{C}_{abcd}\equiv\tilde{R}_{abcd}-\frac{1}{2}(\tilde{g}_{ac}\tilde{R}_{bd}+\tilde{g}_{bd}\tilde{R}_{ac}-\tilde{g}_{bc}\tilde{R}_{ad}-\tilde{g}_{ad}\tilde{R}_{bc})+\frac{\tilde{R}}{6}(\tilde{g}_{ac}\tilde{g}_{bd}-\tilde{g}_{ad}\tilde{g}_{bc}). \tag{19}$$

The electric and magnetic parts of the Weyl curvature are introduced as

$$\tilde{E}_{ab}\equiv\tilde{C}_{acbd}\tilde{u}^c\tilde{u}^d, \quad \tilde{H}_{ab}\equiv\frac{1}{2}\tilde{\eta}_{ac}{}^{ef}\tilde{C}_{efbd}\tilde{u}^c\tilde{u}^d. \tag{20}$$

The energy-momentum tensor is decomposed into fluid quantities based on the four-vector field $\tilde{u}^a$ as

$$\tilde{T}_{ab}\equiv\tilde{\mu}\tilde{u}_a\tilde{u}_b+\tilde{p}(\tilde{g}_{ab}+\tilde{u}_a\tilde{u}_b)+\tilde{q}_a\tilde{u}_b+\tilde{q}_b\tilde{u}_a+\tilde{\pi}_{ab}, \tag{21}$$

where

$$\tilde{u}^a\tilde{q}_a=0=\tilde{u}^a\tilde{\pi}_{ab}, \quad \tilde{\pi}_{ab}=\tilde{\pi}_{ba}, \quad \tilde{\pi}^a_a=0. \tag{22}$$

The variables $\tilde{\mu}$, $\tilde{p}$, $\tilde{q}_a$, and $\tilde{\pi}_{ab}$ are the energy density, the isotropic pressure (including the entropic one), the energy flux, and the anisotropic pressure based on the $\tilde{u}_a$ frame, respectively. We have

$$\tilde{\mu}\equiv\tilde{T}_{ab}\tilde{u}^a\tilde{u}^b, \quad \tilde{p}\equiv\frac{1}{3}\tilde{T}_{ab}\tilde{h}^{ab},$$

$$\tilde{q}_a\equiv-\tilde{T}_{cd}\tilde{u}^c\tilde{h}^d_a, \quad \tilde{\pi}_{ab}\equiv\tilde{T}_{cd}\tilde{h}^c_a\tilde{h}^d_b-\tilde{p}\tilde{h}_{ab}. \tag{23}$$

The specific entropy $\tilde{S}$ can be introduced by $\tilde{T}d\tilde{S}=d\tilde{\varepsilon}+\tilde{p}_Td\tilde{v}$ where $\tilde{\varepsilon}$ is the specific internal energy density with $\tilde{\mu}=\tilde{\varrho}(1+\tilde{\varepsilon})$, $\tilde{p}_T$ the thermodynamic pressure, $\tilde{v}\equiv1/\tilde{\varrho}$ the specific volume, and $\tilde{T}$ the temperature. We have the isotropic pressure $\tilde{p}=\tilde{p}_T+\tilde{\pi}$ where $\tilde{\pi}$ is the entropic pressure. Using Eq. (26) below we can show that

$$\tilde{\varrho}\tilde{T}\dot{\tilde{S}}=-(\tilde{\pi}\tilde{\theta}+\tilde{\pi}^{ab}\tilde{\sigma}_{ab}+\tilde{q}^a{}_{;a}+\tilde{q}^a\tilde{a}_a). \tag{24}$$

Thus, we notice that $\tilde{\pi}$, $\tilde{\pi}^{ab}$, and $\tilde{q}^a$ generate the entropy. Using the four-vector $\tilde{S}^a\equiv\tilde{\varrho}\tilde{u}^a\tilde{S}+(1/\tilde{T})\tilde{q}^a$, which is termed the entropy flow density [19], we can derive

$$\tilde{S}^a{}_{;a}=-\frac{1}{\tilde{T}^2}(\tilde{T}_{,a}+\tilde{T}\tilde{a}_a)\tilde{q}^a-\frac{1}{\tilde{T}}(\tilde{\pi}\tilde{\theta}+\tilde{\pi}^{ab}\tilde{\sigma}_{ab}). \tag{25}$$

The covariant formulation provides a useful complement to the ADM formulation. We summarize the covariant (1+3) set of equations in the following. For details, see [19,20] and the Appendix in [21].

The energy and momentum conservation equations

$$\dot{\tilde{\mu}}+(\tilde{\mu}+\tilde{p})\tilde{\theta}+\tilde{\pi}^{ab}\tilde{\sigma}_{ab}+\tilde{q}^a{}_{;a}+\tilde{q}^a\tilde{a}_a=0, \tag{26}$$

$$(\tilde{\mu}+\tilde{p})\tilde{a}_a+\tilde{h}^b_a(\tilde{p}_{,b}+\tilde{\pi}^c_{b;c}+\dot{\tilde{q}}_b)+\left(\tilde{\omega}_{ab}+\tilde{\sigma}_{ab}+\frac{4}{3}\tilde{\theta}\tilde{h}_{ab}\right)\tilde{q}^b=0. \tag{27}$$

The Raychaudhuri equation

$$\dot{\tilde{\theta}}+\frac{1}{3}\tilde{\theta}^2-\tilde{a}^a{}_{;a}+2(\tilde{\sigma}^2-\tilde{\omega}^2)+4\pi G(\tilde{\mu}+3\tilde{p})-\Lambda=0. \tag{28}$$

Vorticity propagation

$$\tilde{h}^a_b\dot{\tilde{\omega}}{}^b+\frac{2}{3}\tilde{\theta}\tilde{\omega}^a=\tilde{\sigma}^a_b\tilde{\omega}^b+\frac{1}{2}\tilde{\eta}^{abcd}\tilde{u}_b\tilde{a}_{c;d}. \tag{29}$$

Shear propagation

$$\tilde{h}^c_a\tilde{h}^d_b(\dot{\tilde{\sigma}}_{cd}-\tilde{a}_{(c;d)})-\tilde{a}_a\tilde{a}_b+\tilde{\omega}_a\tilde{\omega}_b+\tilde{\sigma}_{ac}\tilde{\sigma}^c_b+\frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ab}-\frac{1}{3}\tilde{h}_{ab}(\tilde{\omega}^2+2\tilde{\sigma}^2-\tilde{a}^c{}_{;c})+\tilde{E}_{ab}-4\pi G\tilde{\pi}_{ab}=0. \tag{30}$$

Three constraint equations

$$\tilde{h}_{ab}\left(\tilde{\omega}^{bc}{}_{;c}-\tilde{\sigma}^{bc}{}_{;c}+\frac{2}{3}\tilde{\theta}^{;b}\right)+(\tilde{\omega}_{ab}+\tilde{\sigma}_{ab})\tilde{a}^b=8\pi G\tilde{q}_a, \tag{31}$$

$$\tilde{\omega}^a{}_{;a}=2\tilde{\omega}^b\tilde{a}_b, \tag{32}$$

$$\tilde{H}_{ab}=2\tilde{a}_{(a}\tilde{\omega}_{b)}-\tilde{h}^c_a\tilde{h}^d_b(\tilde{\omega}_{(c}{}^{e;f}+\tilde{\sigma}_{(c}{}^{e;f})\tilde{\eta}_{d)gef}\tilde{u}^g. \tag{33}$$

Four quasi-Maxwellian equations

$$\tilde{h}^a_b\tilde{h}^c_d\tilde{E}^{bd}{}_{;c}-\tilde{\eta}^{abcd}\tilde{u}_b\tilde{\sigma}^e_c\tilde{H}_{de}+3\tilde{H}^a_b\tilde{\omega}^b=4\pi G\left(\frac{2}{3}\tilde{h}^{ab}\tilde{\mu}_{,b}-\tilde{h}^a_b\tilde{\pi}^{bc}{}_{;c}-3\tilde{\omega}^a{}_b\tilde{q}^b+\tilde{\sigma}^a_b\tilde{q}^b+\tilde{\pi}^a_b\tilde{a}^b-\frac{2}{3}\tilde{\theta}\tilde{q}^a\right), \tag{34}$$

$$\tilde{h}^a_b \tilde{h}^c_d \tilde{H}^{bd}{}_{;c} + \tilde{\eta}^{abcd} \tilde{u}_b \tilde{\sigma}^e_c \tilde{E}_{de} - 3 \tilde{E}^a_b \tilde{\omega}^b = 4\pi G \{ 2(\tilde{\mu} + \tilde{p}) \tilde{\omega}^a + \tilde{\eta}^{abcd} \tilde{u}_b \times [\tilde{q}_{c;d} + \tilde{\pi}_{ce} (\tilde{\omega}^e{}_d + \tilde{\sigma}^e{}_d)] \}, \quad (35)$$

$$\tilde{h}^a_c \tilde{h}^b_d \dot{\tilde{\tilde{E}}}{}^{cd} + (\tilde{H}^f_{d;e} \tilde{h}^{(a}_f - 2\tilde{a}_d \tilde{H}^{(a}_e) \tilde{\eta}^{b)cde} \tilde{u}_c + \tilde{h}^{ab} \tilde{\sigma}^{cd} \tilde{E}_{cd} + \tilde{\theta} \tilde{E}^{ab} - \tilde{E}^{(a}_c (3\tilde{\sigma}^{b)c} + \tilde{\omega}^{b)c}) = 4\pi G \bigg[-(\tilde{\mu} + \tilde{p}) \tilde{\sigma}^{ab} - 2\tilde{a}^{(a} \tilde{q}^{b)} - \tilde{h}^{(a}_c \tilde{h}^{b)}_d (\tilde{q}^{c;d} + \dot{\tilde{\tilde{\pi}}}{}^{cd}) - (\tilde{\omega}_c{}^{(a} + \tilde{\sigma}^{(a}_c) \tilde{\pi}^{b)c} - \frac{1}{3} \tilde{\theta} \tilde{\pi}^{ab} + \frac{1}{3} (\tilde{q}^c{}_{;c} + \tilde{a}_c \tilde{q}^c + \tilde{\pi}^{cd} \tilde{\sigma}_{cd}) \tilde{h}^{ab} \bigg], \quad (36)$$

$$\tilde{h}^a_c \tilde{h}^b_d \dot{\tilde{\tilde{H}}}{}^{cd} - (\tilde{E}^f_{d;e} \tilde{h}^{(a}_f - 2\tilde{a}_d \tilde{E}^{(a}_e) \tilde{\eta}^{b)cde} \tilde{u}_c + \tilde{h}^{ab} \tilde{\sigma}^{cd} \tilde{H}_{cd} + \tilde{\theta} \tilde{H}^{ab} - \tilde{H}^{(a}_c (3\tilde{\sigma}^{b)c} + \tilde{\omega}^{b)c}) = 4\pi G [(\tilde{q}_e \tilde{\sigma}^{(a}_d - \tilde{\pi}^f_{d;e} \tilde{h}^{(a}_f) \tilde{\eta}^{b)cde} \tilde{u}_c + \tilde{h}^{ab} \tilde{\omega}_c \tilde{q}^c - 3\tilde{\omega}^{(a} \tilde{q}^{b)}]. \quad (37)$$

Evaluated in the normal-frame Eqs. (26),(27),(28),(30),(31) reproduce Eqs. (12),(13),(10),(11),(9) in the ADM formulation.

Now, we take the normal-frame vector; thus $\tilde{u}_a = \tilde{n}_a$ with $\tilde{n}_\alpha \equiv 0$ and thus $\tilde{\omega}_{ab} = 0$. The trace and trace-free parts of the Gauss equation give [21]

$$\tilde{R}^{(3)} = 2 \left(-\frac{1}{3} \tilde{\theta}^2 + \tilde{\sigma}^2 + 8\pi G \tilde{\mu} + \Lambda \right), \quad (38)$$

$$\tilde{R}^{(3)}_{ab} - \frac{1}{3} \tilde{R}^{(3)} \tilde{h}_{ab} = \tilde{h}^c_a \tilde{h}^d_b (- \dot{\tilde{\tilde{\sigma}}}_{cd} - \tilde{\theta} \tilde{\sigma}_{cd} + \tilde{a}_{(c;d)}) + \tilde{a}_a \tilde{a}_b - \frac{1}{3} \tilde{h}_{ab} \tilde{a}^c{}_{;c} + 8\pi G \tilde{\pi}_{ab}, \quad (39)$$

where $\tilde{R}^{(3)}_{ab}$ and $\tilde{R}^{(3)}$ are the Ricci and scalar curvatures of the hypersurface normal to $\tilde{n}_a$; for an arbitrary vector $\tilde{V}_a$ we have

$$\tilde{R}^{(3)}_{abcd} \tilde{V}^b \equiv 2 \tilde{\nabla}^{(3)}_{[c} \tilde{\nabla}^{(3)}_{d]} \tilde{V}_a \equiv 2 \tilde{h}^e_c \tilde{h}^f_d \tilde{h}^g_a \tilde{\nabla}_{[e} (\tilde{h}^h_{f]} \tilde{h}^i_g \tilde{\nabla}_h \tilde{V}_i),$$

$$\tilde{R}^{(3)}_{ab} \equiv \tilde{h}^{cd} \tilde{R}^{(3)}_{cadb}, \quad \tilde{R}^{(3)} \equiv \tilde{h}^{ab} \tilde{R}^{(3)}_{ab}. \quad (40)$$

From this we have

$$\tilde{R}^{(3)}_{abcd} = \tilde{h}^e_a \tilde{h}^f_b \tilde{h}^g_c \tilde{h}^h_d \tilde{R}_{efgh} - \tilde{\theta}_{ca} \tilde{\theta}_{db} + \tilde{\theta}_{bc} \tilde{\theta}_{ad}, \quad (41)$$

which is the Gauss equation. We can show that $\tilde{R}^{(3)}_{\alpha\beta\gamma\delta} = R^{(h)}_{\alpha\beta\gamma\delta}$. Equation (39) follows from Eq. (30) evaluated in the normal frame. Using Eqs. (19),(20) we can show that Eq. (39) reproduces Eq. (11) in the ADM formulation. Equation (38) gives Eq. (8) in the ADM formulation.

Compared with the ADM equations in (8)–(13) some of the covariant equations in (29),(32),(33),(34)–(37) look new. In the normal frame Eqs. (29),(32) are identically satisfied; using Eqs. (14),(6),(3) we can show that

$$\tilde{a}_\alpha = (\ln N)_{,\alpha}, \quad (42)$$

thus $\tilde{a}_{[\beta;\gamma]} = 0$. Still, Eqs. (8)–(13) provide a complete set. These additional equations in the covariant form should be regarded as complementary equations which could possibly show certain aspects of the system better. In our perturbation analyses we will use parts of these equations as complementary ones. Although the covariant set of equations is based on the general frame vector, this does not add any new physics which is not covered by the normal-frame taken in the ADM formulation; see Sec. III E.

The covariant equations for the scalar fields, generalized gravity, electromagnetic field, null geodesic, and Boltzmann equation will be introduced individually in the corresponding sections later.

C. Multicomponent situation

In the multicomponent situation we have

$$\tilde{T}_{ab} = \sum_l \tilde{T}_{(l)ab}, \quad \tilde{T}_{(i)a;b}^{\ \ b} \equiv \tilde{I}_{(i)a}, \quad \sum_l \tilde{I}_{(l)a} = 0. \quad (43)$$

Based on the normal-frame vector, we have

$$\tilde{\mu} = \sum_l \tilde{\mu}_{(l)}, \quad \tilde{p} = \sum_l \tilde{p}_{(l)},$$

$$\tilde{q}_a = \sum_l \tilde{q}_{(l)a}, \quad \tilde{\pi}_{ab} = \sum_l \tilde{\pi}_{(l)ab}. \quad (44)$$

The ADM formulation is based on the normal-frame vector $\tilde{u}_a = \tilde{n}_a$. The ADM fluid quantities in Eq. (4) correspond to the fluid quantities based on the normal-frame vector as

$$E = \tilde{\mu}, \quad S = 3\tilde{p}, \quad J_\alpha = \tilde{q}_\alpha, \quad \bar{S}_{\alpha\beta} = \tilde{\pi}_{\alpha\beta}. \quad (45)$$

From Eq. (4) or Eq. (45) we have

$$E = \sum_l E_{(l)}, \quad S = \sum_l S_{(l)},$$

$$J_\alpha = \sum_l J_{(l)\alpha}, \quad S_{\alpha\beta} = \sum_l S_{(l)\alpha\beta}. \quad (46)$$

Equation (43) gives

$$E_{(i),0} N^{-1} - E_{(i),\alpha} N^\alpha N^{-1} - K \left(E_{(i)} + \frac{1}{3} S_{(i)} \right) - \bar{S}^{\alpha\beta}_{(i)} \bar{K}_{\alpha\beta} + N^{-2} (N^2 J^\alpha_{(i)})_{:\alpha} = -\frac{1}{N} (\tilde{I}_{(i)0} - \tilde{I}_{(i)\alpha} N^\alpha), \quad (47)$$

$$J_{(i)\alpha,0}N^{-1}-J_{(i)\alpha:\beta}N^{\beta}N^{-1}-J_{(i)\beta}N^{\beta}{}_{:\alpha}N^{-1}-KJ_{(i)\alpha}+E_{(i)}N_{,\alpha}N^{-1}+S_{(i)\alpha:\beta}^{\beta}+S_{(i)\alpha}^{\beta}N_{,\beta}N^{-1}=\tilde{I}_{(i)\alpha}. \tag{48}$$

The ADM equations in Eqs. (8)–(13) remain valid, with the above additional equations of motion for the individual component. Thus, in the multicomponent situation Eqs. (8)–(13), (46)–(48) provide a complete set.

III. PERTURBED QUANTITIES

A. Metric and connections

We use the following convention for the metric variables:

$$\tilde{g}_{00}\equiv -a^2(1+2A), \quad \tilde{g}_{0\alpha}\equiv -a^2B_{\alpha},$$
$$\tilde{g}_{\alpha\beta}\equiv a^2(g^{(3)}_{\alpha\beta}+2C_{\alpha\beta}), \tag{49}$$

where A, B_{α}, and $C_{\alpha\beta}$ are perturbed order variables and are *assumed* to be based on $g^{(3)}_{\alpha\beta}$ as the metric. To second order, we can write the perturbation variables explicitly as

$$A\equiv A^{(1)}+A^{(2)}, \quad B_{\alpha}\equiv B^{(1)}_{\alpha}+B^{(2)}_{\alpha}, \quad C_{\alpha\beta}\equiv C^{(1)}_{\alpha\beta}+C^{(2)}_{\alpha\beta}. \tag{50}$$

As we are interested in the perturbation to second order, as our ansatz, we include up to second-order (quadratic) terms in the deviation from the Friedmann background. This can be extended to any higher-order perturbation as long as we take the perturbative approach where the lower-order solutions drive (work as sources for) the next higher-order variables. Thus, in this work we ignore the terms that are higher than quadratic (second-order) combinations of the perturbed metric (A, B_{α}, $C_{\alpha\beta}$), the perturbed fluid quantities ($\delta\mu$, δp, Q_{α}, $\Pi_{\alpha\beta}$) to be introduced in Eq. (72), the perturbed field ($\delta\phi$) to be introduced in Eq. (111), etc.

The inverse metric expanded to second order in perturbation variables is

$$\tilde{g}^{00}=\frac{1}{a^2}(-1+2A-4A^2+B_{\alpha}B^{\alpha}),$$
$$\tilde{g}^{0\alpha}=\frac{1}{a^2}(-B^{\alpha}+2AB^{\alpha}+2B_{\beta}C^{\alpha\beta}),$$
$$\tilde{g}^{\alpha\beta}=\frac{1}{a^2}(g^{(3)\alpha\beta}-2C^{\alpha\beta}-B^{\alpha}B^{\beta}+4C^{\alpha}_{\gamma}C^{\beta\gamma}). \tag{51}$$

The connections are

$$\tilde{\Gamma}^{0}_{00}=\frac{a'}{a}+A'-2AA'-A_{,\alpha}B^{\alpha}+B_{\alpha}\left(B^{\alpha\prime}+\frac{a'}{a}B^{\alpha}\right),$$
$$\tilde{\Gamma}^{0}_{0\alpha}=A_{,\alpha}-\frac{a'}{a}B_{\alpha}-2AA_{,\alpha}+2\frac{a'}{a}AB_{\alpha}-B_{\beta}C^{\beta\prime}_{\alpha}+B^{\beta}B_{[\beta|\alpha]},$$
$$\tilde{\Gamma}^{\alpha}_{00}=A^{|\alpha}-B^{\alpha\prime}-\frac{a'}{a}B^{\alpha}+A'B^{\alpha}-2A_{,\beta}C^{\alpha\beta}+2C^{\alpha}_{\beta}\left(B^{\beta\prime}+\frac{a'}{a}B^{\beta}\right),$$
$$\tilde{\Gamma}^{0}_{\alpha\beta}=\frac{a'}{a}g^{(3)}_{\alpha\beta}-2\frac{a'}{a}g^{(3)}_{\alpha\beta}A+B_{(\alpha|\beta)}+C'_{\alpha\beta}+2\frac{a'}{a}C_{\alpha\beta}+\frac{a'}{a}g^{(3)}_{\alpha\beta}(4A^2-B_{\gamma}B^{\gamma})-2A\left(B_{(\alpha|\beta)}+C'_{\alpha\beta}+2\frac{a'}{a}C_{\alpha\beta}\right)-B_{\gamma}(2C^{\gamma}_{(\alpha|\beta)}-C_{\alpha\beta}{}^{|\gamma}),$$
$$\tilde{\Gamma}^{\alpha}_{0\beta}=\frac{a'}{a}\delta^{\alpha}_{\beta}+\frac{1}{2}(B_{\beta}{}^{|\alpha}-B^{\alpha}{}_{|\beta})+C^{\alpha\prime}_{\beta}+B^{\alpha}\left(A_{,\beta}-\frac{a'}{a}B_{\beta}\right)+2C^{\alpha\gamma}(B_{[\gamma|\beta]}-C'_{\gamma\beta}),$$
$$\tilde{\Gamma}^{\alpha}_{\beta\gamma}=\Gamma^{(3)\alpha}{}_{\beta\gamma}+\frac{a'}{a}g^{(3)}_{\beta\gamma}B^{\alpha}+2C^{\alpha}_{(\beta|\gamma)}-C_{\beta\gamma}{}^{|\alpha}-2C^{\alpha}_{\delta}(2C^{\delta}_{(\beta|\gamma)}-C_{\beta\gamma}{}^{|\delta})-2\frac{a'}{a}g^{(3)}_{\gamma\beta}(AB^{\alpha}+B^{\delta}C^{\alpha}_{\delta})+B^{\alpha}\left(B_{(\beta|\gamma)}+C'_{\beta\gamma}+2\frac{a'}{a}C_{\beta\gamma}\right), \tag{52}$$

where a vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$. An index 0 indicates the conformal time η, and a prime indicates a time derivative with respect to η. The components of the frame four-vector $\tilde{u}_a$ are introduced as

$$\tilde{u}^{0}\equiv\frac{1}{a}\left(1-A+\frac{3}{2}A^2+\frac{1}{2}V^{\alpha}V_{\alpha}-V^{\alpha}B_{\alpha}\right), \quad \tilde{u}^{\alpha}\equiv\frac{1}{a}V^{\alpha},$$
$$\tilde{u}_{0}=-a\left(1+A-\frac{1}{2}A^2+\frac{1}{2}V^{\alpha}V_{\alpha}\right),$$
$$\tilde{u}_{\alpha}=a(V_{\alpha}-B_{\alpha}+AB_{\alpha}+2V^{\beta}C_{\alpha\beta}), \tag{53}$$

where V^{α} is based on $g^{(3)}_{\alpha\beta}$.

B. Normal-frame quantities

The normal-frame vector $\tilde{n}_a$ has the property $\tilde{n}_{\alpha}\equiv 0$. Thus we have

$$\tilde{n}^{0}\equiv\frac{1}{a}\left(1-A+\frac{3}{2}A^2-\frac{1}{2}B^{\alpha}B_{\alpha}\right),$$

$$\tilde{n}^\alpha \equiv \frac{1}{a}(B^\alpha - AB^\alpha - 2B^\beta C^\alpha_\beta),$$

$$\tilde{n}_0 = -a\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}B^\alpha B_\alpha\right), \quad \tilde{n}_\alpha = 0. \tag{54}$$

Using Eqs. (2),(3) the ADM metric variables become

$$N = a\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}B^\alpha B_\alpha\right),$$

$$N_\alpha = -a^2 B_\alpha, \quad N^\alpha = -B^\alpha + 2B^\beta C^\alpha_\beta,$$

$$h_{\alpha\beta} = a^2(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}),$$

$$h^{\alpha\beta} = \frac{1}{a^2}(g^{(3)\alpha\beta} - 2C^{\alpha\beta} + 4C^\alpha_\gamma C^{\beta\gamma}). \tag{55}$$

The connection becomes

$$\Gamma^{(h)\gamma}{}_{\alpha\beta} = \Gamma^{(3)\gamma}{}_{\alpha\beta} + (g^{(3)\gamma\delta} - 2C^{\gamma\delta})(C_{\delta\alpha|\beta} + C_{\delta\beta|\alpha} - C_{\alpha\beta|\delta}). \tag{56}$$

The extrinsic curvature in Eq. (5) gives

$$K_{\alpha\beta} = -a\Bigg[\left(\frac{a'}{a}g^{(3)}_{\alpha\beta} + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}\right)(1-A) + \frac{1}{2}\frac{a'}{a}g^{(3)}_{\alpha\beta}(3A^2 - B_\gamma B^\gamma) - B_\gamma(2C^\gamma_{(\alpha|\beta)} - C_{\alpha\beta}{}^{|\gamma})\Bigg],$$

$$K = -\frac{1}{a}\Bigg[\left(3\frac{a'}{a} + B^\alpha{}_{|\alpha} + C^{\alpha\prime}_\alpha\right)(1-A) + \frac{3}{2}\frac{a'}{a} \times (3A^2 - B^\alpha B_\alpha) - B^\beta(2C^\alpha_{\beta|\alpha} - C^\alpha_{\alpha|\beta}) - 2C^{\alpha\beta}(C'_{\alpha\beta} + B_{\alpha|\beta})\Bigg],$$

$$\overline{K}_{\alpha\beta} = -a\Bigg\{(B_{(\alpha|\beta)} + C'_{\alpha\beta})(1-A) - B_\gamma(2C^\gamma_{(\alpha|\beta)} - C_{\alpha\beta}{}^{|\gamma}) - \frac{2}{3}C_{\alpha\beta}(B^\gamma{}_{|\gamma} + C^{\gamma\prime}_\gamma) - \frac{1}{3}g^{(3)}_{\alpha\beta}[(B^\gamma{}_{|\gamma} + C^{\gamma\prime}_\gamma)(1-A) - B^\gamma(2C^\delta_{\gamma|\delta} - C^\delta_{\delta|\gamma}) - 2C^{\gamma\delta}(C'_{\gamma\delta} + B_{\gamma|\delta})]\Bigg\}. \tag{57}$$

The intrinsic curvature in Eq. (7) becomes

$$R^{(h)}_{\alpha\beta} = R^{(3)}_{\alpha\beta} + (g^{(3)\gamma\delta} - 2C^{\gamma\delta})(C_{\delta\alpha|\beta\gamma} + C_{\delta\beta|\alpha\gamma} - C_{\alpha\beta|\delta\gamma} - C_{\delta\gamma|\alpha\beta}) + 2C^{\gamma\delta}{}_{|\beta}C_{\gamma\delta|\alpha} - (2C^\gamma_{\delta|\gamma} - C^\gamma_{\gamma|\delta})(C^\delta_{\alpha|\beta} + C^\delta_{\beta|\alpha} - C_{\alpha\beta}{}^{|\delta}) - (C^\delta_{\alpha|\gamma} + C^\delta_{\gamma|\alpha} - C_{\alpha\gamma}{}^{|\delta})(C^\gamma_{\beta|\delta} + C^\gamma_{\delta|\beta} - C_{\beta\delta}{}^{|\gamma}),$$

$$R^{(h)} = \frac{1}{a^2}[R^{(3)} - 2C^{\alpha\beta}R^{(3)}_{\alpha\beta} + 2C^{\beta|\alpha}_\alpha{}_\beta - 2C^{\alpha|\beta}_\alpha{}_\beta + 4C^\alpha_\gamma C^{\beta\gamma}R^{(3)}_{\alpha\beta} + 4C^{\alpha\beta}(-C^\gamma_{\alpha|\beta\gamma} - C^\gamma_{\alpha|\gamma\beta} + C_{\alpha\beta}{}^{|\gamma}{}_\gamma + C^\gamma_{\gamma|\alpha\beta}) - (2C^\gamma_{\beta|\gamma} - C^\gamma_{\gamma|\beta})(2C^{\alpha\beta}{}_{|\alpha} - C^{\alpha|\beta}_\alpha) + C^{\alpha\beta|\gamma}(3C_{\alpha\beta|\gamma} - 2C_{\alpha\gamma|\beta})], \tag{58}$$

where

$$R^{(3)\alpha}{}_{\beta\gamma\delta} = \frac{1}{6}R^{(3)}(\delta^\alpha_\gamma g^{(3)}_{\beta\delta} - \delta^\alpha_\delta g^{(3)}_{\beta\gamma}),$$

$$R^{(3)}_{\alpha\beta} = \frac{1}{3}R^{(3)}g^{(3)}_{\alpha\beta}, \quad R^{(3)} = 6K, \tag{59}$$

with a normalized $K(=0,\pm1)$, the sign of the background three-space curvature. Thus,

$$\overline{R}^{(h)\alpha}{}_\beta = \frac{1}{a^2}\Bigg\{C^{\alpha\gamma}{}_{|\beta\gamma} + C^{\gamma|\alpha}_\beta{}_\gamma - C^{\alpha|\gamma}_\beta{}_\gamma - C^{\gamma|\alpha}_\gamma{}_\beta - \frac{2}{3}R^{(3)}C^\alpha_\beta - 2C^{\gamma\delta}(C^\alpha_{\delta|\beta\gamma} + C_{\delta\beta}{}^{|\alpha}{}_\gamma - C^\alpha_{\beta|\delta\gamma} - C_{\delta\gamma}{}^{|\alpha}{}_\beta) - 2C^{\alpha\gamma}(C^\delta_{\gamma|\beta\delta} + C^\delta_{\beta|\gamma\delta} - C_{\beta\gamma}{}^{|\delta}{}_\delta - C^\delta_{\delta|\gamma\beta}) + \frac{4}{3}R^{(3)}C^\alpha_\gamma C^\gamma_\beta - (2C^\gamma_{\delta|\gamma} - C^\gamma_{\gamma|\delta})(C^{\alpha\delta}{}_{|\beta} + C^{\delta|\alpha}_\beta - C^{\alpha|\delta}_\beta) + C_{\gamma\delta|\beta}C^{\gamma\delta|\alpha} + 2C^{\alpha\gamma|\delta}(C_{\beta\gamma|\delta} - C_{\beta\delta|\gamma}) - \frac{1}{3}\delta^\alpha_\beta\Bigg[-\frac{2}{3}R^{(3)}C^\gamma_\gamma + 2C^{\delta|\gamma}_\gamma{}_\delta - 2C^{\gamma|\delta}_\gamma{}_\delta + 4C^{\gamma\delta}(-C^\epsilon_{\gamma|\delta\epsilon} - C^\epsilon_{\gamma|\epsilon\delta} + C_{\gamma\delta}{}^{|\epsilon}{}_\epsilon + C^\epsilon_{\epsilon|\gamma\delta}) + \frac{4}{3}R^{(3)}C^\delta_\gamma C^\gamma_\delta - (2C^\epsilon_{\delta|\epsilon} - C^\epsilon_{\epsilon|\delta})(2C^{\gamma\delta}{}_{|\gamma} - C^{\gamma|\delta}_\gamma) + C^{\gamma\delta|\epsilon}(3C_{\gamma\delta|\epsilon} - 2C_{\gamma\epsilon|\delta})\Bigg]\Bigg\}. \tag{60}$$

It is convenient to have

$$B^{\alpha}{}_{|\beta\gamma} = B^{\alpha}{}_{|\gamma\beta} - R^{(3)\alpha}{}_{\delta\beta\gamma} B^{\delta}, \quad B_{\alpha|\beta\gamma} = B_{\alpha|\gamma\beta} + R^{(3)\delta}{}_{\alpha\beta\gamma} B_{\delta}. \tag{61}$$

C. General ($\tilde{u}_a$)-frame quantities

To second-order perturbation, using Eqs. (49),(51),(53), the kinematic quantities in Eq. (14) become

$$\tilde{h}_{\alpha\beta} = a^2 [g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta} + (V_\alpha - B_\alpha)(V_\beta - B_\beta)],$$
$$\tilde{h}_{0\alpha} = -a^2 (V_\alpha + AV_\alpha + 2C_{\alpha\beta} V^\beta), \quad \tilde{h}_{00} = a^2 V^\alpha V_\alpha, \tag{62}$$

$$\tilde{\theta} = a^{-1} \Bigg[3\frac{a'}{a}(1-A) + V^\alpha{}_{|\alpha} + C^{\alpha\prime}_{\alpha} + \frac{9}{2}\frac{a'}{a}A^2 + B^\alpha B'_\alpha - 2C^{\alpha\beta} C'_{\alpha\beta} - AC^{\alpha\prime}_{\alpha} - V^{\alpha\prime} B_\alpha + V^\alpha \left(V'_\alpha + \frac{3}{2}\frac{a'}{a} V_\alpha - B'_\alpha - 3\frac{a'}{a} B_\alpha + A_{,\alpha} + C^\beta_{\beta|\alpha} \right) \Bigg], \tag{63}$$

$$\tilde{a}_\alpha = A_{,\alpha} + V'_\alpha - B'_\alpha + \frac{a'}{a}(V_\alpha - B_\alpha) + A' B_\alpha + A\left(-2A_{,\alpha} + 2B'_\alpha + 2\frac{a'}{a} B_\alpha - V'_\alpha - \frac{a'}{a} V_\alpha \right) + V^\beta (V_{\alpha|\beta} + B_{\beta|\alpha} - B_{\alpha|\beta}) + 2C_{\alpha\beta}\left(V^{\beta\prime} + \frac{a'}{a} V^\beta \right) + 2C'_{\alpha\beta} V^\beta, \quad \tilde{a}_0 = -V^\alpha \tilde{a}_\alpha, \tag{64}$$

$$\tilde{\omega}_{\alpha\beta} = a(V_{[\alpha|\beta]} - B_{[\alpha|\beta]} + AB_{[\alpha|\beta]} - V_{[\alpha} A_{,\beta]} + 2B_{[\alpha} A_{,\beta]} - B_{[\alpha} B'_{\beta]} - V_{[\alpha} V'_{\beta]} + B_{[\alpha} V'_{\beta]} + V_{[\alpha} B'_{\beta]} + 2V_\gamma C^\gamma_{[\alpha|\beta]} + 2C_{\gamma[\alpha} V^\gamma{}_{|\beta]}), \quad \tilde{\omega}_{0\alpha} = V^\beta \tilde{\omega}_{\alpha\beta}, \quad \tilde{\omega}_{00} = 0, \tag{65}$$

$$\tilde{\sigma}_{\alpha\beta} = a\Bigg[V_{(\alpha|\beta)} + C'_{\alpha\beta} - \frac{1}{3} g^{(3)}_{\alpha\beta} (V^\gamma{}_{|\gamma} + C^{\gamma\prime}_{\gamma}) + V_{(\alpha} V'_{\beta)} - V_{(\alpha} B'_{\beta)} - V'_{(\alpha} B_{\beta)} + B_{(\alpha} B'_{\beta)} + V_{(\alpha} A_{,\beta)} + V_\gamma C_{\alpha\beta}{}^{|\gamma} - AC'_{\alpha\beta} + 2C_{\delta(\alpha} V^\delta{}_{|\beta)} - \frac{2}{3} C_{\alpha\beta} (V^\gamma{}_{|\gamma} + C^{\gamma\prime}_{\gamma}) - \frac{1}{3} g^{(3)}_{\alpha\beta} (V^\gamma V'_\gamma - V^\gamma B'_\gamma - V^{\gamma\prime} B_\gamma + B^\gamma B'_\gamma + V^\gamma A_{,\gamma} + V_\delta C^{\gamma|\delta}_{\gamma} - AC^{\gamma\prime}_{\gamma} - 2C^{\delta\gamma} C'_{\delta\gamma}) \Bigg],$$

$$\tilde{\sigma}_{0\alpha} = -V^\beta \tilde{\sigma}_{\alpha\beta}, \quad \tilde{\sigma}_{00} = 0. \tag{66}$$

In the normal frame we have $\tilde{u}_\alpha \equiv 0$; thus $V_\alpha = B_\alpha - AB_\alpha - 2B^\beta C_{\alpha\beta}$. In this frame we have

$$\tilde{h}_{\alpha\beta} = a^2 (g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}), \quad \tilde{h}_{0\alpha} = -a^2 B_\alpha, \quad \tilde{h}_{00} = a^2 B^\alpha B_\alpha, \tag{67}$$

$$\tilde{\theta} = -K, \tag{68}$$

$$\tilde{a}_\alpha = (\ln N)_{,\alpha} = \left(A - A^2 + \frac{1}{2} B^\beta B_\beta \right)_{,\alpha}, \quad \tilde{a}_0 = -B^\alpha A_{,\alpha}, \tag{69}$$

$$\tilde{\sigma}_{\alpha\beta} = -\bar{K}_{\alpha\beta}, \quad \tilde{\sigma}_{0\alpha} = B^\beta \bar{K}_{\alpha\beta}, \quad \tilde{\sigma}_{00} = 0, \tag{70}$$

$$\tilde{\omega}_{ab} = 0. \tag{71}$$

In this frame we have $\tilde{\theta} = -K$ and $\tilde{\sigma}_{\alpha\beta} = -\bar{K}_{\alpha\beta}$. These are natural because K and $\bar{K}_{\alpha\beta}$ are the same as negatives of the expansion scalar and the shear, respectively, of the normal-frame vector field.

D. Fluid quantities

To the perturbed order we decompose the fluid quantities as

$$\tilde{\mu} \equiv \mu + \delta\mu, \quad \tilde{p} \equiv p + \delta p, \quad \tilde{q}_\alpha \equiv aQ_\alpha, \quad \tilde{\pi}_{\alpha\beta} \equiv a^2 \Pi_{\alpha\beta}, \tag{72}$$

where Q_α and $\Pi_{\alpha\beta}$ are based on $g^{(3)}_{\alpha\beta}$. In the Friedmann world model we have $\tilde{\mu} = \mu$ and $\tilde{p} = p$ and zeros for the other fluid quantities. We have

$$\Pi^\alpha_\alpha - 2C^{\alpha\beta} \Pi_{\alpha\beta} = 0, \tag{73}$$

which follows from $\tilde{\pi}^a_a = 0$ or $\bar{S}^\alpha_\alpha = 0$. The perturbed order fluid quantities $\delta\mu$, δp, Q_α, and $\Pi_{\alpha\beta}$ in Eq. (72) can be expanded similarly as in Eq. (50):

$$\delta\mu = \delta\mu^{(1)} + \delta\mu^{(2)}, \quad \delta p = \delta p^{(1)} + \delta p^{(2)},$$
$$Q_\alpha = Q^{(1)}_\alpha + Q^{(2)}_\alpha, \quad \Pi_{\alpha\beta} = \Pi^{(1)}_{\alpha\beta} + \Pi^{(2)}_{\alpha\beta}. \tag{74}$$

In the multicomponent situation, from Eqs. (44),(72) we set

$$\mu = \sum_l \mu_{(l)}, \quad p = \sum_l p_{(l)},$$

$$\delta\mu = \sum_l \delta\mu_{(l)}, \quad \delta p = \sum_l \delta p_{(l)},$$

$$Q_\alpha = \sum_l Q_{(l)\alpha}, \quad \Pi_{\alpha\beta} = \sum_l \Pi_{(l)\alpha\beta}. \tag{75}$$

Thus, from Eq. (45) the ADM fluid variables become

$$E \equiv \mu + \delta\mu,$$

$$S \equiv 3(p + \delta p),$$

$$J_\alpha \equiv a Q_\alpha, \quad J^\alpha = \frac{1}{a}(Q^\alpha - 2C^{\alpha\beta}Q_\beta),$$

$$\overline{S}_{\alpha\beta} \equiv a^2 \Pi_{\alpha\beta}, \quad \overline{S}^\alpha_\beta = \Pi^\alpha_\beta - 2C^{\alpha\gamma}\Pi_{\beta\gamma},$$

$$\overline{S}^{\alpha\beta} = \frac{1}{a^2}(\Pi^{\alpha\beta} - 4C^{\gamma(\alpha}\Pi^{\beta)}_\gamma). \tag{76}$$

From Eqs. (22),(72) we have

$$\tilde{q}_\alpha \equiv a Q_\alpha, \quad \tilde{q}_0 = -aQ_\alpha B^\alpha,$$

$$\tilde{\pi}_{\alpha\beta} \equiv a^2 \Pi_{\alpha\beta}, \quad \tilde{\pi}_{0\alpha} = -a^2 \Pi_{\alpha\beta} B^\beta, \quad \tilde{\pi}_{00} = 0. \tag{77}$$

From Eqs. (21),(72),(54) we have

$$\tilde{T}_{00} = a^2[\mu + \delta\mu + 2\mu A + 2\delta\mu A + (\mu + p)B^\alpha B_\alpha + 2Q_\alpha B^\alpha],$$

$$\tilde{T}_{0\alpha} = -a^2(Q_\alpha + pB_\alpha + \delta p B_\alpha + AQ_\alpha + \Pi_{\alpha\beta}B^\beta),$$

$$\tilde{T}_{\alpha\beta} = a^2(p g^{(3)}_{\alpha\beta} + \delta p g^{(3)}_{\alpha\beta} + \Pi_{\alpha\beta} + 2pC_{\alpha\beta} + 2\delta p C_{\alpha\beta}). \tag{78}$$

From Eqs. (4), (76) we have

$$\mu + \delta\mu = \tilde{T}_{ab}\tilde{n}^a\tilde{n}^b, \quad p + \delta p = \frac{1}{3}h^{\alpha\beta}\tilde{T}_{\alpha\beta},$$

$$Q_\alpha = -\frac{1}{a}\tilde{T}_{\alpha b}\tilde{n}^b, \quad \Pi_{\alpha\beta} = \frac{1}{a^2}(\tilde{T}_{\alpha\beta} - h_{\alpha\beta}\tilde{p}). \tag{79}$$

Equation (24) gives

$$\tilde{\varrho}\tilde{T}\tilde{\tilde{S}} = \tilde{\pi}K + \frac{1}{a^2}\Pi^{\alpha\beta}\overline{K}_{\alpha\beta} - 2\frac{1}{a}Q^\alpha A_{,\alpha} - \frac{1}{a}Q^\alpha{}_{|\alpha} + 2\frac{1}{a}(C^{\alpha\beta}Q_\beta)_{|\alpha} - \frac{1}{a}C^\alpha_{\alpha|\beta}Q^\beta. \tag{80}$$

For the interaction terms in Eq. (43) we set

$$\tilde{I}_{(i)0} \equiv I_{(i)0} + \delta I_{(i)0}, \quad \tilde{I}_{(i)\alpha} \equiv \delta I_{(i)\alpha}, \tag{81}$$

where $\delta I_{(i)\alpha}$ is based in $g^{(3)}_{\alpha\beta}$.

E. Frame choice

The energy-momentum tensor in the general ($\tilde{u}_a$) frame follows from Eqs. (21),(72),(53):

$$\tilde{T}^0_0 = -\mu - \delta\mu - (\mu + p)V^\alpha(V_\alpha - B_\alpha) - Q^\alpha(2V_\alpha - B_\alpha),$$

$$\tilde{T}^0_\alpha = (1 - A)[Q_\alpha + (\mu + p)(V_\alpha - B_\alpha)] + (\mu + p)(AB_\alpha + 2V^\beta C_{\alpha\beta}) + (\delta\mu + \delta p)(V_\alpha - B_\alpha) + (V^\beta - B^\beta)\Pi_{\alpha\beta},$$

$$\tilde{T}^\alpha_\beta = (p + \delta p)\delta^\alpha_\beta + \Pi^\alpha_\beta + V^\alpha[Q_\beta + (\mu + p)(V_\beta - B_\beta)] + Q^\alpha(V_\beta - B_\beta) - 2C^{\alpha\gamma}\Pi_{\beta\gamma}. \tag{82}$$

In the energy frame we set $Q_\alpha \equiv 0$; thus $\tilde{q}_a = 0$. In the normal frame we have $\tilde{u}_\alpha \equiv 0$; thus from Eq. (53) we have

$$\text{energy frame:} \quad Q_\alpha \equiv 0,$$

$$\text{Normal frame:} \quad V_\alpha - B_\alpha + AB_\alpha + 2B^\beta C_{\alpha\beta} \equiv 0. \tag{83}$$

Although we can take infinitely many different combinations of the two frames, the energy and the normal frames are the ones often used in the literature [22]. By choosing a frame (which is a decision about Q_α and V_α) we lose no generality. This is because we have ten independent pieces of information in $\tilde{T}_{ab}$, which can be allocated to the energy density $\tilde{\mu}$ (one), the pressure $\tilde{p}$ (one), the anisotropic stress $\Pi_{\alpha\beta}$ (five, because it is trace-free). The remaining (three) pieces of information can be assigned to either the velocity V_α (three) or the flux Q_α (three); or some combinations of V_α and Q_α with a total of three pieces of information.

Thus, in the normal frame (indicated by a superscript N) we have

$$\tilde{T}^0_0 = -\mu - \delta\mu^N - Q^{N\alpha}B_\alpha,$$
$$\tilde{T}^0_\alpha = (1-A)Q^N_\alpha,$$
$$\tilde{T}^\alpha_\beta = (p+\delta p^N)\delta^\alpha_\beta + \Pi^{N\alpha}{}_\beta + B^\alpha Q^N_\beta - 2C^{\alpha\gamma}\Pi^N_{\beta\gamma}, \quad (84)$$

which is consistent with Eq. (78). To linear order we notice that $\delta\mu$, δp, $\Pi_{\alpha\beta}$ are independent of the frame choice, and $Q_\alpha + (\mu+p)(V_\alpha - B_\alpha)$ is a frame-invariant combination [21]. However, to second order we no longer have such a luxury. As the fluid quantities are defined based on the frame vector as in Eq. (23) the values of $\delta\mu$, δp, and $\Pi_{\alpha\beta}$ are dependent on the frame.

By comparing the energy-momentum tensor in the normal frame in Eq. (84) with the one in the general frame in Eq. (82), we find that by replacing the normal-frame fluid quantities with

$$\delta\mu^N \equiv \delta\mu + (V^\alpha - B^\alpha)[(\mu+p)(V_\alpha - B_\alpha) + 2Q_\alpha],$$
$$\delta p^N \equiv \delta p + \frac{1}{3}(V^\alpha - B^\alpha)[(\mu+p)(V_\alpha - B_\alpha) + 2Q_\alpha],$$
$$Q^N_\alpha \equiv Q_\alpha + (\mu+p)(V_\alpha - B_\alpha) + (\mu+p)(AB_\alpha + 2V^\beta C_{\alpha\beta}) + (\delta\mu + \delta p)(V_\alpha - B_\alpha) + (V^\beta - B^\beta)\Pi_{\alpha\beta},$$
$$\Pi^N_{\alpha\beta} \equiv \Pi_{\alpha\beta} + (\mu+p)(V_\alpha - B_\alpha)(V_\beta - B_\beta) + 2Q_{(\alpha}(V_{\beta)} - B_{\beta)}) - \frac{1}{3}g^{(3)}_{\alpha\beta}(V^\gamma - B^\gamma) \times[(\mu+p)(V_\gamma - B_\gamma) + 2Q_\gamma], \quad (85)$$

we recover the general frame energy-momentum tensor. Thus, by imposing the energy-frame condition ($Q_\alpha = 0$) we recover the fluid quantities in the energy frame.

Now, similarly, by comparing Eq. (82) with the same one evaluated in the energy frame, we find that by replacing the energy-frame fluid quantities (indicated by a superscript E) with

$$\delta\mu^E \equiv \delta\mu - \frac{1}{\mu+p}Q^\alpha Q_\alpha,$$
$$\delta p^E \equiv \delta p - \frac{1}{3}\frac{1}{\mu+p}Q^\alpha Q_\alpha,$$
$$(\mu+p)(V^E_\alpha - B_\alpha) \equiv (\mu+p)(V_\alpha - B_\alpha) + Q_\alpha - 2Q^\beta C_{\alpha\beta} - \frac{\delta\mu + \delta p}{\mu+p}Q_\alpha - Q^\beta\frac{\Pi_{\alpha\beta}}{\mu+p},$$
$$\Pi^E_{\alpha\beta} \equiv \Pi_{\alpha\beta} - \frac{1}{\mu+p}\left(Q_\alpha Q_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}Q^\gamma Q_\gamma\right), \quad (86)$$

we recover the general frame energy-momentum tensor. By imposing the normal-frame condition in Eq. (83) we recover the fluid quantities in the normal frame. Thus, using Eqs. (85),(86) we can transform the fluid quantities in one frame to the other:

$$\delta\mu^N = \delta\mu^E + (\mu+p)(V^{E\alpha} - B^\alpha)(V^E_\alpha - B_\alpha),$$
$$\delta p^N = \delta p^E + \frac{1}{3}(\mu+p)(V^{E\alpha} - B^\alpha)(V^E_\alpha - B_\alpha),$$
$$Q^N_\alpha = (\mu+p)(V^E_\alpha - B_\alpha) + (\mu+p) \times(AB_\alpha + 2V^{E\beta}C_{\alpha\beta}) + (\delta\mu^E + \delta p^E) \times(V^E_\alpha - B_\alpha) + (V^{E\beta} - B^\beta)\Pi^E_{\alpha\beta},$$
$$\Pi^N_{\alpha\beta} = \Pi^E_{\alpha\beta} + (\mu+p)(V^E_\alpha - B_\alpha)(V^E_\beta - B_\beta) - \frac{1}{3}g^{(3)}_{\alpha\beta}(\mu+p)(V^{E\gamma} - B^\gamma)(V^E_\gamma - B_\gamma); \quad (87)$$

$$\delta\mu^E = \delta\mu^N - \frac{1}{\mu+p}Q^{N\alpha}Q^N_\alpha,$$
$$\delta p^E = \delta p^N - \frac{1}{3}\frac{1}{\mu+p}Q^{N\alpha}Q^N_\alpha,$$
$$(\mu+p)(V^E_\alpha - B_\alpha) = Q^N_\alpha - 2Q^{N\beta}C_{\alpha\beta} - \frac{\delta\mu^N + \delta p^N}{\mu+p}Q^N_\alpha - Q^{N\beta}\frac{\Pi^N_{\alpha\beta}}{\mu+p} - (\mu+p)(AB_\alpha + 2B^\beta C_{\alpha\beta}),$$
$$\Pi^E_{\alpha\beta} = \Pi^N_{\alpha\beta} - \frac{1}{\mu+p}\left(Q^N_\alpha Q^N_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}Q^{N\gamma}Q^N_\gamma\right). \quad (88)$$

F. Spacetime curvature to linear order

Although straightforward, it is not an easy task to derive the spacetime curvature to second order. For our purpose, fortunately, it is not necessary to have the forms. Still, it is convenient to have the curvature to linear order and we present them in the following. These follow from Eqs. (49), (51),(52).

The curvature tensors are

$$\tilde{R}^a{}_{b00} = 0, \quad \tilde{R}^0{}_{00\alpha} = -\left(\frac{a'}{a}\right)' B_\alpha, \quad \tilde{R}^0{}_{0\alpha\beta} = 0,$$

$$\tilde{R}^0{}_{\alpha 0\beta} = \left(\frac{a'}{a}\right)' g^{(3)}_{\alpha\beta} - \left[\frac{a'}{a}A' + 2\left(\frac{a'}{a}\right)' A\right] g^{(3)}_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + \frac{a'}{a}B_{(\alpha|\beta)} + C''_{\alpha\beta} + \frac{a'}{a}C'_{\alpha\beta} + 2\left(\frac{a'}{a}\right)' C_{\alpha\beta},$$

$$\tilde{R}^0{}_{\alpha\beta\gamma} = 2\frac{a'}{a} g^{(3)}_{\alpha[\beta} A_{,\gamma]} - B_{\alpha|[\beta\gamma]} + \frac{1}{2}(B_{\gamma|\alpha\beta} - B_{\beta|\alpha\gamma}) - 2C'_{\alpha[\beta|\gamma]},$$

$$\tilde{R}^\alpha{}_{00\beta} = \left(\frac{a'}{a}\right)' \delta^\alpha_\beta - \frac{a'}{a} A' \delta^\alpha_\beta - A^{|\alpha}{}_\beta + \frac{1}{2}(B_\beta{}^{|\alpha} + B^\alpha{}_{|\beta})' + \frac{1}{2}\frac{a'}{a}(B_\beta{}^{|\alpha} + B^\alpha{}_{|\beta}) + C^{\alpha''}_\beta + \frac{a'}{a} C^{\alpha'}_\beta,$$

$$\tilde{R}^\alpha{}_{0\beta\gamma} = 2\frac{a'}{a}\delta^\alpha_{[\beta} A_{,\gamma]} - B_{[\beta}{}^{|\alpha}{}_{\gamma]} + B^\alpha{}_{|[\beta\gamma]} - 2\left(\frac{a'}{a}\right)^2 \delta^\alpha_{[\beta} B_{\gamma]} - 2C^{\alpha'}_{[\beta|\gamma]},$$

$$\tilde{R}^\alpha{}_{\beta 0\gamma} = \frac{a'}{a}(g^{(3)}_{\beta\gamma} A^{,\alpha} - \delta^\alpha_\gamma A_{,\beta}) + \left(\frac{a'}{a}\right)' g^{(3)}_{\beta\gamma} B^\alpha - \left(\frac{a'}{a}\right)^2 (g^{(3)}_{\beta\gamma} B^\alpha - \delta^\alpha_\gamma B_\beta) - \frac{1}{2}(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta})_{|\gamma} + C^{\alpha'}_{\gamma|\beta} - C'_{\beta\gamma}{}^{|\alpha},$$

$$\tilde{R}^\alpha{}_{\beta\gamma\delta} = R^{(3)\alpha}{}_{\beta\gamma\delta} + \left(\frac{a'}{a}\right)^2 (\delta^\alpha_\gamma g^{(3)}_{\beta\delta} - \delta^\alpha_\delta g^{(3)}_{\beta\gamma})(1-2A) + \frac{1}{2}\frac{a'}{a}[g^{(3)}_{\beta\delta}(B_\gamma{}^{|\alpha} + B^\alpha{}_{|\gamma}) - g^{(3)}_{\beta\gamma}(B_\delta{}^{|\alpha} + B^\alpha{}_{|\delta}) + 2\delta^\alpha_\gamma B_{(\beta|\delta)} - 2\delta^\alpha_\delta B_{(\beta|\gamma)}] + \frac{a'}{a}\left[g^{(3)}_{\beta\delta} C^{\alpha'}_\gamma - g^{(3)}_{\beta\gamma} C^{\alpha'}_\delta + \delta^\alpha_\gamma C'_{\beta\delta} - \delta^\alpha_\delta C'_{\beta\gamma} + 2\frac{a'}{a}(\delta^\alpha_\gamma C_{\beta\delta} - \delta^\alpha_\delta C_{\beta\gamma})\right] + 2C^\alpha_{(\beta|\delta)\gamma} - 2C^\alpha_{(\beta|\gamma)\delta} + C_{\beta\gamma}{}^{|\alpha}{}_\delta - C_{\beta\delta}{}^{|\alpha}{}_\gamma, \tag{89}$$

$$\tilde{R}_{00} = -3\left(\frac{a'}{a}\right)' + 3\frac{a'}{a}A' + \Delta A - B^{\alpha'}{}_{|\alpha} - \frac{a'}{a}B^\alpha{}_{|\alpha} - C^{\alpha''}_\alpha - \frac{a'}{a}C^{\alpha'}_\alpha,$$

$$\tilde{R}_{0\alpha} = 2\frac{a'}{a}A_{,\alpha} - \left(\frac{a'}{a}\right)' B_\alpha - 2\left(\frac{a'}{a}\right)^2 B_\alpha + \frac{1}{2}\Delta B_\alpha - \frac{1}{2}B^\beta{}_{|\alpha\beta} - C^{\beta'}_{\beta|\alpha} + C'_{\alpha\beta}{}^{|\beta},$$

$$\tilde{R}_{\alpha\beta} = 2Kg^{(3)}_{\alpha\beta} + \left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right] g^{(3)}_{\alpha\beta}(1-2A) - \frac{a'}{a}A' g^{(3)}_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + 2\frac{a'}{a}B_{(\alpha|\beta)} + \frac{a'}{a}g^{(3)}_{\alpha\beta}B^\gamma{}_{|\gamma} + C''_{\alpha\beta} + 2\frac{a'}{a}C'_{\alpha\beta} + 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right] C_{\alpha\beta} + \frac{a'}{a} g^{(3)}_{\alpha\beta} C^{\gamma'}_\gamma + 2C^\gamma_{(\alpha|\beta)\gamma} - C^\gamma_{\gamma|\alpha\beta} - \Delta C_{\alpha\beta}, \tag{90}$$

$$\tilde{R} = \frac{1}{a^2}\left\{6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] - 6\frac{a'}{a}A' - 12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]A - 2\Delta A + 2B^{\alpha'}{}_{|\alpha} + 6\frac{a'}{a}B^\alpha{}_{|\alpha} + 2C^{\alpha''}_\alpha + 6\frac{a'}{a}C^{\alpha'}_\alpha - 4KC^\alpha_\alpha - 2\Delta C^\alpha_\alpha + 2C^{\alpha\beta}{}_{|\alpha\beta}\right\}. \tag{91}$$

The nonvanishing components of the electric and magnetic parts of the Weyl curvature in Eq. (20) are

$$\tilde{E}_{\alpha\beta} = -\tilde{C}^0{}_{\alpha 0\beta} = \frac{1}{2}A_{,\alpha|\beta} - \frac{1}{2}B'_{(\alpha|\beta)} - \frac{1}{2}C''_{\alpha\beta} - \frac{1}{2}\Delta C_{\alpha\beta} - 2KC_{\alpha\beta} + C^\gamma_{(\alpha|\beta)\gamma} - \frac{1}{2}C^\gamma_{\gamma|\alpha\beta} - \frac{1}{3}g^{(3)}_{\alpha\beta}\left(\frac{1}{2}\Delta A - \frac{1}{2}B^{\gamma'}{}_{|\gamma} - \frac{1}{2}C^{\gamma''}_\gamma - \Delta C^\gamma_\gamma - 2KC^\gamma_\gamma + C^{\gamma\delta}{}_{|\gamma\delta}\right)$$
$$= \frac{1}{2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)\left(\alpha - \varphi - \frac{1}{a}\chi' + \frac{a'}{a^2}\chi\right) - \frac{1}{2}\Psi^{(v)'}_{(\alpha|\beta)} - \frac{1}{2}[C^{(t)''}_{\alpha\beta} + (\Delta - 2K)C^{(t)}_{\alpha\beta}], \tag{92}$$

$$\tilde{H}_{\alpha\beta} = -\frac{1}{2}\tilde{\eta}_{0(\alpha}{}^{\gamma\delta}\tilde{C}^0{}_{\beta)\gamma\delta} = -\eta_{(\alpha}{}^{\gamma\delta}\left(\frac{1}{2}B_{\gamma|\beta)\delta} + C'_{\beta)\gamma|\delta}\right) = -\eta_{(\alpha}{}^{\gamma\delta}\left(\frac{1}{2}\Psi^{(v)}_{\gamma|\beta)\delta} + C^{(t)'}_{\beta)\gamma|\delta}\right), \tag{93}$$

where the symmetrization is only over α and β indices. The last steps are evaluated in decomposed forms which will be introduced in Sec. V. We introduced $\eta^{\alpha\beta\gamma}$ which is based on $g^{(3)}_{\alpha\beta}$ with $\eta^{\alpha\beta\gamma}\equiv\eta^{[\alpha\beta\gamma]}$ and $\eta^{123}\equiv 1/\sqrt{g^{(3)}}$. We have

$$\tilde{\eta}^{0\alpha\beta\gamma}=\frac{1}{a^4\sqrt{1+D}}\eta^{\alpha\beta\gamma},$$

$$\tilde{g}=-a^8(1+D)g^{(3)},$$

$$D\equiv 2A+2C^\alpha_\alpha+4AC^\alpha_\alpha+B_\alpha B^\alpha + 2C^\alpha_\alpha C^\beta_\beta-2C^\alpha_\beta C^\beta_\alpha, \tag{94}$$

which is valid to second order in the perturbation.

Deriving the electric and magnetic parts of the Weyl tensor to second order using Eqs. (19), (20) requires quite lengthy algebra. Instead, using Eqs. (30), (33) we can derive them easily. Evaluated in the normal frame we have

$$\begin{aligned}\tilde{E}_{\alpha\beta}=&\frac{1}{a}(1-A)\left(\overline{K}'_{\alpha\beta}-2\frac{a'}{a}\overline{K}_{\alpha\beta}\right)\\&-\frac{1}{a}(B_{(\alpha}{}^{|\gamma}-B^\gamma{}_{|(\alpha}+2C^{\gamma\prime}_{(\alpha})\overline{K}_{\beta)\gamma}+\frac{1}{a}\overline{K}_{\alpha\beta|\gamma}B^\gamma\\&-\frac{1}{a^2}g^{(3)\gamma\delta}\overline{K}_{\alpha\gamma}\overline{K}_{\beta\delta}-\frac{2}{3}K\overline{K}_{\alpha\beta}\\&+\left(A-A^2+\frac{1}{2}B^\gamma B_\gamma\right)_{,\alpha|\beta}-(2C^\gamma_{(\alpha|\beta)}-C_{\alpha\beta}{}^{|\gamma})A_{,\gamma}\\&+A_{,\alpha}A_{,\beta}-\frac{2}{3}C_{\alpha\beta}\Delta A+4\pi Ga^2\Pi_{\alpha\beta}-\frac{1}{3}g^{(3)}_{\alpha\beta}\\&\times\Bigg[-\frac{1}{a^2}g^{(3)\alpha\gamma}g^{(3)\beta\delta}\overline{K}_{\alpha\beta}\overline{K}_{\gamma\delta}+A_{,\gamma}(A^{,\gamma}-2C^{\gamma\delta}{}_{|\delta}\\&+C^{\delta|\gamma}_\delta)+\Delta\left(A-A^2+\frac{1}{2}B^\gamma B_\gamma\right)-2C^{\gamma\delta}A_{,\gamma|\delta}\Bigg],\end{aligned}\tag{95}$$

$$\begin{aligned}\tilde{H}_{\alpha\beta}=&\{[g^{(3)}_{\delta(\beta}(1-C^\nu_\nu)+2C_{\delta(\beta}]\overline{K}_{\alpha)\gamma|\mu}\\&-(C^\nu_{(\alpha|\mu}+C^\nu_{\mu|(\alpha}-C_{\mu(\alpha}{}^{|\nu})g^{(3)}_{\beta)\delta}\overline{K}_{\gamma\nu}\}\frac{1}{a}\eta^{\delta\gamma\mu},\end{aligned}\tag{96}$$

where K and $\overline{K}_{\alpha\beta}$ are given in Eq. (57); the symmetrization is over only α and β indices. We have $\tilde{E}_{ab}\tilde{u}^b=0=\tilde{H}_{ab}\tilde{u}^b$, and thus $\tilde{E}_{\alpha 0}=-\tilde{E}_{\alpha\beta}B^\beta$, $\tilde{E}_{00}=0$, and similarly for $\tilde{H}_{ab}$.

For later use, it is convenient to have the spacetime scalar curvature expanded to second order. In terms of the ADM notation, using Eq. (6), we have

$$\tilde{R}=R^{(h)}+K^{\alpha\beta}K_{\alpha\beta}+K^2+\frac{2}{N}(-K_{,0}+K_{,\alpha}N^\alpha-N^{:\alpha}{}_\alpha). \tag{97}$$

To second order in the perturbation, using quantities in Sec. III B we have

$$\begin{aligned}\tilde{R}\equiv R+\delta R=&6\left(\frac{K}{a^2}+\dot{H}+2H^2\right)-6H\dot{A}-12(\dot{H}+2H^2)A-2\frac{\Delta}{a^2}A+2\left(\frac{1}{a}B^\alpha{}_{|\alpha}+\dot{C}^\alpha_\alpha\right)^{\cdot}+8H\left(\frac{1}{a}B^\alpha{}_{|\alpha}+\dot{C}^\alpha_\alpha\right)\\&+2\frac{1}{a^2}[C^{\beta|\alpha}_\alpha{}_\beta-(\Delta+2K)C^\alpha_\alpha]+24HA\dot{A}-4A\left(\frac{1}{a}B^\alpha{}_{|\alpha}+\dot{C}^\alpha_\alpha\right)^{\cdot}-2(\dot{A}+8HA)\left(\frac{1}{a}B^\alpha{}_{|\alpha}+\dot{C}^\alpha_\alpha\right)\\&+24(\dot{H}+2H^2)A^2+4A\frac{\Delta}{a^2}A+2\frac{1}{a^2}A^{,\alpha}A_{,\alpha}-6H\frac{1}{a}A_{,\alpha}B^\alpha+4\frac{1}{a^2}A_{,\alpha|\beta}C^{\alpha\beta}+2\frac{1}{a^2}A_{,\beta}(2C^{\alpha\beta}{}_{|\alpha}-C^{\alpha|\beta}_\alpha)\\&+\frac{1}{a^2}[B_{(\alpha|\beta)}B^{\alpha|\beta}+B^\alpha{}_{|\alpha}B^\beta{}_{|\beta}-2(B_\alpha B^{\alpha|\beta})_{|\beta}]-6HB^\alpha\dot{B}_\alpha-6(\dot{H}+2H^2)B^\alpha B_\alpha-2\frac{1}{a}B^\alpha(2C^\beta_{\alpha|\beta}-C^\beta_{\beta|\alpha})^{\cdot}\\&+2\frac{1}{a}B^\alpha{}_{|\alpha}\dot{C}^\beta_\beta-2\frac{1}{a}(2C^\beta_{\alpha|\beta}-C^\beta_{\beta|\alpha})(\dot{B}^\alpha+3HB^\alpha)-4\frac{1}{a}C^{\alpha\beta}\dot{B}_{\alpha|\beta}-2\frac{1}{a}(\dot{C}^{\alpha\beta}+6HC^{\alpha\beta})B_{\alpha|\beta}+2\frac{1}{a}B^\alpha\left(\frac{1}{a}B^\beta{}_{|\beta}+\dot{C}^\beta_\beta\right)_{|\alpha}\\&+\dot{C}^\alpha_\alpha\dot{C}^\beta_\beta-3\dot{C}^{\alpha\beta}\dot{C}_{\alpha\beta}-4C^{\alpha\beta}\left(\ddot{C}_{\alpha\beta}+4H\dot{C}_{\alpha\beta}-2\frac{K}{a^2}C_{\alpha\beta}\right)+4\frac{1}{a^2}C^{\alpha\beta}(-C^\gamma_{\alpha|\beta\gamma}-C^\gamma_{\alpha|\gamma\beta}+\Delta C_{\alpha\beta}+C^\gamma_{\gamma|\alpha\beta})\\&-\frac{1}{a^2}(2C^\gamma_{\beta|\gamma}-C^\gamma_{\gamma|\beta})(2C^{\alpha\beta}{}_{|\alpha}-C^{\alpha|\beta}_\alpha)+\frac{1}{a^2}C^{\alpha\beta|\gamma}(3C_{\alpha\beta|\gamma}-2C_{\alpha\gamma|\beta}).\end{aligned}\tag{98}$$

An overdot indicates a time derivative with respect to t, with $dt=ad\eta$.

IV. PERTURBED EQUATIONS

A. Basic equations with general fluids

In the following we present complete sets of equations valid up to second order in the perturbation without fixing the gauge conditions. As the basic set we consider Eqs. (5),(8)–(13),(47),(48) in the ADM formulation.

The definition of δK is

$$\begin{aligned}&\overline{K}+3H+\delta K-3HA+\dot{C}^{\alpha}_{\alpha}+\frac{1}{a}B^{\alpha}{}_{|\alpha}\\&\quad=-A\left(\frac{9}{2}HA-\dot{C}^{\alpha}_{\alpha}-\frac{1}{a}B^{\alpha}{}_{|\alpha}\right)+\frac{3}{2}HB^{\alpha}B_{\alpha}\\&\qquad+\frac{1}{a}B^{\alpha}(2C^{\beta}_{\alpha|\beta}-C^{\beta}_{\beta|\alpha})+2C^{\alpha\beta}\left(\dot{C}_{\alpha\beta}+\frac{1}{a}B_{\alpha|\beta}\right)\\&\quad\equiv N_0,\end{aligned}\tag{99}$$

where $K\equiv\overline{K}+\delta K$ and K is read from Eq. (57).

The energy constraint equation is

$$\begin{aligned}&16\pi G\mu+2\Lambda-6H^2-\frac{1}{a^2}R^{(3)}+16\pi G\,\delta\mu+4H\,\delta K-\frac{1}{a^2}\left(2C^{\beta|\alpha}_{\alpha}{}_{\beta}-2C^{\alpha|\beta}_{\alpha}{}_{\beta}-\frac{2}{3}R^{(3)}C^{\alpha}_{\alpha}\right)\\&\quad=\frac{2}{3}\delta K^2-\left(\dot{C}_{\alpha\beta}+\frac{1}{a}B_{(\alpha|\beta)}\right)\left(\dot{C}^{\alpha\beta}+\frac{1}{a}B^{\alpha|\beta}\right)+\frac{1}{3}\left(\dot{C}^{\alpha}_{\alpha}+\frac{1}{a}B^{\alpha}{}_{|\alpha}\right)^2\\&\qquad+\frac{1}{a^2}\Big[4C^{\alpha\beta}(-C^{\gamma}_{\alpha|\beta\gamma}-C^{\gamma}_{\alpha|\gamma\beta}+C_{\alpha\beta}{}^{|\gamma}{}_{\gamma}+C^{\gamma}_{\gamma|\alpha\beta})+\frac{4}{3}R^{(3)}C^{\alpha}_{\gamma}C^{\gamma}_{\alpha}\\&\qquad-(2C^{\gamma}_{\beta|\gamma}-C^{\gamma}_{\gamma|\beta})(2C^{\alpha\beta}{}_{|\alpha}-C^{\alpha|\beta}_{\alpha})\quad+C^{\alpha\beta|\gamma}(3C_{\alpha\beta|\gamma}-2C_{\alpha\gamma|\beta})\Big]\\&\quad\equiv N_1.\end{aligned}\tag{100}$$

The momentum constraint equation is

$$\begin{aligned}&\left[\dot{C}^{\beta}_{\alpha}+\frac{1}{2a}(B^{\beta}{}_{|\alpha}+B_{\alpha}{}^{|\beta})\right]_{|\beta}-\frac{1}{3}\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)_{,\alpha}+\frac{2}{3}\delta K_{,\alpha}+8\pi GaQ_{\alpha}\\&\quad=A\left(-\frac{2}{3}\delta K_{,\alpha}-8\pi GaQ_{\alpha}\right)+A_{,\beta}\left[\dot{C}^{\beta}_{\alpha}+\frac{1}{2a}(B^{\beta}{}_{|\alpha}+B_{\alpha}{}^{|\beta})\right]+(2C^{\beta}_{\gamma|\beta}-C^{\beta}_{\beta|\gamma})\left[\dot{C}^{\gamma}_{\alpha}+\frac{1}{2a}(B_{\alpha}{}^{|\gamma}+B^{\gamma}{}_{|\alpha})\right]\\&\qquad+2C^{\beta\gamma}\left(\dot{C}_{\alpha\gamma}+\frac{1}{a}B_{(\alpha|\gamma)}\right)_{|\beta}+\frac{1}{a}[B_{\gamma}(C^{\beta\gamma}{}_{|\alpha}+C^{\gamma|\beta}_{\alpha}-C^{\beta|\gamma}_{\alpha})]_{|\beta}+\frac{1}{3}C^{\gamma}_{\beta|\alpha}\left(\dot{C}^{\beta}_{\gamma}+\frac{1}{a}B^{\beta}{}_{|\gamma}\right)\\&\qquad-\frac{1}{3}\left\{A_{,\alpha}\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)+2C^{\gamma\delta}\left(\dot{C}_{\gamma\delta}+\frac{1}{a}B_{\gamma|\delta}\right)_{|\alpha}+\frac{1}{a}[B^{\delta}(2C^{\gamma}_{\delta|\gamma}-C^{\gamma}_{\gamma|\delta})]_{|\alpha}\right\}\\&\quad\equiv N_{2\alpha}.\end{aligned}\tag{101}$$

The trace of the ADM propagation equation is

$$\begin{aligned}&-[3\dot{H}+3H^2+4\pi G(\mu+3p)-\Lambda]+\delta\dot{K}+2H\,\delta K-4\pi G(\delta\mu+3\,\delta p)+\frac{1}{a^2}A^{|\alpha}{}_{\alpha}+3\dot{H}A\\&\quad=A\,\delta\dot{K}-\frac{1}{a}\delta K_{,\alpha}B^{\alpha}+\frac{1}{3}\delta K^2+\frac{3}{2}\dot{H}(3A^2-B^{\alpha}B_{\alpha})+\frac{1}{a^2}[2AA^{|\alpha}{}_{\alpha}+A_{,\alpha}A^{,\alpha}-B^{\beta}B_{\beta}{}^{|\alpha}{}_{\alpha}-B^{\beta|\alpha}B_{\beta|\alpha}+A^{,\alpha}(2C^{\beta}_{\alpha|\beta}-C^{\beta}_{\beta|\alpha})\\&\qquad+2C^{\alpha\beta}A_{,\alpha|\beta}]+\left(\dot{C}_{\alpha\beta}+\frac{1}{a}B_{(\alpha|\beta)}\right)\left(\dot{C}^{\alpha\beta}+\frac{1}{a}B^{\alpha|\beta}\right)-\frac{1}{3}\left(\dot{C}^{\alpha}_{\alpha}+\frac{1}{a}B^{\alpha}{}_{|\alpha}\right)^2\\&\quad\equiv N_3.\end{aligned}\tag{102}$$

The trace-free ADM propagation equation is

$$
\begin{aligned}
&\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]^{\cdot}+3H\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]-\frac{1}{a^{2}}A^{|\alpha}{}_{\beta}-\frac{1}{3}\delta^{\alpha}_{\beta}\left[\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)^{\cdot}+3H\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)-\frac{1}{a^{2}}A^{|\gamma}{}_{\gamma}\right]\\
&\quad+\frac{1}{a^{2}}\left[C^{\alpha\gamma}{}_{|\beta\gamma}+C^{\gamma}_{\beta}{}^{|\alpha}{}_{\gamma}-C^{\alpha|\gamma}_{\beta}{}_{\gamma}-C^{\gamma|\alpha}_{\gamma}{}_{\beta}-\frac{2}{3}R^{(3)}C^{\alpha}_{\beta}-\frac{1}{3}\delta^{\alpha}_{\beta}\left(2C^{\delta|\gamma}_{\gamma}{}_{\delta}-2C^{\gamma|\delta}_{\gamma}{}_{\delta}-\frac{2}{3}R^{(3)}C^{\gamma}_{\gamma}\right)\right]-8\pi G\Pi^{\alpha}_{\beta}\\
&=\left\{\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]A+2C^{\alpha\gamma}\left(\dot{C}_{\beta\gamma}+\frac{1}{a}B_{(\beta|\gamma)}\right)+\frac{1}{a}B_{\gamma}(C^{\alpha\gamma}{}_{|\beta}+C^{\gamma|\alpha}_{\beta}-C^{\alpha|\gamma}_{\beta})\right\}^{\cdot}\\
&\quad+3H\left\{\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]A+2C^{\alpha\gamma}\left(\dot{C}_{\beta\gamma}+\frac{1}{a}B_{(\beta|\gamma)}\right)+\frac{1}{a}B_{\gamma}(C^{\alpha\gamma}{}_{|\beta}+C^{\gamma|\alpha}_{\beta}-C^{\alpha|\gamma}_{\beta})\right\}\\
&\quad+\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]^{\cdot}A-\frac{1}{a}\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]_{|\gamma}B^{\gamma}+\delta K\left[\dot{C}^{\alpha}_{\beta}+\frac{1}{2a}(B^{\alpha}{}_{|\beta}+B_{\beta}{}^{|\alpha})\right]\\
&\quad+\frac{1}{a^{2}}\left[-AA^{|\alpha}{}_{\beta}+\frac{1}{2}(-A^{2}+B^{\gamma}B_{\gamma})^{|\alpha}{}_{\beta}-2C^{\alpha\gamma}A_{,\beta|\gamma}-(C^{\alpha\gamma}{}_{|\beta}+C^{\gamma|\alpha}_{\beta}-C^{\alpha|\gamma}_{\beta})A_{,\gamma}\right]\\
&\quad-\frac{1}{3}\delta^{\alpha}_{\beta}\left\{\left[\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)A+2C^{\gamma\delta}\left(\dot{C}_{\gamma\delta}+\frac{1}{a}B_{\gamma|\delta}\right)+\frac{1}{a}B^{\delta}(2C^{\gamma}_{\delta|\gamma}-C^{\gamma}_{\gamma|\delta})\right]^{\cdot}\right.\\
&\quad+3H\left[\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)A+2C^{\gamma\delta}\left(\dot{C}_{\gamma\delta}+\frac{1}{a}B_{\gamma|\delta}\right)+\frac{1}{a}B^{\delta}(2C^{\gamma}_{\delta|\gamma}-C^{\gamma}_{\gamma|\delta})\right]+\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)^{\cdot}A-\frac{1}{a}\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)_{|\delta}B^{\delta}\\
&\quad\left.+\delta K\left(\dot{C}^{\gamma}_{\gamma}+\frac{1}{a}B^{\gamma}{}_{|\gamma}\right)+\frac{1}{a^{2}}\left[-AA^{|\gamma}{}_{\gamma}+\frac{1}{2}(-A^{2}+B^{\delta}B_{\delta})^{|\gamma}{}_{\gamma}-2C^{\gamma\delta}A_{,\gamma|\delta}-(2C^{\gamma\delta}{}_{|\gamma}-C^{\gamma|\delta}_{\gamma})A_{,\delta}\right]\right\}\\
&\quad+\frac{1}{a}B^{\alpha}{}_{|\gamma}\left[\dot{C}^{\gamma}_{\beta}+\frac{1}{2a}(B^{\gamma}{}_{|\beta}+B_{\beta}{}^{|\gamma})\right]-\frac{1}{a}B^{\gamma}{}_{|\beta}\left[\dot{C}^{\alpha}_{\gamma}+\frac{1}{2a}(B^{\alpha}{}_{|\gamma}+B_{\gamma}{}^{|\alpha})\right]\\
&\quad+\frac{1}{a^{2}}\left\{2C^{\gamma\delta}(C^{\alpha}_{\delta|\beta\gamma}+C_{\delta\beta}{}^{|\alpha}{}_{\gamma}-C^{\alpha}_{\beta|\delta\gamma}-C_{\delta\gamma}{}^{|\alpha}{}_{\beta})+2C^{\alpha\gamma}(C^{\delta}_{\gamma|\beta\delta}+C^{\delta}_{\beta|\gamma\delta}-C_{\beta\gamma}{}^{|\delta}{}_{\delta}-C^{\delta}_{\delta|\gamma\beta})\right.\\
&\quad-\frac{4}{3}R^{(3)}C^{\alpha}_{\gamma}C^{\gamma}_{\beta}+(2C^{\gamma}_{\delta|\gamma}-C^{\gamma}_{\gamma|\delta})(C^{\alpha\delta}{}_{|\beta}+C^{\delta|\alpha}_{\beta}-C^{\alpha|\delta}_{\beta})-C_{\gamma\delta|\beta}C^{\gamma\delta|\alpha}+2C^{\alpha\gamma|\delta}(C_{\beta\delta|\gamma}-C_{\beta\gamma|\delta})-\frac{1}{3}\delta^{\alpha}_{\beta}\left[4C^{\gamma\delta}(C^{\epsilon}_{\gamma|\delta\epsilon}+C^{\epsilon}_{\gamma|\epsilon\delta}\right.\\
&\quad\left.\left.-C_{\gamma\delta}{}^{|\epsilon}{}_{\epsilon}-C^{\epsilon}_{\epsilon|\gamma\delta})-\frac{4}{3}R^{(3)}C^{\delta}_{\gamma}C^{\gamma}_{\delta}+(2C^{\epsilon}_{\delta|\epsilon}-C^{\epsilon}_{\epsilon|\delta})(2C^{\gamma\delta}{}_{|\gamma}-C^{\gamma|\delta}_{\gamma})+C^{\gamma\delta|\epsilon}(2C_{\gamma\epsilon|\delta}-3C_{\gamma\delta|\epsilon})\right]\right\}-16\pi GC^{\alpha\gamma}\Pi_{\beta\gamma}\\
&\equiv N_{4}{}^{\alpha}_{\beta}.
\end{aligned}
\tag{103}
$$

The energy conservation equation is

$$
\begin{aligned}
&\dot{\mu}+3H(\mu+p)+\delta\dot{\mu}+3H(\delta\mu+\delta p)-(\mu+p)(\delta K-3HA)+\frac{1}{a}Q^{\alpha}{}_{|\alpha}\\
&=-\frac{1}{a}\delta\mu_{,\alpha}B^{\alpha}+(\delta\mu+\delta p)(\delta K-3HA)+(\mu+p)\left[A\delta K+\frac{3}{2}H(A^{2}-B^{\alpha}B_{\alpha})\right]\\
&\quad-\frac{1}{a}[AQ^{\alpha}{}_{|\alpha}+Q^{\alpha}(2A_{,\alpha}+C^{\beta}_{\beta|\alpha}-2C^{\beta}_{\alpha|\beta})-2C^{\alpha\beta}Q_{\alpha|\beta}]-\Pi^{\alpha\beta}\left(\dot{C}_{\alpha\beta}+\frac{1}{a}B_{\alpha|\beta}\right)\\
&\equiv N_{5}.
\end{aligned}
\tag{104}
$$

The momentum conservation equation is

$$\dot{Q}_\alpha + 4HQ_\alpha + \frac{1}{a}[(\mu+p)A_{,\alpha} + \delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta}]$$
$$= Q_\alpha(\delta K - 3HA) + \frac{1}{a}\{-Q_{\alpha|\beta}B^\beta - Q_\beta B^\beta{}_{|\alpha} - (\delta\mu + \delta p)A_{,\alpha} + A[(\mu+p)A_{,\alpha} - \delta p_{,\alpha} - \Pi^\beta_{\alpha|\beta}]$$
$$-(\mu+p)B^\beta B_{\beta|\alpha} + 2(C^{\gamma\beta}\Pi_{\alpha\gamma})_{|\beta} - \Pi^\gamma_\alpha C^\beta_{\beta|\gamma} + \Pi^\beta_\gamma C^\gamma_{\beta|\alpha} - A_{,\beta}\Pi^\beta_\alpha\}$$
$$\equiv N_{6\alpha}. \tag{105}$$

In the multicomponent situation we additionally have the energy and momentum conservation of individual components in Eqs. (47),(48).

The energy conservation equation for the ith component is

$$\dot{\mu}_{(i)} + 3H(\mu_{(i)} + p_{(i)}) + \frac{1}{a}I_{(i)0} + \delta\dot{\mu}_{(i)} + 3H(\delta\mu_{(i)} + \delta p_{(i)}) - (\mu_{(i)} + p_{(i)})(\delta K - 3HA) + \frac{1}{a}Q^\alpha_{(i)|\alpha} + \frac{1}{a}\delta I_{(i)0}$$
$$= -\frac{1}{a}\delta\mu_{(i),\alpha}B^\alpha + (\delta\mu_{(i)} + \delta p_{(i)})(\delta K - 3HA) + (\mu_{(i)} + p_{(i)})A\,\delta K + \frac{3}{2}H(\mu_{(i)} + p_{(i)})(A^2 - B^\alpha B_\alpha)$$
$$+ \frac{1}{a}[-Q^\alpha_{(i)|\alpha}A + 2(C^{\alpha\beta}Q_{(i)\beta})_{|\alpha} - C^{\alpha|\beta}_\alpha Q_{(i)\beta} - 2A_{,\alpha}Q^\alpha_{(i)}] - \Pi^{\alpha\beta}_{(i)}\left(\frac{1}{a}B_{\alpha|\beta} + \dot{C}_{\alpha\beta}\right) - \frac{1}{a}\delta I_{(i)\alpha}B^\alpha$$
$$\equiv N_{(i)5}. \tag{106}$$

The momentum conservation equation for the ith component is

$$\dot{Q}_{(i)\alpha} + 4HQ_{(i)\alpha} + \frac{1}{a}(\mu_{(i)} + p_{(i)})A_{,\alpha} + \frac{1}{a}(\delta p_{(i),\alpha} + \Pi_{(i)\alpha|\beta}^{\ \ \beta} - \delta I_{(i)\alpha})$$
$$= \frac{1}{a}\{-(\delta p_{(i),\alpha} + \Pi_{(i)\alpha|\beta}^{\ \ \beta} - \delta I_{(i)\alpha})A - (\delta\mu_{(i)} + \delta p_{(i)})A_{,\alpha} + (\mu_{(i)} + p_{(i)})(AA_{,\alpha} - B^\beta B_{\beta|\alpha})$$
$$-Q_{(i)\alpha|\beta}B^\beta - Q_{(i)\beta}B^\beta{}_{|\alpha} + a(\delta K - 3HA)Q_{(i)\alpha} + 2(C^{\beta\gamma}\Pi_{(i)\alpha\gamma})_{|\beta} - C^\beta_{\beta|\gamma}\Pi_{(i)\alpha}{}^\gamma + C^\gamma_{\beta|\alpha}\Pi_{(i)\gamma}{}^\beta - A_{,\beta}\Pi_{(i)\alpha}{}^\beta\}$$
$$\equiv N_{(i)6\alpha}. \tag{107}$$

The collective fluid quantities are given in Eq. (75). The equations are presented with the quadratic combination of the linear order terms located on the right-hand side. Still, notice that the equations are presented to second order without separating the background order part.

Equations (99)–(107) provide a complete set valid for the Einstein gravity with an imperfect fluid; thus the most general form of energy-momentum tensor. We have not imposed any condition like a gauge condition. In the following subsections we will consider the cases of minimally coupled scalar fields, an electromagnetic field, and a broad class of generalized gravity theories. We emphasize that even in these additional fields or generalized gravity the above equations *remain valid*, with the fluid quantities reinterpreted to absorb the contributions from the fields and the generalized gravity.

B. Scalar field

1. Covariant equations

The action for a minimally coupled scalar field is given by

$$S = \int \sqrt{-\tilde{g}}\left[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\tilde{\phi}^{;c}\tilde{\phi}_{,c} - \tilde{V}(\tilde{\phi})\right]d^4x. \tag{108}$$

The equation of motion follows from the variation in $\tilde{\phi}$:

$$\tilde{\phi}^{;c}{}_{c} - \tilde{V}_{,\tilde{\phi}} = 0, \tag{109}$$

where $\tilde{V}_{,\tilde{\phi}} \equiv \partial \tilde{V}/\partial \tilde{\phi}$. From $\delta_{\tilde{g}_{ab}} \mathcal{L}_M \equiv \frac{1}{2}\sqrt{-\tilde{g}}\tilde{T}^{ab}\delta \tilde{g}_{ab}$ we have the energy-momentum tensor

$$\tilde{T}^{(\phi)}_{ab} = \tilde{\phi}_{,a}\tilde{\phi}_{,b} - \frac{1}{2}\tilde{g}_{ab}\tilde{\phi}^{,c}\tilde{\phi}_{,c} - \tilde{g}_{ab}\tilde{V}(\tilde{\phi}). \tag{110}$$

2. Perturbations

We decompose

$$\tilde{\phi} \equiv \phi + \delta\phi. \tag{111}$$

The equation of motion becomes

$$\begin{aligned}
&\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + \delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi + V_{,\phi\phi}\delta\phi - 2A\ddot{\phi} - \dot{\phi}\left(\dot{A} + 6HA - \frac{1}{a}B^{\alpha}{}_{|\alpha} - \dot{C}^{\alpha}_{\alpha}\right)\\
&= 2A\left[\delta\ddot{\phi} + 3H\delta\dot{\phi} - 2A\ddot{\phi} - \dot{\phi}\left(2\dot{A} + 6HA - \frac{1}{a}B^{\alpha}{}_{|\alpha} - \dot{C}^{\alpha}_{\alpha}\right)\right] + \delta\dot{\phi}\left(\dot{A} - \frac{1}{a}B^{\alpha}{}_{|\alpha} - \dot{C}^{\alpha}_{\alpha}\right)\\
&\quad - 2\frac{1}{a}B^{\alpha}\delta\dot{\phi}_{,\alpha} + \frac{1}{a}\delta\phi_{,\alpha}\left(\frac{1}{a}A^{,\alpha} - \dot{B}^{\alpha} - 2HB^{\alpha} - 2\frac{1}{a}C^{\alpha\beta}{}_{|\beta} + \frac{1}{a}C^{\beta|\alpha}_{\beta}\right) - 2\frac{1}{a^2}\delta\phi_{,\alpha|\beta}C^{\alpha\beta} - \frac{1}{2}V_{,\phi\phi\phi}\delta\phi^2 + \ddot{\phi}B_{\alpha}B^{\alpha}\\
&\quad + \dot{\phi}\left[\frac{1}{a}A_{,\alpha}B^{\alpha} + B_{\alpha}\dot{B}^{\alpha} + 3HB^{\alpha}B_{\alpha} + \frac{1}{a}B_{\beta}(2C^{\alpha\beta}{}_{|\alpha} - C^{\alpha|\beta}_{\alpha}) + 2C^{\alpha\beta}\left(\frac{1}{a}B_{\alpha|\beta} + \dot{C}_{\alpha\beta}\right)\right]\\
&\equiv N_{\phi}.
\end{aligned} \tag{112}$$

The energy-momentum tensor gives

$$\begin{aligned}
\tilde{T}^{(\phi)}_{00} &= \frac{1}{2}\phi'^2 + a^2V + \phi'\delta\phi' + a^2(V_{,\phi}\delta\phi + 2VA) + \frac{1}{2}\delta\phi'^2 + \frac{1}{2}\delta\phi_{,\alpha}\delta\phi^{,\alpha}\\
&\quad + \frac{1}{2}a^2V_{,\phi\phi}\delta\phi^2 + 2a^2V_{,\phi}\delta\phi A - \phi'\delta\phi_{,\alpha}B^{\alpha} + \frac{1}{2}\phi'^2B_{\alpha}B^{\alpha},\\
\tilde{T}^{(\phi)}_{0\alpha} &= \phi'\delta\phi_{,\alpha} - \left(\frac{1}{2}\phi'^2 - a^2V\right)B_{\alpha} + \delta\phi'\delta\phi_{,\alpha} + (-\phi'\delta\phi' + a^2V_{,\phi}\delta\phi + \phi'^2A)B_{\alpha},\\
\tilde{T}^{(\phi)}_{\alpha\beta} &= g^{(3)}_{\alpha\beta}\left(\frac{1}{2}\phi'^2 - a^2V\right) + g^{(3)}_{\alpha\beta}(\phi'\delta\phi' - a^2V_{,\phi}\delta\phi - \phi'^2A) + (\phi'^2 - 2a^2V)C_{\alpha\beta} + \delta\phi_{,\alpha}\delta\phi_{,\beta}\\
&\quad + 2[\phi'\delta\phi' - a^2V_{,\phi}\delta\phi - \phi'^2A]C_{\alpha\beta} - \frac{1}{2}g^{(3)}_{\alpha\beta}[-\delta\phi'^2 + \delta\phi_{,\gamma}\delta\phi^{,\gamma} + a^2V_{,\phi\phi}\delta\phi^2 + 4\phi'\delta\phi'A\\
&\quad - 2\phi'\delta\phi_{,\gamma}B^{\gamma} + \phi'^2(-4A^2 + B_{\gamma}B^{\gamma})].
\end{aligned} \tag{113}$$

Fluid quantities can be read from Eq. (79) as

$$\begin{aligned}
\mu^{(\phi)} + \delta\mu^{(\phi)} &= \frac{1}{2}\dot{\phi}^2 + V + \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2A + V_{,\phi}\delta\phi + \frac{1}{2}\delta\dot{\phi}^2\\
&\quad + \frac{1}{2a^2}\delta\phi_{,\alpha}\delta\phi^{,\alpha} + \frac{1}{2}V_{,\phi\phi}\delta\phi^2 - 2\dot{\phi}\delta\dot{\phi}A\\
&\quad + \frac{1}{a}\dot{\phi}\delta\phi_{,\alpha}B^{\alpha} + 2\dot{\phi}^2A^2 - \frac{1}{2}\dot{\phi}^2B^{\alpha}B_{\alpha},
\end{aligned}$$

$$\begin{aligned}
p^{(\phi)} + \delta p^{(\phi)} &= \frac{1}{2}\dot{\phi}^2 - V + \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2A - V_{,\phi}\delta\phi + \frac{1}{2}\delta\dot{\phi}^2\\
&\quad - \frac{1}{6a^2}\delta\phi_{,\alpha}\delta\phi^{,\alpha} - \frac{1}{2}V_{,\phi\phi}\delta\phi^2 - 2\dot{\phi}\delta\dot{\phi}A\\
&\quad + \frac{1}{a}\dot{\phi}\delta\phi_{,\alpha}B^{\alpha} + 2\dot{\phi}^2A^2 - \frac{1}{2}\dot{\phi}^2B^{\alpha}B_{\alpha},
\end{aligned}$$

$$Q^{(\phi)}_{\alpha} = -\frac{1}{a}[\dot{\phi}\delta\phi_{,\alpha} + \delta\phi_{,\alpha}(\delta\dot{\phi} - \dot{\phi}A)],$$

$$\Pi^{(\phi)}_{\alpha\beta} = \frac{1}{a^2}\left(\delta\phi_{,\alpha}\delta\phi_{,\beta} - \frac{1}{3}g^{(3)}_{\alpha\beta}\delta\phi_{,\gamma}\delta\phi^{,\gamma}\right). \tag{114}$$

We indicate the quadratic parts (the quadratic combinations of two linear-order terms) as $\delta\mu^{(q)}$, $\delta p^{(q)}$, $Q^{(q)}_\alpha$, and $\Pi^{(q)}_{\alpha\beta}$.

C. Scalar fields

1. Covariant equations

The action for multiple components of minimally coupled scalar fields is

$$S = \int \sqrt{-\tilde{g}}\left[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\sum_k \tilde{\phi}_{(k)}{}^{,c}\tilde{\phi}_{(k),c} - \tilde{V}(\tilde{\phi}_{(l)})\right]d^4x, \tag{115}$$

where $i,j,\ldots = 1,2,\ldots,n$ indicate the n scalar fields. The equation of motion for the ith component is

$$\tilde{\phi}_{(i)}{}^{;c}{}_{c} - \tilde{V}_{,\tilde{\phi}_{(i)}} = 0. \tag{116}$$

The energy-momentum tensor is

$$\tilde{T}^{(\phi)}_{ab} = \sum_k \left(\tilde{\phi}_{(k),a}\tilde{\phi}_{(k),b} - \frac{1}{2}\tilde{g}_{ab}\tilde{\phi}_{(k)}{}^{;c}\tilde{\phi}_{(k),c}\right) - \tilde{g}_{ab}\tilde{V}(\tilde{\phi}_{(l)}). \tag{117}$$

2. Perturbations

We introduce

$$\tilde{\phi}_{(i)} \equiv \phi_{(i)} + \delta\phi_{(i)}. \tag{118}$$

The equation of motion for the ith component becomes

$$\begin{aligned}
&\ddot{\phi}_{(i)} + 3H\dot{\phi}_{(i)} + V_{,\phi_{(i)}} + \delta\ddot{\phi}_{(i)} + 3H\delta\dot{\phi}_{(i)} - \frac{\Delta}{a^2}\delta\phi_{(i)} + \sum_k V_{,\phi_{(i)}\phi_{(k)}}\delta\phi_{(k)} - 2A\ddot{\phi}_{(i)} - \dot{\phi}_{(i)}\left(\dot{A} + 6HA - \frac{1}{a}B^\alpha{}_{|\alpha} - \dot{C}^\alpha_\alpha\right)\\
&= 2A\left[\delta\ddot{\phi}_{(i)} + 3H\delta\dot{\phi}_{(i)} - 2A\ddot{\phi}_{(i)} - \dot{\phi}_{(i)}\left(2\dot{A} + 6HA - \frac{1}{a}B^\alpha{}_{|\alpha} - \dot{C}^\alpha_\alpha\right)\right] + \delta\dot{\phi}_{(i)}\left(\dot{A} - \frac{1}{a}B^\alpha{}_{|\alpha} - \dot{C}^\alpha_\alpha\right) - 2\frac{1}{a}B^\alpha\delta\dot{\phi}_{(i),\alpha}\\
&\quad + \frac{1}{a}\delta\phi_{(i),\alpha}\left(\frac{1}{a}A^{,\alpha} - \dot{B}^\alpha - 2HB^\alpha - 2\frac{1}{a}C^{\alpha\beta}{}_{|\beta} + \frac{1}{a}C^{\beta|\alpha}_\beta\right) - 2\frac{1}{a^2}\delta\phi_{(i),\alpha|\beta}C^{\alpha\beta} - \frac{1}{2}\sum_{k,l} V_{,\phi_{(i)}\phi_{(k)}\phi_{(l)}}\delta\phi_{(k)}\delta\phi_{(l)} + \ddot{\phi}_{(i)}B_\alpha B^\alpha\\
&\quad + \dot{\phi}_{(i)}\left[\frac{1}{a}A_{,\alpha}B^\alpha + B_\alpha\dot{B}^\alpha + 3HB^\alpha B_\alpha + \frac{1}{a}B_\beta(2C^{\alpha\beta}{}_{|\alpha} - C^{\alpha|\beta}_\alpha) + 2C^{\alpha\beta}\left(\frac{1}{a}B_{\alpha|\beta} + \dot{C}_{\alpha\beta}\right)\right] \equiv N_{\phi_{(i)}}.
\end{aligned} \tag{119}$$

The energy-momentum tensor gives

$$\begin{aligned}
\tilde{T}^{(\phi)}_{00} &= \frac{1}{2}\sum_k \phi'^2_{(k)} + a^2V + \sum_k (\phi'_{(k)}\delta\phi'_{(k)} + a^2V_{,\phi_{(k)}}\delta\phi_{(k)}) + 2a^2VA + \sum_k \left(\frac{1}{2}\delta\phi'^2_{(k)} + \frac{1}{2}\delta\phi_{(k),\alpha}\delta\phi_{(k)}{}^{,\alpha}\right.\\
&\quad \left. + \frac{1}{2}a^2\sum_l V_{,\phi_{(k)}\phi_{(l)}}\delta\phi_{(k)}\delta\phi_{(l)} + 2a^2V_{,\phi_{(k)}}\delta\phi_{(k)}A - \phi'_{(k)}\delta\phi_{(k),\alpha}B^\alpha + \frac{1}{2}\phi'^2_{(k)}B_\alpha B^\alpha\right),
\end{aligned}$$

$$\tilde{T}^{(\phi)}_{0\alpha} = \sum_k \phi'_{(k)}\delta\phi_{(k),\alpha} - \left(\frac{1}{2}\sum_k \phi'^2_{(k)} - a^2V\right)B_\alpha + \sum_k [\delta\phi'_{(k)}\delta\phi_{(k),\alpha} + (-\phi'_{(k)}\delta\phi'_{(k)} + a^2V_{,\phi_{(k)}}\delta\phi_{(k)} + \phi'^2_{(k)}A)B_\alpha],$$

$$\begin{aligned}
\tilde{T}^{(\phi)}_{\alpha\beta} &= g^{(3)}_{\alpha\beta}\left(\frac{1}{2}\sum_k \phi'^2_{(k)} - a^2V\right) + g^{(3)}_{\alpha\beta}\sum_k (\phi'_{(k)}\delta\phi'_{(k)} - a^2V_{,\phi_{(k)}}\delta\phi_{(k)} - \phi'^2_{(k)}A) + \left(\sum_k \phi'^2_{(k)} - 2a^2V\right)C_{\alpha\beta}\\
&\quad + \sum_k \left\{\delta\phi_{(k),\alpha}\delta\phi_{(k),\beta} + 2[\phi'_{(k)}\delta\phi'_{(k)} - a^2V_{,\phi_{(k)}}\delta\phi_{(k)} - \phi'^2_{(k)}A]C_{\alpha\beta} - \frac{1}{2}g^{(3)}_{\alpha\beta}\left[-\delta\phi'^2_{(k)} + \delta\phi_{(k),\gamma}\delta\phi_{(k)}{}^{,\gamma}\right.\right.\\
&\quad \left.\left. + a^2\sum_l V_{,\phi_{(k)}\phi_{(l)}}\delta\phi_{(k)}\delta\phi_{(l)} + 4\phi'_{(k)}\delta\phi'_{(k)}A - 2\phi'_{(k)}\delta\phi_{(k),\gamma}B^\gamma + \phi'^2_{(k)}(-4A^2 + B_\gamma B^\gamma)\right]\right\}.
\end{aligned} \tag{120}$$

Fluid quantities can be read from Eq. (79) as

$$\mu^{(\phi)}+\delta\mu^{(\phi)}=\frac{1}{2}\sum_k \dot{\phi}_{(k)}^2+V+\sum_k (\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)}-\dot{\phi}_{(k)}^2A+V_{,\phi_{(k)}}\delta\phi_{(k)})+\sum_k\left[\frac{1}{2}\delta\dot{\phi}_{(k)}^2+\frac{1}{2a^2}\delta\phi_{(k),\alpha}\delta\phi_{(k)}{}^{,\alpha}\right.$$
$$\left.+\frac{1}{2}\sum_l V_{,\phi_{(k)}\phi_{(l)}}\delta\phi_{(k)}\delta\phi_{(l)}-2\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)}A+\frac{1}{a}\dot{\phi}_{(k)}\delta\phi_{(k),\alpha}B^\alpha+\left(2A^2-\frac{1}{2}B^\alpha B_\alpha\right)\dot{\phi}_{(k)}^2\right],$$

$$p^{(\phi)}+\delta p^{(\phi)}=\sum_k \frac{1}{2}\dot{\phi}_{(k)}^2-V+\sum_k (\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)}-\dot{\phi}_{(k)}^2A-V_{,\phi_{(k)}}\delta\phi_{(k)})+\sum_k\left[\frac{1}{2}\delta\dot{\phi}_{(k)}^2-\frac{1}{6a^2}\delta\phi_{(k),\alpha}\delta\phi_{(k)}{}^{,\alpha}\right.$$
$$\left.-\frac{1}{2}\sum_l V_{,\phi_{(k)}\phi_{(l)}}\delta\phi_{(k)}\delta\phi_{(l)}-2\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)}A+\frac{1}{a}\dot{\phi}_{(k)}\delta\phi_{(k),\alpha}B^\alpha+\left(2A^2-\frac{1}{2}B^\alpha B_\alpha\right)\dot{\phi}_{(k)}^2\right],$$

$$Q_\alpha^{(\phi)}=-\frac{1}{a}\sum_k [\dot{\phi}_{(k)}\delta\phi_{(k),\alpha}+(\delta\dot{\phi}_{(k)}-\dot{\phi}_{(k)}A)\delta\phi_{(k),\alpha}],$$

$$\Pi_{\alpha\beta}^{(\phi)}=\frac{1}{a^2}\sum_k\left(\delta\phi_{(k),\alpha}\delta\phi_{(k),\beta}-\frac{1}{3}g_{\alpha\beta}^{(3)}\delta\phi_{(k),\gamma}\delta\phi_{(k)}{}^{,\gamma}\right). \qquad (121)$$

We indicate the quadratic parts as $\delta\mu^{(q)}$, $\delta p^{(q)}$, $Q_\alpha^{(q)}$, and $\Pi_{\alpha\beta}^{(q)}$.

D. Generalized gravity theories

1. Covariant equations

As the action for a class of generalized gravity theories we consider

$$S=\int\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{f}(\tilde{\phi}^K,\tilde{R})-\frac{1}{2}\tilde{g}_{IJ}(\tilde{\phi}^K)\tilde{\phi}^{I,c}\tilde{\phi}^J_{,c}\right.$$
$$\left.-\tilde{V}(\tilde{\phi}^K)+\tilde{L}_m\right]d^4x. \qquad (122)$$

$\tilde{\phi}^I$ is the Ith component of N scalar fields. The capital indices $I,J,K,\ldots=1,2,3,\ldots,N$ indicate the scalar fields, and the summation convention is used for repeated indices. $\tilde{f}(\tilde{\phi}^K,\tilde{R})$ is a general algebraic function of $\tilde{R}$ and the scalar fields $\tilde{\phi}^I$, and $\tilde{g}_{IJ}(\tilde{\phi}^K)$ and $\tilde{V}(\tilde{\phi}^K)$ are general algebraic functions of the scalar fields. We include a nonlinear sigma-type kinetic term where the kinetic matrix $\tilde{g}_{IJ}$ is considered as a Riemannian metric on the manifold with the coordinates $\tilde{\phi}^I$. The matter part Lagrangian $\tilde{L}_m$ includes the fluids, the kinetic components, and the interaction with the fields, as well. We introduced the general action in Eq. (122) in [23,12] as a toy model which allows quite general handling of various different generalized gravity theories in a unified manner (see [24]). Our generalized gravity includes as a subset $f(R)$ gravity, which includes R^2 gravity, the scalar-tensor theory, which includes the Jordan-Brans-Dicke theory, the nonminimally coupled scalar field, the induced gravity, the low-energy effective action of string theory, etc., and various combinations of such gravity theories with additional multiple fields and fluids. It does not, however, include higher-derivative theories with terms like $R^{ab}R_{ab}$; see [25] for its role.

The gravitational field equation and the equation of motion become

$$\tilde{G}_{ab}=\frac{1}{\tilde{F}}\left[\tilde{T}_{ab}+\tilde{g}_{IJ}\left(\tilde{\phi}^I_{,a}\tilde{\phi}^J_{,b}-\frac{1}{2}\tilde{g}_{ab}\tilde{\phi}^{I,c}\tilde{\phi}^J_{,c}\right)\right.$$
$$\left.+\frac{1}{2}(\tilde{f}-\tilde{R}\tilde{F}-2\tilde{V})\tilde{g}_{ab}+\tilde{F}_{,a;b}-\tilde{g}_{ab}\tilde{F}^{;c}{}_c\right]$$
$$\equiv 8\pi G\tilde{T}_{ab}^{(\mathrm{eff})}, \qquad (123)$$

$$\tilde{\phi}^{I;c}{}_c+\frac{1}{2}(\tilde{f}-2\tilde{V})^{,I}+\tilde{\Gamma}^I_{JK}\tilde{\phi}^{J,c}\tilde{\phi}^K_{,c}=-\tilde{L}_m{}^{,I}\equiv\tilde{\Gamma}^I, \qquad (124)$$

$$\tilde{T}^b_{a;b}=\tilde{L}_{m,J}\tilde{\phi}^J_{,a}, \qquad (125)$$

where $\tilde{F}\equiv\partial\tilde{f}/\partial\tilde{R}$; $\tilde{g}^{IJ}$ is the inverse metric of $\tilde{g}_{IJ}$, $\tilde{\Gamma}^I_{JK}\equiv\frac{1}{2}\tilde{g}^{IL}(\tilde{g}_{LJ,K}+\tilde{g}_{LK,J}-\tilde{g}_{JK,L})$, and $\tilde{V}_{,\tilde{I}}\equiv\partial\tilde{V}/(\partial\tilde{\phi}^I)$. Introduction of the effective energy-momentum tensor $\tilde{T}_{ab}^{(\mathrm{eff})}$ provides a useful trick for deriving and handling the perturbed set of equations [23]. It allows the equations derived in Einstein gravity to remain valid with the energy-momentum parts replaced by the effective ones.

We note that the gravity theory in Eq. (122) can be transformed to Einstein's gravity through a conformal rescaling of the metric and rescaling of one of the fields. As the result we have Einstein's gravity sector with complications appearing only in the modified form of the field potential; the nonlinear sigma-type couplings in the kinetic part also remain. We studied the conformal transformation properties to linear-order perturbation in [23,26], and in the most general form in Appendix A of [29]. Extention to second-order perturbation is trivial.

2. Perturbed equations

The perturbed set of equations can be derived similarly as in the previous sections on the scalar fields. We set

$$\tilde{F} \equiv F + \delta F, \quad \tilde{\Gamma}^I \equiv \Gamma^I + \delta \Gamma^I. \tag{126}$$

Thus,

$$\delta F = F_{,I}\delta\phi^I + F_{,R}\delta R + \frac{1}{2}F_{,IJ}\delta\phi^I\delta\phi^J + F_{,IR}\delta\phi^I\delta R + \frac{1}{2}F_{,RR}\delta R^2. \tag{127}$$

The equation of motion in Eq. (124) gives

$$\begin{aligned}
&\ddot{\phi}^I + 3H\dot{\phi}^I - \frac{1}{2}g^{IJ}(f_{,J} - 2V_{,J}) + \Gamma^I_{JK}\dot{\phi}^J\dot{\phi}^K + \Gamma^I + \delta\ddot{\phi}^I + 3H\delta\dot{\phi}^I - \frac{\Delta}{a^2}\delta\phi^I - 2A\ddot{\phi}^I + \dot{\phi}^I\left(-\dot{A} - 6HA + \frac{1}{a}B^\alpha{}_{|\alpha} + \dot{C}^\alpha_\alpha\right)\\
&- \frac{1}{2}g^{IJ}[F_{,J}\delta R + (f_{,LJ} - 2V_{,LJ})\delta\phi^L] - \frac{1}{2}g^{IJ}{}_{,L}\delta\phi^L(f_{,J} - 2V_{,J}) + 2\Gamma^I_{JK}(\dot{\phi}^J\delta\dot{\phi}^K - A\dot{\phi}^J\dot{\phi}^K) + \Gamma^I_{JK,L}\delta\phi^L\dot{\phi}^J\dot{\phi}^K + \delta\Gamma^I\\
&= 2A\left[\delta\ddot{\phi}^I + 3H\delta\dot{\phi}^I - 2A\ddot{\phi}^I - \dot{\phi}^I\left(2\dot{A} + 6HA - \frac{1}{a}B^\alpha{}_{|\alpha} - \dot{C}^\alpha_\alpha\right) + 2\Gamma^I_{JK}(\dot{\phi}^J\delta\dot{\phi}^K - A\dot{\phi}^J\dot{\phi}^K) + \Gamma^I_{JK,L}\delta\phi^L\dot{\phi}^J\dot{\phi}^K\right]\\
&+ B^\alpha B_\alpha(\ddot{\phi}^I + 3H\dot{\phi}^I + \Gamma^I_{JK}\dot{\phi}^J\dot{\phi}^K) + \left(\dot{A} - \frac{1}{a}B^\alpha{}_{|\alpha} - \dot{C}^\alpha_\alpha\right)\delta\dot{\phi}^I - 2\frac{1}{a}B^\alpha\delta\dot{\phi}^I{}_{,\alpha} - 2\frac{1}{a^2}C^{\alpha\beta}\delta\phi^I{}_{,\alpha|\beta}\\
&+ \frac{1}{a}\delta\phi^I{}_{,\alpha}\left(\frac{1}{a}A^{,\alpha} - \dot{B}^\alpha - 2HB^\alpha - 2\frac{1}{a}C^{\alpha\beta}{}_{|\beta} + \frac{1}{a}C^{\beta|\alpha}_\beta\right) + \dot{\phi}^I\left[\frac{1}{a}A_{,\alpha}B^\alpha + B^\alpha\dot{B}_\alpha + \frac{1}{a}B_\alpha(2C^{\alpha\beta}{}_{|\beta} - C^{\beta|\alpha}_\beta)\right.\\
&\left. + 2C^{\alpha\beta}\left(\frac{1}{a}B_{\alpha|\beta} + \dot{C}_{\alpha\beta}\right)\right] + \frac{1}{4}g^{IJ}[F_{,RJ}\delta R^2 + 2F_{,LJ}\delta R\delta\phi^L + (f_{,LMJ} - 2V_{,LMJ})\delta\phi^L\delta\phi^M]\\
&+ \frac{1}{2}g^{IJ}{}_{,L}\delta\phi^L[F_{,J}\delta R + (f_{,LJ} - 2V_{,LJ})\delta\phi^L] + \frac{1}{4}g^{IJ}{}_{,LM}\delta\phi^L\delta\phi^M(f_{,J} - 2V_{,J})\\
&+ \Gamma^I_{JK}\left(-\delta\dot{\phi}^J\delta\dot{\phi}^K - 2\frac{1}{a}\dot{\phi}^J\delta\phi^K{}_{,\alpha}B^\alpha + \frac{1}{a^2}\delta\phi^{J|\alpha}\delta\phi^K{}_{,\alpha}\right) - 2\Gamma^I_{JK,L}\delta\phi^L\dot{\phi}^J\delta\dot{\phi}^K - \frac{1}{2}\Gamma^I_{JK,LM}\delta\phi^L\delta\phi^M\dot{\phi}^J\dot{\phi}^K \equiv N_g.
\end{aligned} \tag{128}$$

δR can be read from Eq. (98). From Eq. (123) the effective energy-momentum tensor gives

$$\begin{aligned}
\tilde{T}^{(\text{eff})}_{00} = \frac{1}{8\pi G\tilde{F}}\Bigg\{&\tilde{T}_{00} + \frac{1}{2}g_{IJ}\phi^{I\prime}\phi^{J\prime} - \frac{1}{2}a^2(f - RF - 2V) - 3\frac{a'}{a}F' + g_{IJ}\phi^{I\prime}\delta\phi^{J\prime} + \frac{1}{2}g_{IJ,L}\delta\phi^L\phi^{I\prime}\phi^{J\prime}\\
&- \frac{1}{2}a^2[(f_{,L} - 2V_{,L})\delta\phi^L - R\delta F] - a^2A(f - RF - 2V) - 3\frac{a'}{a}\delta F' + \Delta\delta F - (B^\alpha{}_{|\alpha} + C^{\alpha\prime}_\alpha)F'\\
&+ g_{IJ}\left(\frac{1}{2}\delta\phi^{I\prime}\delta\phi^{J\prime} + \frac{1}{2}\delta\phi^{I,\alpha}\delta\phi^J{}_{,\alpha} + \frac{1}{2}B^\alpha B_\alpha\phi^{I\prime}\phi^{J\prime} - B^\alpha\phi^{I\prime}\delta\phi^J{}_{,\alpha}\right) + g_{IJ,L}\delta\phi^L\phi^{I\prime}\delta\phi^{J\prime}\\
&+ \frac{1}{4}g_{IJ,LM}\delta\phi^L\delta\phi^M\phi^{I\prime}\phi^{J\prime} + \frac{1}{4}a^2[F_{,R}\delta R^2 - (f_{,LM} - 2V_{,LM})\delta\phi^L\delta\phi^M] - a^2A[(f_{,L} - 2V_{,L})\delta\phi^L - R\delta F]\\
&- (B^\alpha{}_{|\alpha} + C^{\alpha\prime}_\alpha)\delta F' - 2B^\alpha\delta F'_{,\alpha} - \left(\frac{a'}{a}B^\alpha + 2C^{\alpha\beta}{}_{|\beta} - C^{\beta|\alpha}_\beta\right)\delta F_{,\alpha} + 2A\Delta\delta F - 2C^{\alpha\beta}\delta F_{,\alpha|\beta} + B^\alpha B_\alpha\left(F'' + \frac{a'}{a}F'\right)\\
&+ [2A_{,\alpha}B^\alpha + B_\alpha(2C^{\alpha\beta}{}_{|\beta} - C^{\beta|\alpha}_\beta) + 2C^{\alpha\beta}(B_{\alpha|\beta} + C'_{\alpha\beta})]F'\Bigg\},
\end{aligned}$$

$$\tilde{T}^{(\text{eff})}_{0\alpha}=\frac{1}{8\pi G\tilde{F}}\Bigg\{\tilde{T}_{0\alpha}+g_{IJ}\phi^{I\prime}\delta\phi^{J}{}_{,\alpha}-\frac{1}{2}B_\alpha[g_{IJ}\phi^{I\prime}\phi^{J\prime}+a^2(f-RF-2V)]+\delta F'_{,\alpha}-\frac{a'}{a}\delta F_{,\alpha}-B_\alpha F''$$
$$-\left(A_{,\alpha}+\frac{a'}{a}B_\alpha\right)F'+g_{IJ}[\delta\phi^{I\prime}\delta\phi^{J}{}_{,\alpha}+B_\alpha(-\phi^{I\prime}\delta\phi^{J\prime}+A\phi^{I\prime}\phi^{J\prime})]+g_{IJ,L}\delta\phi^L\left(\phi^{I\prime}\delta\phi^{J}{}_{,\alpha}-\frac{1}{2}B_\alpha\phi^{I\prime}\phi^{J\prime}\right)$$
$$-\frac{1}{2}a^2B_\alpha[(f_{,L}-2V_{,L})\delta\phi^L-R\delta F]-B_\alpha\delta F''-\left(A_{,\alpha}+\frac{a'}{a}B_\alpha\right)\delta F'-\left(\frac{1}{2}B_\alpha{}^{|\beta}-\frac{1}{2}B^\beta{}_{|\alpha}+C^{\beta\prime}_\alpha\right)\delta F_{,\beta}+B_\alpha\Delta\delta F$$
$$+2AB_\alpha F''+\left[2AA_{,\alpha}+2\frac{a'}{a}AB_\alpha+B_\beta C^{\beta\prime}_\alpha-B^\beta B_{[\beta|\alpha]}+B_\alpha(A'-B^\beta{}_{|\beta}-C^{\beta\prime}_\beta)\right]F'\Bigg\},$$

$$\tilde{T}^{(\text{eff})}_{\alpha\beta}=\frac{1}{8\pi G\tilde{F}}\Bigg\{\tilde{T}_{\alpha\beta}+\frac{1}{2}g^{(3)}_{\alpha\beta}\left[g_{IJ}\phi^{I\prime}\phi^{J\prime}+a^2(f-RF-2V)+2F''+2\frac{a'}{a}F'\right]+C_{\alpha\beta}\Big[g_{IJ}\phi^{I\prime}\phi^{J\prime}+a^2(f-RF-2V)$$
$$+2F''+2\frac{a'}{a}F'\Big]+\delta F_{,\alpha|\beta}-(B_{(\alpha|\beta)}+C'_{\alpha\beta})F'+g^{(3)}_{\alpha\beta}\Big[g_{IJ}(\phi^{I\prime}\delta\phi^{J\prime}-A\phi^{I\prime}\phi^{J\prime})+\frac{1}{2}g_{IJ,L}\delta\phi^L\phi^{I\prime}\phi^{J\prime}$$
$$+\frac{1}{2}a^2[(f_{,L}-2V_{,L})\delta\phi^L-R\delta F]+\delta F''+\frac{a'}{a}\delta F'-\Delta\delta F-2AF''-\left(A'+2\frac{a'}{a}A-B^\gamma{}_{|\gamma}-C^{\gamma\prime}_\gamma\right)F'\Big]$$
$$+g_{IJ}\delta\phi^I{}_{,\alpha}\delta\phi^J{}_{,\beta}+C_{\alpha\beta}\Big[2g_{IJ}(\phi^{I\prime}\delta\phi^{J\prime}-A\phi^{I\prime}\phi^{J\prime})+g_{IJ,L}\delta\phi^L\phi^{I\prime}\phi^{J\prime}+a^2[(f_{,L}-2V_{,L})\delta\phi^L-R\delta F]$$
$$+2\left[\delta F''+\frac{a'}{a}\delta F'-\Delta\delta F-2AF''-\left(A'+2\frac{a'}{a}A-B^\gamma{}_{|\gamma}-C^{\gamma\prime}_\gamma\right)F'\right]\Big]-(B_{(\alpha|\beta)}+C'_{\alpha\beta})(\delta F'-2AF')$$
$$-(2C^\gamma_{(\alpha|\beta)}-C_{\alpha\beta}{}^{|\gamma})(\delta F_{,\gamma}-B_\gamma F')+g^{(3)}_{\alpha\beta}\Big[\frac{1}{2}g_{IJ}[\delta\phi^{I\prime}\delta\phi^{J\prime}-4A\phi^{I\prime}\delta\phi^{J\prime}+(4A^2-B^\gamma B_\gamma)\phi^{I\prime}\phi^{J\prime}+2B^\gamma\phi^{I\prime}\delta\phi^J{}_{,\gamma}$$
$$-\delta\phi^{I,\gamma}\delta\phi^J{}_{,\gamma}]+g_{IJ,L}\delta\phi^L(\phi^{I\prime}\delta\phi^{J\prime}-A\phi^{I\prime}\phi^{J\prime})+\frac{1}{4}g_{IJ,LM}\delta\phi^L\delta\phi^M\phi^{I\prime}\phi^{J\prime}+\frac{1}{4}a^2[-F_{,R}\delta R^2+(f_{,LM}$$
$$-2V_{,LM})\delta\phi^L\delta\phi^M]-2A\delta F''-\left(A'+2\frac{a'}{a}A-B^\gamma{}_{|\gamma}-C^{\gamma\prime}_\gamma\right)\delta F'+2B^\gamma\delta F'_{,\gamma}+\Big(-A^{,\gamma}+B^{\gamma\prime}+\frac{a'}{a}B^\gamma$$
$$+2C^{\gamma\delta}{}_{|\delta}-C^{\delta|\gamma}_\delta\Big)\delta F_{,\gamma}+2C^{\gamma\delta}\delta F_{,\gamma|\delta}+(4A^2-B^\gamma B_\gamma)\left(F''+\frac{a'}{a}F'\right)+[4AA'-A_{,\gamma}B^\gamma-B^\gamma B'_\gamma-2A(B^\gamma{}_{|\gamma}+C^{\gamma\prime}_\gamma)$$
$$-B_\gamma(2C^{\gamma\delta}{}_{|\delta}-C^{\delta|\gamma}_\delta)-2C^{\gamma\delta}(B_{\gamma|\delta}+C'_{\gamma\delta})]F'\Big]\Bigg\}. \tag{129}$$

The fluid quantities follow from Eq. (79):

$$\mu^{(\text{eff})}+\delta\mu^{(\text{eff})}=\frac{1}{8\pi G\tilde{F}}\Bigg\{\mu+\frac{1}{2}g_{IJ}\dot\phi^I\dot\phi^J-\frac{1}{2}(f-RF-2V)-3H\dot F+\delta\mu+g_{IJ}(\dot\phi^I\delta\dot\phi^J-A\dot\phi^I\dot\phi^J)+\frac{1}{2}g_{IJ,L}\delta\phi^L\dot\phi^I\dot\phi^J$$
$$-\frac{1}{2}(f_{,L}-2V_{,L})\delta\phi^L-3H\delta\dot F+\left(\frac{1}{2}R+\frac{\Delta}{a^2}\right)\delta F-\left(-6HA+\frac{1}{a}B^\alpha{}_{|\alpha}+\dot C^\alpha_\alpha\right)\dot F+\frac{1}{2}g_{IJ}\Big[\delta\dot\phi^I\delta\dot\phi^J+\frac{1}{a^2}\delta\phi^{I,\alpha}$$
$$\times\delta\phi^J{}_{,\alpha}-4A\dot\phi^I\delta\dot\phi^J+2\frac{1}{a}B^\alpha\dot\phi^I\delta\phi^J{}_{,\alpha}+(4A^2-B^\alpha B_\alpha)\dot\phi^I\dot\phi^J\Big]+g_{IJ,L}\delta\phi^L(\dot\phi^I\delta\dot\phi^J-A\dot\phi^I\dot\phi^J)+\frac{1}{4}g_{IJ,LM}\delta\phi^L$$
$$\times\delta\phi^M\dot\phi^I\dot\phi^J+\frac{1}{4}[F_{,R}\delta R^2-(f_{,LM}-2V_{,LM})\delta\phi^L\delta\phi^M]-\left(-6HA+\frac{1}{a}B^\alpha{}_{|\alpha}+\dot C^\alpha_\alpha\right)(\delta\dot F-2A\dot F)$$
$$-\left[3HB^\alpha+\frac{1}{a}(2C^{\alpha\beta}{}_{|\beta}-C^{\beta|\alpha}_\beta)\right]\left(\frac{1}{a}\delta F_{,\alpha}-B_\alpha\dot F\right)-2\frac{1}{a^2}C^{\alpha\beta}\delta F_{,\alpha|\beta}+2\dot F C^{\alpha\beta}\left(\frac{1}{a}B_{\alpha|\beta}+\dot C_{\alpha\beta}\right)\Bigg\},$$

$$p^{(\mathrm{eff})}+\delta p^{(\mathrm{eff})}=\frac{1}{8\pi G\tilde{F}}\Bigg\{p+\frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J+\frac{1}{2}(f-RF-2V)+\ddot{F}+2H\dot{F}+\delta p+g_{IJ}(\dot{\phi}^I\delta\dot{\phi}^J-A\dot{\phi}^I\dot{\phi}^J)+\frac{1}{2}g_{IJ,L}\delta\phi^L\dot{\phi}^I\dot{\phi}^J$$
$$+\frac{1}{2}(f_{,L}-2V_{,L})\delta\phi^L+\delta\ddot{F}+2H\delta\dot{F}-\left(\frac{1}{2}R+\frac{2}{3}\frac{\Delta}{a^2}\right)\delta F-2A\ddot{F}-\left[\dot{A}+4HA-\frac{2}{3}\left(\frac{1}{a}B^\alpha{}_{|\alpha}+\dot{C}^\alpha_\alpha\right)\right]\dot{F}$$
$$+g_{IJ}\left[\frac{1}{2}\delta\phi^I\delta\dot{\phi}^J-2A\dot{\phi}^I\delta\dot{\phi}^J+\frac{1}{2}(4A^2-B^\alpha B_\alpha)\dot{\phi}^I\dot{\phi}^J+\frac{1}{a}B^\alpha\dot{\phi}^I\delta\phi^J{}_{,\alpha}-\frac{1}{6a^2}\delta\phi^{I,\alpha}\delta\phi^J{}_{,\alpha}\right]$$
$$+g_{IJ,L}\delta\phi^L(\dot{\phi}^I\delta\dot{\phi}^J-A\dot{\phi}^I\dot{\phi}^J)+\frac{1}{4}g_{IJ,LM}\delta\phi^L\delta\phi^M\dot{\phi}^I\dot{\phi}^J-\frac{1}{4}[F_{,R}\delta R^2-(f_{,LM}-2V_{,LM})\delta\phi^L\delta\phi^M]$$
$$-2A\delta\ddot{F}-\left[\dot{A}+4HA-\frac{2}{3}\left(\frac{1}{a}B^\alpha{}_{|\alpha}+\dot{C}^\alpha_\alpha\right)\right](\delta\dot{F}-2A\dot{F})+2\frac{1}{a}B^\alpha\delta\dot{F}_{,\alpha}$$
$$+\left[-\frac{1}{a}A^{,\alpha}+\dot{B}^\alpha+HB^\alpha+\frac{2}{3a}(2C^{\alpha\beta}{}_{|\beta}-C^{\beta|\alpha}_\beta)\right]\left(\frac{1}{a}\delta F_{,\alpha}-B_\alpha\dot{F}\right)+\frac{4}{3a^2}C^{\alpha\beta}\delta F_{,\alpha|\beta}+(4A^2-B^\alpha B_\alpha)\ddot{F}$$
$$+\left[2A\dot{A}-\frac{2}{a}A_{,\alpha}B^\alpha-HB^\alpha B_\alpha-\frac{4}{3}C^{\alpha\beta}\left(\frac{1}{a}B_{\alpha|\beta}+\dot{C}_{\alpha\beta}\right)\right]\dot{F}\Bigg\},$$

$$Q^{(\mathrm{eff})}_\alpha=\frac{1}{8\pi G\tilde{F}}\Bigg\{Q_\alpha-\frac{1}{a}g_{IJ}\dot{\phi}^I\delta\phi^J{}_{,\alpha}+\frac{1}{a}(-\delta\dot{F}_{,\alpha}+H\delta F_{,\alpha})+\frac{1}{a}A_{,\alpha}\dot{F}-\frac{1}{a}g_{IJ}(\delta\dot{\phi}^I-A\dot{\phi}^I)\delta\phi^J{}_{,\alpha}$$
$$-\frac{1}{a}g_{IJ,L}\delta\phi^L\dot{\phi}^I\delta\phi^J{}_{,\alpha}+A\left[-3\frac{1}{a}\dot{F}A_{,\alpha}+\frac{1}{a}(\delta\dot{F}_{,\alpha}-H\delta F_{,\alpha})\right]$$
$$+\frac{1}{a}A_{,\alpha}\delta\dot{F}+\frac{1}{a}\left[\frac{1}{2a}(B_\alpha{}^{|\beta}-B^\beta{}_{|\alpha})+\dot{C}^\beta_\alpha\right]\delta F_{,\beta}-\frac{1}{a^2}B^\beta\delta F_{,\alpha|\beta}+\frac{1}{a}B^\beta B_{\alpha|\beta}\dot{F}\Bigg\},$$

$$\Pi^{(\mathrm{eff})}_{\alpha\beta}=\frac{1}{8\pi G\tilde{F}}\Bigg\{\Pi_{\alpha\beta}+\frac{1}{a^2}\delta F_{,\alpha|\beta}-\left(\frac{1}{a}B_{(\alpha|\beta)}+\dot{C}_{\alpha\beta}\right)\dot{F}-\frac{1}{3}g^{(3)}_{\alpha\beta}\left[\frac{\Delta}{a^2}\delta F-\left(\frac{1}{a}B^\gamma{}_{|\gamma}+\dot{C}^\gamma_\gamma\right)\dot{F}\right]+\frac{1}{a^2}g_{IJ}\delta\phi^I{}_{,\alpha}\delta\phi^J{}_{,\beta}$$
$$-\frac{2}{3}C_{\alpha\beta}\left[\frac{\Delta}{a^2}\delta F-\left(\frac{1}{a}B^\gamma{}_{|\gamma}+\dot{C}^\gamma_\gamma\right)\dot{F}\right]-\left(\frac{1}{a}B_{(\alpha|\beta)}+\dot{C}_{\alpha\beta}\right)(\delta\dot{F}-2A\dot{F})-\frac{1}{a}(2C^\gamma_{(\alpha|\beta)}-C_{\alpha\beta}{}^{|\gamma})\left(\frac{1}{a}\delta F_{,\gamma}-B_\gamma\dot{F}\right)$$
$$-\frac{1}{3}g^{(3)}_{\alpha\beta}\left[\frac{1}{a^2}g_{IJ}\delta\phi^{I,\gamma}\delta\phi^J{}_{,\gamma}+2C^{\gamma\delta}\left(\frac{1}{a}B_{\gamma|\delta}+\dot{C}_{\gamma\delta}\right)\dot{F}-\frac{2}{a^2}C^{\gamma\delta}\delta F_{,\gamma|\delta}\right.$$
$$\left.-\left(\frac{1}{a}B^\gamma{}_{|\gamma}+\dot{C}^\gamma_\gamma\right)(\delta\dot{F}-2A\dot{F})-\frac{1}{a}(2C^{\gamma\delta}{}_{|\delta}-C^{\delta|\gamma}_\delta)\left(\frac{1}{a}\delta F_{,\gamma}-B_\gamma\dot{F}\right)\right]\Bigg\}. \quad (130)$$

We have used Eq. (78) for the energy-momentum tensor. We indicate the quadratic parts as $\delta\mu^{(\mathrm{eff},q)}$, $\delta p^{(\mathrm{eff},q)}$, $Q^{(\mathrm{eff},q)}_\alpha$, and $\Pi^{(\mathrm{eff},q)}_{\alpha\beta}$. We note again that in this generalized gravity the basic equations in Sec. IV A *remain valid* with the fluid quantities replaced by the effective ones.

E. Electromagnetic field

1. Covariant equations

The Lagrangian of the electromagnetic field is given as

$$\mathcal{L}_{\mathrm{e.m.}}=-\frac{1}{4}\sqrt{-\tilde{g}}\tilde{F}^{ab}\tilde{F}_{ab}, \quad (131)$$

where $\tilde{F}_{ab}\equiv\tilde{A}_{a,b}-\tilde{A}_{b,a}$. The energy-momentum tensor becomes

$$\tilde{T}^{(\mathrm{e.m.})}_{ab}=\tilde{F}_{ac}\tilde{F}_b{}^c-\frac{1}{4}\tilde{g}_{ab}\tilde{F}_{cd}\tilde{F}^{cd}. \quad (132)$$

We introduce [20]

$$\tilde{F}_{ab}=\tilde{u}_a\tilde{E}_b-\tilde{u}_b\tilde{E}_a-\tilde{\eta}_{abcd}\tilde{u}^c\tilde{H}^d,$$

$$\tilde{E}_a\equiv\tilde{F}_{ab}\tilde{u}^b,\quad \tilde{H}_a\equiv\frac{1}{2}\tilde{\eta}_{abcd}\tilde{u}^b\tilde{F}^{cd},$$

$$\tilde{E}^2\equiv\tilde{E}^a\tilde{E}_a,\quad \tilde{H}^2\equiv\tilde{H}^a\tilde{H}_a,$$

$$\tilde{q}\equiv-\tilde{j}^a\tilde{u}_a,\quad \tilde{J}^a\equiv\tilde{h}^a_b\tilde{j}^b. \tag{133}$$

Then we have

$$\tilde{T}^{(\text{e.m.})}_{ab}=\frac{1}{2}\tilde{u}_a\tilde{u}_b(\tilde{E}^2+\tilde{H}^2)+2\tilde{u}_{(a}\tilde{\eta}_{b)cgd}\tilde{u}^c\tilde{E}^g\tilde{H}^d-\tilde{E}_a\tilde{E}_b$$
$$-\tilde{H}_a\tilde{H}_b+\frac{1}{2}\tilde{h}_{ab}(\tilde{E}^2+\tilde{H}^2),$$

$$\tilde{\mu}^{(\text{e.m.})}=\frac{1}{2}(\tilde{E}^2+\tilde{H}^2),$$

$$\tilde{p}^{(\text{e.m.})}=\frac{1}{6}(\tilde{E}^2+\tilde{H}^2),$$

$$\tilde{q}^{(\text{e.m.})}_a=\tilde{\eta}_{acgd}\tilde{u}^c\tilde{E}^g\tilde{H}^d,$$

$$\tilde{\pi}^{(\text{e.m.})}_{ab}=-\tilde{E}_a\tilde{E}_b-\tilde{H}_a\tilde{H}_b+\frac{1}{3}\tilde{h}_{ab}(\tilde{E}^2+\tilde{H}^2). \tag{134}$$

From the Maxwell equations and the conservation equations

$$\tilde{F}^{ab}{}_{;b}=\tilde{j}^a,\quad \tilde{F}_{[ab;c]}=0,\quad \tilde{j}^a{}_{;a}=0, \tag{135}$$

we can derive the covariant forms of the relativistic Maxwell equations [20]:

$$\tilde{E}^a{}_{;b}\tilde{h}^b_a+2\tilde{H}_a\tilde{\omega}^a=\tilde{q}, \tag{136}$$

$$\tilde{H}^a{}_{;b}\tilde{h}^b_a-2\tilde{E}_a\tilde{\omega}^a=0, \tag{137}$$

$$\tilde{h}^a_b\tilde{E}^b{}_{;c}\tilde{u}^c=\tilde{E}^b\left(\tilde{\omega}^a{}_b+\tilde{\sigma}^a{}_b-\frac{2}{3}\tilde{\theta}\tilde{h}^a_b\right)$$
$$+\tilde{\eta}^{abcd}u_b(\tilde{a}_c\tilde{H}_d-\tilde{H}_{c;d})-\tilde{J}^a, \tag{138}$$

$$\tilde{h}^a_b\tilde{H}^b{}_{;c}\tilde{u}^c=\tilde{H}^b\left(\tilde{\omega}^a{}_b+\tilde{\sigma}^a{}_b-\frac{2}{3}\tilde{\theta}\tilde{h}^a_b\right)$$
$$+\tilde{\eta}^{abcd}\tilde{u}_b(\tilde{a}_c\tilde{E}_d-\tilde{E}_{c;d})-\tilde{J}^a, \tag{139}$$

$$\tilde{q}_{,a}\tilde{u}^a+\tilde{\theta}\tilde{q}+\tilde{h}^a_b\tilde{J}^b{}_{;a}+\tilde{J}^a\tilde{a}_a=0. \tag{140}$$

2. Perturbations

We take the normal frame; thus $\tilde{u}_a=\tilde{n}_a$ and thus $\tilde{\omega}_{ab}=0$. Due to the high symmetry the Friedmann background does not support an electric or magnetic field. Thus, $\tilde{E}_a$ and $\tilde{H}_a$ are already at perturbed order. We set

$$\tilde{E}_\alpha\equiv E_\alpha,\quad \tilde{H}_\alpha\equiv H_\alpha, \tag{141}$$

where E_α and H_α are based on $g^{(3)}_{\alpha\beta}$. Thus, $\tilde{E}_0=-E_\alpha B^\alpha$ (which follows from $\tilde{E}_a\tilde{n}^a=0$), etc. For η^{abcd} see Eq. (94). Equations (136)–(140) become

$$E^\alpha{}_{|\alpha}-a^2\delta q=2(C^{\alpha\beta}E_\beta)_{|\alpha}-C^\alpha_{\alpha|\beta}E^\beta, \tag{142}$$

$$H^\alpha{}_{|\alpha}=2(C^{\alpha\beta}H_\beta)_{|\alpha}-C^\alpha_{\alpha|\beta}H^\beta, \tag{143}$$

$$\dot{E}^\alpha+HE^\alpha+\frac{1}{a}\eta^{\alpha\beta\gamma}H_{\beta|\gamma}+J^\alpha$$
$$=A(\dot{E}^\alpha+HE^\alpha)-\frac{1}{a}E^\alpha{}_{|\beta}B^\beta$$
$$+E^\beta\left(\frac{1}{a}B^\alpha{}_{|\beta}+2\dot{C}^\alpha_\beta\right)-E^\alpha\left(\frac{1}{a}B^\beta{}_{|\beta}+\dot{C}^\beta_\beta\right)+\frac{1}{a}\eta^{\alpha\beta\gamma}$$
$$\times(H_\gamma A_{,\beta}-H_{\beta|\gamma}C^\delta_\delta)\quad-\frac{2}{a}\eta^{\beta\gamma\delta}C^\alpha_\beta H_{\gamma|\delta}, \tag{144}$$

$$(H_\alpha\Leftrightarrow E_\alpha), \tag{145}$$

$$\delta\dot{q}+3H\delta q=-3HA\delta q-\frac{1}{a}\delta q_{,\alpha}B^\alpha+\delta K\delta q$$
$$-\frac{1}{a^2}[(J^\alpha-2C^{\alpha\beta}J_\beta)_{|\alpha}+J^\alpha(C^\beta_{\beta|\alpha}+A_{,\alpha})], \tag{146}$$

where we set

$$\tilde{q}\equiv q+\delta q,\quad \tilde{J}_\alpha\equiv J_\alpha, \tag{147}$$

with J_α based on $g^{(3)}_{\alpha\beta}$; J_α in this subsection differs from the flux term in ADM notation used in the other sections. We have $q=0$.

The energy-momentum tensor becomes

$$\tilde{T}^{(\text{e.m.})}_{00}=\frac{1}{2}(E^\alpha E_\alpha+H^\alpha H_\alpha),$$

$$\tilde{T}^{(\text{e.m.})}_{0\alpha}=-\eta_{\alpha\beta\gamma}E^\beta H^\gamma,$$

$$\tilde{T}^{(\text{e.m.})}_{\alpha\beta}=-E_\alpha E_\beta-H_\alpha H_\beta+\frac{1}{2}g^{(3)}_{\alpha\beta}(E^\gamma E_\gamma+H^\gamma H_\gamma). \tag{148}$$

The fluid quantities can be read from Eq. (79) as

$$\delta\mu^{(\text{e.m.})}=3\delta p^{(\text{e.m.})}=\frac{1}{2a^2}(E^\alpha E_\alpha+H^\alpha H_\alpha),$$

$$Q^{(\text{e.m.})}_\alpha=\frac{1}{a^2}\eta_{\alpha\beta\gamma}E^\beta H^\gamma,$$

$$\Pi^{(\text{e.m.})}_{\alpha\beta}=-\frac{1}{a^2}\Bigg[E_\alpha E_\beta+H_\alpha H_\beta$$
$$-\frac{1}{3}g^{(3)}_{\alpha\beta}(E^\gamma E_\gamma+H^\gamma H_\gamma)\Bigg]. \tag{149}$$

We have $\mu^{(\text{e.m.})}=0=p^{(\text{e.m.})}$.

F. Null geodesic and temperature anisotropy

We introduce the photon four-velocity as

$$\tilde{k}^0 \equiv \frac{1}{a}(\nu + \delta\nu), \quad \tilde{k}^\alpha \equiv -\frac{\nu}{a}(e^\alpha + \delta e^\alpha),$$

$$\tilde{k}_0 = -a\nu\left(1 + \frac{\delta\nu}{\nu} + 2A - B_\alpha e^\alpha + 2A\frac{\delta\nu}{\nu} - B_\alpha \delta e^\alpha\right),$$

$$\tilde{k}_\alpha = -a\nu\left(e_\alpha + \delta e_\alpha + B_\alpha + 2C_{\alpha\beta}e^\beta + B_\alpha\frac{\delta\nu}{\nu} + 2C_{\alpha\beta}\delta e^\beta\right), \tag{150}$$

where e^α and δe^α are based on $g^{(3)}_{\alpha\beta}$, and ν and e^α are assumed to be the background order. We have

$$\frac{d}{d\lambda} = \frac{\partial x^a}{\partial\lambda}\frac{\partial}{\partial x^a} = \tilde{k}^a\partial_a = \frac{\nu}{a}\left(\partial_0 - e^\alpha\partial_\alpha + \frac{\delta\nu}{\nu}\partial_0 - \delta e^\alpha\partial_\alpha\right). \tag{151}$$

Thus,

$$\frac{d}{dy} \equiv \partial_0 - e^\alpha\partial_\alpha \tag{152}$$

can be considered as a derivative along the background photon four-velocity. The null equation $\tilde{k}^a\tilde{k}_a = 0$ gives

$$\tilde{k}^a\tilde{k}_a = \nu^2\Bigg[e^\alpha e_\alpha - 1 + 2\left(e^\alpha\delta e_\alpha - \frac{\delta\nu}{\nu} - A + B_\alpha e^\alpha + C_{\alpha\beta}e^\alpha e^\beta\right) + \delta e^\alpha\delta e_\alpha - \frac{\delta\nu^2}{\nu^2} - 2\frac{\delta\nu}{\nu}(2A - B_\alpha e^\alpha) + 2(B_\alpha + 2C_{\alpha\beta}e^\beta)\delta e^\alpha\Bigg] = 0. \tag{153}$$

The geodesic equation $\tilde{k}^a{}_{;b}\tilde{k}^b = 0$, using Eq. (52), gives

$$\begin{aligned}\tilde{k}^0{}_{;b}\tilde{k}^b = \frac{\nu^2}{a^2}\Bigg\{&\frac{(a\nu)'}{a\nu} + \left(\frac{\delta\nu}{\nu}\right)' + 2\frac{\nu'}{\nu}\frac{\delta\nu}{\nu} - \frac{\delta\nu_{,\alpha}}{\nu}e^\alpha + 2\frac{a'}{a}e^\alpha\delta e_\alpha + A' - 2\frac{a'}{a}A + \left(B_{\alpha|\beta} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}\right)e^\alpha e^\beta \\ &- 2\left(A_{,\alpha} - \frac{a'}{a}B_\alpha\right)e^\alpha + \frac{\delta\nu}{\nu}\frac{\delta\nu'}{\nu} - \frac{\delta\nu_{,\alpha}}{\nu}\delta e^\alpha + 2\frac{\delta\nu}{\nu}A' - 2\frac{\delta\nu}{\nu}\left(A_{,\alpha} - \frac{a'}{a}B_\alpha\right)e^\alpha + \frac{a'}{a}\delta e^\alpha\delta e_\alpha - 2\delta e^\alpha\left(A_{,\alpha} - \frac{a'}{a}B_\alpha\right) \\ &- 4\frac{a'}{a}e^\alpha\delta e_\alpha A + 2e^\alpha\delta e^\beta\left(C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta} + B_{(\alpha|\beta)}\right) - \left[A\left(2B_{\alpha|\beta} + 2C'_{\alpha\beta} + 4\frac{a'}{a}C_{\alpha\beta}\right) + B_\gamma(2C^\gamma_{\alpha|\beta} - C_{\alpha\beta}{}^{|\gamma})\right]e^\alpha e^\beta \\ &+ \frac{a'}{a}(4A^2 - B^\alpha B_\alpha) + 2\left(2AA_{,\alpha} - 2\frac{a'}{a}AB_\alpha + B_\beta C^{\beta\prime}_\alpha + B^\beta B_{[\alpha|\beta]}\right)e^\alpha - 2AA' - A_{,\alpha}B^\alpha + B^\alpha\left(B'_\alpha + \frac{a'}{a}B_\alpha\right)\Bigg\} = 0,\end{aligned} \tag{154}$$

$$\begin{aligned}\tilde{k}^\alpha{}_{;b}\tilde{k}^b = \frac{\nu^2}{a^2}\Bigg\{&-e^{\alpha\prime} + e^\beta e^\alpha{}_{|\beta} - \delta e^{\alpha\prime} - \frac{\delta\nu}{\nu}e^{\alpha\prime} + \delta e^\alpha{}_{|\beta}e^\beta + \delta e^\beta e^\alpha{}_{|\beta} + (2C^\alpha_{\beta|\gamma} - C_{\beta\gamma}{}^{|\alpha})e^\beta e^\gamma - (B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta} + 2C^{\alpha\prime}_\beta)e^\beta \\ &+ A^{,\alpha} - B^{\alpha\prime} - \frac{\delta\nu}{\nu}\delta e^{\alpha\prime} + 2\frac{\delta\nu}{\nu}(A^{,\alpha} - B^{\alpha\prime}) - \left(\delta e^\beta + \frac{\delta\nu}{\nu}e^\beta\right)(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta} + 2C^{\alpha\prime}_\beta) \\ &+ \delta e^\beta\delta e^\alpha{}_{|\beta} + 2e^\beta\delta e^\gamma(2C^\alpha_{\beta|\gamma} - C_{\beta\gamma}{}^{|\alpha}) + A'B^\alpha - 2A_{,\beta}C^{\alpha\beta} + 2C^\alpha_\beta B^{\beta\prime} - 2B^\alpha A_{,\beta}e^\beta \\ &+ 4C^{\alpha\gamma}(B_{[\beta|\gamma]} + C'_{\beta\gamma})e^\beta - 2C^\alpha_\delta(2C^\delta_{\beta|\gamma} - C_{\beta\gamma}{}^{|\delta})e^\beta e^\gamma + B^\alpha(B_{\beta|\gamma} + C'_{\beta\gamma})e^\beta e^\gamma\Bigg\} = 0,\end{aligned} \tag{155}$$

where we used the null equation in Eq. (153). To the background order Eqs. (153)–(155) give

$$e^\alpha e_\alpha = 1, \quad \nu \propto a^{-1}, \quad e^{\alpha\prime} = e^\beta e^\alpha{}_{|\beta}. \tag{156}$$

Using Eqs. (152),(153), Eq. (154) becomes

$$\frac{d}{dy}\left(\frac{\delta\nu}{\nu}+A\right)-A_{,\alpha}e^{\alpha}+(B_{\alpha|\beta}+C'_{\alpha\beta})e^{\alpha}e^{\beta}$$
$$=-\frac{\delta\nu}{\nu}\frac{\delta\nu'}{\nu}-\frac{a'}{a}\frac{\delta\nu^2}{\nu^2}-2\frac{\delta\nu}{\nu}\left(A'+2\frac{a'}{a}A\right)+\frac{\delta\nu_{,\alpha}}{\nu}\delta e^{\alpha}+2\frac{\delta\nu}{\nu}A_{,\alpha}e^{\alpha}+2\,\delta e^{\alpha}A_{,\alpha}+4\frac{a'}{a}e^{\alpha}\delta e_{\alpha}A-2e^{\alpha}\delta e^{\beta}(B_{(\alpha|\beta)}+C'_{\alpha\beta})$$
$$+\left[2A\left(B_{\alpha|\beta}+C'_{\alpha\beta}+2\frac{a'}{a}C_{\alpha\beta}\right)+B_{\gamma}(2C^{\gamma}_{\alpha|\beta}-C_{\alpha\beta}{}^{|\gamma})\right]e^{\alpha}e^{\beta}-\frac{a'}{a}(4A^2-B^{\alpha}B_{\alpha})-2\left(2AA_{,\alpha}-2\frac{a'}{a}AB_{\alpha}+B_{\beta}C^{\beta\prime}_{\alpha}\right.$$
$$\left.+B^{\beta}B_{[\alpha|\beta]}\right)e^{\alpha}+2AA'+A_{,\alpha}B^{\alpha}-B^{\alpha}\left(B'_{\alpha}+\frac{a'}{a}B_{\alpha}\right)\equiv N_{\nu}. \tag{157}$$

Thus, we have

$$\left(\frac{\delta\nu}{\nu}+A\right)\bigg|_E^O=\int_E^O[A_{,\alpha}e^{\alpha}-(B_{\alpha|\beta}+C'_{\alpha\beta})e^{\alpha}e^{\beta}+N_{\nu}]dy, \tag{158}$$

where the integral is along the ray's null-geodesic path from E, the emitted event at the intersection of the ray and the last scattering surface, to O, the observed event here and now.

The temperatures of the CMB at two different points (O and E) along a single null-geodesic ray in a given observational direction are [27,17]

$$\frac{\tilde{T}_O}{\tilde{T}_E}\equiv\frac{1}{1+\tilde{z}}\equiv\frac{(\tilde{k}^a\tilde{u}_a)_O}{(\tilde{k}^b\tilde{u}_b)_E}, \tag{159}$$

where $\tilde{u}_a$ at O and E are the local four-velocities of the observer and the emitter, respectively. Thus, $\tilde{u}_a$ should be considered as the one based on the energy frame which sets $\tilde{q}_a\equiv 0$; or equivalently in a general frame vector which absorbs the flux term to the frame vector to second order.

Using Eqs. (53),(150) we have

$$\tilde{k}^a\tilde{u}_a=-\nu\left[1+\frac{\delta\nu}{\nu}+A+(V^E_{\alpha}-B_{\alpha})e^{\alpha}+\frac{\delta\nu}{\nu}A\right.$$
$$+\delta e^{\alpha}(V^E_{\alpha}-B_{\alpha})+(AB_{\alpha}+2C_{\alpha\beta}V^{E\beta})e^{\alpha}$$
$$\left.+\frac{1}{2}V^{E\alpha}V^E_{\alpha}-\frac{1}{2}A^2\right]. \tag{160}$$

We have denoted the energy-frame nature by replacing V_{α} with V^E_{α}; if we consider Eq. (86), Eq. (160) in this form is valid in the general frame. Since the calculations in the rest of this paper are based on the normal-frame vector we use Eq. (88) to derive the result in the normal frame. We have

$$\tilde{k}^a\tilde{u}_a=-\nu\left[1+\frac{\delta\nu}{\nu}+A+\frac{1}{\mu+p}Q_{\alpha}e^{\alpha}\right.$$
$$-\frac{(\delta\mu+\delta p)Q_{\alpha}+\Pi_{\alpha\beta}Q^{\beta}}{(\mu+p)^2}e^{\alpha}+\frac{\delta\nu}{\nu}A+\delta e^{\alpha}\frac{Q_{\alpha}}{\mu+p}$$
$$\left.+\frac{1}{2}\left(B_{\alpha}+\frac{Q_{\alpha}}{\mu+p}\right)\left(B^{\alpha}+\frac{Q^{\alpha}}{\mu+p}\right)-\frac{1}{2}A^2\right]. \tag{161}$$

Using Eq. (159) we have

$$\left(1-\frac{T_E}{T_O}\frac{\nu_O}{\nu_E}\right)\left(1+\frac{\delta T_E}{T_E}\right)+\frac{\delta T}{T}\bigg|_E^O$$
$$=\frac{T_E}{T_O}\frac{\nu_O}{\nu_E}\left[\frac{\delta\nu}{\nu}+A+\frac{1}{\mu+p}Q_{\alpha}e^{\alpha}\right.$$
$$-\frac{(\delta\mu+\delta p)Q_{\alpha}+\Pi_{\alpha\beta}Q^{\beta}}{(\mu+p)^2}e^{\alpha}+\frac{\delta\nu}{\nu}A+\delta e^{\alpha}\frac{Q_{\alpha}}{\mu+p}$$
$$\left.+\frac{1}{2}\left(B_{\alpha}+\frac{Q_{\alpha}}{\mu+p}\right)\left(B^{\alpha}+\frac{Q^{\alpha}}{\mu+p}\right)-\frac{1}{2}A^2\right]\bigg|_E^O$$
$$\times\left[1-\left(\frac{\delta\nu}{\nu}+A+\frac{1}{\mu+p}Q_{\gamma}e^{\gamma}\right)+\frac{\delta T}{T}\right]\bigg|_E$$
$$\equiv\frac{T_E}{T_O}\frac{\nu_O}{\nu_E}\left(\frac{\delta\nu}{\nu}+A+\frac{1}{\mu+p}Q_{\alpha}e^{\alpha}\right)\bigg|_E^O+N_T, \tag{162}$$

where $(\delta T/T)|_E^O\equiv(\delta T/T)|_O-(\delta T/T)|_E$ and $(\delta T/T)|_E\equiv\delta T/T$ at E. Thus, if we take $T_O/T_E=\nu_O/\nu_E$, Eqs. (162), (158) give

$$\frac{\delta T}{T}\bigg|_O = \frac{\delta T}{T}\bigg|_E + \frac{1}{\mu+p} Q_\alpha e^\alpha \bigg|_E^O + \int_E^O [A_{,\alpha} e^\alpha - (B_{\alpha|\beta} + C'_{\alpha\beta}) e^\alpha e^\beta + N_\nu] dy + N_T . \tag{163}$$

On the large angular scale we are considering (larger than the horizon size at the last scattering era), the detailed dynamics at last scattering is not important. The physical processes of last scattering are important at the small angular scale where we need to solve the Boltzmann equation for the photon distribution function (see Sec. IV G).

G. Boltzmann equation

1. Covariant equations

The relativistic Boltzmann equation is [28]

$$\frac{d}{d\lambda}\tilde{f} = \frac{dx^a}{d\lambda}\frac{\partial \tilde{f}}{\partial x^a} + \frac{d\tilde{p}^a}{d\lambda}\frac{\partial \tilde{f}}{\partial \tilde{p}^a} = \tilde{p}^a \frac{\partial \tilde{f}}{\partial x^a} - \tilde{\Gamma}^a_{bc}\tilde{p}^b\tilde{p}^c \frac{\partial \tilde{f}}{\partial \tilde{p}^a} = \tilde{C}[\tilde{f}], \tag{164}$$

where $\tilde{f}(x^a,\tilde{p}^b)$ is a distribution function with the phase space variables x^a and $\tilde{p}^a \equiv dx^a/d\lambda$, and $\tilde{C}[\tilde{f}]$ is the collision term. The energy-momentum tensor of the collisionless (or collisional) component is

$$\tilde{T}^{(c)}_{ab} = \int 2\theta(\tilde{p}^0)\,\delta(\tilde{p}^c\tilde{p}_c + m^2)\tilde{p}_a\tilde{p}_b\tilde{f}\frac{d^4\tilde{p}_{0123}}{\sqrt{-\tilde{g}}} = \int 2\theta(\tilde{p}^0)\,\delta(\tilde{p}^c\tilde{p}_c + m^2)\tilde{p}_a\tilde{p}_b\tilde{f}\sqrt{-\tilde{g}}\,d^4\tilde{p}^{0123}, \tag{165}$$

where

$$\delta(\tilde{p}^c\tilde{p}_c + m^2) = \delta(\tilde{g}_{00}\tilde{p}^0\tilde{p}^0 + 2\tilde{g}_{0\alpha}\tilde{p}^0\tilde{p}^\alpha + \tilde{g}_{\alpha\beta}\tilde{p}^\alpha\tilde{p}^\beta + m^2) = \frac{\delta(\text{mass shell})}{|2\tilde{g}_{00}\tilde{p}^0 + 2\tilde{g}_{0\alpha}\tilde{p}^\alpha|} = \frac{\delta(\text{mass shell})}{2|\tilde{p}_0|}. \tag{166}$$

Thus, after integrating over $\tilde{p}^0$, we have

$$\tilde{T}^{(c)}_{ab} = \int \frac{\sqrt{-g}\,d^3\tilde{p}^{123}}{|\tilde{p}_0|}\tilde{p}_a\tilde{p}_b\tilde{f}, \tag{167}$$

with the mass-shell condition $\tilde{p}^a\tilde{p}_a + m^2 = 0$.

2. Perturbed equations

Under our metric, using $\tilde{p}^a$ as the phase space variable, we have

$$\begin{aligned}
&\tilde{p}^0\tilde{f}' + \tilde{p}^\alpha\tilde{f}_{,\alpha} - \Bigg\{\frac{a'}{a}(\tilde{p}^0\tilde{p}^0 + g^{(3)}_{\alpha\beta}\tilde{p}^\alpha\tilde{p}^\beta) + A'\tilde{p}^0\tilde{p}^0 + 2\left(A_{,\alpha} - \frac{a'}{a}B_\alpha\right)\tilde{p}^0\tilde{p}^\alpha + \left(-2\frac{a'}{a}g^{(3)}_{\alpha\beta}A + B_{\alpha|\beta} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}\right)\tilde{p}^\alpha\tilde{p}^\beta \\
&\quad + \left(-2AA' - A_{,\alpha}B^\alpha + B^\alpha B'_\alpha + \frac{a'}{a}B^\alpha B_\alpha\right)\tilde{p}^0\tilde{p}^0 + 2\left(-2AA_{,\alpha} + 2\frac{a'}{a}AB_\alpha - B_\beta C^{\beta\prime}_\alpha + B^\beta B_{[\beta|\alpha]}\right)\tilde{p}^0\tilde{p}^\alpha \\
&\quad + \left[\frac{a'}{a}g^{(3)}_{\alpha\beta}(4A^2 - B^\gamma B_\gamma) - 2A\left(B_{\alpha|\beta} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}\right) - B_\gamma(2C^\gamma_{\alpha|\beta} - C_{\alpha\beta}{}^{|\gamma})\right]\tilde{p}^\alpha\tilde{p}^\beta\Bigg\}\frac{\partial\tilde{f}}{\partial\tilde{p}^0} \\
&\quad - \Bigg[\left(2\frac{a'}{a}\tilde{p}^0\tilde{p}^\alpha + \Gamma^{(3)\alpha}{}_{\beta\gamma}\tilde{p}^\beta\tilde{p}^\gamma\right) + \left(A^{,\alpha} - B^{\alpha\prime} - \frac{a'}{a}B^\alpha\right)\tilde{p}^0\tilde{p}^0 + (B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta} + 2C^{\alpha\prime}_\beta)\tilde{p}^0\tilde{p}^\beta \\
&\quad + \left(\frac{a'}{a}g^{(3)}_{\beta\gamma}B^\alpha + 2C^\alpha_{\beta|\gamma} - C_{\beta\gamma}{}^{|\alpha}\right)\tilde{p}^\beta\tilde{p}^\gamma\Bigg]\frac{\partial\tilde{f}}{\partial\tilde{p}^\alpha} = \tilde{C}[\tilde{f}].
\end{aligned} \tag{168}$$

As the phase space variable it is convenient to use $(q,\ \gamma^\alpha)$ introduced as

$$q \equiv a\sqrt{a^2(\tilde{p}^0)^2 - m^2}\left[1 + \frac{a^2(\tilde{p}^0)^2}{a^2(\tilde{p}^0)^2 - m^2}\left(A + \frac{1}{2}B^\alpha B_\alpha\right) - \frac{1}{2}\frac{a^4(\tilde{p}^0)^4}{[a^2(\tilde{p}^0)^2 - m^2]^2}A^2\right],$$

$$\gamma^\alpha \equiv \frac{a}{\sqrt{a^2(\tilde{p}^0)^2 - m^2}}\Bigg[\tilde{p}^\alpha - \frac{a^2(\tilde{p}^0)^2}{a^2(\tilde{p}^0)^2 - m^2}A\tilde{p}^\alpha - B^\alpha\tilde{p}^0 + C^\alpha_\beta\tilde{p}^\beta + \frac{3}{2}\frac{a^4(\tilde{p}^0)^4}{[a^2(\tilde{p}^0)^2 - m^2]^2}A^2\tilde{p}^\alpha$$
$$+ \frac{a^2(\tilde{p}^0)^2}{a^2(\tilde{p}^0)^2 - m^2}\left(AB^\alpha\tilde{p}^0 - AC^\alpha_\beta\tilde{p}^\beta - \frac{1}{2}B^\beta B_\beta\tilde{p}^\alpha\right) + C^\alpha_\beta B^\beta\tilde{p}^0 - \frac{1}{2}C^\gamma_\beta C^\alpha_\gamma\tilde{p}^\beta\Bigg],$$
$$\tilde{p}^0 = \frac{1}{a^2}\sqrt{q^2 + m^2a^2}\left(1 - A + \frac{3}{2}A^2 - \frac{1}{2}B^\alpha B_\alpha\right),$$
$$\tilde{p}^\alpha = \frac{1}{a^2}\left[q\gamma^\alpha + \sqrt{q^2+m^2a^2}B^\alpha - qC^\alpha_\beta\gamma^\beta - \sqrt{q^2+m^2a^2}(AB^\alpha + 2C^\alpha_\beta B^\beta) + \frac{3}{2}qC^\gamma_\beta C^\alpha_\gamma\gamma^\beta\right], \tag{169}$$

where γ^α is based on $g^{(3)}_{\alpha\beta}$. The mass-shell condition gives

$$(\tilde{p}^0)^2 - g^{(3)}_{\alpha\beta}\tilde{p}^\alpha\tilde{p}^\beta - m^2/a^2 + 2A(\tilde{p}^0)^2 + 2B_\alpha\tilde{p}^\alpha\tilde{p}^0 - 2C_{\alpha\beta}\tilde{p}^\alpha\tilde{p}^\beta = 0, \tag{170}$$

and we can show $\gamma^\alpha\gamma_\alpha = 1$. The Boltzmann equation becomes

$$\tilde{f}' + \frac{q}{\sqrt{q^2+m^2a^2}}\left(\gamma^\alpha\delta f_{,\alpha} - \Gamma^{(3)\alpha}_{\beta\gamma}\gamma^\beta\gamma^\gamma\frac{\partial\delta f}{\partial\gamma^\alpha}\right) - \left[\frac{\sqrt{q^2+m^2a^2}}{q}\gamma^\alpha A_{,\alpha} + (B_{\alpha|\beta} + C'_{\alpha\beta})\gamma^\alpha\gamma^\beta\right]q\frac{\partial\tilde{f}}{\partial q}$$
$$= -\frac{1}{\sqrt{q^2+m^2a^2}}(qA\gamma^\alpha + \sqrt{q^2+m^2a^2}B^\alpha - qC^\alpha_\beta\gamma^\beta)\delta f_{,\alpha} - \Bigg\{\frac{\sqrt{q^2+m^2a^2}}{q}[A_{,\alpha}(A\gamma^\alpha + C^\alpha_\beta\gamma^\beta) - B^\beta B_{\beta|\alpha}\gamma^\alpha]$$
$$+ 2C^\beta_\gamma(C'_{\alpha\beta} + B_{(\alpha|\beta)})\gamma^\alpha\gamma^\gamma + B_\gamma(2C^\gamma_{\alpha|\beta} - C_{\alpha\beta}{}^{|\gamma})\gamma^\alpha\gamma^\beta\Bigg\}q\frac{\partial f}{\partial q} + \Bigg[\Gamma^{(3)\alpha}{}_{\beta\gamma}\Bigg(\frac{q}{\sqrt{q^2+m^2a^2}}A\gamma^\beta\gamma^\gamma + B^\beta\gamma^\gamma$$
$$- \frac{q}{\sqrt{q^2+m^2a^2}}C^\gamma_\delta\gamma^\beta\gamma^\delta\Bigg) + \frac{\sqrt{q^2+m^2a^2}}{q}(A^{|\alpha} - A_{,\beta}\gamma^\beta\gamma^\alpha) + (B_\beta{}^{|\alpha} + C^{\alpha\prime}_\beta)\gamma^\beta - (B_{\beta|\gamma} + C'_{\beta\gamma})\gamma^\beta\gamma^\gamma\gamma^\alpha$$
$$+ \frac{q}{\sqrt{q^2+m^2a^2}}(C^\alpha_{\beta|\gamma} - C_{\beta\gamma}{}^{|\alpha})\gamma^\beta\gamma^\gamma\Bigg]\frac{\partial\delta f}{\partial\gamma^\alpha} + \frac{a^2}{\sqrt{q^2+m^2a^2}}\left(1 + A - \frac{1}{2}A^2 + \frac{1}{2}B^\alpha B_\alpha\right)\tilde{C}[\tilde{f}] \equiv N_c. \tag{171}$$

For convenience we located the collision term in N_c. The energy-momentum tensor becomes

$$\tilde{T}^{(c)}_{ab} = \frac{1}{a^2}\int \tilde{p}_a\tilde{p}_b\tilde{f}\frac{q^2dqd\Omega_q}{\sqrt{q^2+m^2a^2}}. \tag{172}$$

Thus, using $\int\gamma_\alpha\gamma_\beta d\Omega_q = \frac{1}{3}g^{(3)}_{\alpha\beta}$, we have

$$\tilde{T}^{(c)}_{00} = \frac{1}{a^2}\int \sqrt{q^2+m^2a^2}q^2dqd\Omega_q\Bigg[f(1+2A) + \delta f(1+2A)$$
$$+ \left(1 + \frac{1}{3}\frac{q^2}{q^2+m^2a^2}\right)B^\alpha B_\alpha f + \frac{2q}{\sqrt{q^2+m^2a^2}}B_\alpha\gamma^\alpha\delta f\Bigg],$$

$$\tilde{T}^{(c)}_{\alpha 0} = -\frac{1}{a^2}\int \frac{q^4dqd\Omega_q}{\sqrt{q^2+m^2a^2}}\Bigg\{\frac{1}{3}B_\alpha f + \Bigg[\frac{\sqrt{q^2+m^2a^2}}{q}$$
$$\times(\gamma_\alpha + A\gamma_\alpha + C_{\alpha\beta}\gamma^\beta) + B_\beta\gamma^\beta\gamma_\alpha\Bigg]\delta f\Bigg\},$$

$$\tilde{T}^{(c)}_{\alpha\beta} = \frac{1}{a^2}\int \frac{q^4dqd\Omega_q}{\sqrt{q^2+m^2a^2}}\Bigg(\frac{1}{3}g^{(3)}_{\alpha\beta}f + \delta f\gamma_\alpha\gamma_\beta + \frac{2}{3}fC_{\alpha\beta}$$
$$+ 2\delta f\gamma_{(\alpha}C_{\beta)\gamma}\gamma^\gamma\Bigg). \tag{173}$$

From Eq. (79) the fluid quantities become

$$\mu^{(c)} + \delta\mu^{(c)} = \frac{1}{a^4}\int \sqrt{q^2+m^2a^2}q^2dqd\Omega_q(f + \delta f),$$

$$p^{(c)}+\delta p^{(c)}=\frac{1}{3a^4}\int\frac{q^4dqd\Omega_q}{\sqrt{q^2+m^2a^2}}(f+\delta f),$$

$$Q^{(c)}_\alpha=\frac{1}{a^4}\int q^3dqd\Omega_q(\gamma_\alpha+C_{\alpha\beta}\gamma^\beta)\delta f,$$

$$\Pi^{(c)}_{\alpha\beta}=\frac{1}{a^4}\int\frac{q^4dqd\Omega_q}{\sqrt{q^2+m^2a^2}}\left(\gamma_\alpha\gamma_\beta-\frac{1}{3}g^{(3)}_{\alpha\beta}+2\gamma_{(\alpha}C_{\beta)\gamma}\gamma^\gamma-\frac{2}{3}C_{\alpha\beta}\right)\delta f. \tag{174}$$

If we have multiple components each described by the Boltzmann equation, all equations in this subsection remain valid for any component with $\tilde{f}$ replaced by $\tilde{f}_{(i)}$, etc., and the total (collective) fluid quantities as the sum of the individual ones.

V. DECOMPOSITION

A. Three perturbation types

We decompose the perturbation variables as follows:

$$A\equiv\alpha,$$

$$B_\alpha\equiv\beta_{,\alpha}+B^{(v)}_\alpha,$$

$$C_{\alpha\beta}\equiv\varphi g^{(3)}_{\alpha\beta}+\gamma_{,\alpha|\beta}+C^{(v)}_{(\alpha|\beta)}+C^{(t)}_{\alpha\beta},$$

$$Q_\alpha\equiv Q_{,\alpha}+Q^{(v)}_\alpha\equiv(\mu+p)(-v_{,\alpha}+v^{(v)}_\alpha),$$

$$\Pi_{\alpha\beta}\equiv\frac{1}{a^2}\left(\Pi_{,\alpha|\beta}-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\Pi\right)+\frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)}+\Pi^{(t)}_{\alpha\beta}, \tag{175}$$

with the properties

$$B^{(v)\alpha}{}_{|\alpha}\equiv0,\quad C^{(v)\alpha}{}_{|\alpha}\equiv0,\quad v^{(v)\alpha}{}_{|\alpha}\equiv0,\quad \Pi^{(v)\alpha}{}_{|\alpha}\equiv0,$$

$$C^{(t)\alpha}{}_\alpha\equiv0,\quad \Pi^{(t)\alpha}{}_\alpha\equiv0,\quad C^{(t)\beta}{}_{\alpha|\beta}\equiv0,\quad \Pi^{(t)\beta}{}_{\alpha|\beta}\equiv0. \tag{176}$$

$\mu+p$ appearing in the decomposition of Q_α is assumed to be the background order quantity. The vector- and tensor-type perturbations are denoted by superscripts (v) and (t), respectively. We assume all these variables are based on $g^{(3)}_{\alpha\beta}$. The decomposed variables can also be expressed in terms of the original variables. For example, we have $\beta=\Delta^{-1}\nabla^\alpha B_\alpha$ and $B^{(v)}_\alpha=B_\alpha-\nabla_\alpha\Delta^{-1}\nabla^\beta B_\beta$, etc., where ∇_α means $\nabla^{(3)}_\alpha$. For the fluid quantities we have

$$Q=\Delta^{-1}\nabla^\alpha Q_\alpha,$$

$$Q^{(v)}_\alpha=Q_\alpha-\nabla_\alpha\Delta^{-1}\nabla^\beta Q_\beta,$$

$$\Pi=\frac{3}{2}a^2\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}\Delta^{-1}\nabla^\alpha\nabla^\beta\Pi_{\alpha\beta},$$

$$\Pi^{(v)}_\alpha=2a\left(\Delta+\frac{1}{3}R^{(3)}\right)^{-1}(\nabla^\beta\Pi_{\alpha\beta}-\nabla_\alpha\Delta^{-1}\nabla^\beta\nabla^\gamma\Pi_{\beta\gamma}),$$

$$\Pi^{(t)}_{\alpha\beta}=\Pi_{\alpha\beta}-\frac{3}{2}\left(\nabla_\alpha\nabla_\beta-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}\times\Delta^{-1}\nabla^\gamma\nabla^\delta\Pi_{\gamma\delta}-2\nabla_{(\alpha}\left(\Delta+\frac{1}{3}R^{(3)}\right)^{-1}\times(\nabla^\gamma\Pi_{\beta)\gamma}-\nabla_{\beta)}\Delta^{-1}\nabla^\gamma\nabla^\delta\Pi_{\gamma\delta}). \tag{177}$$

We introduce

$$\chi\equiv a(\beta+a\dot{\gamma}),\quad \Psi^{(v)}_\alpha\equiv B^{(v)}_\alpha+a\dot{C}^{(v)}_\alpha, \tag{178}$$

and let

$$\kappa\equiv\delta K. \tag{179}$$

In the multicomponent situation we have Eq. (75). For the individual components we have

$$Q_{(i)\alpha}\equiv(\mu_{(i)}+p_{(i)})(-v_{(i),\alpha}+v^{(v)}_{(i)\alpha}),$$

$$\Pi_{(i)\alpha\beta}\equiv\frac{1}{a^2}\left(\Pi_{(i),\alpha|\beta}-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\Pi_{(i)}\right)+\frac{1}{a}\Pi^{(v)}_{(i)(\alpha|\beta)}+\Pi^{(t)}_{(i)\alpha\beta}, \tag{180}$$

with

$$v^{(v)\alpha}_{(i)}{}_{|\alpha}\equiv0,\quad \Pi^{(v)\alpha}_{(i)}{}_{|\alpha}\equiv0,\quad \Pi^{(t)\alpha}_{(i)\alpha}\equiv0,\quad \Pi^{(t)\beta}_{(i)\alpha|\beta}\equiv0, \tag{181}$$

and

$$\delta I_{(i)\alpha}\equiv\delta I_{(i),\alpha}+\delta I^{(v)}_{(i)\alpha},\quad \delta I^{(v)\alpha}_{(i)}{}_{|\alpha}\equiv0. \tag{182}$$

The definitions for the scalar-type perturbation variables are introduced to match our notation used in the linear analysis [7,12,29]; compared with our previous definitions in the linear theory our v and $v_{(i)}$ correspond to v/k and $v_{(i)}/k$ in [29] where k is the wave number. These are the notations introduced by Bardeen in 1988 [7]. A complete set of equations written separately for the three perturbation types will be presented in Eqs. (195)–(210), where the quadratic combinations of the linear-order variables contribute to the second-order perturbations. Thus, to second order the three perturbation types couple with each other through quadratic combinations of the linear-order terms. If needed we may decompose the perturbed order quantities explicitly as in Eq. (50):

$$\alpha\equiv\alpha^{(1)}+\alpha^{(2)},\quad \varphi\equiv\varphi^{(1)}+\varphi^{(2)}, \tag{183}$$

etc.

B. Background equations

To background order, Eqs. (100),(102),(104),(112) give

$$H^2=\frac{8\pi G}{3}\mu-\frac{K}{a^2}+\frac{\Lambda}{3}, \tag{184}$$

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(\mu+3p)+\frac{\Lambda}{3}, \tag{185}$$

$$\dot{\mu}+3H(\mu+p)=0, \tag{186}$$

$$\ddot{\phi}+3H\dot{\phi}+V_{,\phi}=0. \tag{187}$$

In the multicomponent situations from Eqs. (106),(119) we have

$$\dot{\mu}_{(i)}+3H(\mu_{(i)}+p_{(i)})=-\frac{1}{a}I_{(i)0}, \tag{188}$$

$$\ddot{\phi}_{(i)}+3H\dot{\phi}_{(i)}+V_{,\phi_{(i)}}=0, \tag{189}$$

with

$$\mu_{(\phi)}=\frac{1}{2}\sum_k \dot{\phi}^2_{(k)}+V, \quad p_{(\phi)}=\frac{1}{2}\sum_k \dot{\phi}^2_{(k)}-V, \tag{190}$$

which follow from Eq. (121).

In the generalized gravity considered in Sec. IV D, Eqs. (184),(185) remain valid by replacing the fluid quantities to the effective one in Eq. (130):

$$\mu^{(\mathrm{eff})}=\frac{1}{8\pi GF}\left[\mu+\frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J-\frac{1}{2}(f-RF-2V)-3H\dot{F}\right],$$

$$p^{(\mathrm{eff})}=\frac{1}{8\pi GF}\left[p+\frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J+\frac{1}{2}(f-RF-2V)+\ddot{F}+2H\dot{F}\right]. \tag{191}$$

For the equation of motion, Eq. (128) gives

$$\ddot{\phi}^I+3H\dot{\phi}^I-\frac{1}{2}g^{IJ}(f_{,J}-2V_{,J})+\Gamma^I_{JK}\dot{\phi}^J\dot{\phi}^K+\Gamma^I=0. \tag{192}$$

The null-geodesic equations are presented in Eq. (156). The Boltzmann equation in Eq. (171) gives

$$f'=\frac{a^2}{\sqrt{q^2+m^2a^2}}C[f], \tag{193}$$

and from Eq. (174) we have

$$\mu^{(c)}=\frac{1}{a^4}\int \sqrt{q^2+m^2a^2}\,q^2dqd\Omega_q f,$$

$$p^{(c)}=\frac{1}{3a^4}\int \frac{q^4dqd\Omega_q}{\sqrt{q^2+m^2a^2}}f. \tag{194}$$

C. Decomposed equations

We summarize a complete set of equations necessary to analyze each perturbation type. We decompose the perturbation variables according to Eq. (175). Algebraic manipulations are made which can be recognized by examining the right-hand sides of the following equations.

For the scalar-type perturbation,

$$\kappa-3H\alpha+3\dot{\varphi}+\frac{\Delta}{a^2}\chi=N_0, \tag{195}$$

$$4\pi G\delta\mu+H\kappa+\frac{\Delta+3K}{a^2}\varphi=\frac{1}{4}N_1, \tag{196}$$

$$\kappa+\frac{\Delta+3K}{a^2}\chi-12\pi G(\mu+p)av=\frac{3}{2}\Delta^{-1}\nabla^\alpha N_{2\alpha}\equiv N_2^{(s)}, \tag{197}$$

$$\dot{\kappa}+2H\kappa-4\pi G(\delta\mu+3\delta p)+\left(3\dot{H}+\frac{\Delta}{a^2}\right)\alpha=N_3, \tag{198}$$

$$\dot{\chi}+H\chi-\varphi-\alpha-8\pi G\Pi=\frac{3}{2}a^2(\Delta+3K)^{-1}\Delta^{-1}\nabla^\alpha\nabla_\beta N_{4\alpha}{}^\beta \equiv N_4^{(s)}, \tag{199}$$

$$\delta\dot{\mu}+3H(\delta\mu+\delta p)-(\mu+p)\left(\kappa-3H\alpha+\frac{1}{a}\Delta v\right)=N_5, \tag{200}$$

$$\frac{[a^4(\mu+p)v]^{\cdot}}{a^4(\mu+p)}-\frac{1}{a}\alpha-\frac{1}{a(\mu+p)}\left(\delta p+\frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)=-\frac{1}{\mu+p}\Delta^{-1}\nabla^\alpha N_{6\alpha}\equiv N_6^{(s)}, \tag{201}$$

$$\delta\dot{\mu}_{(i)}+3H(\delta\mu_{(i)}+\delta p_{(i)})-(\mu_{(i)}+p_{(i)})\times\left(\kappa-3H\alpha+\frac{1}{a}\Delta v_{(i)}\right)+\frac{1}{a}\delta I_{(i)0}=N_{5(i)}, \tag{202}$$

$$\frac{[a^4(\mu_{(i)}+p_{(i)})v_{(i)}]^{\cdot}}{a^4(\mu_{(i)}+p_{(i)})}-\frac{1}{a}\alpha-\frac{1}{a(\mu_{(i)}+p_{(i)})}\times\left(\delta p_{(i)}+\frac{2}{3}\frac{\Delta+3K}{a^2}\Pi_{(i)}-\delta I_{(i)}\right)=-\frac{1}{\mu_{(i)}+p_{(i)}}\Delta^{-1}\nabla^\alpha N_{6(i)\alpha}\equiv N_{6(i)}^{(s)}, \tag{203}$$

$$\delta\ddot{\phi}+3H\delta\dot{\phi}-\frac{\Delta}{a^2}\delta\phi+V_{,\phi\phi}\delta\phi-\dot{\phi}(\kappa+\dot{\alpha})-(2\ddot{\phi}+3H\dot{\phi})\alpha$$
$$=N_\phi-\dot{\phi}N_0, \tag{204}$$

$$\delta\ddot{\phi}_{(i)}+3H\delta\dot{\phi}_{(i)}-\frac{\Delta}{a^2}\delta\phi_{(i)}+\sum_k V_{,\phi_{(i)}\phi_{(k)}}\delta\phi_{(k)}$$
$$-\dot{\phi}_{(i)}(\kappa+\dot{\alpha})-(2\ddot{\phi}_{(i)}+3H\dot{\phi}_{(i)})\alpha$$
$$=N_{\phi_{(i)}}-\dot{\phi}_{(i)}N_0. \tag{205}$$

For the vector-type perturbation,

$$\frac{\Delta+2K}{2a^2}\Psi^{(v)}_\alpha+8\pi G(\mu+p)v^{(v)}_\alpha=\frac{1}{a}(N_{2\alpha}-\nabla_\alpha\Delta^{-1}\nabla^\beta N_{2\beta})$$
$$\equiv N^{(v)}_{2\alpha}, \tag{206}$$

$$\dot{\Psi}^{(v)}_\alpha+2H\Psi^{(v)}_\alpha-8\pi G\Pi^{(v)}_\alpha$$
$$=2a(\Delta+2K)^{-1}(\nabla_\beta N_{4\alpha}^{\ \ \beta}-\nabla_\alpha\Delta^{-1}\nabla^\gamma\nabla_\beta N_{4\gamma}^{\ \ \beta})\equiv N^{(v)}_{4\alpha}, \tag{207}$$

$$\frac{[a^4(\mu+p)v^{(v)}_\alpha]^\cdot}{a^4(\mu+p)}+\frac{\Delta+2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\mu+p}$$
$$=\frac{1}{\mu+p}(N_{6\alpha}-\nabla_\alpha\Delta^{-1}\nabla^\beta N_{6\beta})\equiv N^{(v)}_{6\alpha}, \tag{208}$$

$$\frac{[a^4(\mu_{(i)}+p_{(i)})v^{(v)}_{(i)\alpha}]^\cdot}{a^4(\mu_{(i)}+p_{(i)})}+\frac{\Delta+2K}{2a^2}\frac{\Pi^{(v)}_{(i)\alpha}}{\mu_{(i)}+p_{(i)}}-\frac{1}{a}\frac{\delta I^{(v)}_{(i)\alpha}}{\mu_{(i)}+p_{(i)}}$$
$$=\frac{1}{\mu_{(i)}+p_{(i)}}(N_{6(i)\alpha}-\nabla_\alpha\Delta^{-1}\nabla^\beta N_{6(i)\beta})\equiv N^{(v)}_{6(i)\alpha}. \tag{209}$$

For the tensor-type perturbation,

$$\ddot{C}^{(t)}_{\alpha\beta}+3H\dot{C}^{(t)}_{\alpha\beta}-\frac{\Delta-2K}{a^2}C^{(t)}_{\alpha\beta}-8\pi G\Pi^{(t)}_{\alpha\beta}$$
$$=N_{4\alpha\beta}-\frac{3}{2}\left(\nabla_\alpha\nabla_\beta-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)(\Delta+3K)^{-1}\Delta^{-1}\nabla^\gamma$$
$$\times\nabla_\delta N_{4\gamma}^{\ \ \delta}-2\nabla_{(\alpha}(\Delta+2K)^{-1}$$
$$\times(\nabla^\gamma N_{4\beta)\gamma}-\nabla_{\beta)}\Delta^{-1}\nabla^\gamma\nabla_\delta N_{4\gamma}^{\ \ \delta})\equiv N^{(t)}_{4\alpha\beta}. \tag{210}$$

In order to derive Eqs. (199),(207),(210) it is convenient to show that

$$\frac{1}{a^2}\left(\nabla_\alpha\nabla_\beta-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)(\dot{\chi}+H\chi-\varphi-\alpha-8\pi G\Pi)$$
$$+\frac{1}{a^3}(a^2\Psi^{(v)}_{(\alpha|\beta)})^\cdot-8\pi G\frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)}+\ddot{C}^{(t)}_{\alpha\beta}+3H\dot{C}^{(t)}_{\alpha\beta}$$
$$-\frac{\Delta-2K}{a^2}C^{(t)}_{\alpha\beta}-8\pi G\Pi^{(t)}_{\alpha\beta}=N_{4\alpha\beta}, \tag{211}$$

which follows from Eq. (103). In our perturbative approach, the second-order perturbations are sourced by the quadratic combinations of all three types of linear-order terms.

For the scalar field, from Eq. (121), we have

$$\delta\mu^{(\phi)}=\sum_k(\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)}-\dot{\phi}^2_{(k)}\alpha+V_{,\phi_{(k)}}\delta\phi_{(k)})+\delta\mu^{(q)},$$

$$\delta p^{(\phi)}=\sum_k(\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)}-\dot{\phi}^2_{(k)}\alpha-V_{,\phi_{(k)}}\delta\phi_{(k)})+\delta p^{(q)},$$

$$Q^{(\phi)}=-(\mu^{(\phi)}+p^{(\phi)})v^{(\phi)}=-\frac{1}{a}\sum_k\dot{\phi}_{(k)}\delta\phi_{(k)}$$
$$+\Delta^{-1}\nabla^\alpha Q^{(q)}_\alpha,$$

$$Q^{(\phi,v)}_\alpha=(\mu^{(\phi)}+p^{(\phi)})v^{(\phi,v)}_\alpha=Q^{(q)}_\alpha-\nabla_\alpha\Delta^{-1}\nabla^\beta Q^{(q)}_\beta. \tag{212}$$

The anisotropic pressure follows from Eqs. (121),(177).

In the generalized gravity theory in Sec. IV D, Eq. (128) gives

$$\delta\ddot{\phi}^I+3H\delta\dot{\phi}^I-\frac{\Delta}{a^2}\delta\phi^I-2\alpha\ddot{\phi}^I+\dot{\phi}^I\left(3\dot{\varphi}-\dot{\alpha}-6H\alpha+\frac{\Delta}{a^2}\chi\right)$$
$$-\frac{1}{2}g^{IJ}[F_{,J}\delta R+(f_{,LJ}-2V_{,LJ})\delta\phi^L]$$
$$-\frac{1}{2}g^{IJ}_{\ \ ,L}\delta\phi^L(f_{,J}-2V_{,J})+2\Gamma^I_{JK}(\dot{\phi}^J\delta\dot{\phi}^K-A\dot{\phi}^J\dot{\phi}^K)$$
$$+\Gamma^I_{JK,L}\delta\phi^L\dot{\phi}^J\dot{\phi}^K+\delta\Gamma^I=N_g. \tag{213}$$

Equations (195)–(201),(206)–(208),(210) remain valid even in generalized gravity by replacing the fluid quantities with the effective ones. The decomposed effective fluid quantities follow from Eqs. (130),(177) as

$$\delta\mu^{(\rm eff)}=\frac{1}{8\pi GF}\Bigg[\delta\mu+g_{IJ}(\dot\phi^I\delta\dot\phi^J-\alpha\dot\phi^I\dot\phi^J)+\frac{1}{2}g_{IJ,L}\delta\phi^L\dot\phi^I\dot\phi^J-\frac{1}{2}(f_{,L}-2V_{,L})\delta\phi^L-3H\delta\dot F+\left(\frac{1}{2}R+\frac{\Delta}{a^2}\right)\delta F+\left(6H\alpha-\frac{\Delta}{a^2}\chi-3\dot\varphi\right)\dot F-8\pi G\mu^{(\rm eff)}\delta F\Bigg]+\delta\mu^{(\rm eff,q)},$$

$$\delta p^{(\rm eff)}=\frac{1}{8\pi GF}\Bigg[\delta p+g_{IJ}(\dot\phi^I\delta\dot\phi^J-\alpha\dot\phi^I\dot\phi^J)+\frac{1}{2}g_{IJ,L}\delta\phi^L\dot\phi^I\dot\phi^J+\frac{1}{2}(f_{,L}-2V_{,L})\delta\phi^L+\delta\ddot F+2H\delta\dot F-\left(\frac{1}{2}R+\frac{2}{3}\frac{\Delta}{a^2}\right)\delta F-2\alpha\ddot F-\left(\dot\alpha+4H\alpha-\frac{2}{3}\frac{\Delta}{a^2}\chi-2\dot\varphi\right)\dot F-8\pi Gp^{(\rm eff)}\delta F\Bigg]+\delta p^{(\rm eff,q)},$$

$$Q^{(\rm eff)}=\frac{1}{8\pi GF}\left[Q-\frac{1}{a}g_{IJ}\dot\phi^I\delta\phi^J+\frac{1}{a}(-\delta\dot F+H\delta F)+\frac{1}{a}A\dot F\right]+Q^{(\rm eff,q)},$$

$$Q_\alpha^{(\rm eff,v)}=\frac{1}{8\pi GF}Q_\alpha^{(v)}+Q_\alpha^{(\rm eff,v,q)},$$

$$\Pi^{(\rm eff)}=\frac{1}{8\pi GF}(\Pi+\delta F-\chi\dot F)+\Pi^{(\rm eff,q)},$$

$$\Pi_\alpha^{(\rm eff,v)}=\frac{1}{8\pi GF}(\Pi_\alpha^{(v)}-\Psi_\alpha^{(v)}\dot F)+\Pi_\alpha^{(\rm eff,v,q)},$$

$$\Pi_{\alpha\beta}^{(\rm eff,t)}=\frac{1}{8\pi GF}(\Pi_{\alpha\beta}^{(t)}-\dot C_{\alpha\beta}^{(t)}\dot F)+\Pi_{\alpha\beta}^{(\rm eff,t,q)},\qquad(214)$$

where the quadratic parts follow from Eq. (177). As an example, from Eqs. (210),(214), the gravitational wave equation in generalized gravity becomes

$$\ddot C_{\alpha\beta}^{(t)}+\left(3H+\frac{\dot F}{F}\right)\dot C_{\alpha\beta}^{(t)}-\frac{\Delta-2K}{a^2}C_{\alpha\beta}^{(t)}=\frac{1}{F}\Pi_{\alpha\beta}^{(t)}+8\pi G\Pi_{\alpha\beta}^{(\rm eff,t,q)}+N_{4\alpha\beta}^{(t)}.\qquad(215)$$

For the electromagnetic field we can decompose

$$E_\alpha\equiv E_{,\alpha}^{(\rm e.m.)}+E_\alpha^{(v)},\quad H_\alpha\equiv H_{,\alpha}^{(\rm e.m.)}+H_\alpha^{(v)},\quad E_\alpha^{(v)|\alpha}\equiv 0\equiv H_\alpha^{(v)|\alpha}.\qquad(216)$$

The decomposed forms of fluid quantities can be read from Eqs. (149),(177). Similarly, for the null-geodesic equations we decompose

$$\delta e_\alpha\equiv\delta e_{,\alpha}+\delta e_\alpha^{(v)},\quad \delta e_\alpha^{(v)|\alpha}\equiv 0.\qquad(217)$$

For the temperature anisotropy, Eq. (163) gives

$$\frac{\delta T}{T}\bigg|_O=\frac{\delta T}{T}\bigg|_E-v_{,\alpha}e^\alpha\bigg|_E^O+\int_E^O\left(-\varphi'+\alpha_{,\alpha}e^\alpha-\frac{1}{a}\chi_{,\alpha|\beta}e^\alpha e^\beta\right)dy+v_\alpha^{(v)}e^\alpha\bigg|_E^O-\int_E^O\Psi_{\alpha|\beta}^{(v)}e^\alpha e^\beta dy-\int_E^O C_{\alpha\beta}^{(t)\prime}e^\alpha e^\beta dy+\int_E^O N_\nu dy+N_T.\qquad(218)$$

To linear order this result was first presented by Sachs and Wolfe [17]; for further analyses using our notation, see [30].

For the Boltzmann equation, Eq. (171) becomes

$$\tilde f'+\frac{q}{\sqrt{q^2+m^2a^2}}\left(\gamma^\alpha\delta f_{,\alpha}-\Gamma^{(3)\alpha}{}_{\beta\gamma}\gamma^\beta\gamma^\gamma\frac{\partial\delta f}{\partial\gamma^\alpha}\right)-\left[\varphi'+\frac{\sqrt{q^2+m^2a^2}}{q}\gamma^\alpha\alpha_{,\alpha}+\left(\frac{1}{a}\chi_{,\alpha|\beta}+\Psi_{\alpha|\beta}^{(v)}+C_{\alpha\beta}^{(t)\prime}\right)\gamma^\alpha\gamma^\beta\right]q\frac{\partial\tilde f}{\partial q}=N_c.\qquad(219)$$

The fluid quantities can be read from Eqs. (174),(177).

We emphasize that all the equations up to this point are presented without fixing the gauge conditions. In order to solve the equations in a given situation, we can choose *any* allowed gauge conditions suitable for the situation. In this sense, the equations are presented in a *gauge-ready* form.

VI. GAUGE ISSUE

A. Gauge transformation

We consider the following transformation between two coordinates x^a and $\hat x^a$:

$$\hat{x}^a \equiv x^a + \tilde{\xi}^a(x^e) \equiv x^a + \tilde{\zeta}^a + \frac{1}{2}\tilde{\zeta}^a{}_{,b}\tilde{\zeta}^b. \tag{220}$$

The variables $\tilde{\xi}^a$ and $\tilde{\zeta}^a$ are perturbed order quantities. To second order we may have

$$\tilde{\xi}^a \equiv \tilde{\xi}^{(1)a} + \tilde{\xi}^{(2)a}, \tag{221}$$

and similarly for $\tilde{\zeta}^a$. For any tensor quantity we use the tensor transformation property between x^a and $\hat{x}^a$ spacetimes:

$$\tilde{\phi}(x^e) = \hat{\tilde{\phi}}(\hat{x}^e), \quad \tilde{v}_a(x^e) = \frac{\partial \hat{x}^b}{\partial x^a}\hat{\tilde{v}}_b(\hat{x}^e),$$

$$\tilde{t}_{ab}(x^e) = \frac{\partial \hat{x}^c}{\partial x^a}\frac{\partial \hat{x}^d}{\partial x^b}\hat{\tilde{t}}_{cd}(\hat{x}^e). \tag{222}$$

Comparing the tensor quantities at the same spacetime point x^a, we can derive the gauge transformation property of the tensor quantity. We can show that a tensor quantity $\mathbf{t}$ transforms as [31]

$$\hat{\mathbf{t}}(x^c) = \mathbf{t}(x^c) - £_{\tilde{\zeta}}\mathbf{t} + \frac{1}{2}£^2_{\tilde{\zeta}}\mathbf{t}, \tag{223}$$

where $£_{\tilde{\zeta}}$ is a Lie derivative along $\tilde{\zeta}^a$. We have

$$\hat{\tilde{\phi}}(x^e) = \tilde{\phi}(x^e) - \tilde{\phi}_{,c}\tilde{\xi}^c + \tilde{\phi}_{,b}\tilde{\xi}^b{}_{,c}\tilde{\xi}^c + \frac{1}{2}\tilde{\phi}_{,bc}\tilde{\xi}^b\tilde{\xi}^c, \tag{224}$$

$$\hat{\tilde{v}}_a(x^e) = \tilde{v}_a(x^e) - \tilde{v}_{a,b}\tilde{\xi}^b - \tilde{v}_b\tilde{\xi}^b{}_{,a} + \frac{1}{2}\tilde{v}_{a,bc}\tilde{\xi}^b\tilde{\xi}^c + \tilde{v}_{a,b}\tilde{\xi}^b{}_{,c}\tilde{\xi}^c + \tilde{v}_{b,c}\tilde{\xi}^b{}_{,a}\tilde{\xi}^c + \tilde{v}_b\tilde{\xi}^b{}_{,ac}\tilde{\xi}^c + \tilde{v}_c\tilde{\xi}^c{}_{,b}\tilde{\xi}^b{}_{,a}, \tag{225}$$

$$\hat{\tilde{t}}_{ab}(x^e) = \tilde{t}_{ab}(x^e) - 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)} - \tilde{t}_{ab,c}\tilde{\xi}^c + 2\tilde{t}_{c(a}\tilde{\xi}^d{}_{,b)}\tilde{\xi}^c{}_{,d} + \tilde{t}_{cd}\tilde{\xi}^c{}_{,a}\tilde{\xi}^d{}_{,b} + \tilde{\xi}^d\left(2\tilde{\xi}^c{}_{,(b}\tilde{t}_{a)c,d} + 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)d} + \frac{1}{2}\tilde{t}_{ab,cd}\tilde{\xi}^c + \tilde{t}_{ab,c}\tilde{\xi}^c{}_{,d}\right). \tag{226}$$

We define

$$\tilde{\xi}^0 \equiv \xi^0, \quad \tilde{\xi}^\alpha \equiv \xi^\alpha, \tag{227}$$

where ξ^α is based on $g^{(3)}_{\alpha\beta}$. In terms of $\tilde{\zeta}^a$ we set $\tilde{\zeta}^0 \equiv \zeta^0$ and $\tilde{\zeta}^\alpha \equiv \zeta^\alpha$ where ζ^α is based on $g^{(3)}_{\alpha\beta}$. Thus, we have

$$\xi^0 = \zeta^0 + \frac{1}{2}\zeta^{0\prime}\zeta^0 + \frac{1}{2}\zeta^0{}_{,\alpha}\zeta^\alpha, \quad \xi^\alpha = \zeta^\alpha + \frac{1}{2}\zeta^{\alpha\prime}\zeta^0 + \frac{1}{2}\zeta^\alpha{}_{,\beta}\zeta^\beta. \tag{228}$$

From the gauge transformation property of $\tilde{g}_{ab}$ and the definitions of our perturbation variables we can derive

$$\hat{A} = A - \left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right) - A'\xi^0 - 2A\left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right) - A_{,\alpha}\xi^\alpha - B_\alpha\xi^{\alpha\prime} + \frac{3}{2}\xi^{0\prime}\xi^{0\prime} + \xi^0{}_{,\alpha}\xi^{\alpha\prime} + \xi^\alpha\left(\xi^{0\prime}{}_{,\alpha} + \frac{a'}{a}\xi^0{}_{,\alpha}\right) + \xi^0\left[\xi^{0\prime\prime} + 3\frac{a'}{a}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\xi^0\right] - \frac{1}{2}\xi^{\alpha\prime}\xi'_\alpha, \tag{229}$$

$$\hat{B}_\alpha = B_\alpha - \xi^0{}_{,\alpha} + \xi'_\alpha - 2A\xi^0{}_{,\alpha} - \left(B'_\alpha + 2\frac{a'}{a}B_\alpha\right)\xi^0 - B_\alpha\xi^{0\prime} - B_{\alpha,\beta}\xi^\beta - B_\beta\xi^\beta{}_{,\alpha} + 2C_{\alpha\beta}\xi^{\beta\prime} - \xi'_\alpha\xi^{0\prime} + 2\xi^{0\prime}\xi^0{}_{,\alpha} + \xi^0{}_{,\beta}\xi^\beta{}_{,\alpha} + \xi^\gamma\xi^0{}_{,\alpha\gamma} - \xi^0\left(\xi''_\alpha + 2\frac{a'}{a}\xi'_\alpha - \xi^{0\prime}{}_{,\alpha} - 2\frac{a'}{a}\xi^0{}_{,\alpha}\right) - \xi^\beta{}_{,\alpha}\xi'_\beta - g^{(3)}_{\alpha\beta}\xi^\beta{}_{,\gamma}\xi^{\gamma\prime} - \xi^\gamma(g^{(3)}_{\alpha\beta,\gamma}\xi^{\beta\prime} + g^{(3)}_{\alpha\beta}\xi^{\beta\prime}{}_{,\gamma}), \tag{230}$$

$$\hat{C}_{\alpha\beta} = C_{\alpha\beta} - \frac{a'}{a}\xi^0 g^{(3)}_{\alpha\beta} - \frac{1}{2}g^{(3)}_{\alpha\beta,\gamma}\xi^\gamma - g^{(3)}_{\gamma(\alpha}\xi^\gamma{}_{,\beta)} + B_{(\alpha}\xi^0{}_{,\beta)} - \left(C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}\right)\xi^0 - C_{\alpha\beta,\gamma}\xi^\gamma - 2C_{\gamma(\alpha}\xi^\gamma{}_{,\beta)} + \xi'_{(\alpha}\xi^0{}_{,\beta)} - \frac{1}{2}\xi^0{}_{,\alpha}\xi^0{}_{,\beta} + \frac{a'}{a}g^{(3)}_{\alpha\beta}\xi^\gamma\xi^0{}_{,\gamma} + \xi^0\left[\frac{a'}{a}g^{(3)}_{\alpha\beta}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)g^{(3)}_{\alpha\beta}\xi^0 + \left(\frac{1}{2}\xi^{\gamma\prime} + \frac{a'}{a}\xi^\gamma\right)g^{(3)}_{\alpha\beta,\gamma} + 2\frac{a'}{a}g^{(3)}_{\gamma(\alpha}\xi^\gamma{}_{,\beta)} + g^{(3)}_{\gamma(\alpha}\xi^{\gamma\prime}{}_{,\beta)}\right] + \xi^\delta{}_{,(\beta}g^{(3)}_{\alpha)\gamma}\xi^\gamma{}_{,\delta} + \frac{1}{2}g^{(3)}_{\gamma\delta}\xi^\gamma{}_{,\alpha}\xi^\delta{}_{,\beta} + \xi^\delta\left(\frac{1}{2}g^{(3)}_{\alpha\beta,\gamma}\xi^\gamma{}_{,\delta} + \xi^\gamma{}_{,(\beta}g^{(3)}_{\alpha)\gamma,\delta} + \frac{1}{4}g^{(3)}_{\alpha\beta,\gamma\delta}\xi^\gamma + g^{(3)}_{\gamma(\alpha}\xi^\gamma{}_{,\beta)\delta}\right). \tag{231}$$

From the gauge transformation property of $\tilde{T}_{ab}$ and using the definitions of our perturbed fluid variables in the normal frame we can derive

$$\delta\hat{\mu} = \delta\mu - (\mu' + \delta\mu')\xi^0 - \delta\mu_{,\alpha}\xi^\alpha + \mu'(\xi^0\xi^{0\prime} + \xi^0{}_{,\alpha}\xi^\alpha) + \frac{1}{2}\mu''\xi^0\xi^0 + [2Q^\alpha + (\mu+p)\xi^{0,\alpha}]\xi^0{}_{,\alpha}, \tag{232}$$

$$\delta\hat{p} = \delta p - (p' + \delta p')\xi^0 - \delta p_{,\alpha}\xi^\alpha + p'(\xi^0\xi^{0\prime} + \xi^0{}_{,\alpha}\xi^\alpha) + \frac{1}{2}p''\xi^0\xi^0 + \frac{1}{3}[2Q^\alpha + (\mu+p)\xi^{0,\alpha}]\xi^0{}_{,\alpha}, \tag{233}$$

$$\begin{aligned}\hat{Q}_\alpha = {}& Q_\alpha + (\mu+p)\xi^0{}_{,\alpha} - Q_\beta\xi^\beta{}_{,\alpha} - Q_{\alpha,\beta}\xi^\beta \\ & - \left(Q'_\alpha + \frac{a'}{a}Q_\alpha\right)\xi^0 + (\mu+p)A\xi^0{}_{,\alpha} \\ & + [\delta\mu + \delta p - (\mu'+p')\xi^0]\xi^0{}_{,\alpha} + \Pi^\beta_\alpha\xi^0{}_{,\beta} - (\mu+p) \\ & \times\left[\xi^0{}_{,\alpha}\xi^{0\prime} + \xi^\beta{}_{,\alpha}\xi^0{}_{,\beta} + \xi^0\left(\xi^{0\prime}{}_{,\alpha} + \frac{a'}{a}\xi^0{}_{,\alpha}\right) \right.\\ & \left. + \xi^\beta\xi^0{}_{,\alpha\beta}\right],\end{aligned} \tag{234}$$

$$\begin{aligned}\hat{\Pi}_{\alpha\beta} = {}& \Pi_{\alpha\beta} - 2\Pi_{\gamma(\alpha}\xi^\gamma{}_{,\beta)} - \left(\Pi'_{\alpha\beta} + 2\frac{a'}{a}\Pi_{\alpha\beta}\right)\xi^0 - \Pi_{\alpha\beta,\gamma}\xi^\gamma \\ & + [2Q_{(\alpha} + (\mu+p)\xi^0{}_{,(\alpha}]\xi^0{}_{,\beta)} \\ & - \frac{1}{3}g^{(3)}_{\alpha\beta}[2Q^\gamma + (\mu+p)\xi^{0,\gamma}]\xi^0{}_{,\gamma}.\end{aligned} \tag{235}$$

Under the gauge transformation the individual fluid quantities $\delta\mu_{(i)}$, $\delta p_{(i)}$, $Q_{(i)\alpha}$, $\Pi_{(i)\alpha\beta}$, and $\delta\phi_{(i)}$ transform just like the corresponding collective fluid quantities in Eqs. (232)–(239) with all the fluid quantities changed into those for the individual one. Using the vector nature of $\tilde{I}_{(i)a}$ we have

$$\begin{aligned}\delta\hat{I}_{(i)0} = {}& \delta I_{(i)0} - (I_{(i)0}\xi^0)' - (\delta I_{(i)0}\xi^0)' - \delta I_{(i)0,\alpha}\xi^\alpha \\ & + \delta I_{(i)\alpha}\xi^{\alpha\prime} + I'_{(i)0}\xi^{0\prime}\xi^0 + [I_{(i)0}(\xi^{0\prime}\xi^0 + \xi^0{}_{,\alpha}\xi^\alpha)]',\end{aligned} \tag{236}$$

$$\begin{aligned}\delta\hat{I}_{(i)\alpha} = {}& \delta I_{(i)\alpha} - I_{(i)0}\xi^0{}_{,\alpha} - \delta I_{(i)0}\xi^0{}_{,\alpha} - \delta I'_{(i)\alpha}\xi^0 - \delta I_{(i)\alpha,\beta}\xi^\beta \\ & - \delta I_{(i)\beta}\xi^\beta{}_{,\alpha} + I'_{(i)0}\xi^0{}_{,\alpha}\xi^0 + I_{(i)0} \\ & \times[(\xi^0{}_{,\alpha}\xi^0)' + (\xi^0{}_{,\beta}\xi^\beta)_{,\alpha}].\end{aligned} \tag{237}$$

The fluid quantities we use in this work are based on the normal-frame four-vector where $\tilde{n}_\alpha = 0$ [see Eq. (83)]. It is convenient to have the gauge-transformation properties of the fluid quantities in the energy frame where we set $Q_\alpha = 0$. These can be derived either by applying the frame-transformation rule presented in Eqs. (87),(88) or directly from the gauge transformation property of the energy-momentum tensor in Eq. (82) with $Q_\alpha = 0$ in the energy frame. We have

$$\delta\hat{\mu}^E = \delta\mu^E - (\mu' + \delta\mu^{E\prime})\xi^0 - \delta\mu^E{}_{,\alpha}\xi^\alpha + \frac{1}{2}\mu''\xi^0\xi^0 + \mu'(\xi^0\xi^{0\prime} + \xi^\alpha\xi^0{}_{,\alpha}),$$

$$\delta\hat{p}^E = \delta p^E - (p' + \delta p^{E\prime})\xi^0 - \delta p^E{}_{,\alpha}\xi^\alpha + \frac{1}{2}p''\xi^0\xi^0 + p'(\xi^0\xi^{0\prime} + \xi^\alpha\xi^0{}_{,\alpha}),$$

$$\begin{aligned}\hat{V}^E_\alpha - \hat{B}_\alpha = {}& V^E_\alpha - B_\alpha + \xi^0{}_{,\alpha} - (V^E_\alpha - B_\alpha)'\xi^0 + \frac{a'}{a}(V^E_\alpha - B_\alpha)\xi^0 \\ & - (V^E_\beta - B_\beta)\xi^\beta{}_{,\alpha} - (V^E_\alpha - B_\alpha)_{,\beta}\xi^\beta + (V^{E\beta} + \xi^{\beta\prime}) \\ & \times(g^{(3)}_{\alpha\beta,\gamma}\xi^\gamma + 2g^{(3)}_{\gamma(\alpha}\xi^\gamma{}_{,\beta)}) + \left(A - \xi^{0\prime} - \frac{a'}{a}\xi^0\right) \\ & \times(2\xi^0{}_{,\alpha} - \xi'_\alpha) + B_\alpha\left(\xi^{0\prime} + 3\frac{a'}{a}\xi^0\right) - 2C_{\alpha\beta}\xi^{\beta\prime} \\ & - \xi^\beta{}_{,\alpha}\xi^0{}_{,\beta} - \xi^0\xi^{0\prime}{}_{,\alpha} - \xi^\beta\xi^0{}_{,\alpha\beta} + 2\frac{a'}{a}\xi^0\xi'_\alpha,\end{aligned}$$

$$\hat{\Pi}^E_{\alpha\beta} = \Pi^E_{\alpha\beta} - \left(\Pi^{E\prime}_{\alpha\beta} + 2\frac{a'}{a}\Pi^E_{\alpha\beta}\right)\xi^0 - \Pi^E_{\alpha\beta,\gamma}\xi^\gamma - 2\Pi^E_{\gamma(\alpha}\xi^\gamma{}_{,\beta)}. \tag{238}$$

From the gauge transformation of $\tilde{\phi}$ we have

$$\delta\hat{\phi} = \delta\phi - (\phi' + \delta\phi')\xi^0 - \delta\phi_{,\alpha}\xi^\alpha + \phi'(\xi^{0\prime}\xi^0 + \xi^0{}_{,\alpha}\xi^\alpha) + \frac{1}{2}\phi''\xi^0\xi^0. \tag{239}$$

Using the gauge-transformation property of the vector quantity $\tilde{k}^a$ similar to Eqs. (222),(225), and using the definition of $\tilde{k}^a$ in Eq. (150), we can derive

$$\frac{\delta\hat{\nu}}{\nu}=\frac{\delta\nu}{\nu}+\xi^{0\prime}+2\frac{a'}{a}\xi^0-\xi^0{}_{,\alpha}e^\alpha-\frac{\delta\nu'}{\nu}\xi^0+\frac{\delta\nu}{\nu}\left(\xi^{0\prime}+\frac{a'}{a}\xi^0\right)-\frac{\delta\nu_{,\alpha}}{\nu}\xi^\alpha-\xi^0{}_{,\alpha}\delta e^\alpha+\xi^0\left[-\xi^{0\prime\prime}-\left(\frac{a''}{a}-3\frac{a'^2}{a^2}\right)\xi^0+\xi^{0\prime}{}_{,\alpha}e^\alpha\right.$$
$$\left.+\xi^0{}_{,\alpha}\left(e^{\alpha\prime}-2\frac{a'}{a}e^\alpha\right)\right]+\xi^\alpha\left(-\xi^{0\prime}{}_{,\alpha}-2\frac{a'}{a}\xi^0{}_{,\alpha}+\xi^0{}_{,\alpha\beta}e^\beta+\xi^0{}_{,\beta}e^\beta{}_{,\alpha}\right), \quad (240)$$

$$\delta\hat{e}^\alpha=\delta e^\alpha-\xi^0\left(e^{\alpha\prime}-2\frac{a'}{a}e^\alpha\right)-\xi^{\alpha\prime}+\xi^\alpha{}_{,\beta}e^\beta-e^\alpha{}_{,\beta}\xi^\beta-\xi^0\left(\delta e^{\alpha\prime}-2\frac{a'}{a}\delta e^\alpha\right)-\xi^{\alpha\prime}\frac{\delta\nu}{\nu}+\xi^\alpha{}_{,\beta}\delta e^\beta-\delta e^\alpha{}_{,\beta}\xi^\beta$$
$$+\xi^0\left\{\xi^{0\prime}\left(e^{\alpha\prime}-2\frac{a'}{a}e^\alpha\right)+\xi^0\left[\frac{1}{2}e^{\alpha\prime\prime}-2\frac{a'}{a}e^{\alpha\prime}-\left(\frac{a''}{a}-3\frac{a'^2}{a^2}\right)e^\alpha\right]+\xi^{\alpha\prime\prime}-2\frac{a'}{a}\xi^{\alpha\prime}-\xi^{\alpha\prime}{}_{,\beta}e^\beta+e^\alpha{}_{,\beta}\xi^{\beta\prime}\right.$$
$$\left.+\xi^\beta\left(e^{\alpha\prime}{}_{,\beta}-2\frac{a'}{a}e^\alpha{}_{,\beta}\right)-\xi^\alpha{}_{,\beta}\left(e^{\beta\prime}-2\frac{a'}{a}e^\beta\right)\right\}+\xi^\beta\left[\frac{1}{2}e^\alpha{}_{,\beta\gamma}\xi^\gamma-\xi^\alpha{}_{,\gamma}e^\gamma{}_{,\beta}+e^\alpha{}_{,\gamma}\xi^\gamma{}_{,\beta}\right.$$
$$\left.+\xi^{\alpha\prime}{}_{,\beta}-\xi^\alpha{}_{,\beta\gamma}e^\gamma+\xi^0{}_{,\beta}\left(e^{\alpha\prime}-2\frac{a'}{a}e^\alpha\right)\right]. \quad (241)$$

Using the scalar nature of the temperature $\tilde{T}$ and Eq. (224) we can show that

$$\delta\hat{T}(x^e)=\delta T(x^e)-(T'+\delta T')\xi^0-\delta T_{,\alpha}\xi^\alpha$$
$$+T'(\xi^{0\prime}\xi^0+\xi^0{}_{,\alpha}\xi^\alpha)+\frac{1}{2}T''\xi^0\xi^0. \quad (242)$$

Using the vector nature of the electric and magnetic vectors and Eq. (225) we can show that

$$\hat{E}_\alpha(x^e)=E_\alpha(x^e)-E'_\alpha\xi^0-E_{\alpha,\beta}\xi^\beta-E_\beta\xi^\beta{}_{,\alpha}, \quad (243)$$

and similarly for H_α. Thus, E_α and H_α are gauge invariant to linear order.

Since $\tilde{p}^a\equiv dx^a/d\lambda$, under the gauge transformation we have $\hat{\tilde{p}}^a=\tilde{p}^a+\tilde{\xi}^a{}_{,b}\tilde{p}^b$. Using the definitions of q and γ^α in Eq. (169) we can derive

$$\hat{q}=q\left\{1+\frac{a'}{a}\xi^0+\frac{\sqrt{q^2+m^2a^2}}{q}\xi^0{}_{,\alpha}\gamma^\alpha+\frac{\sqrt{q^2+m^2a^2}}{q}\right.$$
$$\times\xi^0{}_{,\alpha}\left[\gamma^\alpha\left(A-\xi^{0\prime}+\frac{a'}{a}\xi^0\right)-C^\alpha_\beta\gamma^\beta\right]+\frac{q^2+m^2a^2}{q^2}$$
$$\times\left[-A'\xi^0-A_{,\alpha}\xi^\alpha+\frac{1}{2}\xi^0{}_{,\alpha}\xi^{0|\alpha}+\xi^\alpha\left(\xi^{0\prime}{}_{,\alpha}+\frac{a'}{a}\xi^0{}_{,\alpha}\right)\right.$$
$$\left.+\xi^0\xi^{0\prime\prime}+\frac{a'}{a}\xi^0\xi^{0\prime}+\left(\frac{1}{2}\frac{a''}{a}\frac{3q^2+2m^2a^2}{q^2+m^2a^2}-\frac{a'^2}{a^2}\right)\xi^0\xi^0\right]$$
$$\left.-\frac{1}{2}\frac{m^2a^2}{q^2}\xi^0{}_{,\alpha}\xi^0{}_{,\beta}\gamma^\alpha\gamma^\beta\right\}, \quad (244)$$

$$\hat{\gamma}^\alpha=\gamma^\alpha+\frac{\sqrt{q^2+m^2a^2}}{q}(\xi^{0,\alpha}-\xi^0{}_{,\beta}\gamma^\beta\gamma^\alpha)$$
$$+\left(\xi^\alpha{}_{,\beta}-\frac{1}{2}\xi^\alpha{}_{|\beta}-\frac{1}{2}\xi_\beta{}^{|\alpha}\right)\gamma^\beta. \quad (245)$$

As $\hat{\gamma}^\alpha$ always appears together with perturbed order terms multiplied, it is evaluated only to linear order. From the scalar nature of $\tilde{f}$ we have

$$\tilde{f}(x^e,q,\gamma^\epsilon)=\hat{\tilde{f}}(\hat{x}^e,\hat{q},\hat{\gamma}^\epsilon)=\hat{\tilde{f}}(x^e+\tilde{\xi}^e,q+\delta q,\gamma^\epsilon+\delta\gamma^\epsilon). \quad (246)$$

At the same momentum space and spacetime point, we have

$$\delta\hat{f}=\delta f-\tilde{f}'\xi^0-\tilde{f}_{,q}\delta q+f'\xi^0\xi^{0\prime}+\frac{1}{2}f''\xi^0\xi^0+2f'{}_{,q}\xi^0\delta q$$
$$+f_{,q}\delta q_{,q}\delta q+\frac{1}{2}f_{,qq}\delta q^2-\delta f_{,\alpha}\xi^\alpha-\delta f_{,\gamma^\alpha}\delta\gamma^\alpha. \quad (247)$$

Using Eqs. (244),(245),(229) we have

$$\delta\hat{f}=\delta f-\tilde{f}'\xi^0-q\tilde{f}_{,q}\left(\frac{a'}{a}\xi^0+\frac{\sqrt{q^2+m^2a^2}}{q}\xi^0{}_{,\alpha}\gamma^\alpha\right)-qf_{,q}\Bigg\{\frac{\sqrt{q^2+m^2a^2}}{q}\xi^0{}_{,\alpha}[\gamma^\alpha(A-\xi^{0\prime})-C^\alpha_\beta\gamma^\beta]$$
$$+\frac{q^2+m^2a^2}{q^2}\left[-A'\xi^0-A_{,\alpha}\xi^\alpha+\xi^\alpha\left(\xi^{0\prime}{}_{,\alpha}+\frac{a'}{a}\xi^0{}_{,\alpha}\right)+\xi^0\xi^{0\prime\prime}+\frac{1}{2}\frac{a''}{a}\frac{3q^2+2m^2a^2}{q^2+m^2a^2}\xi^0\xi^0+\frac{a'}{a}\xi^0\xi^{0\prime}+\frac{1}{2}\xi^0{}_{,\alpha}\xi^{0,\alpha}\right]$$
$$-\frac{1}{2}\frac{2q^2+m^2a^2}{q^2}\xi^0{}_{,\alpha}\xi^0{}_{,\beta}\gamma^\alpha\gamma^\beta-\frac{2q^2+m^2a^2}{q^2}\frac{a'^2}{a^2}\xi^0\xi^0-\frac{q}{\sqrt{q^2+m^2a^2}}\frac{a'}{a}\xi^0\xi^0{}_{,\alpha}\gamma^\alpha\Bigg\}+\frac{1}{2}q^2f_{,qq}$$
$$\times\left(\frac{a'^2}{a^2}\xi^0\xi^0+2\frac{\sqrt{q^2+m^2a^2}}{q}\frac{a'}{a}\xi^0\xi^0{}_{,\alpha}\gamma^\alpha+\frac{q^2+m^2a^2}{q^2}\xi^0{}_{,\alpha}\xi^0{}_{,\beta}\gamma^\alpha\gamma^\beta\right)+2qf'{}_{,q}\xi^0\left(\frac{a'}{a}\xi^0+\frac{\sqrt{q^2+m^2a^2}}{q}\xi^0{}_{,\alpha}\gamma^\alpha\right)$$
$$+f'\xi^0\xi^{0\prime}+\frac{1}{2}f''\xi^0\xi^0-\delta f_{,\alpha}\xi^\alpha-\delta f_{,\gamma^\alpha}\left[\frac{\sqrt{q^2+m^2a^2}}{q}(\xi^{0,\alpha}-\xi^0{}_{,\beta}\gamma^\beta\gamma^\alpha)+\left(\xi^\alpha{}_{,\beta}-\frac{1}{2}\xi^\alpha{}_{|\beta}-\frac{1}{2}\xi_\beta{}^{|\alpha}\right)\gamma^\beta\right]. \tag{248}$$

Notice that with our phase space variables introduced in Eq. (169) the distribution function f is spatially gauge invariant to linear order. We can check that the gauge transformation property of δf in Eq. (248) is consistent with the gauge transformation properties of the fluid quantities identified in Eq. (174).

We further decompose ξ_α (and similarly for ζ_α) into the scalar and vector types as

$$\xi_\alpha\equiv\frac{1}{a}\xi_{,\alpha}+\xi^{(v)}_\alpha, \tag{249}$$

with $\xi^{(v)\alpha}{}_{|\alpha}\equiv 0$. In order to fix the gauge we can impose three conditions on three variables such that these conditions can fix ξ^0, ξ, and $\xi^{(v)}_\alpha$. We call these conditions fixing ξ^0, ξ, and $\xi^{(v)}_\alpha$ the temporal, spatial, and rotational gauge-fixing conditions, respectively.

The decomposed variables in Eq. (175) and others transform as

$$\hat{\alpha}=\alpha-\frac{1}{a}(a\xi^0)'+A_\xi,$$

$$\hat{\beta}=\beta-\xi^0+\left(\frac{1}{a}\xi\right)'+\Delta^{-1}\nabla^\alpha B_{\xi\alpha},$$

$$\hat{B}^{(v)}_\alpha=B^{(v)}_\alpha+\xi^{(v)\prime}_\alpha+B_{\xi\alpha}-\nabla_\alpha\Delta^{-1}\nabla^\beta B_{\xi\beta},$$

$$\hat{\gamma}=\gamma-\frac{1}{a}\xi+\frac{1}{2}\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}(3\Delta^{-1}\nabla^\alpha\nabla^\beta \times C_{\xi\alpha\beta}-C_{\xi\alpha}^{\ \ \alpha}),$$

$$\hat{\varphi}=\varphi-\frac{a'}{a}\xi^0+\frac{1}{3}C_{\xi\alpha}^{\ \ \alpha}-\frac{1}{6}\Delta\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}\times(3\Delta^{-1}\nabla^\alpha\nabla^\beta C_{\xi\alpha\beta}-C_{\xi\alpha}^{\ \ \alpha}),$$

$$\hat{C}^{(v)}_\alpha=C^{(v)}_\alpha-\xi^{(v)}_\alpha+2\left(\Delta+\frac{1}{3}R^{(3)}\right)^{-1}\times(\nabla^\beta C_{\xi\alpha\beta}-\nabla_\alpha\Delta^{-1}\nabla^\gamma\nabla^\beta C_{\xi\gamma\beta}),$$

$$\hat{C}^{(t)}_{\alpha\beta}=C^{(t)}_{\alpha\beta}-C_{\xi\alpha\beta}-\frac{1}{3}C_{\xi\gamma}^{\ \ \gamma}g^{(3)}_{\alpha\beta}-\frac{1}{2}\left(\nabla_\alpha\nabla_\beta-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)\times\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}(3\Delta^{-1}\nabla^\gamma\nabla^\delta C_{\xi\gamma\delta}-C_{\xi\gamma}^{\ \ \gamma})-\nabla_{(\alpha}\left(\Delta+\frac{1}{3}R^{(3)}\right)^{-1}(\nabla^\gamma C_{\xi\beta)\gamma}-\nabla_{\beta)}\times\Delta^{-1}\nabla^\gamma\nabla^\delta C_{\xi\gamma\delta}),$$

$$\delta\hat{\mu}=\delta\mu-\mu'\xi^0+\delta\mu_\xi,$$

$$\delta\hat{p}=\delta p-p'\xi^0+\delta p_\xi,$$

$$\hat{v}=v-\xi^0-\frac{1}{\mu+p}\Delta^{-1}\nabla^\alpha Q_{\xi\alpha},$$

$$\hat{v}^{(v)}_\alpha=v^{(v)}_\alpha+\frac{1}{\mu+p}(Q_{\xi\alpha}-\nabla_\alpha\Delta^{-1}\nabla^\beta Q_{\xi\beta}),$$

$$\hat{\Pi}=\Pi+\frac{3}{2}a^2\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}\Delta^{-1}\nabla^\alpha\nabla^\beta\Pi_{\xi\alpha\beta},$$

$$\hat{\Pi}^{(v)}_\alpha=\Pi^{(v)}_\alpha+2a\left(\Delta+\frac{1}{3}R^{(3)}\right)^{-1}(\nabla^\beta\Pi_{\xi\alpha\beta}-\nabla_\alpha\Delta^{-1}\nabla^\beta\nabla^\gamma\Pi_{\xi\beta\gamma}),$$

$$\hat{\Pi}^{(t)}_{\alpha\beta}=\Pi^{(t)}_{\alpha\beta}-\Pi_{\xi\alpha\beta}-\frac{3}{2}\left(\nabla_\alpha\nabla_\beta-\frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)\times\left(\Delta+\frac{1}{2}R^{(3)}\right)^{-1}\Delta^{-1}\nabla^\gamma\nabla^\delta\Pi_{\xi\gamma\delta}-\nabla_{(\alpha}\left(\Delta+\frac{1}{3}R^{(3)}\right)^{-1}(\nabla^\gamma\Pi_{\xi\beta)\gamma}-\nabla_{\beta)}\times\Delta^{-1}\nabla^\gamma\nabla^\delta\Pi_{\xi\gamma\delta}),$$

$$\delta\hat{\phi}=\delta\phi-\phi'\xi^0+\delta\phi_\xi,$$

$$\frac{\delta\hat{\nu}}{\nu}=\frac{\delta\nu}{\nu}+\xi^{0\prime}+2\frac{a'}{a}\xi^0-\xi^0{}_{,\alpha}e^\alpha+\frac{\delta\nu_\xi}{\nu},$$

$$\delta\hat{e}=\delta e-\frac{1}{a}\xi'+\frac{a'}{a^2}\xi-\Delta^{-1}\nabla_\alpha\left[\xi^0\left(e^{\alpha\prime}-2\frac{a'}{a}e^\alpha\right)\right]+\Delta^{-1}\left[\frac{1}{a}\Delta(\xi_{,\alpha})e^\alpha+2K\xi^{(v)}_\alpha e^\alpha-e^\alpha{}_{|\beta\alpha}\times\left(\frac{1}{a}\xi^{,\beta}+\xi^{(v)\beta}\right)+\nabla_\alpha\delta e^\alpha_\xi\right],$$

$$\delta\hat{e}^{(v)}_\alpha=\delta e_\alpha-\delta e_{,\alpha},$$

$$\delta\hat{T}=\delta T-T'\xi^0+\delta T_\xi,$$

$$\hat{E}^{(\mathrm{e.m.})}=E^{(\mathrm{e.m.})}+\Delta^{-1}\nabla^\alpha E_{\alpha\xi},$$

$$\hat{E}^{(v)}_\alpha=E^{(v)}_\alpha+E_{\alpha\xi}-\nabla_\alpha\Delta^{-1}\nabla^\beta E_{\beta\xi},$$

$$\delta\tilde{f}=\delta f-q\frac{\partial f}{\partial q}\left(\frac{a'}{a}\xi^0+\frac{\sqrt{q^2+m^2a^2}}{q}\xi^0{}_{,\alpha}\gamma^\alpha\right)+\delta f_\xi, \tag{250}$$

where A_ξ indicates the quadratic parts of Eq. (229), and similarly for other variables. For δf we used $f'=0$ which follows from Eq. (171) for $C[f]=0$ to background order.

Using t instead of η (indicated as 0) as the time variable, from the definition $dt\equiv ad\eta$ we can show that

$$\xi^0=\frac{1}{a}\xi^t\left(1-\frac{1}{2}H\xi^t\right). \tag{251}$$

B. Linear order

From Eq. (250) we find that the decomposed metric and matter variables transform to linear order as

$$\hat{\alpha}=\alpha-\dot{\xi}^t,\quad \hat{\beta}=\beta-\frac{1}{a}\xi^t+a\left(\frac{\xi}{a}\right)^{\cdot},\quad \hat{\gamma}=\gamma-\frac{1}{a}\xi,$$

$$\hat{\varphi}=\varphi-H\xi^t,\quad \hat{\chi}=\chi-\xi^t,\quad \hat{\kappa}=\kappa+\left(3\dot{H}+\frac{\Delta}{a^2}\right)\xi^t,$$

$$\delta\hat{\mu}=\delta\mu-\dot{\mu}\xi^t,\quad \delta\hat{p}=\delta p-\dot{p}\xi^t,\quad \hat{v}=v-\frac{1}{a}\xi^t,$$

$$\hat{\Pi}=\Pi,\quad \delta\hat{\phi}=\delta\phi-\dot{\phi}\xi^t,$$

$$\hat{B}^{(v)}_\alpha=B^{(v)}_\alpha+a\dot{\xi}^{(v)}_\alpha,\quad \hat{C}^{(v)}_\alpha=C^{(v)}_\alpha-\xi^{(v)}_\alpha,$$

$$\hat{\Psi}^{(v)}_\alpha=\Psi^{(v)}_\alpha,\quad \hat{v}^{(v)}_\alpha=v^{(v)}_\alpha,\quad \hat{\Pi}^{(v)}_\alpha=\Pi^{(v)}_\alpha,$$

$$\hat{C}^{(t)}_{\alpha\beta}=C^{(t)}_{\alpha\beta},\quad \hat{\Pi}^{(t)}_{\alpha\beta}=\Pi^{(t)}_{\alpha\beta}. \tag{252}$$

1. Temporal gauge conditions

The temporal gauge-fixing condition, fixing ξ^t, applies only to a scalar-type perturbation. To linear order, we can impose any one of the following temporal gauge conditions to be valid at any spacetime point:

$$\begin{aligned}
&\text{synchronous gauge:} && \alpha\equiv 0 \ \rightarrow\ \xi^t(\mathbf{x}),\\
&\text{comoving gauge:} && v\equiv 0 \ \rightarrow\ \xi^t=0,\\
&\text{zero-shear gauge:} && \chi\equiv 0 \ \rightarrow\ \xi^t=0,\\
&\text{uniform-expansion gauge:} && \kappa\equiv 0 \ \rightarrow\ \xi^t=0,\\
&\text{uniform-curvature gauge:} && \varphi\equiv 0 \ \rightarrow\ \xi^t=0,\\
&\text{uniform-density gauge:} && \delta\mu\equiv 0 \ \rightarrow\ \xi^t=0,\\
&\text{uniform-pressure gauge:} && \delta p\equiv 0 \ \rightarrow\ \xi^t=0,\\
&\text{uniform-field gauge:} && \delta\phi\equiv 0 \ \rightarrow\ \xi^t=0.
\end{aligned} \tag{253}$$

Except for the synchronous gauge condition, each of the other temporal gauge-fixing conditions completely removes the temporal gauge mode. In the multicomponent situations in addition we can choose one of the following conditions as the proper temporal gauge condition, which also removes the temporal gauge mode completely:

$$\delta\mu_{(i)}\equiv 0,\quad \delta p_{(i)}\equiv 0,\quad v_{(i)}\equiv 0,\quad \delta\phi_{(i)}\equiv 0. \tag{254}$$

All these variables which can be used to fix the temporal gauge freedom in fact do not depend on the spatial gauge transformation ξ and thus are naturally spatially gauge invariant.

The following are some examples of combinations of variables that are temporally gauge invariant:

$$\delta\mu_v\equiv\delta\mu-\dot{\mu}av,\quad \varphi_\chi\equiv\varphi-H\chi,\quad v_\chi\equiv v-\frac{1}{a}\chi,$$

$$\varphi_v\equiv\varphi-aHv,\quad \varphi_{\delta\phi}\equiv\varphi-\frac{H}{\dot{\phi}}\delta\phi\equiv-\frac{H}{\dot{\phi}}\delta\phi_\varphi. \tag{255}$$

These are completely (i.e., both spatially and temporally) gauge invariant to linear order. Any variable under any gauge condition in Eqs. (253),(254) (except for the synchronous gauge) has a unique equivalent gauge-invariant combination. For example, we have

$$\varphi_\chi = \varphi|_{\chi\equiv 0}. \tag{256}$$

Thus, φ_χ is *the same* as the φ variable in the zero-shear gauge where we set $\chi\equiv 0$.

All the equations in Secs. IV and V are presented without imposing any gauge conditions. The equations are arranged using the above variables in Eqs. (253), (254) which can be used in fixing the temporal gauge condition. This allows us to use the various temporal gauge conditions optimally depending on the situation; thus the equations are presented in a sort of *gauge-ready* manner. Usually we do not know the most suitable gauge condition *a priori*. In order to take advantage of the gauge choice in the most optimal way it is desirable to use the gauge-ready form equations presented in this paper. Our set of equations is arranged so that we can easily impose various fundamental gauge conditions in Eqs. (253), (254), and their suitable combinations as well. As we have so many different ways of fixing the temporal gauge conditions it is convenient to denote the gauge condition, or equivalently, the gauge-invariant combination, we are using. Our notation for gauge-invariant combinations proposed in Eq. (255) is convenient for this purpose in the spirit of our gauge strategy [12,29]. The notation is also practically convenient for connecting solutions in different gauge conditions as well as tracing the associated gauge conditions easily. Compared with the notations for gauge-invariant variables which were introduced by Bardeen [6,7], we have

$$\epsilon_m = \delta_v \equiv \delta\mu_v/\mu, \quad \Psi_H = \varphi_\chi, \quad v_s^{(0)} = kv_\chi,$$
$$p\pi_L^{(0)} = \delta p, \quad p\pi_T^{(0)} = -\frac{\Delta}{a^2}\Pi, \quad \zeta \equiv \varphi_\delta, \tag{257}$$

etc.; we ignored the harmonic functions used in [6]. The perturbed curvature variable in the comoving gauge $\mathcal{R}$ often used in the literature is the same as our φ_v, which is the same as $\varphi_{\delta\phi}$ in the scalar field.

2. Spatial gauge conditions

The spatial gauge transformations ξ and $\xi_\alpha^{(v)}$ affect the scalar- and vector-type perturbations, respectively. Due to spatial homogeneity of the background we have natural spatial gauge-fixing conditions to choose [7]. We have two natural spatial gauge-fixing conditions. From Eq. (252) we can see that

$$B \text{ gauge:} \beta\equiv 0, \quad B_\alpha^{(v)}\equiv 0 \quad \rightarrow \quad \xi(\mathbf{x},t)\propto a, \quad \xi_\alpha^{(v)}(\mathbf{x}), \tag{258}$$

$$C \text{ gauge:} \gamma\equiv 0, \quad C_\alpha^{(v)}\equiv 0 \quad \rightarrow \quad \xi = 0, \quad \xi_\alpha^{(v)} = 0. \tag{259}$$

For β we have considered a situation where the temporal gauge condition has already completely removed ξ^t. We call the spatial gauge-fixing conditions in Eqs. (258),(259) the B gauge and the C gauge, respectively [5]. These gauge conditions are imposed so that we have

$$B \text{ gauge:} \quad B_\alpha \equiv 0,$$
$$C \text{ gauge:} \quad C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + C^{(t)}_{\alpha\beta}. \tag{260}$$

Apparently, the B gauge conditions fail to fix the spatial and rotational gauge modes completely; thus, even after imposing the gauge conditions we still have remaining gauge modes. In contrast, the C-gauge conditions successfully remove the gauge modes. To linear order, the variables χ and $\Psi_\alpha^{(v)}$ introduced in Eq. (178) are natural and unique spatially gauge-invariant combinations. Notice that in the C gauge χ is the same as $a\beta$, and $\Psi^{(v)}$ is the same as $B_\alpha^{(v)}$. Thus, the β and $B_\alpha^{(v)}$ variables in the C gauge conditions are equivalent to the corresponding (spatially and rotationally) gauge-invariant combinations χ/a and $\Psi_\alpha^{(v)}$, respectively.

C. Second order

1. Gauge conditions

If we use any one of the gauge conditions which completely fixes both the temporal and spatial gauge modes to linear order, the gauge transformation properties of the second-order variables, say $\varphi^{(2)}$ in Eq. (183), follow *exactly the same forms* as their linear counterparts. Using the transformation of $\delta\phi$ in Eq. (239) as an example, to linear order we have

$$\delta\hat{\phi}^{(1)} = \delta\phi^{(1)} - \phi'\xi^{0(1)}. \tag{261}$$

If we take gauge conditions which remove (fix) ξ^0 and ξ^α completely to linear order we have $\xi^{0(1)}=0=\xi^{\alpha(1)}$. Thus, from Eq. (239) we have

$$\delta\hat{\phi}^{(2)} = \delta\phi^{(2)} - \phi'\xi^{0(2)}, \tag{262}$$

which shows exactly the same form as in Eq. (261). Thus, the gauge conditions in Eqs. (252),(254) apply to second-order perturbation variables as well, and we can impose similar gauge conditions even to second order. For example, in the zero-shear gauge we impose $\chi=0$ as the gauge condition to second order and thus $\chi^{(1)}=0=\chi^{(2)}$; unless otherwise mentioned, we always take the C gauge for the spatial and rotational ones. In this gauge condition the gauge transformation properties are completely fixed, and the gauge modes do not appear. Thus, we anticipate that each variable in that gauge condition has a unique corresponding gauge-invariant combination of variables. Thus, using φ, we have that $\varphi|_{\chi=0,C\text{ gauge}}$ is free of gauge modes. We denote the corresponding gauge-invariant combination as

$$\varphi_\chi \tag{263}$$

with the C gauge condition assumed always. To linear order we have $\varphi_\chi \equiv \varphi - H\chi$, but to second order we need correction terms to make φ_χ gauge invariant. Construction of such a gauge-invariant combination will be shown below.

From Eqs. (57),(58), assuming the pure scalar mode in the C gauge we can show that

$$\overline{K}_{\alpha\beta}=-(1-\alpha)\chi_{,\alpha|\beta}+2\chi_{,(\alpha}\varphi_{,\beta)} -\frac{1}{3}g^{(3)}_{\alpha\beta}[-(1-\alpha)\Delta\chi+2\chi^{,\gamma}\varphi_{,\gamma}], \tag{264}$$

$$R^{(h)}=\frac{1}{a^2}\left[R^{(3)}-4\left(\Delta+\frac{1}{2}R^{(3)}\right)\varphi +16\varphi\left(\Delta+\frac{1}{4}R^{(3)}\right)\varphi+6\varphi^{,\alpha}\varphi_{,\alpha}\right]. \tag{265}$$

Thus, the gauge condition $\chi\equiv 0$ implies $\overline{K}_{\alpha\beta}=0$, justifying its name as the zero-shear gauge to second order. Similarly, the gauge condition $\varphi\equiv 0$ implies $R^{(h)}=(1/a^2)R^{(3)}$ (we also have $R^{(h)}_{\alpha\beta}=\frac{1}{3}R^{(3)}g^{(3)}_{\alpha\beta}$), justifying its name as the uniform-curvature gauge to second order. We can show that the names of gauge conditions in Eq. (253) remain valid to second order.

In the perturbative approach, apparently, this method can be similarly applied to any higher-order perturbations. As long as we work in any of these gauge conditions, the gauge modes are completely removed and the behavior of all the variables is equivalently gauge invariant. As the variables are free of gauge modes, these can be considered as physically important ones in the particular gauge conditions we choose. We can also choose different gauge conditions in second order compared with the ones imposed to linear order. Examples will be shown below.

2. Constructing gauge-invariant combinations

Let us explain a method to derive the gauge-invariant combinations using an example. Since the gauge transformation properties of $\delta\mu$ and $\delta\phi$ are available in convenient forms in Eqs. (232),(239) we consider the gauge-invariant combinations involving these two variables to second order. Thus, we consider the case with a scalar field. To linear order we can construct various gauge-invariant combinations involving $\delta\mu$, and as examples we consider two cases

$$\delta\mu_{\delta\phi}\equiv\delta\mu-\frac{\mu'}{\phi'}\delta\phi,\quad \delta\mu_{\varphi}\equiv\delta\mu-\phi'\frac{a}{a'}\varphi. \tag{266}$$

Clearly, the combinations in Eqs. (266) are not gauge invariant to second order. In order to construct the gauge-invariant combination in the gauge with $\delta\phi^{(2)}=0$, we construct $\delta\hat{\mu}-(\mu'/\phi')\delta\hat{\phi}$ using Eqs. (232),(239). Then, on the right-hand side (RHS) we have a quadratic combination of linear-order terms involving ξ^0 and ξ^α. As the spatial and rotational gauge we consider the C gauge conditions, which remove the corresponding gauge modes completely. This can be achieved by taking

$$\xi^\alpha=-(\hat{\gamma}^{,\alpha}+\hat{C}^{(v)\alpha})+\gamma^{,\alpha}+C^{(v)\alpha}, \tag{267}$$

which follows from Eqs. (252),(249), and moving terms with carets to the LHS. Now, coming to the temporal gauge freedom, if we want to consider the uniform-field gauge we take

$$\xi^0=-\frac{1}{\phi'}(\delta\hat{\phi}-\delta\phi), \tag{268}$$

which follows from Eq. (252), and move terms with carets to the LHS. Then we have the gauge-invariant combination

$$\begin{aligned}\delta\mu_{\delta\phi}\equiv{}&\delta\mu-\frac{\mu'}{\phi'}\delta\phi-\left(\delta\mu-\frac{\mu'}{\phi'}\delta\phi\right)_{,\alpha}(\gamma^{,\alpha}+C^{(v)\alpha})\\ &-\frac{1}{\phi'}\left(\delta\mu-\frac{\mu'}{\phi'}\delta\phi\right)'\delta\phi-\frac{1}{2}\left(\frac{\mu'}{\phi'}\right)'\frac{1}{\phi'}\delta\phi^2\\ &-\frac{1}{a^2}\delta\phi^{,\alpha}\delta\phi_{,\alpha}\\ \equiv{}&\delta\mu-\frac{\mu'}{\phi'}\delta\phi+\delta\mu^{(q)}_{\delta\phi}.\end{aligned} \tag{269}$$

We have $\delta\mu_{\delta\phi}=\delta\mu|_{\delta\phi^{(1)}=0=\delta\phi^{(2)},C\ \mathrm{gauge}}$; thus $\delta\mu_{\delta\phi}$ is the same as $\delta\mu$ under the gauge conditions $\delta\phi^{(1)}=0=\delta\phi^{(2)}$ and the C gauges. If we want to take the uniform-curvature gauge to linear order we take

$$\xi^0=-\frac{a}{a'}(\hat{\varphi}-\varphi), \tag{270}$$

which follows from Eq. (252), and move terms with carets to the LHS. Then we can identify the gauge-invariant combination

$$\begin{aligned}&\delta\mu-\frac{\mu'}{\phi'}\delta\phi-\left(\delta\mu-\frac{\mu'}{\phi'}\delta\phi\right)_{,\alpha}(\gamma^{,\alpha}+C^{(v)\alpha})\\ &-\frac{a}{a'}\left(\delta\mu-\frac{\mu'}{\phi'}\delta\phi\right)'\varphi\\ &-\left(\frac{\mu'}{\phi'}\right)'\frac{a}{a'}\left(\delta\phi-\frac{1}{2}\phi'\frac{a}{a'}\varphi\right)\varphi\\ &-2\frac{1}{a^2}\phi'\frac{a}{a'}\left(\delta\phi-\frac{1}{2}\phi'\frac{a}{a'}\varphi\right)^{,\alpha}\varphi_{,\alpha}.\end{aligned} \tag{271}$$

This combination is equivalent to $\delta\mu$ in the following gauge conditions: $\delta\phi=0$ in the linear and pure second-order part [i.e., $\delta\mu^{(2)}-(\mu'/\phi')\delta\phi^{(2)}$], and $\varphi=0$ in the quadratic parts, and the C gauges. By replacing the linear-order part of Eq. (271) with $\delta\mu^{(1)}-\phi'(a/a')\varphi^{(1)}$ we can make another gauge-invariant combination:

$$\delta\mu_{\varphi^{(1)},\delta\phi^{(2)}} \equiv \delta\mu^{(1)} - \phi' \frac{a}{a'}\varphi^{(1)} + \delta\mu^{(2)} - \frac{\mu'}{\phi'}\delta\phi^{(2)} - \left(\delta\mu - \frac{\mu'}{\phi'}\delta\phi\right)_{,\alpha}(\gamma^{,\alpha} + C^{(v)\alpha}) - \frac{a}{a'}\left(\delta\mu - \frac{\mu'}{\phi'}\delta\phi\right)'\varphi - \left(\frac{\mu'}{\phi'}\right)'\frac{a}{a'}\left(\delta\phi - \frac{1}{2}\phi'\frac{a}{a'}\varphi\right)\varphi - 2\frac{1}{a^2}\phi'\frac{a}{a'}\left(\delta\phi - \frac{1}{2}\phi'\frac{a}{a'}\varphi\right)^{,\alpha}\varphi_{,\alpha}, \tag{272}$$

which is the same as $\delta\mu|_{\varphi^{(1)}=0=\delta\phi^{(2)},C\ \mathrm{gauge}}$. The calculation becomes simpler if we take the C gauge condition: this sets $\gamma\equiv 0 \equiv C_{\alpha}^{(v)}$, thus we can simply set $\xi_{\alpha}\equiv 0$ ($\xi \equiv 0 \equiv \xi_{\alpha}^{(v)}$). Similarly, we can construct diverse combinations of the gauge-invariant variables: several useful gauge-invariant combinations will be presented in the next subsection.

In the following, as in Eq. (269), a gauge-invariant notation, say, φ_v, indicates a combination that is equivalent to φ in the comoving gauge ($v=0$) to all orders (thus, $v^{(1)}=0=v^{(2)}$) and in the C gauge. In order to denote gauge-invariant combinations valid to second order, we introduce the following notation:

$$\varphi_v \equiv \varphi - aHv + \varphi_v^{(q)}, \quad \varphi_{\chi} \equiv \varphi - H\chi + \varphi_{\chi}^{(q)},$$

$$\delta\mu_v \equiv \delta\mu - \dot{\mu}av + \delta\mu_v^{(q)}, \quad v_{\chi} \equiv v - \frac{1}{a}\chi + v_{\chi}^{(q)}, \tag{273}$$

etc., where the upper (q) index indicates the quadratic combinations of linear-order terms. In the following we always take the spatial C gauge. We note that $\delta\mu_v^{(q)}$ is the quadratic correction term to make $\delta\mu_v$ a gauge-invariant combination to second order; thus it differs from, say, $\delta\mu^{(q)} - \dot{\mu}av^{(q)}$. As φ_v is the same as φ in the $v=0$ gauge, we have that $\varphi_v^{(q)}$ vanishes under the $v=0$ gauge, i.e.,

$$\varphi_v^{(q)}|_v = \varphi_v^{(q)}|_{v=0} = 0. \tag{274}$$

Using the definition of our gauge-invariant combinations we can show, for example, that

$$\varphi_v = \varphi - aHv + \varphi_v^{(q)} = \varphi_{\chi} - aHv_{\chi} + \varphi_v^{(q)}|_{\chi}, \tag{275}$$

where in the second step we have evaluated the first step in the zero-shear gauge. Thus

$$\varphi_v^{(q)}|_{\chi} = \varphi_v - (\varphi_{\chi} - aHv_{\chi}) = \varphi_v^{(q)} - (\varphi_{\chi}^{(q)} - aHv_{\chi}^{(q)}), \tag{276}$$

and similarly for other correction terms.

3. Gauge-invariant variables

Now we present several useful gauge-invariant combinations explicitly. We assume $R^{(3)}=0$ and pure scalar-type perturbations. We take the spatial C gauge, $\gamma\equiv 0$. As long as we take the temporal gauge which fixes ξ^0 completely, we can set $\xi_{\alpha}\equiv 0$. The metric becomes

$$A = \alpha, \quad B_{\alpha} = \frac{1}{a}\chi_{,\alpha}, \quad C_{\alpha\beta} = \varphi g^{(3)}_{\alpha\beta}. \tag{277}$$

From Eqs. (229)–(235),(239) we have

$$\hat{\alpha} = \alpha - \frac{1}{a}(a\xi^0)' - \alpha'\xi^0 - 2\alpha\left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right) + \frac{3}{2}\xi^{0\prime}\xi^{0\prime} + \xi^0\left[\xi^{0\prime\prime} + 3\frac{a'}{a}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\xi^0\right],$$

$$\hat{\varphi} = \varphi - \frac{a'}{a}\xi^0 + \xi^0\left[-\varphi' - 2\frac{a'}{a}\varphi + \frac{a'}{a}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\xi^0\right] + \frac{1}{2}\left(\frac{1}{a}\chi^{,\alpha}\xi^0_{,\alpha} - \frac{1}{2}\xi^{0,\alpha}\xi^0_{,\alpha}\right) - \frac{1}{2}\Delta^{-1}\nabla^{\alpha}\nabla^{\beta}\left(\frac{1}{a}\chi_{,\alpha}\xi^0_{,\beta} - \frac{1}{2}\xi^0_{,\alpha}\xi^0_{,\beta}\right),$$

$$\hat{\chi} = \chi - a\xi^0 + a\xi^0\left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right) + a\Delta^{-1}\nabla^{\alpha} \times\left[-2\alpha\xi^0_{,\alpha} - \frac{1}{a}\left(\chi' + \frac{a'}{a}\chi\right)_{,\alpha}\xi^0 - \frac{1}{a}\chi_{,\alpha}\xi^{0\prime} + \xi^{0\prime}\xi^0_{,\alpha}\right] - \frac{a}{2}\Delta^{-1}\left[\frac{1}{a}\chi^{,\alpha}\xi^0_{,\alpha} - \frac{1}{2}\xi^{0,\alpha}\xi^0_{,\alpha} - 3\Delta^{-1}\nabla^{\alpha}\nabla^{\beta}\left(\frac{1}{a}\chi_{,\alpha}\xi^0_{,\beta} - \frac{1}{2}\xi^0_{,\alpha}\xi^0_{,\beta}\right)\right]',$$

$$\hat{\kappa} = \kappa + \left(3\frac{a''}{a^2} - 6\frac{a'^2}{a^3} + \frac{\Delta}{a}\right)\xi^0 + \text{quadratic terms},$$

$$\delta\hat{\mu} = \delta\mu - \mu'\xi^0 - \delta\mu'\xi^0 + \mu'\xi^{0\prime}\xi^0 + \frac{1}{2}\mu''\xi^0\xi^0 + (\mu+p)(-2v + \xi^0)^{,\alpha}\xi^0_{,\alpha},$$

$$\hat{v} = v - \xi^0 + \xi^0\xi^{0\prime} + \frac{1}{2}\left(\frac{a'}{a} + \frac{\mu'+p'}{\mu+p}\right)\xi^0\xi^0 - \left[v' + \left(4\frac{a'}{a} + \frac{\mu'+p'}{\mu+p}\right)v\right]\xi^0 - \Delta^{-1}\nabla^{\alpha}\left[-3\frac{a'}{a}v_{,\alpha}\xi^0 + \frac{\delta\mu}{\mu+p}\xi^0_{,\alpha} + \frac{1}{a^2}\frac{1}{\mu+p}(\Pi^{,\beta}{}_{\alpha} - \delta^{\beta}_{\alpha}\Delta\Pi)\xi^0_{,\beta}\right],$$

$$\delta\hat{\phi}=\delta\phi-\phi'\xi^0-\delta\phi'\xi^0+\phi'\xi^{0\prime}\xi^0+\frac{1}{2}\phi''\xi^0\xi^0. \tag{278}$$

In the transformation of v we have used Eq. (201). We have ignored the quadratic terms in the transformation of κ; this can be read from the definition of κ:

$$\kappa\equiv-\frac{1}{a}\left[3\varphi'-3\frac{a'}{a}\alpha+\frac{\Delta\chi}{a}-(\alpha+2\varphi)\left(3\varphi'+\frac{\Delta\chi}{a}\right)+\frac{3a'}{2a}\left(3\alpha^2-\frac{1}{a^2}\chi^{,\alpha}\chi_{,\alpha}\right)-\frac{1}{a}\chi^{,\alpha}\varphi_{,\alpha}\right], \tag{279}$$

which follows from Eqs. (179),(57).

Following the prescription in the previous subsection, from Eq. (278) we can construct the following gauge-invariant combinations:

$$\varphi_\chi\equiv\varphi-H\chi-(\dot{\varphi}_\chi+2H\varphi_\chi)\chi-\frac{1}{2}(\dot{H}+H^2)\chi^2+\frac{1}{4a^2}[\chi^{,\alpha}\chi_{,\alpha}-\Delta^{-1}\nabla^\alpha\nabla^\beta(\chi_{,\alpha}\chi_{,\beta})]+H\Delta^{-1}\nabla^\alpha[2\alpha_\chi\chi_{,\alpha}+(\dot{\chi}-H\chi)\chi_{,\alpha}]+\frac{1}{4}a^2H\Delta^{-1}\left[\frac{1}{a^2}\chi^{,\alpha}\chi_{,\alpha}-3\frac{1}{a^2}\Delta^{-1}\nabla^\alpha\nabla^\beta(\chi_{,\alpha}\chi_{,\beta})\right]^{\cdot}, \tag{280}$$

$$\varphi_v\equiv\varphi-aHv-(\dot{\varphi}_v+2H\varphi_v)av-\frac{1}{2}\left(\dot{H}+2H^2-H\frac{\dot{\mu}+\dot{p}}{\mu+p}\right)a^2v^2+\frac{1}{4a}(2\chi-av)^{,\alpha}v_{,\alpha}-\frac{1}{4a}\Delta^{-1}\nabla^\alpha\nabla^\beta[(2\chi-av)_{,\alpha}v_{,\beta}]+aH\Delta^{-1}\nabla^\alpha\left[\frac{\delta\mu_v}{\mu+p}v_{,\alpha}+\frac{1}{a^2}\frac{1}{\mu+p}(\Pi^{,\beta}{}_\alpha-\delta^\beta_\alpha\Delta\Pi)v_{,\beta}\right], \tag{281}$$

$$\delta_v\equiv\delta-\frac{\dot{\mu}}{\mu}av-\frac{\delta\dot{\mu}_v}{\mu}av-\frac{1}{2}\frac{\dot{\mu}}{\mu}\frac{\dot{H}}{H}a^2v^2-\frac{\mu+p}{\mu}v^{,\alpha}v_{,\alpha}+\frac{\dot{\mu}}{\mu}a\Delta^{-1}\nabla^\alpha\left[\frac{\delta\mu_v}{\mu+p}v_{,\alpha}+\frac{1}{a^2}\frac{1}{\mu+p}(\Pi^{,\beta}{}_\alpha-\delta^\beta_\alpha\Delta\Pi)v_{,\beta}\right], \tag{282}$$

$$v_\chi\equiv v-\frac{1}{a}\chi-\left[\dot{v}_\chi+\left(H+\frac{\dot{\mu}+\dot{p}}{\mu+p}\right)v_\chi\right]\chi+\frac{1}{2a}\left(H-\frac{\dot{\mu}+\dot{p}}{\mu+p}\right)\times\chi^2+\frac{1}{a}\Delta^{-1}\nabla^\alpha\left[2\alpha_\chi\chi_{,\alpha}+\chi_{,\alpha}(\dot{\chi}-H\chi)-\frac{\delta\mu_v}{\mu+p}\chi_{,\alpha}-\frac{1}{a^2}\frac{1}{\mu+p}(\Pi^{,\beta}{}_\alpha-\delta^\beta_\alpha\Delta\Pi)\chi_{,\beta}\right]+\frac{a}{4}\Delta^{-1}\left[\frac{1}{a^2}\chi^{,\alpha}\chi_{,\alpha}-\frac{3}{a^2}\Delta^{-1}\nabla^\alpha\nabla^\beta(\chi_{,\alpha}\chi_{,\beta})\right]^{\cdot}, \tag{283}$$

$$\chi_v\equiv\chi-av-\frac{1}{2}a^2\left(H-\frac{\dot{\mu}+\dot{p}}{\mu+p}\right)v^2+a\Delta^{-1}\nabla^\alpha\left[-2\alpha_v v_{,\alpha}-(\dot{\chi}_v+H\chi_v)_{,\alpha}v-\chi_{,\alpha}\dot{v}+\frac{\delta\mu_v}{\mu+p}v_{,\alpha}+\frac{1}{a^2}\frac{1}{\mu+p}(\Pi^{,\beta}{}_\alpha-\delta^\beta_\alpha\Delta\Pi)v_{,\beta}\right]+\frac{1}{2}a^2\Delta^{-1}\left[-\left(\frac{1}{a}\chi^{,\alpha}v_{,\alpha}-\frac{1}{2}v^{,\alpha}v_{,\alpha}\right)+3\Delta^{-1}\nabla^\alpha\nabla^\beta\left(\frac{1}{a}\chi_{,\alpha}v_{,\beta}-\frac{1}{2}v_{,\alpha}v_{,\beta}\right)\right]^{\cdot}. \tag{284}$$

To linear order, δ_v (equivalently, δ in the comoving gauge) behaves like a Newtonian density perturbation, and v_χ and $-\varphi_\chi$ (equivalently, v and $-\varphi$ in the zero-shear gauge) behave like the Newtonian velocity and the gravitational potential. Also to linear order φ_v is known to be the best conserved quantity on the super-sound-horizon scale. For extensions of these results to second order, see Secs. VII C and VII D, respectively.

In the case of a scalar field we have

$$\varphi_{\delta\phi}\equiv\varphi-\frac{H}{\dot{\phi}}\delta\phi-\frac{1}{a^2}(a^2\varphi_{\delta\phi})^{\cdot}\frac{\delta\phi}{\dot{\phi}}-\frac{1}{2a^2}\left(\frac{a^2H}{\dot{\phi}}\right)^{\cdot}\frac{\delta\phi^2}{\dot{\phi}}+\frac{1}{2a^2\dot{\phi}}\left[\chi^{,\alpha}\delta\phi_{,\alpha}-\frac{1}{2\dot{\phi}}\delta\phi^{,\alpha}\delta\phi_{,\alpha}-\Delta^{-1}\nabla^\alpha\nabla^\beta\left(\chi_{,\alpha}\delta\phi_{,\beta}-\frac{1}{2\dot{\phi}}\delta\phi_{,\alpha}\delta\phi_{,\beta}\right)\right], \tag{285}$$

$$\delta\phi_\varphi\equiv\delta\phi-\frac{\dot{\phi}}{H}\varphi-\frac{1}{H}\delta\dot{\phi}_\varphi\varphi+\frac{\dot{\phi}^2}{2a^2H^3}\left(\frac{a^2H}{\dot{\phi}}\right)^{\cdot}\varphi^2-\frac{\dot{\phi}}{2a^2H^2}\left[\chi^{,\alpha}\varphi_{,\alpha}-\frac{1}{2H}\varphi^{,\alpha}\varphi_{,\alpha}-\Delta^{-1}\nabla^\alpha\nabla^\beta\left(\chi_{,\alpha}\varphi_{,\beta}-\frac{1}{2H}\varphi_{,\alpha}\varphi_{,\beta}\right)\right]. \tag{286}$$

Thus, to second order we have $\varphi_{\delta\phi}\neq-(H/\dot{\phi})\delta\phi_\varphi$. By evaluating the RHSs of Eqs. (285),(286) in the $\varphi=0$ gauge and $\delta\phi=0$ gauge, respectively, we have the following relations between the two gauge-invariant variables:

$$\varphi_{\delta\phi}=-\frac{H}{\dot{\phi}}\delta\phi_\varphi+\frac{H}{\dot{\phi}^2}\delta\phi_\varphi\delta\dot{\phi}_\varphi+\frac{1}{2a^2\dot{\phi}}\left(\frac{a^2H}{\dot{\phi}}\right)^{\cdot}\delta\phi_\varphi^2+\frac{1}{2a^2\dot{\phi}}\left[\chi_\varphi{}^{,\alpha}\delta\phi_{\varphi,\alpha}-\frac{1}{2\dot{\phi}}\delta\phi_\varphi{}^{,\alpha}\delta\phi_{\varphi,\alpha}-\Delta^{-1}\nabla^\alpha\nabla^\beta\left(\chi_{\varphi,\alpha}\delta\phi_{\varphi,\beta}-\frac{1}{2\dot{\phi}}\delta\phi_{\varphi,\alpha}\delta\phi_{\varphi,\beta}\right)\right], \tag{287}$$

$$\delta\phi_\varphi=-\frac{\dot{\phi}}{H}\varphi_{\delta\phi}+\frac{\dot{\phi}}{H^2}\varphi_{\delta\phi}\dot{\varphi}_{\delta\phi}+\frac{1}{2a^2H}\left(\frac{a^2\dot{\phi}}{H}\right)^{\cdot}\varphi_{\delta\phi}^2-\frac{\dot{\phi}}{2a^2H^2}\left[\chi_{\delta\phi}{}^{,\alpha}\varphi_{\delta\phi,\alpha}-\frac{1}{2H}\varphi_{\delta\phi}{}^{,\alpha}\varphi_{\delta\phi,\alpha}-\Delta^{-1}\nabla^\alpha\nabla^\beta\left(\chi_{\delta\phi,\alpha}\varphi_{\delta\phi,\beta}-\frac{1}{2H}\varphi_{\delta\phi,\alpha}\varphi_{\delta\phi,\beta}\right)\right]. \tag{288}$$

To linear order $\delta\phi_\varphi$ (equivalently, $\delta\phi$ in the uniform-curvature gauge) most closely resembles the scalar-field equation in the fixed cosmological background metric [14]. Since $\delta\phi\equiv0$ implies $Q_\alpha^{(\phi)}=0$ (thus $v^{(\phi)}=0$), we have $\varphi_{\delta\phi}=\varphi_v$; from Eq. (114) we see that this is valid to second order.

4. Spatial gradient variable

The covariant density gradient variable

$$\tilde{\Delta}_a\equiv\frac{1}{\tilde{\mu}}\tilde{h}_a^b\tilde{\mu}_{,b} \tag{289}$$

is gauge invariant to linear order [32]. In [32] the energy frame is taken. Using the $\tilde{u}_a$ frame in Eq. (53) we have

$$\Delta_\alpha=\frac{1}{1+\delta}\left\{\delta^E{}_{,\alpha}+\frac{\mu'}{\mu}(V_\alpha^E-B_\alpha+AB_\alpha+2V^{E\beta}C_{\alpha\beta})+\left[\delta'+\frac{\mu'}{\mu}(\delta-A)\right](V_\alpha^E-B_\alpha)\right\}, \tag{290}$$

where we set $\tilde{\Delta}_\alpha\equiv\Delta_\alpha$ with Δ_α based on $g^{(3)}_{\alpha\beta}$; from $\tilde{u}^a\tilde{\Delta}_a=0$ we have $\tilde{\Delta}_0=-V^\alpha\Delta_\alpha$. From Eqs. (85),(86) we see that δ is frame invariant to linear order, and we ignore the superscript E in such cases. Using the prescription in Eq. (88) we can express Δ_α in the normal frame as

$$\Delta_\alpha=\frac{1}{1+\delta}\left\{\delta^N{}_{,\alpha}+\frac{\mu'}{\mu}\frac{Q_\alpha^N}{\mu+p}+\left[\delta'+\frac{\mu'}{\mu}(\delta-A)\right]\frac{Q_\alpha^N}{\mu+p}-\frac{(Q^{N\beta}Q_\beta^N)_{,\alpha}}{\mu(\mu+p)}-\frac{\mu'}{\mu}\frac{1}{(\mu+p)^2}[(\delta\mu+\delta p)Q_\alpha^N+\Pi_{\alpha\beta}Q^{N\beta}]\right\}. \tag{291}$$

Thus, under the condition $Q_\alpha=0$, we have $\Delta_\alpha=\delta_{,\alpha}/(1+\delta)$; $Q_\alpha=0$ can be achieved by the comoving gauge condition ($Q\equiv0$) and the irrotational condition ($Q_\alpha^{(v)}\equiv0$); see [33] and the note added in proof of [21]. Under the gauge transformation, either using Eqs. (232)–(235) for Eq. (291) or using Eq. (238) for Eq. (290), we can show that

$$\hat{\Delta}_\alpha=\Delta_\alpha-\Delta'_\alpha\xi^0-(\Delta_\beta\xi^\beta)_{,\alpha}-2\Delta_{[\alpha,\beta]}\xi^\beta. \tag{292}$$

Thus, Δ_α is not gauge invariant to second order. To linear order, the scalar-type part becomes $\Delta_\alpha=\delta_{v,\alpha}$ where the gauge-invariant combination δ_v is the same as δ in the comoving gauge.

VII. APPLICATIONS

A. Closed form equations

From Eqs. (196),(197), Eqs. (197),(200),(201), Eqs. (199), Eqs. (199),(201), and Eqs. (195),(197),(199) we can derive, respectively,

$$\frac{\Delta+3K}{a^2}\varphi_\chi+4\pi G\delta\mu_v=\frac{\Delta+3K}{a^2}\varphi_\chi^{(q)}+4\pi G\delta\mu_v^{(q)}+\frac{1}{4}N_1-HN_2^{(s)}, \tag{293}$$

$$\delta\dot{\mu}_v+3H\delta\mu_v-\frac{\Delta+3K}{a^2}[a(\mu+p)v_\chi+2H\Pi]=\delta\dot{\mu}_v^{(q)}+3H\delta\mu_v^{(q)}-\frac{\Delta+3K}{a^2}a(\mu+p)v_\chi^{(q)}+N_5+(\mu+p)(N_2^{(s)}+3aHN_6^{(s)}), \tag{294}$$

$$\varphi_\chi+\alpha_\chi+8\pi G\Pi=\varphi_\chi^{(q)}+\alpha_\chi^{(q)}-N_4^{(s)}$$

or

$$\varphi_\chi+\alpha_\chi+8\pi G\Pi_\chi=-N_{4\chi}^{(s)}, \tag{295}$$

$$\dot{v}_\chi + H v_\chi - \frac{1}{a}\left(\alpha_\chi + \frac{\delta p_v}{\mu+p} + \frac{2}{3}\frac{\Delta+3K}{a^2}\frac{\Pi}{\mu+p}\right) = \dot{v}_\chi^{(q)} + H v_\chi^{(q)} - \frac{1}{a}\left(\alpha_\chi^{(q)} + \frac{\delta p_v^{(q)}}{\mu+p}\right) + N_6^{(s)}, \tag{296}$$

$$\dot{\varphi}_\chi + H\varphi_\chi + 4\pi G(\mu+p)a v_\chi + 8\pi G H \Pi = \dot{\varphi}_\chi^{(q)} + H\varphi_\chi^{(q)} + 4\pi G(\mu+p)a v_\chi^{(q)} + \frac{1}{3}(N_0 - N_2^{(s)}) - H N_4^{(s)}. \tag{297}$$

These equations are presented using mixed gauge-invariant variables. We note that to linear order $\delta\mu_v$, $-\varphi_\chi$, and $k v_\chi$ ($\sim \nabla v_\chi$) closely resemble the Newtonian density perturbation, the perturbed gravitational potential, and the perturbed velocity perturbation, respectively [34,35,13]. To linear order these equations were presented by Bardeen in 1980 [6]; see Eqs. (4.3),(4.8),(4.4),(4.5),(4.7) in [6] and compare with our notation; see Eq. (257). Using Eq. (297) and Eqs. (293),(296),(297) we can show that

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu+p)}\varphi_\chi = \frac{H^2}{4\pi G(\mu+p)a}\left[\frac{a}{H}(\varphi_\chi - \varphi_\chi^{(q)})\right]^{\cdot} + 2H^2\frac{\Pi}{\mu+p} + \Phi^{(q)} + N_\Phi, \tag{298}$$

$$\dot{\Phi} = \frac{H c_s^2 \Delta}{4\pi G(\mu+p)a^2}(\varphi_\chi - \varphi_\chi^{(q)}) - \frac{H}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) + \dot{\Phi}^{(q)} + N_{\dot{\Phi}}, \tag{299}$$

where

$$\Phi^{(q)} \equiv \varphi_v^{(q)} - \frac{K/a^2}{4\pi G(\mu+p)}\varphi_\chi^{(q)},$$

$$N_\Phi \equiv \frac{H^2}{4\pi G(\mu+p)}\left[N_4^{(s)} + \frac{1}{3H}(N_2^{(s)} - N_0)\right],$$

$$N_{\dot{\Phi}} \equiv \frac{1}{3}\left(1 - \frac{K/a^2}{4\pi G(\mu+p)}\right)(N_0 - N_2^{(s)}) - \frac{H c_s^2}{4\pi G(\mu+p)}\left(\frac{1}{4}N_1 - H N_2^{(s)}\right) + \frac{K/a^2}{4\pi G(\mu+p)} H N_4^{(s)} - a H N_6^{(s)}. \tag{300}$$

We have introduced an entropic perturbation e by

$$\delta p \equiv c_s^2 \delta\mu + e, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}. \tag{301}$$

Defined in this way, e is not necessarily gauge invariant to second order. To linear order we have $e = \tilde{\pi}$ introduced above Eq. (24). Combining Eqs. (298),(299), we can derive

$$\frac{H^2 c_s^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{H^2 c_s^2}\dot{\Phi}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\Phi = \frac{H c_s}{a^3\sqrt{\mu+p}}\left[v'' - \left(\frac{z''}{z} + c_s^2\Delta\right)v\right] = \frac{H^2 c_s^2}{(\mu+p)a^3}\left\{\frac{(\mu+p)a^3}{H^2 c_s^2}\left[-\frac{H}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) + \dot{\Phi}^{(q)} + N_{\dot{\Phi}}\right]\right\}^{\cdot} - c_s^2\frac{\Delta}{a^2}\left(2H^2\frac{\Pi}{\mu+p} + \Phi^{(q)} + N_\Phi\right), \tag{302}$$

$$\frac{\mu+p}{H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_\chi = \frac{\sqrt{\mu+p}}{a^2}\left[u'' - \left(\frac{(1/\bar{z})''}{1/\bar{z}} + c_s^2\Delta\right)u\right] = \frac{4\pi G(\mu+p)}{H}\left[-\frac{H}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) - 2\left(H^2\frac{\Pi}{\mu+p}\right)^{\cdot} + N_{\dot{\Phi}} - \dot{N}_\Phi\right] + \frac{\mu+p}{H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a}{H}\varphi_\chi^{(q)}\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_\chi^{(q)}, \tag{303}$$

where we used

$$v \equiv z\Phi, \quad u \equiv \frac{1}{z}\frac{a}{H}\varphi_\chi, \quad c_s z \equiv \frac{a\sqrt{\mu+p}}{H} \equiv \bar{z}. \tag{304}$$

[v in Eqs. (302)–(304) differs from the perturbed velocity related variable used in the rest of this paper.] The equation using v in the linear theory was first derived by Field and Shepley in 1968 [36]; see also [37,21]. Using Eq. (293), Eq. (303) gives an equation for δ_v. Using Eqs. (293)–(296) we can derive an equation for δ_v in *another* form:

$$\frac{\mu+p}{a^2\mu H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a^3\mu}{H}\delta_v\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\delta_v = \frac{\Delta+3K}{a^2}\left[\frac{e}{\mu} + \frac{2}{3}\frac{\Delta}{a^2}\frac{\Pi}{\mu} + 2\frac{\mu+p}{\mu H}\left(\frac{H^2}{\mu+p}\Pi\right)^{\cdot}\right] + \frac{\mu+p}{a^2\mu H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a^3\mu}{H}\delta_v^{(q)}\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\delta_v^{(q)}$$

$$+\frac{\mu+p}{\mu}\left\{-\frac{1}{4}N_1+HN_2^{(s)}+\frac{\Delta+3K}{a^2}(aN_6^{(s)}-N_4^{(s)})\right.$$
$$\left.+\frac{1}{a^2}\left[a^2\left(N_2^{(s)}+\frac{N_5}{\mu+p}+3aHN_6^{(s)}\right)\right]^{\cdot}\right\}. \quad (305)$$

The above set of equations is valid for a general imperfect fluid. A minimally coupled scalar field can be regarded as an imperfect fluid with special fluid quantities. We additionally have an equation of motion of the field which is actually included in the energy and momentum conservation equations. In fact, the above set of equations is valid even for multicomponent fluids and fields. In such cases, the fluid quantities become collective fluid quantities and we additionally need the energy and momentum conservation equations for the individual fluids and the equations of motion for the individual fields.

In the single scalar field case, from Eqs. (212),(190),(293),(301) we have

$$e=-\frac{1-c_s^2}{4\pi G}\frac{\Delta+3K}{a^2}(\varphi_\chi-\varphi_\chi^{(q)})+N_e,$$
$$N_e\equiv\frac{1-c_s^2}{4\pi G}\left(\frac{1}{4}N_1-HN_2^{(s)}\right)+\delta p^{(q)}-\delta\mu^{(q)}$$
$$+3H(1-c_s^2)a\Delta^{-1}\nabla^\alpha Q_\alpha^{(q)}. \quad (306)$$

Equation (298) remains valid, and Eq. (299) becomes

$$\dot{\Phi}=\frac{Hc_A^2\Delta}{4\pi G(\mu+p)a^2}(\varphi_\chi-\varphi_\chi^{(q)})-\frac{H}{\mu+p}\left(N_e+\frac{2}{3}\frac{\Delta}{a^2}\Pi\right)$$
$$+\dot{\Phi}^{(q)}+N_{\dot{\Phi}}, \quad (307)$$

where

$$c_A^2\Delta\equiv\Delta+3(1-c_s^2)K. \quad (308)$$

Therefore, Eqs. (302),(303),(304) remain valid with c_s and e replaced by c_A and N_e; in Eq. (302) one can show that we can ignore the operator nature of Δ^{-1} in c_A^2.

The rotational perturbation and the gravitational wave are described by Eqs. (208),(210), respectively:

$$\frac{[a^4(\mu+p)v_\alpha^{(v)}]^{\cdot}}{a^4(\mu+p)}=-\frac{\Delta+2K}{2a^2}\frac{\Pi_\alpha^{(v)}}{\mu+p}+N_{6\alpha}^{(v)}, \quad (309)$$

$$\ddot{C}_{\alpha\beta}^{(t)}+3H\dot{C}_{\alpha\beta}^{(t)}-\frac{\Delta-2K}{a^2}C_{\alpha\beta}^{(t)}$$
$$=\frac{1}{a^3}\left[v_{\alpha\beta}^{(t)\prime\prime}-\left(\frac{a''}{a}+\Delta-2K\right)v_{\alpha\beta}^{(t)}\right]$$
$$=8\pi G\Pi_{\alpha\beta}^{(t)}+N_{4\alpha\beta}^{(t)}, \quad (310)$$

where $v_{\alpha\beta}^{(t)}\equiv aC_{\alpha\beta}^{(t)}$. We note that all the equations in this section are valid for *general K*.

B. Solutions to linear order

1. Scalar type

We consider a single component ideal fluid. Several known solutions in the literature are the following.

(i) In the large-scale limit (the super-sound-horizon scale), i.e., ignoring the $c_s^2\Delta$ term compared with the z''/z and $(1/\overline{z})''/(1/\overline{z})$ terms in Eqs. (302),(303), we have the general solutions[1]

$$\Phi(k,t)=C(k)-d(k)\frac{k^2}{4\pi G}\int^t\frac{c_s^2H^2}{a^3(\mu+p)}dt, \quad (311)$$

$$\varphi_\chi(k,t)=4\pi GC(k)\frac{H}{a}\int^t\frac{a(\mu+p)}{H^2}dt+d(k)\frac{H}{a}. \quad (312)$$

$C(k)$ and $d(k)$ are the coefficients of the growing and decaying solutions (in an expanding medium), respectively. To second order in the large-scale expansion we have

$$\Phi=C\left\{1+k^2\left[\int^\eta\overline{z}^2\left(\int^\eta\frac{d\eta}{z^2}\right)d\eta-\int^\eta\overline{z}^2d\eta\int^\eta\frac{d\eta}{z^2}\right]\right\}$$
$$-d\frac{k^2}{4\pi G}\int^\eta\frac{d\eta}{z^2}, \quad (313)$$

$$\varphi_\chi=4\pi GC\frac{H}{a}\int^\eta\overline{z}^2d\eta+d\frac{H}{a}$$
$$\times\left\{1+k^2\left[\int^\eta\frac{1}{z^2}\left(\int^\eta\overline{z}^2d\eta\right)d\eta-\int^\eta\overline{z}^2d\eta\int^\eta\frac{d\eta}{z^2}\right]\right\}. \quad (314)$$

We emphasize that these solutions are valid for *general K* and Λ, and a time-varying equation of state.

(ii) In the small-scale limits $[c_s^2k^2\gg z''/z,\ (1/\overline{z})''/(1/\overline{z})]$, if we further assume that c_s is constant in time, Eqs. (302),(303) give the general solutions

$$v=z\Phi\propto e^{\pm ic_sk\eta},\quad u=\frac{1}{\sqrt{\mu+p}}\varphi_\chi\propto e^{\pm ic_sk\eta}. \quad (315)$$

(iii) For $K=0=\Lambda$ and $w\equiv p/\mu=$const we have exact solutions [6,38]. For the background, from Eqs. (184),(186), we have

[1] $\mathbf{k}$ is the wave vector with $k\equiv|\mathbf{k}|$. With the wave number k appearing in the equation the variables can be regarded as the Fourier transformed ones. To *linear order* each Fourier mode decouples from the other modes and evolves independently. The same equations in configuration space remain valid in Fourier space as well. Thus, we ignore specific symbols distinguishing the variables in the two spaces. Only in this subsection concerning the linear theory do we use the Fourier transformation.

$$a \propto t^{2/3(1+w)} \propto \eta^{2/(1+3w)}, \quad aH\eta = \frac{2}{1+3w}. \tag{316}$$

Equation (304) gives $z \propto \bar{z} \propto a$; thus

$$\frac{z''}{z} = \frac{2(1-3w)}{(1+3w)^2}\frac{1}{\eta^2}, \quad \frac{(1/\bar{z})''}{(1/\bar{z})} = \frac{6(1+w)}{(1+3w)^2}\frac{1}{\eta^2}. \tag{317}$$

In this case, Eqs. (302),(303) become Bessel's equations with solutions:

$$v = z\Phi \propto \sqrt{\eta}(J_\nu(x), Y_\nu(x)), \quad x \equiv c_s k|\eta|, \quad \nu \equiv \frac{3(1-w)}{2(1+3w)}, \tag{318}$$

$$u = \frac{1}{\sqrt{\mu+p}}\varphi_\chi \propto \sqrt{\eta}(J_{\bar{\nu}}(x), Y_{\bar{\nu}}(x)), \quad \bar{\nu} \equiv \frac{5+3w}{2(1+3w)}. \tag{319}$$

We have $\bar{\nu} = \nu + 1$. Using Eqs. (298),(299) we can normalize the solutions as

$$\Phi \equiv c_1(k)\frac{J_\nu(x)}{x^\nu} + c_2(k)\frac{Y_\nu(x)}{x^\nu}, \tag{320}$$

$$\varphi_\chi = \frac{3(1+w)}{1+3w}\left(c_1(k)\frac{J_{\bar{\nu}}(x)}{x^{\bar{\nu}}} + c_2(k)\frac{Y_{\bar{\nu}}(x)}{x^{\bar{\nu}}}\right). \tag{321}$$

Equation (293) gives

$$\delta_v = \frac{(1+3w)^2}{6w}x^2\varphi_\chi. \tag{322}$$

In the large-scale limit ($x \ll 1$) we have

$$\Phi = \frac{c_1}{2^\nu\Gamma(\nu+1)} - 2^\nu\frac{\Gamma(\nu)}{\pi}x^{-2\nu}c_2, \tag{323}$$

where for $\nu=0$ we have an additional $2\ln x$ factor in the c_2 mode. By matching the general large-scale solution in Eq. (311) we can identify

$$c_1 = 2^\nu\Gamma(\nu+1)C, \quad c_2 = -\frac{1}{3(1+w)}\frac{\pi}{2^\nu\Gamma(\bar{\nu})}\frac{x^{2\bar{\nu}}}{a^2\eta}d. \tag{324}$$

In the large-scale limit ($x \ll 1$) we have

$$\Phi \propto C, \ \ da^{-3(1-w)/2}, \tag{325}$$

$$\varphi_\chi \propto C, \ \ da^{-(5+3w)/2}, \tag{326}$$

$$\begin{aligned}\delta_v &\propto Ca^{1+3w}, \ \ da^{-3(1-w)/2} \\ &\propto Ct^{2(1+3w)/3(1+w)}, \ \ dt^{-(1-w)/(1+w)} \\ &\propto C\eta^2, \ \ d\eta^{-3(1-w)/(1+3w)}.\end{aligned} \tag{327}$$

Equation (327) follows from Eq. (293) which gives $\delta_v \propto a^{1+3w}\varphi_\chi \propto \eta^2\varphi_\chi$ in general. Equation (327) includes the well known solutions in the matter- ($w=0$) and radiation- ($w=\frac{1}{3}$) dominated eras MDE and RDE [1]:

$$\begin{aligned}&\text{MDE:} \quad \delta_v \propto Ca, \ \ da^{-3/2} \propto Ct^{2/3}, \ \ dt^{-1} \propto C\eta^2, \ \ d\eta^{-3}, \\ &\text{RDE:} \quad \delta_v \propto Ca^2, \ \ da^{-1} \propto Ct, \ \ dt^{-1/2} \propto C\eta^2, \ \ d\eta^{-1}.\end{aligned} \tag{328}$$

If we consider only the C mode, which is the relatively growing mode in an expanding phase, we have

$$\Phi(\mathbf{x},t) = C(\mathbf{x}), \tag{329}$$

$$\varphi_\chi(\mathbf{x},t) = \frac{3+3w}{5+3w}C(\mathbf{x}). \tag{330}$$

The nontransient mode of Φ remains constant on the supersound-horizon scale, whereas that of φ_χ jumps as the background equation of state changes. Still, it is $-\varphi_\chi$, not $-\varphi_v$, which closely resembles the perturbed Newtonian gravitational potential [13].

The asymptotic solutions (i) and (ii) remain valid for the scalar field with $K=0$; in this case we have c_s^2 replaced by 1. The background solutions for the $w=$const. case considered in (iii), Eq. (316), are valid for a scalar field with an exponential potential of the following form [39]:

$$V = \frac{1-w}{12\pi G(1+w)^2}e^{-\sqrt{24\pi G(1+w)}\phi},$$

$$\phi = \sqrt{\frac{1}{6\pi G(1+w)}}\ln t. \tag{331}$$

For the perturbation, from Eq. (308) and the prescription below it, the same equations of the fluid remain valid with the coefficient of Laplacian term replaced by 1 (for the field) instead of c_s^2 (for the fluid) [14]. Thus, the perturbation solutions in the fluid remain valid for such a scalar field with $x=k|\eta|$, instead of $x=c_s k|\eta|$ for the fluid case.

We emphasize that, if a solution is known in a given gauge condition the rest of the variables in all gauge conditions can be derived through simple algebra; such solutions are presented in tabular forms for an ideal fluid and a scalar field in [38,40,15]. Solutions in the situations of generalized gravity theories considered in Sec. IV D can be found in [24]. The cases with multiple components of the fluids and fields were analyzed in [41,42].

2. Vector type

The rotational perturbation is described by Eq. (309). If we assume $\Pi_\alpha^{(v)}=0$, we have the general solution

$$a \cdot a^3(\mu+p)\cdot v_\alpha^{(v)}(\mathbf{x},t) = L_\alpha^{(v)}(\mathbf{x}). \tag{332}$$

Thus, to linear order the rotational perturbation is described by the conservation of angular momentum and is transient in expanding media. We note that Eq. (309) follows from

$T^b_{\alpha;b}=0$ and is thus *independent* of the gravitational field equation. Thus the presence of scalar fields or the generalized gravity theories considered in Sec. IV D do not affect the vector-type perturbation of the fluids [23].

3. Tensor type

Now, we consider the gravitational wave with $K=0$ and $\Pi^{(t)}_{\alpha\beta}=0$. The basic equation is presented in Eq. (310).

(i) The general large-scale ($k^2 \ll a''/a$) solution is

$$C^{(t)}_{\alpha\beta}(k,t)=c^{(t)}_{\alpha\beta}(k)+d^{(t)}_{\alpha\beta}(k)\int^t \frac{dt}{a^3}. \tag{333}$$

Thus, ignoring the transient mode ($d_{\alpha\beta}$) in an expanding phase the tensor-type perturbation is characterized by its conserved amplitude $c^{(t)}_{\alpha\beta}(k)$.

(ii) In the small-scale limit ($k^2 \gg a''/a$) we have the general solution

$$C^{(t)}_{\alpha\beta}(k,t) \propto \frac{1}{a} e^{\pm ik\eta}. \tag{334}$$

Thus the gravitational wave redshifts away.

(iii) For $K=0=\Lambda$ and $w=$const, we have the exact solution

$$v^{(t)}_{\alpha\beta}=aC^{(t)}_{\alpha\beta} \propto \sqrt{\eta}(J_\nu(x),Y_\nu(x)), \quad \nu\equiv\frac{3(1-w)}{2(1+3w)}, \quad x\equiv k|\eta|. \tag{335}$$

Thus, we set

$$C^{(t)}_{\alpha\beta}=c^{(t)}_{1\alpha\beta}\frac{J_\nu(x)}{x^\nu}+c^{(t)}_{2\alpha\beta}\frac{Y_\nu(x)}{x^\nu}. \tag{336}$$

In the large-scale limit ($x \ll 1$) we have

$$C^{(t)}_{\alpha\beta}=\frac{1}{2^\nu\Gamma(\nu+1)}c^{(t)}_{1\alpha\beta}-2^\nu\frac{\Gamma(\nu)}{\pi}x^{-2\nu}c^{(t)}_{2\alpha\beta}, \tag{337}$$

where for $\nu=0$ we have an additional $2\ln x$ factor in the $c^{(t)}_{2\alpha\beta}$ mode. By matching with the general large-scale solution in Eq. (333) we can identify

$$c^{(t)}_{1\alpha\beta}=2^\nu\Gamma(\nu+1)c^{(t)}_{\alpha\beta}, \quad c^{(t)}_{2\alpha\beta}=\frac{\pi}{2^{\nu+1}\Gamma(\nu+1)}\frac{\eta x^{2\nu}}{a^2}d^{(t)}_{\alpha\beta}. \tag{338}$$

Thus,

$$\begin{aligned} C^{(t)}_{\alpha\beta} &\propto c^{(t)}_{\alpha\beta}, d^{(t)}_{\alpha\beta}a^{-3(1-w)/2} \\ &\propto c^{(t)}_{\alpha\beta}, d^{(t)}_{\alpha\beta}t^{-(1-w)/(1+w)} \\ &\propto c^{(t)}_{\alpha\beta}, d^{(t)}_{\alpha\beta}\eta^{-3(1-w)/(1+3w)}. \end{aligned} \tag{339}$$

C. Pressureless irrotational fluid

The equation of δ_v was derived in Eq. (305) or can be derived from Eqs. (303),(296). In the pressureless case, a simpler route is to use the basic equations in Eqs. (195)–(201).

We *consider* a pressureless fluid, thus $\delta p=0=\Pi_{\alpha\beta}$, and *ignore* the vector-type perturbation. For the spatial gauge we take $\gamma\equiv 0$ and thus $\beta=\chi/a$. If we take the temporal comoving gauge ($v\equiv 0$) we have $Q_\alpha=0$. Equation (201) gives $\alpha_v=-aN^{(s)}_{6v}=-\frac{1}{2}\beta_{v,\alpha}\beta_v{}^{,\alpha}$ and thus α vanishes to linear order; in the pressureless medium, to linear order, the comoving particle follows a geodesic and thus $v=0$ implies $\alpha=0$. From Eqs. (200), (198), first evaluating these in the comoving gauge, we can derive

$$\dot{\delta}_v=\kappa_v+\frac{1}{a}\nabla\cdot(\delta_v\nabla v_\chi), \tag{340}$$

$$\begin{aligned} &\dot{\kappa}_v+2H\kappa_v-4\pi G\mu\delta_v \\ &\quad=\frac{1}{a^2}[(\nabla v_\chi)\cdot(\nabla v_\chi)^{,\alpha}]_{|\alpha}+\dot{C}^{(t)}_{\alpha\beta}\left(\dot{C}^{(t)\alpha\beta}+\frac{2}{a^2}\chi_v{}^{,\alpha|\beta}\right), \end{aligned} \tag{341}$$

where we have used $\chi_v=-av_\chi$ following from Eq. (255), and $\kappa_v=(1/a)\Delta v_\chi$ following from Eq. (197) with $K=0$, both to linear order.

In order to compare with a Newtonian analysis we introduce $\mathbf{u}=-\nabla v_\chi$ to linear order. By combining Eqs. (340), (341) we have

$$\begin{aligned} \ddot{\delta}_v+2H\dot{\delta}_v-4\pi G\bar{\mu}\delta_v=&-\frac{1}{a^2}[a\nabla\cdot(\delta_v\mathbf{u})]^{\cdot}+\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \\ &+\dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}\nabla^\alpha u^\beta+\dot{C}^{(t)\alpha\beta}\right). \end{aligned} \tag{342}$$

We note that, to linear order, the growing solution of the gravitational wave remains constant in time on the superhorizon scale, whereas it redshifts away ($C^{(t)}_{\alpha\beta}\propto a^{-1}$) on the subhorizon scale; see Sec. VII B 3. Ignoring the gravitational wave we reproduce correctly the corresponding Newtonian equation

$$\ddot{\delta}+2H\dot{\delta}-4\pi G\bar{\varrho}\delta=-\frac{1}{a^2}[a\nabla\cdot(\delta\mathbf{u})]^{\cdot}+\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}). \tag{343}$$

We note that our Eq. (342) is valid on the super-sound-horizon (Jeans) scale, which is negligible in the pressureless medium, and thus is valid even on the super-(visual)-horizon scale. In the Newtonian context Eq. (343) is *valid to all orders* in perturbation, and follows from the mass conservation, the momentum conservation, and the Poisson equation given by [10]

$$\dot{\delta}+\frac{1}{a}\nabla\cdot\mathbf{u}=-\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{344}$$

$$\dot{\mathbf{u}}+H\mathbf{u}+\frac{1}{a}\nabla\delta\Phi=-\frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{345}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi=4\pi G\bar{\varrho}\delta. \tag{346}$$

To linear order these equations can be compared with the relativistic version in Eqs. (294),(296),(293) with Eq. (295). To second order, however, although the final result in Eq. (342) coincides with the Newtonian one in Eq. (343), we notice some difference between Eqs. (340),(341),(294)–(296) from Eq. (295) and Eqs. (344)–(346). From Eq. (197), to second order, we have

$$\kappa_v-\frac{\Delta}{a}v_\chi=N_{2v}^{(s)}-\frac{\Delta}{a^2}(\chi_v^{(q)}+av_\chi^{(q)}). \tag{347}$$

Since the RHSs of this equation and of Eq. (293) do not vanish we cannot directly relate $-\varphi_\chi$ and $-\nabla v_\chi$ (or $a\Delta^{-1}\nabla\kappa_v$) to the Newtonian counterparts $\delta\Phi$ and $\mathbf{u}$, respectively. Still, we emphasize that the final equation identified in Eq. (342) coincides exactly with the Newtonian one in Eq. (343). For a similar conclusion in the relativistic situation, see [43]. For analyses of Eq. (343) in the Newtonian context, see [44].

Using $\beta=0$ as the spatial gauge condition Kasai [45] has derived a different equation compared with ours in Eq. (342). Kasai [45] took both the comoving gauge $v\equiv 0$ and the original synchronous gauge, which takes $\alpha=0=\beta$. As we showed above Eq. (340) in a pressureless medium, the comoving gauge $v=0$ implies $\alpha=-\frac{1}{2}\beta_{,\alpha}\beta^{,\alpha}$ and thus vanishes if we take $\beta=0$ as the spatial gauge condition. However, in that gauge condition (the spatial B gauge) the spatial gauge mode is incompletely fixed. Thus, comparison with the Newtonian analyses is not tranparent in that gauge condition.

1. Nonlinear equation based on (3+1) formulation

The general equation of the pressureless and irrotational ideal fluid can be derived from Eqs. (10),(12),(13). The pressureless ideal fluid implies $S_{\alpha\beta}=0$. We take the temporal comoving gauge condition $v=0$. Together with the irrotational condition we have $Q_\alpha=0$ and thus $J_\alpha=0$. Equation (13) gives $N_{,\alpha}=0$; if we use the normalization in Eq. (55), we have $N=a$. Equations (10),(12) give

$$K=\frac{\dot{E}}{E}-\frac{1}{N}\frac{E_{,\alpha}}{E}N^\alpha, \tag{348}$$

$$\dot{K}-\frac{1}{N}K_{,\beta}N^\beta-\frac{1}{3}K^2=4\pi GE-\Lambda+\bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta}. \tag{349}$$

Apparently, the spatial B gauge condition $B_\alpha\equiv 0$ leads to $N_\alpha=0$, thus simplifying the equations. However, such a gauge condition leaves the spatial gauge mode removed incompletely. We prefer to take the spatial C gauge condition, which fixes the spatial gauge modes completely; in this way the analyses can be equivalently considered as spatially gauge-invariant ones. From Eqs. (348),(349) we can derive Eq. (342) to second order. We notice that, in contrast with the Newtonian case, in general we anticipate having infinite perturbation series expansion, and Eq. (342) looks valid only to second order. If we have the higher-order terms nonvanishing, these can be regarded as purely relativistic effects.

2. Nonlinear equation based on (1+3) formulation

Assuming the pressureless condition, Eqs. (26),(27),(28) in the energy frame ($\tilde{q}_a=0$) become

$$\tilde{\dot{\tilde{\mu}}}+\tilde{\theta}\tilde{\mu}=0, \tag{350}$$

$$\tilde{a}_a=0, \tag{351}$$

$$\tilde{\dot{\tilde{\theta}}}+\frac{1}{3}\tilde{\theta}^2+4\pi G\tilde{\mu}-\Lambda+2(\tilde{\sigma}^2-\tilde{\omega}^2)=0. \tag{352}$$

In the energy frame the frame vector follows the possible energy flux; thus, the energy flux term $\tilde{q}_a$ vanishes. Equations (350) and (352) can be combined to give

$$\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^{\tilde{\cdot}}-\frac{1}{3}\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^2-4\pi G\tilde{\mu}+\Lambda-2(\tilde{\sigma}^2-\tilde{\omega}^2)=0. \tag{353}$$

If we set $\tilde{\mu}\equiv\mu(1+\delta)$, where μ is the background energy density, Eq. (353) becomes

$$\tilde{\ddot{\delta}}-\frac{2}{3}\frac{\tilde{\dot{\mu}}}{\mu}\tilde{\dot{\delta}}-4\pi G\mu(1+\delta)\delta-\frac{4}{3}\frac{\tilde{\dot{\delta}}^2}{1+\delta}-2(\tilde{\sigma}^2-\tilde{\omega}^2)(1+\delta)$$
$$+(1+\delta)\left[\left(\frac{\tilde{\dot{\mu}}}{\mu}\right)^{\cdot}-\frac{1}{3}\left(\frac{\tilde{\dot{\mu}}}{\mu}\right)^2-4\pi G\mu+\Lambda\right]=0. \tag{354}$$

This is a completely nonlinear equation.

Now, we *assume* an irrotational fluid. In the energy frame, the comoving gauge condition leads to $\tilde{u}_\alpha\equiv 0$; this is equivalent to taking the normal frame with vanishing energy flux. The momentum conservation equation in Eq. (351) implies that our frame vector follows a geodesic path. In the comoving gauge Eqs. (351),(69) lead to $A=-\frac{1}{2}B^\alpha B_\alpha$ to second order. Thus, to linear order we have $A=0$ which coincides with taking the synchronous gauge. We have

$$\tilde{\dot{\tilde{\mu}}}=\tilde{\mu}_{,a}\tilde{u}^a=\partial_t\tilde{\mu}+\frac{1}{a}\tilde{\mu}_{,\alpha}B^\alpha,\quad \tilde{\dot{\mu}}=\partial_t\mu,\quad \tilde{\dot{\delta}}=\partial_t\delta+\frac{1}{a}\delta_{,\alpha}B^\alpha. \tag{355}$$

Only in the comoving gauge condition does the covariant derivative along $\tilde{u}^a$ simplify to second order as in Eq. (355). In this derivation the pressureless condition is used essentially. Thus, in this comoving gauge Eq. (354) becomes

$$\ddot{\delta}+2H\dot{\delta}-4\pi G\mu\delta=\frac{1}{a^2}(a^2\delta\dot{\delta})^{\cdot}+\frac{1}{3}\dot{\delta}^2-\frac{1}{a^2}(a\delta_{,\alpha}B^{\alpha})^{\cdot}-\frac{1}{a}\dot{\delta}_{,\alpha}B^{\alpha}+2\tilde{\sigma}^2, \quad (356)$$

to second order. From this we can derive Eq. (342) where $\tilde{\sigma}^2$ follows from Eqs. (16),(57),(71).

3. Gravitational wave as a source

From Eq. (342) considering the pure gravitational wave as the source for density perturbation we have

$$\ddot{\delta}_v+2H\dot{\delta}_v-4\pi G\mu\delta_v=\dot{C}^{(t)}_{\alpha\beta}\dot{C}^{(t)\alpha\beta}\equiv S. \quad (357)$$

In the matter dominated era an exact solution is given in Eq. (336) with $\nu=\frac{3}{2}$; in the large-scale limit, considering the relatively growing mode, S vanishes, whereas, in the small-scale limit it decays proportional to $1/(at)^2\propto a^{-5}$ [see Eq. (334)]. If δ_g and δ_d denote two linear-order solutions, the general solution can be written as

$$\delta_v(\mathbf{x},t)=\delta_g(\mathbf{x},t)+\delta_d(\mathbf{x},t)+\int^t S(\mathbf{x},t')\times\frac{\delta_g(\mathbf{x},t')\delta_d(\mathbf{x},t)-\delta_d(\mathbf{x},t')\delta_g(\mathbf{x},t)}{\delta_g(\mathbf{x},t')\dot{\delta}_d(\mathbf{x},t')-\delta_d(\mathbf{x},t')\dot{\delta}_g(\mathbf{x},t')}dt'. \quad (358)$$

The particular solution is proportional to $a^{-5}t^2\propto a^{-2}$ and decays more rapidly even compared with the decaying mode in the linear theory which behaves as t^{-1}.

D. Pure scalar-type perturbation

Equation (302) can be written as

$$\frac{H^2c_s^2}{(\mu+p)a^3}\left\{\frac{(\mu+p)a^3}{H^2c_s^2}\left[\dot{\Phi}-\dot{\Phi}^{(q)}-N_{\dot{\Phi}}+\frac{H}{\mu+p}\left(e+\frac{2}{3}\frac{\Delta}{a^2}\Pi\right)\right]\right\}^{\cdot}=c_s^2\frac{\Delta}{a^2}\left(\Phi-\Phi^{(q)}-2H^2\frac{\Pi}{\mu+p}-N_{\Phi}\right). \quad (359)$$

In the large-scale limit, if we ignore the second-order spatial derivative terms, we have

$$\dot{\Phi}-\dot{\Phi}^{(q)}-N_{\dot{\Phi}}+\frac{H}{\mu+p}e\propto\frac{H^2c_s^2}{(\mu+p)a^3}. \quad (360)$$

Now, we consider $K=0$ and ideal fluids; thus $e=0$. From Eq. (298) we have $\Phi=\varphi_v$. Ignoring the second-order spatial derivatives, from Eqs. (300),(195)–(201),(99)–(105) we can show that $N_{\dot{\Phi}}=(\varphi_v^2)^{\cdot}$. In the comoving gauge we have $\Phi^{(q)}|_v=\varphi_v^{(q)}|_v=0$; see Eq. (274). Thus, we have

$$\varphi_v-\varphi_v^2=C(\mathbf{x})+d(\mathbf{x})\int^t\frac{H^2c_s^2}{(\mu+p)a^3}dt. \quad (361)$$

Therefore, ignoring the transient mode in the expanding phase, we have

$$\varphi_v-\varphi_v^2=C(\mathbf{x}), \quad (362)$$

which remains constant even to second order in perturbations [46,47]. Thus, φ_v is conserved to second order in the large-scale (super-sound-horizon) limit.

Equation (359) is valid for $p\neq 0$. For $p=0$ we have a simpler form in Eq. (299), which gives

$$\dot{\Phi}-\dot{\Phi}^{(q)}-N_{\dot{\Phi}}+\frac{H}{\mu}\left(e+\frac{2}{3}\frac{\Delta}{a^2}\Pi\right)=0. \quad (363)$$

Equation (360) includes this as a case in the large-scale (super-sound-horizon) limit. Thus, the above results in Eqs. (360)–(362) remain valid for general p.

From Eq. (114) we notice that for a minimally coupled scalar field $\delta\phi=0$ implies $v=0$ to second order; thus

$$\varphi_v=\varphi_{\delta\phi}, \quad (364)$$

and the uniform-field gauge coincides with the comoving gauge. Thus, the above analyses are valid even for a minimally coupled scalar field.

E. Pure rotation

In the case of pure rotation Eqs. (208),(105) provide a complete set for a single component fluid; for the multicomponent case see Eqs. (209),(107). Assuming $\Pi^{(v)}_{\alpha}=0$ we have

$$\frac{[a^4(\mu+p)v^{(v)}_{\alpha}]^{\cdot}}{a^4(\mu+p)}=N^{(v)}_{6\alpha},$$

$$N^{(v)}_{6\alpha}=-\frac{1}{a}[v^{(v)}_{\alpha|\beta}B^{(v)\beta}+v^{(v)}_{\beta}B^{(v)\beta}{}_{|\alpha}-\nabla_{\alpha}\Delta^{-1}\nabla^{\beta}(v^{(v)}_{\beta|\gamma}B^{(v)\gamma}+v^{(v)}_{\gamma}B^{(v)\gamma}{}_{|\beta})]. \quad (365)$$

As a simple exercise, using Eq. (250), one can check the gauge transformation properties of both sides. In the C gauge condition ($C^{(v)}_{\alpha}\equiv 0$) we have $B^{(v)}_{\alpha}=\Psi^{(v)}_{\alpha}$ which is gauge invariant. From Eqs. (332),(206), to linear order we have $\Psi^{(v)}_{\alpha}\propto a^2(\mu+p)v^{(v)}_{\alpha}\propto a^{-2}$; thus from Eq. (365) we have

$$[a^4(\mu+p)v^{(v)}_{\alpha}]^{\cdot}=a^{-3}[a^3\cdot a^4(\mu+p)N^{(v)}_{6\alpha}]\propto a^{-3}. \quad (366)$$

Thus, the additional second-order perturbation sourced by the RHS of Eq. (365) behaves as

$$a^4(\mu+p)v_\alpha^{(v)}=L_\alpha^{(v)}(\mathbf{x})+[a^3\cdot a^4(\mu+p)N_{6\alpha}^{(v)}]\int^t\frac{dt}{a^3}. \tag{367}$$

The time-dependent nonlinear solution is proportional to $\int^t dt/a^3$; for $w=$const it is proportional to $a^{-3(1-w)/2}$; thus it always decays (in expanding phase) for $w<1$. The lower bound of integration which could give a temporally constant nonlinear solution can be absorbed to $L_\alpha^{(v)}$.

As we explained in Sec. VI B 2, to linear order, the C gauge condition removes the rotational gauge mode completely, whereas the B gauge condition fails to fix it completely. That is, even after imposing the gauge condition we have some modes which are coordinate effects; under the B gauge, from Eq. (252) we have $\xi_\alpha^{(v)}=\xi_\alpha^{(v)}(\mathbf{x})$. Then, in Eq. (365) we notice an ironic situation where the B gauge condition gives vanishing quadratic terms, whereas these terms do not vanish in the C gauge condition. That is, although we anticipate that the nonlinear solution in the C gauge in Eq. (367) is physical, in the B gauge condition the RHS of Eq. (366) vanishes, and we do not have the nonlinear solution in Eq. (367). We can check this situation by using the gauge transformation property of the $v_\alpha^{(v)}$ variable in the two gauge conditions.

Considering pure vector-type perturbations, from Eqs. (175),(177),(234),(249) we have

$$\hat{v}_\alpha^{(v)}=v_\alpha^{(v)}-v_\beta^{(v)}\xi^{(v)\beta}{}_{,\alpha}-v_{\alpha,\beta}^{(v)}\xi^{(v)\beta}+\nabla_\alpha\Delta^{-1}\nabla^\beta(v_\gamma^{(v)}\xi^{(v)\gamma}{}_{,\beta}+v_{\beta,\gamma}^{(v)}\xi^{(v)\gamma}). \tag{368}$$

Now, let the variables with and without carets correspond to the ones in the B and C gauge conditions, respectively. As the $\xi_\alpha^{(v)}$s appear in quadratic combination, we need them only to linear order. From Eq. (252), we have $\hat{B}_\alpha^{(v)}=B_\alpha^{(v)}+a\dot{\xi}_\alpha^{(v)}$. Since the caret indicates the B gauge, we have $a\dot{\xi}_\alpha^{(v)}=-B_\alpha^{(v)}$; thus $\xi_\alpha^{(v)}=-\int^t(B_\alpha^{(v)}/a)dt$. Thus, Eq. (368) gives

$$a^4(\mu+p)\hat{v}_\alpha^{(v)}|_{B\text{ gauge}}=\left\{a^4(\mu+p)v_\alpha^{(v)}-[a^3\cdot a^4(\mu+p)N_{6\alpha}^{(v)}]\int^t\frac{dt}{a^3}\right\}\Bigg|_{C\text{ gauge}}. \tag{369}$$

Therefore, in the B gauge the nonlinear solution in the C gauge in Eq. (367) disappears exactly. We note, however, that the solution in Eq. (367) is the physical (gauge-invariant) one in the C gauge.

F. Pure gravitational wave

In the case of pure gravitational wave Eqs. (210),(103) provide a complete set. In the large-scale limit, thus ignoring second-order spatial derivative terms, and assuming $K=0$ and $\Pi_{\alpha\beta}^{(t)}=0$, we have

$$\ddot{C}^{(t)\alpha}{}_\beta+3H\dot{C}^{(t)\alpha}{}_\beta=N^{(t)}_4{}^\alpha{}_\beta,$$

$$N^{(t)}_4{}^\alpha{}_\beta=N_4{}^\alpha_\beta-\Delta^{-1}(\nabla^\alpha\nabla_\gamma N_4{}^\gamma_\beta+\nabla_\beta\nabla^\gamma N_4{}^\alpha_\gamma)+\frac{1}{2}(\Delta^{-1}\nabla^\alpha\nabla_\beta+\delta^\alpha_\beta)\Delta^{-1}\nabla^\gamma\nabla_\delta N_4{}^\delta_\gamma,$$

$$N_4{}^\alpha_\beta=2\left(\dot{C}^{(t)\alpha\gamma}\dot{C}^{(t)}_{\beta\gamma}-\frac{1}{3}\delta^\alpha_\beta\dot{C}^{(t)\gamma\delta}\dot{C}^{(t)}_{\gamma\delta}\right). \tag{370}$$

Notice that in this large-scale limit we have $C_{\alpha\beta}^{(t)}=$const as the relatively growing solution (in the expanding phase) even to second-order perturbation. In this sense, ignoring the transient mode in the expanding phase, the amplitude of $C_{\alpha\beta}^{(t)}$ remains constant even to second order in perturbations [46,47].

G. Action formulation

We consider the action expanded to second order in perturbations which will give the equations of motion to linear order in the perturbation [48,37,9]. We consider the action for a scalar field in Eq. (108). The perturbed action can be derived by using Eqs. (94), (98) and the ADM quantities presented in Sec. III. To background order, ignoring the surface terms, we have

$$S_{\rm BG}=\frac{1}{16\pi G}\int\sqrt{g^{(3)}}a^3\left[-6\left(\frac{\dot{a}}{a}\right)^2+\frac{6K}{a^2}+16\pi G\left(\frac{1}{2}\dot{\phi}^2-V\right)\right]dtd^3x. \tag{371}$$

To the second-order perturbation, ignoring the surface terms, the pure gravitational wave part becomes

$$S_{\rm GW}=\frac{1}{16\pi G}\int\sqrt{g^{(3)}}a^3\left(\dot{C}^{(t)\alpha\beta}\dot{C}^{(t)}_{\alpha\beta}-\frac{1}{a^2}C^{(t)\alpha\beta|\gamma}C^{(t)}_{\alpha\beta|\gamma}-\frac{2K}{a^2}C^{(t)\alpha\beta}C^{(t)}_{\alpha\beta}\right)dtd^3x. \tag{372}$$

This action is valid for an arbitrary number of scalar fields and fluids with vanishing tensor-type anisotropic stress. Now we consider the pure scalar-type perturbation. We assume $K=0$. To the second-order perturbation, ignoring the surface terms, we have

$$S_{\rm scalar}=\frac{1}{2}\int a^3\left\{\delta\dot{\phi}_\varphi^2-\frac{1}{a^2}\delta\phi_\varphi{}^{,\alpha}\delta\phi_{\varphi,\alpha}+\frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot}\delta\phi_\varphi^2\right\}dtd^3x$$
$$=\frac{1}{2}\int a^3\frac{\dot{\phi}^2}{H^2}\left(\dot{\varphi}_{\delta\phi}^2-\frac{1}{a^2}\varphi_{\delta\phi}{}^{,\alpha}\varphi_{\delta\phi,\alpha}\right)dtd^3x. \tag{373}$$

In this case we used the linear-order equations of motions; thus it is an on-shell action. In the second step we used Eq. (255).

Maldacena has considered the perturbed action to third order in perturbations, which is needed to have equations of motion valid to the second order [47]. For the temporal gauge he used two gauge conditions, the uniform-field gauge ($\delta\phi\equiv 0$) and the uniform-curvature gauge ($\varphi\equiv 0$), and the C gauge for the spatial and rotational ones. Compared with our notation we have

$$\zeta_{\rm Maldacena}=\varphi_{\delta\phi}-\varphi_{\delta\phi}^2,\quad \varphi_{\rm Maldacena}=\delta\phi_\varphi. \tag{374}$$

Thus, $\zeta_{\rm Maldacena}$ is conserved in the large-scale limit; see Eqs. (362),(364). To linear order, from Eq. (255) we have $\varphi_{\delta\phi}=-(H/\dot{\phi})\delta\phi_\varphi$; thus $\zeta_{\rm Maldacena}=-(H/\dot{\phi})\varphi_{\rm Maldacena}$.

VIII. DISCUSSION

We have presented the basic equations to investigate the second-order perturbation of the Friedmann world model. In order to serve as a convenient reference for future studies and applications we have presented some useful relations and quantities needed for the second-order perturbations. The present study is, clearly, not entirely new in this rich field of cosmology and large-scale structure formation. In the late 1960s Tomita presented a series of work on the subject in the context of a fluid [49]. Studies in the context of the ideal fluid or the minimally coupled field can be found in [31,50,47]. In the case of a pressureless irrotational fluid, see [45,43]. The case with the null-geodesic equations was studied in [51], and the case with the Boltzmann equation was considered in [52].

Compared with the previous work, perhaps we could emphasize the following as the new points in our work.

(i) We present the complete sets of perturbed equations in a gauge-ready form, so that we can easily apply the equations to any gauge conditions, which make the mathematical analyses of given problems as simple as possible.

(ii) We consider the most general Friedmann background with K and Λ. Previous studies considered the flat Friedmann background only.

(iii) We consider the most general imperfect fluid situation which includes multiple imperfect fluids with general interactions among them. In addition, we also include minimally coupled scalar fields, a class of generalized gravity theories, the electromagnetic fields, the null geodesic, and the relativistic Boltzmann equation.

(iv) In Sec. VII A we present closed form equations, which are simlar to the ones known in linear theory.

(v) In Sec. VII C we show that up to second order in perturbations the relativistic pressureless fluid coincides exactly with the Newtonian one. We note that suitable choices and combinations of different gauges (thus gauge-invariant combinations) are important to show the equivalence.

(vi) In Sec. VII D we have derived the large-scale (super-sound-horizon) conserved quantity to second order, φ_v, directly from the differential equation governing its evolution. This conserved variable was first studied by Salopek and Bond.

Our equations are suitable for handling nonlinear evolution in the perturbative manner. If we have the solutions to linear order (see Sec. VII B for some examples), the evolution of second-order perturbations can be derived using the quadratic combination of the linear variables as sources; our basic sets of equations in Sec. V C and some closed forms in Sec. VII A are presented with this purpose. As long as we take such a perturbative approach, our formulation in this work can be trivially extended to any higher-order perturbation; except for the fact that, of course, the needed algebra would be quite demanding. We have also shown in Sec. VI that the gauge issue can be similarly handled even in such a higher-order perturbation.

Our formulation can be applied using several different methods as follows.

(i) Quasilinear analyses using Fourier analyses as often used in the Newtonian case [44]. In this approach the quadratic combination of the linear-order terms will lead to mode-mode coupling among different scales, as well as among different types of perturbations.

(ii) Nonlinear back reaction. In our approach we have assumed the presence of a ''fictitious'' background metric which is spatially homogeneous and isotropic. As the basic equations of Einstein gravity are nonlinear, the nonlinear fluctuations in the metric and the matter can affect the background world model. One anticipates recovering the background Friedmann world model through averaging the more realistic lumpy world model and finding the best fit to the idealized world model [53].

(iii) Fitting and averaging. Our basic equations in Sec. IV are presented without separating the background order quantities from the perturbed order ones. Thus, the equations are suitable for the operation of averaging. Using our formulation we could apply and check the various different averaging prescriptions suggested in the literature [53,54,45].

Our perturbative formulation would be a useful complement to the following formulations aiming to investigate the nonlinear evolution of cosmological structures.

(i) The large-scale (long wavelength) approximation or the spatial gradient expansion studied by Salopek, Tomita, Deruelle and others in [46,55].

(ii) The cosmological post-Newtonian formulation studied by Futamase, Tomita, and others in [56].

(iii) The relativistic Zel'dovich approximation studied in [45,43,57].

(iv) General (spatially inhomogeneous and anisotropic) solutions near singularity where the large-scale conditions are well met; in such a situation it was shown that the spatially different points decouple and evolve separately. These were studied by Belinsky, Lifshitz, Khalatnikov, and others in [58].

Our general formulation can be used to study the following situations anticipated during the evolution of our universe.

(i) We can check the limit of linear theory. Current cosmological observations can be successfully explained within the current standard theoretical paradigm. In that paradigm linear perturbation theory plays a significant role in explaining the quantum generation stage in the early universe and in the classical evolution processes on the large scale and in the early era. The linear theory provides a self-consistent explanation of some important aspects of the origin and evolution of large-scale structures. However, the limit of the linear theory *cannot* be estimated within the linear theory. We expect that the second-order perturbation theory could provide a meaningful ways to investigate such limits.

(ii) We can investigate the quasilinear process in the relativistic context. In the literature it is commonly assumed that the relativistic linear perturbation theory is sufficient to handle the large-scale structure, and the nonlinear processes occur only in the Newtonian context which are often handled by the numerical simulations. The quasilinear evolution would be useful to investigate the transition regions between linear and nonlinear evolutions. Our perturbative approach may have its own limit, because if we find the importance of second-order contributions it may naturally follow that higher-order contributions would immediately become important as well. Thus, we anticipate, if successful, that relativistic quasilinear analyses can be developed similarly to the Newtonian cosmological quasilinear analyses studied in [44].

(iii) We can examine the fate of fluctuations in the collapsing phase, and possibly through a bounce. The fluctuations of a single component medium and the gravitational wave are described by second-order differential equations. In the linear stage and in the large-scale limit, we have general solutions in Eqs. (311),(312),(333). In an expanding phase, the C mode is growing relatively and the d mode is decaying and thus transient. If the initial conditions (say, generated from the quantum fluctuations) are imposed in the early expanding phase, the d mode parts disappear in a few e-folding times of the scale factor increase and are thus uninteresting. The relatively growing modes for both the scalar- and tensor-type perturbations are characterized by a conserved amplitude of certain gauge-invariant variables, φ_v (and the curvature variables in other gauges) and $C^{(t)}_{\alpha\beta}$, in the large-scale limit. In the collapsing phase, however, the roles of growing and decaying modes are switched. In the collapsing phase the d mode, and the vector mode as well, grows quite rapidly [see Eqs. (325),(339),(332)]; our solutions in Sec. VII B also cover the collapsing phase by considering $t \rightarrow |t|$ with t approaching 0 (see [59]). Thus, the linear perturbations grow rapidly and inevitably reach the nonlinear stage [6,60]. Such growth would cause the transition of our simple (spatially homogeneous and isotropic) background world model to the anisotropic and inhomogeneous ones studied in [58]. Although we anticipate that the perturbations would become quite nonlinear, we hope we can investigate the transition region based on our second-order perturbation formulation. One simplifying fact is that in the collapsing phase the local range covered by the dynamical time scale $\sim H^{-1}$ shrinks relative to the comoving scale, and thus effectively the scales we are interested in satisfy the conditions of the large-scale limit.[2] Such large-scale conditions are well met for a given comoving scale during the early evolution stage (near singularity) and as the background model approaches the singularity in the collapsing phase. Investigation of situations in the collapsing and subsequent bouncing background is left for future study; for evolution under the linear assumption, see [59]. For the general cosmological investigation near singularity, see [58].

(iv) It is well known that the nonlinear effect (either in quantum generation or in classical evolution processes) could lead to non-Gaussian effects in the observed quantities of CMB anisotropies and large-scale galaxy distribution and motion. Maldacena has recently investigated such an effect on the CMB based on second-order perturbation theory (see [47]). The first year WMAP data show no positive detection of non-Gaussian nature of the CMB sky maps under a couple of non-Gaussianity tests [61].

(v) We can check evolution on the superhorizon scale where the scale is larger than the causal domain during the dynamic time scale. See Secs. VII D and VII F for the conserved quantities to second order which were found by Salopek and Bond in [46].

(vi) In Sec. VII C we showed that to second order a pressureless fluid with a pure scalar-type perturbation reproduces the Newtonian result. It is likely that the relativistic effect appears in higher-order perturbations; this is left for future investigation.

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