

Dynamical shift conditions for the Z4 and BSSN formalisms

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A class of dynamical shift conditions is shown to lead to a pseudohyperbolic evolution system, both in the Z4 and in the BSSN numerical relativity formalisms. This is done by using a plane-wave analysis which can be viewed as an extension of the standard Fourier analysis for this kind of systems. The proposed class generalizes the harmonic shift condition, where light speed is the only nontrivial characteristic speed, and it is contained into the multiparameter family of minimal distortion shift conditions recently proposed by Lindblom and Scheel. The relationship with the analogous “dynamical freezing” shift conditions used in black hole simulations is discussed.

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I. INTRODUCTION

General covariance is a characteristic property of Einstein’s theory of gravitation. There are four coordinate degrees of freedom in the field equations, allowing us to freely choose the spacetime coordinates x^μ or, in the framework of the 3+1 decomposition, the lapse function α and the shift vector β^i .

This gauge freedom can be used, like in the early years of general relativity, to set up a complete evolution system (consisting of the field equations plus the gauge conditions) with a well posed Cauchy problem. The well known harmonic coordinate conditions

$$\square x^\mu = (g^{\rho\sigma} \nabla_\rho \nabla_\sigma) x^\mu = 0 \quad (1)$$

provide a simple way to get a well posed initial value problem [1] with an extremely simple principal part, consisting of one wave equation for every component of the four-dimensional metric tensor:

$$\square g_{\mu\nu} = \dots \quad (2)$$

[2,3], where the ellipsis stands for terms not belonging to the principal part.

Although the resulting system is still currently used in analytical approximations, its use in numerical relativity is very limited, mainly because the four conditions (1) completely exhaust the gauge degrees of freedom, and there is no flexibility left that could be used to fit the peculiarities of the specific systems one wants to model (but see also some generalizations in Refs. [4–7]).

The current alternative, represented by the “new hyperbolic formalisms” (Refs. [8–18]), is to use somehow the momentum constraint as a tool for ensuring hyperbolicity, instead of the three space coordinate conditions in Eq. (1). In this way, only the time gauge condition

$$\square x^0 = 0 \quad (3)$$

is kept, or one of its generalizations (harmonic slicing). This happens to be very convenient for numerical simulations, because Eq. (3) implies a direct relationship between the lapse α and the spatial volume element $\sqrt{\gamma}$, which can be

used to avoid collapse singularities [19]. The main advantage is that one can use now the shift degrees of freedom to simplify the system, either using normal coordinates (zero shift) or any other kinematical choice adapted to the specific problem under study.

Recently, there has been a renewed interest in the use of dynamical shift vectors. In the context of the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism [20,21], some shift conditions have been proposed [22] that manage to “freeze” black hole dynamics near the horizon, leading to long term numerical simulations although. As we will see later, the harmonic shift condition given by Eq. (1) is similar to (but not contained in) the ones discussed in Ref. [22].

It is clear that the principal part of the original BSSN system is modified by the choice of a dynamical shift, although no hyperbolicity analysis of the modified system has been yet published, to the best of our knowledge. This is not surprising because the BSSN formalism, like the Arnowitt-Deser-Misner (ADM) one, is of a mixed type: first order in time, but second order in space, and therefore the standard Fourier analysis would lead to the conclusion that the mixed order system is not hyperbolic [23]. This has been explicitly shown by Frittelli and Gomez for both the ADM and BSSN formalisms [24]. We will present here an alternative plane-wave analysis, based on the underlying physics, in order to reveal a related property, which we will call “pseudohyperbolicity” to avoid confusion. As far as the underlying physics does not change when passing from the fully second order system to the mixed order version of the same equations, pseudohyperbolicity can be seen as the imprint left on the mixed order system by the true hyperbolicity of the fully second order version which was at the starting point.

From a different point of view, a generalization of the minimal distortion shift condition has been proposed by Lindblom and Scheel [25] in the context of the first order Kidder-Scheel-Teukolsky (KST) formalism [17]. Surprisingly enough, in spite of the fact that the resulting system contains at least twenty-two free parameters, none of the cases discussed in [25] verifies the condition that all the non-trivial characteristic speeds do coincide with light speed. The surprise comes from the fact that one would expect this condition to be ensured by the use of the full set of harmonic

coordinates (1), which are actually a particular case of the gauge conditions proposed in [25].

As a contribution to clarify this issues, we will present here a family of dynamical shift conditions, which is closely related with the ones presented in [22] and [25]. We will do so first in the framework of the general-covariant Z4 formalism [26], extending then the results to the BSSN case, which can be derived from the Z4 one in a simple way [27]. We will perform a complete plane-wave analysis of both systems, analogous to the one presented in Ref. [28]. The resulting characteristic speeds are directly related with the main free parameters of the proposed family. The further requirement that all the nontrivial characteristic speeds do coincide with light speed, allows one to recover the harmonic case.

II. THE Z4 SYSTEM

A. The evolution equations

The Z4 covariant formalism introduces a four-dimensional vector as a supplementary dynamical field Z_μ . The evolution equations are obtained by adding the (symmetrized) covariant derivatives of Z_μ to Einstein's field equations:

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \quad (4)$$

In the 3+1 decomposition the line element is written as:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (5)$$

where α and β^i are the lapse and the shift, respectively, and γ_{ij} is the spatial 3-metric. Using this decomposition, the general covariant equations (4) do consist of a system of pure evolution equations:

$$(\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\alpha K_{ij} \quad (6)$$

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta) K_{ij} = & -\nabla_i \alpha_j + \alpha \left[{}^{(3)}R_{ij} + \nabla_i Z_j + \nabla_j Z_i - 2K_{ij}^2 \right. \\ & \left. + (\text{tr} K - 2\Theta) K_{ij} - S_{ij} + \frac{1}{2} (\text{tr} S - \tau) \gamma_{ij} \right] \end{aligned} \quad (7)$$

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta) \Theta = & \frac{\alpha}{2} [{}^{(3)}R + 2\nabla_k Z^k \\ & + (\text{tr} K - 2\Theta) \text{tr} K - \text{tr}(K^2) - 2Z^k \alpha_k / \alpha - 2\tau] \end{aligned} \quad (8)$$

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta) Z_i = & \alpha [\nabla_j (K_i^j - \delta_i^j \text{tr} K) + \partial_i \Theta - 2K_i^j Z_j \\ & - \Theta \alpha_i / \alpha - S_i] \end{aligned} \quad (9)$$

where we have noted

$$\Theta \equiv \alpha Z^0, \quad \tau \equiv 8\pi \alpha^2 T^{00}, \quad S_i \equiv 8\pi \alpha T^0_i, \quad S_{ij} \equiv 8\pi T_{ij}. \quad (10)$$

In a recent work [27], a symmetry breaking mechanism is proposed that, starting from the Z4 system (6)–(9), allows one to recover an evolution system which is equivalent, up to quadratic source terms, to the BSSN system [20,21] (partial symmetry breaking). Also, in the first order case, the same mechanism allows one to recover the multiparameter KST system [17] (full symmetry breaking) or, to be more precise, a “live gauge” version of the same [18].

B. Gauge evolution equations

The harmonic gauge conditions (1) can be easily expressed in the 3+1 formalism as

$$(\partial_t - \beta^r \partial_r) \alpha = -\alpha^2 \text{tr} K \quad (11)$$

$$(\partial_t - \beta^r \partial_r) \beta^i = -\alpha^2 [\partial^i \ln(\alpha \sqrt{\gamma}) + \partial_j \gamma^{ij}], \quad (12)$$

where the first equation is the (harmonic) slicing condition, whereas the second one provides the (harmonic) shift once the slicing is known.

The harmonic slicing condition (11) has been generalized in the context of the Z4 system as follows [27]:

$$(\partial_t - \beta^r \partial_r) \alpha = -\alpha^2 f [\text{tr} K - m \Theta], \quad (13)$$

where the parameter f is directly related with the gauge propagation speed, whereas m provides a coupling with the energy-constraint-violating modes, represented by the quantity Θ . We will see in the following analysis that, if one wants the gauge speed to coincide with light speed ($f=1$), then a pseudohyperbolic system is obtained only if $m=2$, so the coupling given by the m parameter cannot be neglected. This conclusion coincides with the result of Ref. [27], where it was confirmed by the robust stability numerical test.

The harmonic shift evolution equation (12) can be generalized along the same lines:

$$(\partial_t - \beta^r \partial_r) \beta^i = -\alpha^2 [2\mu V^i + a \partial^i \ln \alpha - d \partial^i \ln \sqrt{\gamma}] - \eta \beta^i \quad (14)$$

where we have defined

$$V_i = \partial_i \ln \sqrt{\gamma} - \frac{1}{2} \partial^j \gamma_{ji} - Z_i. \quad (15)$$

Notice that the advection term on the left-hand side, which was absent in Ref. [22], is needed if one wants to recover the harmonic shift as a particular case. As we will see in the following analysis, the parameters μ and d are directly related with the characteristic speeds of the longitudinal and transverse shift components, respectively, in the same way as f is related with gauge speed. The parameter a , instead, has no direct relationship with the characteristic speeds: its role is very similar to the parameter m in the lapse condition, as we will see that specific values of a will be required to ensure pseudohyperbolicity in degenerate cases, so one cannot just neglect this kind of coupling. The parameter η , in turn, corresponds instead to a damping term which has shown to be crucial to get stable long term simulations [22]. We have

not included, however, the η term in our analysis to avoid masking the genuine wave propagation effects with the artificial damping produced by this kind of terms.

III. LINEAR PLANE-WAVE ANALYSIS

The system (6)–(9) is of a mixed type: first order in time but second order in space. This means that, according to the standard methods [23], based on the Fourier analysis of the principal part, it cannot be classified as hyperbolic. This is also the case of the original ADM system [24], where the quantities (Θ, Z_k) are supposed to be zero. In what follows, we will present an alternative plane-wave analysis, starting with the ADM case first and including the supplementary quantities (Θ, Z_k) later.

It is well known that any metric can be written down at a given spacetime point P in a locally inertial coordinate system such that the first derivatives of the metric coefficients vanish at P. We will take advantage of this to write down the line element at P as

$$ds^2 = -\alpha_0^2 dt^2 + \gamma_{ij}^0(dx^i + \beta_0^i dt)(dx^j + \beta_0^j dt). \quad (16)$$

It is clear that the validity of the expression (16) is strictly local: second and higher order derivatives of the metric coefficients at P cannot be supposed to vanish: they are rather related to one another by the field equations. This suggests the splitting of the metric into two components:

A uniform static background of the form (16).

A dynamical perturbation which, when superimposed to the background in a linear way, allows one to recover the full metric.

It makes sense then to decompose the dynamical perturbation into plane waves, with a space dependence given by

$$\alpha - \alpha_0 = e^{i\omega \cdot x} \tilde{\alpha}(\omega, t) \quad (17)$$

$$\beta^k - \beta_0^k = e^{i\omega \cdot x} \tilde{\beta}^k(\omega, t) \quad (18)$$

$$\gamma_{ij} - \gamma_{ij}^0 = 2e^{i\omega \cdot x} \tilde{\gamma}_{ij}(\omega, t) \quad (19)$$

where $\omega_k = \omega n_k$, $\gamma_{ij}^0 n^i n^j = 1$.

Up to here, we have followed the standard Fourier analysis. Now we will depart from the standard path by decomposing the dynamical variable K_{ij} in a form which is consistent with the exact evolution equation (6), namely

$$K_{ij} = i\omega e^{i\omega \cdot x} \tilde{K}_{ij}(\omega, t), \quad (20)$$

where one must notice the $i\omega$ factor on the right-hand side. This extra factor is not present in the standard hyperbolicity analysis, where only the principal part of the system is used, breaking in that way the direct relationship (6) between the metric and the extrinsic curvature.

Note that Eq. (6) is crucial to relate the original (second order in time) version of the field equations (4) with the resulting (first order in time) 3+1 version (6)–(9). This is why we will choose the alternative decomposition (20) in order to look for the imprint in the 3+1 system of the hyperbolicity properties of the original second order version.

The same thing can be done with the supplementary quantities, which can be considered as an additional perturbation of the Einstein's equations background. The original system (4) is of first order in Z_μ , but it can be viewed alternatively as being of second order in some ‘‘potential’’ quantities Y_μ which time derivative can be defined to be precisely Z_μ . In this way Eq. (4) could be seen as a fully second order system in $(\gamma_{\mu\nu}, Y_\mu)$ and the same arguments as before would justify the following plane-wave decomposition of the supplementary quantities

$$\Theta = i\omega e^{i\omega \cdot x} \tilde{\Theta}(\omega, t) \quad (21)$$

$$Z_k = i\omega e^{i\omega \cdot x} \tilde{Z}_k(\omega, t), \quad (22)$$

where the $i\omega$ factors still appear on the right-hand side.

IV. PSEUDOHYPERBOLICITY OF THE Z4 SYSTEM WITH DYNAMICAL SHIFT

We will perform here a linear plane-wave analysis of the Z4 system. This means to substitute the perturbations described in the preceding section into the evolution equations (6)–(9), (13), (14), keeping only the linear terms. We get

$$(\partial_t - i\omega\beta_0^n)\tilde{\alpha} = -i\omega\alpha_0^2[\text{tr}\tilde{K} - m\tilde{\Theta}] \quad (23)$$

$$(\partial_t - i\omega\beta_0^n)\tilde{\beta}^i = -i\omega\alpha_0^2[(2\mu - d)n^i\text{tr}\tilde{\gamma} - 2\mu(\tilde{\gamma}^{ni} + \tilde{Z}^i) + an^i\tilde{\alpha}/\alpha_0] \quad (24)$$

$$(\partial_t - i\omega\beta_0^n)\tilde{\gamma}_{ij} = -i\omega\left[\alpha_0\tilde{K}_{ij} - \frac{1}{2}(n_i\tilde{\beta}_j + n_j\tilde{\beta}_i)\right] \quad (25)$$

$$(\partial_t - i\omega\beta_0^n)\tilde{\Theta} = -i\omega\alpha_0[\text{tr}\tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n] \quad (26)$$

$$(\partial_t - i\omega\beta_0^n)\tilde{Z}_i = -i\omega\alpha_0[n_i(\text{tr}\tilde{K} - \tilde{\Theta}) - \tilde{K}_i^n] \quad (27)$$

$$(\partial_t - i\omega\beta_0^n)\tilde{K}_{ij} = -i\omega\alpha_0[\tilde{\gamma}_{ij} + n_i n_j(\text{tr}\tilde{\gamma} + \tilde{\alpha}/\alpha_0) - n_i(\tilde{\gamma}_{nj} + \tilde{Z}_j) - n_j(\tilde{\gamma}_{ni} + \tilde{Z}_i)] \quad (28)$$

where the letter n replacing one index means contracting that index with n_i .

Notice that we have kept the linear source terms in Eqs. (23), (25), in contrast with the usual practice in the standard Fourier analysis, where only the principal part of the system is considered. In fact, our plane wave analysis includes (up to the linear order) all the source terms (with the only exception of the artificial damping one for the shift, as discussed before). Therefore, the underlying physics is accounted for in a consistent way. In particular, the direct relationship between the metric and the extrinsic curvature is fully preserved. This means that the characteristic speeds we are going to compute should be the same ones that could be obtained from either the fully second order or the fully first order versions, where the standard Fourier analysis can be applied in a way which is consistent with the underlying physics of the problem.

The system (23)–(28) can be written in a compact way as

$$\partial_t \tilde{u} = -i\omega[\mathbf{A} - \beta_0^n \mathbf{I}] \tilde{u}, \quad (29)$$

where \tilde{u} is the perturbation array. The geometric properties of matrix on the right-hand side (characteristic matrix) are obviously related with the dynamics of the plane-wave perturbations. Allowing for the trivial structure of the shift term, the pseudohyperbolicity of the evolution system (29) will depend on the properties of the main matrix \mathbf{A} . We will say that the system (29) is “pseudohyperbolic” if and only if \mathbf{A} has real eigenvalues and a complete set of eigenvectors, that is, the number of independent eigenvectors must be the same as the number of independent dynamical fields (20 in our case). This is the analogous of the strong hyperbolicity property of first order systems. The use of the term “pseudohyperbolicity” is just to avoid confusion.

We can start computing the eigenmodes which do not contain shift terms. In this list we have:

Energy cone. There are 2 Θ -related eigenmodes, propagating with light speed, $v = -\beta_0^n \pm \alpha_0$:

$$\tilde{\Theta} \pm (\text{tr } \tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n). \quad (30)$$

Light cones. There are 10 more eigenmodes propagating with light speed, $v = -\beta_0^n \pm \alpha_0$:

$$\tilde{K}_{na} \pm \tilde{\gamma}_{na} \quad (31)$$

$$\tilde{K}_{ab} \pm \tilde{\gamma}_{ab}, \quad (32)$$

where the letters a, b replacing an index mean the projection orthogonal to n_i .

Lapse cone. There are 2 α -related eigenmodes propagating with speed $v = -\beta_0^n \pm \alpha_0 \sqrt{f}$:

$$\sqrt{f}[\text{tr } \tilde{K} - B_1 \tilde{\Theta}] \pm [\tilde{\alpha}/\alpha_0 + (2 - B_1)(\text{tr } \tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n)], \quad (33)$$

where we have used the shortcut

$$B_1 \equiv \frac{mf - 2}{f - 1}. \quad (34)$$

The factor f must be greater than zero for pseudohyperbolicity. Notice that, in the degenerate case $f = 1$ (harmonic slicing), a well defined pair of eigenmodes is obtained only if $m = 2$, so that the parameter B_1 can take any value (arbitrary mixing with the energy cone).

The shift-related cones are:

Transverse shift cones. There are 4 eigenmodes propagating with speeds $v = -\beta_0^n \pm \alpha_0 \sqrt{\mu}$,

$$(\tilde{\beta}_a/\alpha_0) \pm 2\sqrt{\mu}(\tilde{\gamma}_{na} + \tilde{Z}_a). \quad (35)$$

The factor μ must be greater than zero for pseudohyperbolicity. Notice that, in the vanishing shift case, they reduce to the second term, which would correspond to standing eigenmodes (zero characteristic speed).

Longitudinal shift mode. There are 2 eigenmodes propagating with speed $v = -\beta_0^n \pm \alpha_0 \sqrt{d}$,

$$\sqrt{d}[\tilde{\beta}^n/\alpha_0 + B_2 \text{tr } \tilde{K} + B_3 \tilde{\Theta}] \pm [(a + B_2)\tilde{\alpha}/\alpha_0 - d \text{tr } \tilde{\gamma} + (2\mu + 2B_2 + B_3)(\text{tr } \tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n)]. \quad (36)$$

We have used again the shortcuts

$$B_2 \equiv \frac{d - af}{f - d}, \quad B_3 \equiv \frac{(2 - mf)B_2 + 2\mu - amf}{d - 1}. \quad (37)$$

A necessary condition for pseudohyperbolicity is that the factor d should be greater than zero. This condition is also sufficient in the generic case where d is different from both 1 and f . Notice that in the degenerate cases one would need to impose additional conditions on the free parameters in order to get a well defined pair of eigenmodes. For instance, if $d = f$ one must have $a = 1$, so that the parameter B_2 can take any value (arbitrary mixing with the gauge cone). If we have further degeneracy, that is $d = f = 1$ (remember that $f = 1$ implies $m = 2$), then it follows from Eq. (37) that $\mu = 1$ also, so that one gets the harmonic shift case.

In summary, there are 20 fields in the evolution system and we have got real characteristic speeds and 20 independent eigenvectors, provided that all the characteristic speed parameters f, μ, d are greater than zero. The system is then pseudohyperbolic in the generic case, although degenerate cases, where different characteristic speeds actually coincide, require additional conditions on the remaining parameters a, m .

V. PSEUDOHYPERBOLICITY ANALYSIS FOR THE BSSN SYSTEM WITH DYNAMICAL SHIFT

A pseudospectral analysis [29] of the original BSSN system [20,21] has been done recently [30]. We will proceed here instead with the linear plane-wave analysis of the complete system, including the dynamical shift terms. We can take advantage of the symmetry breaking mechanism proposed in Ref. [27]. Starting from the Z4 system equations (6) to (9), we will take the following steps:

- (1) Perform the dynamical fields recombination

$$K'_{ij} \equiv K_{ij} - \frac{n}{2} \Theta \gamma_{ij} \quad (38)$$

- (2) Suppress Θ as a dynamical quantity, setting its value equal to zero wherever it appears in the evolution equations.

This process alters the evolution equation for the extrinsic curvature K_{ij} , even if one has $K_{ij} = K'_{ij}$ after the second step. One gets as a result a one parameter family of evolution systems, with different principal parts for every value of the n parameter, namely:

$$(\partial_t - \beta^r \partial_r) \alpha = -\alpha^2 f \text{tr } K \quad (39)$$

$$(\partial_t - \beta^r \partial_r) \beta^i = -\alpha^2 [2\mu V^i + a \partial^i \ln \alpha - d \partial^i \ln \sqrt{\gamma}] - \eta \beta^i \quad (40)$$

$$(\partial_t - \beta^r \partial_r) \gamma_{ij} = -2\alpha K_{ij} + \gamma_{ik}(\partial_j \beta^k) + \gamma_{jk}(\partial_i \beta^k) \quad (41)$$

$$(\partial_t - \beta^r \partial_r) Z_i + \partial_k [\alpha (\delta_i^k \text{tr} K - K^k_i)] = \dots \quad (42)$$

$$(\partial_t - \beta^r \partial_r) K_{ij} + \partial_k [\alpha \lambda^k_{ij}] = \dots \quad (43)$$

where we have noted for short

$$2\lambda^k_{ij} = \delta^k \gamma_{ij} + \delta_i^k (\partial_j \ln \alpha + \partial_j \ln \sqrt{\gamma} + 2V_j) + \delta_j^k (\partial_i \ln \alpha + \partial_i \ln \sqrt{\gamma} + 2V_i) - n V^k \gamma_{ij}. \quad (44)$$

In particular, it has been shown in [27] that the principal part of the system obtained when $n=4/3$ can be rewritten, by further rearranging the dynamical fields, as that of the BSSN system. We will name this system Z3-BSSN to avoid confusion. Both systems are then linearly equivalent (equivalent up to quadratic source terms, see Ref. [31]), so that showing pseudohyperbolicity for the Z3-BSSN system, as we will do in the present section, amounts to show the same for the original BSSN system.

As far as the second step of the symmetry breaking procedure suppresses the dynamical field Θ , the linear plane-wave analysis must be repeated from scratch, although the results of the preceding section provide a useful guide. We obtain the following list of eigenvectors and eigenvalues:

Standing mode. There is 1 mode propagating along the normal lines, that is with speed $v = -\beta_0^n$:

$$\text{tr } \tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n. \quad (45)$$

This is what is left of the energy cone in the Z4 system after suppressing Θ as a dynamical field.

Light cones. There are 10 modes propagating with speed $v = -\beta_0^n \pm \alpha_0$, namely

$$\tilde{K}_{ab} \pm \left[\tilde{\gamma}_{ab} - \frac{2}{3} (\text{tr } \tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n) \tilde{\gamma}_{ab}^0 \right] \quad (46)$$

$$\tilde{K}_{na} \pm \tilde{\gamma}_{na}. \quad (47)$$

Lapse cone. There are 2 modes propagating with speed $v = -\beta_0^n \pm \alpha_0 \sqrt{f}$,

$$\sqrt{f} \text{tr } \tilde{K} \pm \tilde{\alpha} / \alpha_0. \quad (48)$$

Transverse shift cones. There are 4 modes propagating with speed $v = -\beta_0^n \pm \alpha_0 \sqrt{\mu}$,

$$(\tilde{B}_a / \alpha_0) \pm 2\sqrt{\mu} (\tilde{\gamma}_{na} + \tilde{Z}_a). \quad (49)$$

Longitudinal shift cone. There are 2 modes propagating with speed $v = -\beta_0^n \pm \alpha_0 \sqrt{d}$:

$$\sqrt{d} [\tilde{B}^n / \alpha_0 + B_2 \text{tr } \tilde{K}] \pm [(a + B_2) \tilde{\alpha} / \alpha_0 - d \text{tr } \tilde{\gamma} + 2\mu (\text{tr } \tilde{\gamma} - \tilde{\gamma}^{nn} - \tilde{Z}^n)] \quad (50)$$

where the parameter B_2 is the same that appears in the Z4 system, as defined in Eq. (37). Notice that, in the degenerate

case $d=f$, a well defined pair of eigenmodes is obtained only if $a=1$, so that B_2 can take any value (arbitrary mixing with the lapse cone).

In summary, there are 19 fields in the evolution system and we have got real characteristic speeds and 19 independent eigenvectors, provided that all the characteristic speed parameters f, μ, d are greater than zero. The system is then pseudohyperbolic in the generic case, although the degenerate case $d=f$ requires the additional condition $a=1$. Notice that the harmonic case is recovered precisely when

$$d=f=\mu=1. \quad (51)$$

Comparing with Ref. [22], the only relevant term in their ‘‘Gamma driver’’ shift conditions is the $\tilde{\Gamma}^i$ one. It is clear that this corresponds to our parameter choices

$$a=0, \quad \mu = \frac{3}{4}d \quad (52)$$

so that pseudohyperbolicity is ensured provided that $d \neq f$. The main difference, as stated before, is that our shift conditions (14) are a generalization of the harmonic ones (12). This is only possible by keeping, as we do, the advection term in the shift evolution equation (14). This term was suppressed in Ref. [22] in order to freeze the dynamics in the neighborhood of the black hole’s apparent horizon.

VI. CONCLUSIONS

In this paper, we have studied a multiparameter family of dynamical gauge conditions (13), (14), which generalizes the harmonic gauge conditions (1) along the ways sketched in Refs. [22,25] (Gamma-driver shift conditions). Starting with (the second order version of) the general covariant Z4 formalism, we have computed explicitly all the eigenmodes, identifying the choices of the gauge parameters $\{f, m, \mu, a, d\}$ that make the full evolution system pseudohyperbolic. The relationship between the gauge parameters and the characteristic speeds is direct and simple. Depending on the parameters choice, gauge (lapse and shift) propagation speeds can be made to coincide or not with light speed.

The same kind of analysis has been done with the corresponding gauge conditions, for the second order system (39)–(43), which is linearly equivalent to that of the BSSN system [31]. The Gamma-driver shift condition which has been used in Ref. [22] corresponds to the parameters choice (52), but with the advection term in the left-hand side of the shift equation (40) suppressed. One could redo the analysis without that term, just by adding it to the right-hand side of the same equation. This would affect the shift cones (49), (50), which would then intersect the other ones, leading to a more involved causal structure. We have chosen to keep instead the shift advection term in place, so that the original harmonic gauge condition (1) is kept inside our family.

Finally, we will briefly discuss the relevance of our results in connection with the ones presented in Ref. [25], in the context of the (first order) KST formalism. In that paper, a 22-parameter family of gauge conditions was presented

which contains Eqs. (39), (40) as a sub-family, but no parameter choice was found ensuring both symmetric hyperbolicity and the additional requirement that all the nontrivial characteristic speeds should coincide with light speed. Is this because these two conditions are actually incompatible in the KST formalism . . . or is it rather because there are only few “good” choices (maybe just one), hidden in the huge 22-parameter space, just waiting to be identified?

We cannot fully answer this question here because we are dealing in this paper just with pseudohyperbolicity, not with the stronger condition of symmetric hyperbolicity for first order systems. However, there is a direct relationship between the Z3 systems proposed in Ref. [31] and the KST formalism. This means that we can at least provide necessary conditions that can be helpful by reducing parameter space in the quest for a definitive answer:

The harmonic case is the only one in which we can get both pseudohyperbolicity and light speed as the only nontrivial characteristic speed. We can easily identify the harmonic case in the family of gauge conditions proposed in Ref. [25], getting the following restrictions on their gauge parameters:

$$\mu_S=2, \mu_L=2\sigma=1, \epsilon_S=-1, \epsilon_L=0, \lambda=-1. \quad (53)$$

As far as the damping terms containing $\{\kappa_S, \kappa_L\}$ are not relevant for the hyperbolicity analysis, this completely fixes the gauge parameters.

As seen in the preceding sections, the light cones can always be obtained independently of the details of the lapse

and shift cones. This means that the results presented in Ref. [31] for the zero shift case, where the nontrivial characteristic speeds were given explicitly in terms of the KST parameters, can be applied as such. It follows that the light speed condition imposes the following additional restrictions on the KST parameters:

$$\eta=4(1+\chi), \quad \chi=2\gamma(1+\chi). \quad (54)$$

This means that ζ and γ [which amounts to the parameter n in Eq. (43): $n=-2\gamma$] are the only remaining (independent) KST parameters, although $\zeta=-1$ is usually required for symmetric hyperbolicity [26].

There are, of course, the 10 additional coupling parameters ψ introduced in Ref. [25], so that the question cannot be completely solved here. Nevertheless, we hope that the conditions we provide, which actually reduce by half the 22 parameter space, will pave the way to a definitive answer.

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