First-order cosmological phase transitions in the radiation-dominated era

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We consider first-order phase transitions of the Universe in the radiation-dominated era. We argue that in general the velocity of interfaces is nonrelativistic due to the interaction with the plasma and the release of latent heat. We study the general evolution of such slow phase transitions, which comprise essentially a short reheating stage and a longer phase equilibrium stage. We perform a completely analytical description of both stages. Some rough approximations are needed for the first stage, due to the nontrivial relations between the quantities that determine the variation of temperature with time. The second stage, instead, is considerably simplified by the fact that it develops at a constant temperature, close to the critical one. Indeed, in this case the equations can be solved exactly, including back reaction on the expansion of the Universe. This treatment also applies to phase transitions mediated by impurities. We also investigate the relations between the different parameters that govern the characteristics of the phase transition and its cosmological consequences, and discuss the dependence of these parameters with the particle content of the theory.

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I. INTRODUCTION

It is well known that the Universe could have undergone several phase transitions in the early stages of its history, most of them associated with the spontaneous symmetry breaking of some symmetry. Some examples are the quark-hadron phase transition at the QCD scale, the phase transitions associated to the electroweak $SU(2)\times U(1)$ symmetry breaking or to grand unified theories, and the Peccei-Quinn phase transition, related to the axion field and the strong *CP* problem. Cosmological phase transitions generically produce cosmic relics, such as topological defects, magnetic fields, or baryon number asymmetries, with potentially important cosmological consequences. The mechanisms for generating these relics build on the dynamics of the phase transition.

In a first-order phase transition the dynamics is essentially determined by the nucleation and expansion of bubbles. At zero temperature, when a true vacuum bubble nucleates, it rapidly begins to expand with almost the velocity of light [1]. On the contrary, at high temperature, the bubble expands in a hot plasma, which is perturbed by the motion of the bubble walls. The plasma thus opposes a resistance to the expansion, that depends on the wall velocity. As a consequence, bubble walls feel a friction force, which prevents them to accelerate indefinitely. Then, the velocity quickly reaches a stationary value, determined by the viscosity of the plasma and the pressure difference between the low temperature phase and the supercooled one. These quantities depend on the model, and it is known that the friction can be large enough to prevent the wall from acquiring relativistic velocities [2,3]. Furthermore, the release of latent heat at the interfaces of the phase transition reheats the surrounding plasma up to a temperature that in most cases is close to the critical temperature. Consequently, the pressure difference that drives bubble expansion may decrease considerably, causing a drastic slowdown of the phase transition [4,5]. Therefore, at the radiation-dominated era bubble walls generically undergo nonrelativistic motion. Since the heat liberated at the interfaces is taken away by sound waves, the temperature can be assumed to be homogeneous, which simplifies the analysis.

In order to have a first-order phase transition, the free energy must allow the coexistence of two phases. Therefore, we will assume that the free energy density $\mathcal{F}$ depends on some order parameter (usually a Higgs field) $\phi(x)$. In a certain range of temperatures, the free energy bears two minima separated by a barrier; one of them at $\phi=0$, which corresponds to the symmetric phase, and the other at a nonzero value $\phi_m(T)$, corresponding to the broken symmetry phase. The difference in free energy density $V(\phi,T)$ between some value ϕ of the order parameter and $\phi=0$, is generally given by a finite-temperature effective potential. The free energy difference between the two minima is thus given by

$$\mathcal{F}_b-\mathcal{F}_u=V(\phi_m(T),T)\equiv V(T). \tag{1}$$

At the critical temperature the two minima are degenerate, $V(\phi_m(T_c),T_c)=0$. Above the critical temperature the symmetric phase is the stable one, while below T_c it becomes metastable, ϕ_m being the absolute minimum. Finally, at some temperature $T_0<T_c$, the barrier between the minima disappears, and the symmetric phase becomes unstable. At some stage between the temperatures T_c and T_0, bubbles of the broken-symmetry phase will be formed in the sea of symmetric phase. A bubble can be described as a configuration in which the order parameter is nonvanishing inside a spherical region (see e.g. Ref. [6]). After being nucleated, a bubble will grow with a velocity that depends on the pressure difference at the interface, $\Delta p=-V(T)$, and on the viscosity of the hot plasma in which it expands.

In this work we will be concerned with first-order phase transitions in the radiation-dominated epoch. Our aim is to study the development of such phase transitions within a completely analytical approach. Here we concentrate on the

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determination of the parameters that govern relic formation (e.g. number density of bubbles, bubble wall velocity, etc.). An analytic study of cosmological consequences will be addressed in Ref. [7], where the results of the present analysis will be used. In Sec. II we briefly review the influence of phase transition dynamics on the mechanisms for generating cosmic remnants in first-order phase transitions.

In Sec. III we study a phase transition that completely develops at $T=T_c$, with the two phases in equilibrium. This is a good approximation in the case of inhomogeneous nucleation in the presence of impurities. In the case of homogeneous nucleation of bubbles, this approximation correctly describes the evolution of the phase transition after some latent heat has been released. An interesting feature of phase equilibrium is that it is simple enough to solve analytically, including back reaction on the expansion of the Universe. This is due to the fact that temperature is constant all the way through the phase transition. We thus can obtain the fraction of volume of the Universe that is occupied by the low-temperature phase as a function of time, with no need of any numerical calculations.

Section IV is devoted to the analysis of the phase transition in the case of homogeneous nucleation. The main difference with the previous case is the initial stage of supercooling and quick reheating back to the critical temperature. Section V contains an analytical study of the phase transition dynamics. In Secs. VI and VII we analyze the relations between the different physical parameters involved in the dynamics of the phase transition, leaving some technical discussions to the Appendixes. Our conclusions are summarized in Sec. VIII.

II. COSMOLOGICAL CONSEQUENCES OF PHASE TRANSITIONS

Several cosmological objects may be formed in a phase transition of the Universe. Their abundance and characteristics depend on details of the development of the phase transition. Due to the complexity of the mechanisms by which these objects are created and the difficulty of describing the phase transition, several details of the dynamics (e.g. the variation of the nucleation rate or the wall velocity during the transition) are often disregarded for the sake of simplicity. An analytical study of the phase transition is thus important since analytical expressions will help taking into account the dynamics in a more rigorous way. In this section we review how phase transition dynamics affects the cosmological remnants.

Electroweak baryogenesis. During the last two decades there has been much interest in the possibility that the electroweak phase transition could be the framework for the generation of the baryon number asymmetry of the Universe (BAU). A first-order electroweak phase transition provides the three Sakharov's conditions for the generation of a BAU, although physics beyond the minimal standard model (SM) is mandatory in order to obtain a quantitatively satisfactory result (for reviews on electroweak baryogenesis see Ref. [8]). Due to *CP* violating interactions of particles with the bubble walls, an asymmetry between left handed quarks and their antiparticles is generated in front of the walls of expanding bubbles. This asymmetry biases the baryon number violating sphaleron processes in the symmetric phase. The resulting baryon asymmetry is caught by the walls and enter the bubbles, where baryon number violating processes are turned off.

It is important that the sphaleron processes be suppressed in the broken symmetry phase, in order to avoid the washout of the generated BAU when equilibrium is established after the phase transition. This requirement imposes a condition on the value of the Higgs field at the temperature of the transition [9], $\phi_m(T_t)/T_t \gtrsim 1$. Since ϕ_m is the order parameter, this is a condition on the strength of the first-order phase transition.

The resulting BAU depends also on the bubble wall velocity. On one hand, if the velocity is too large, the left-handed density perturbation will pass so quickly through a given point in space that sphaleron processes will not have enough time to produce baryons; thus the resulting BAU will be small. On the other hand, for very small velocities thermal equilibrium will be restored and the baryon asymmetry will be erased by sphalerons; thus the BAU will be small again. As a consequence, the generated baryon number will have a peak at a given wall velocity, which is of order 10^{-2} [10–12].

Both values of ϕ_m and v_w depend on T and vary during the phase transition. The dependence of the wall velocity is more critical, since reheating may cause it to descend two orders of magnitude before the transition completes. Baryogenesis may be either enhanced or suppressed by this effect [5,13], depending on which side of the peak the initial velocity lies.

Baryon inhomogeneities. A general feature of cosmological phase transitions is the difference of particle masses between the high- and low-temperature phases. These mass differences give rise to different number densities in the two phases. At the QCD phase transition, for instance, baryons are much heavier in the hadron phase than in the deconfined quark phase. As the hadron phase expands, baryons are pushed into the quark phase region, leading to localized clumps of high density surrounded by large voids of low baryon density [14–16]. An important consequence is that large amplitude, small scale density fluctuations may survive until the nucleosynthesis epoch, affecting the standard scenario of big bang nucleosynthesis. Therefore, inhomogeneous nucleosynthesis may put constraints on the quark-hadron phase transition (see e.g. Ref. [17]). Moreover, if the quark phase reaches sufficiently high density, its pressure may balance that of the hadron phase. The quark matter trapped in small regions of space forms quark plasma objects that may survive until the present epoch [14,18].

Baryon inhomogeneities may also arise at the electroweak phase transition, since the amount of baryons produced through electroweak baryogenesis depends drastically on the wall velocity, and the latter has a considerable variation during the phase transition [5]. The geometry of the electroweak inhomogeneities is in general quite different from the QCD case. If the BAU peaks at a certain wall velocity, then the high density regions will form spherical walls, whose radius depends on the moment in the bubble evolution at which the peak velocity is attained. Furthermore, baryon number densities with the wrong sign may arise in some regions of space, depending on the baryogenesis scenario [12,13,19]. This gives rise to the interesting possibility of nucleosynthesis in the presence of antibaryons (see for example Ref. [20]).

However, due to baryon diffusion and "neutrino inflation," baryon inhomogeneities generated at the electroweak phase transition hardly survive until the nucleosynthesis time (see e.g. Refs. [5,21]). Nevertheless, they may survive until the QCD scale. In that case, electroweak scale inhomogeneities can act as impurities for the quark-hadron phase transition (see the next section).

In summary, we can say that the amplitude and scale of baryon inhomogeneities generically depend on the mean nucleation distance and on the variation of the velocity of bubble expansion.

Topological defects and magnetic fields. If a global $U(1)$ symmetry is spontaneously broken at a first-order phase transition, the phase angle θ of the Higgs field within each nucleated bubble is essentially constant, but phases in different bubbles are uncorrelated. When bubbles collide, the discontinuity in the Higgs phase is smoothed out to become a continuous variation. The so called "geodesic rule" states that (for energetic reasons) the shortest path between the two phases is chosen [23]. When three bubbles meet, a vortex (in two spatial dimensions) or a string (in three dimensions) may be trapped between them. This mechanism is obviously generalized to higher symmetry groups and other kinds of topological defects. Ignoring the dynamics of phase equilibration, it is easy to see that the number density of defects is proportional to the number density of bubbles. However, the final number of defects will depend strongly on the velocity of bubble expansion [24]. If the latter is much less than the velocity of light, then phase equilibration between two bubbles will have probably completed before they encounter a third bubble, thus reducing the chances of trapping a string.

The above picture is in fact a rough simplification of the defect-formation problem. One complication is due to dissipation, since the Higgs field is coupled to the other fields in the thermal bath. Another complication arises when considering a gauge symmetry, and is caused by the fact that the phase of the Higgs field is not a gauge-invariant quantity, so it is convenient to define a gauge-invariant phase difference between two bubbles [24]. The phase difference is thus linked to the gauge field. In this case, dissipation can be taken into account by introducing the conductivity of the plasma. Then one can model the collision of three bubbles and calculate the evolution of the phase difference and gauge field. One can say that a vortex is formed whenever a quantum of magnetic flux is trapped in the unbroken-symmetry region between the three bubbles.

From the above it is clear that the formation of local vortices is associated with the generation of magnetic fields. Therefore bubble collision constitutes also a mechanism for generating the cosmic magnetic fields (e.g., see Ref. [25]). Of course, the magnetic field that is formed in this way corresponds to a spontaneously broken symmetry which cannot be the electromagnetic $U(1)_{\rm em}$. Nevertheless, this mechanism can take place at the electroweak phase transition, where unstable cosmic strings[1] and hypermagnetic fields may be formed. The latter are subsequently converted to $U(1)_{\rm em}$ magnetic fields. It is interesting to note that the presence of magnetic fields may affect the dynamics of the electroweak phase transition (see e.g. Ref. [27]).

Calculating the magnitude of the magnetic fields and the density of defects that are left at the end of the phase transition involves the passage from three-bubble collision simulations to the computation of the full phase transition. Evidently, this is a difficult task. Although some simulations have been made (e.g. Ref. [28]), several simplifications are generally required, which include forgetting about variations in the nucleation rate and the velocity of expansion of bubbles during the transition. An analytical investigation of phase transition dynamics may therefore clarify the picture and provide useful tools for the calculation of defect formation and magnetic field generation.

III. PHASE EQUILIBRIUM

We begin by considering the limiting case in which the first-order phase transition is as slowest as possible, namely, that of coexistence of the high- and low-temperature phases at the critical temperature T_c. Such a cosmic separation of phases has been studied for the QCD phase transition [14,16,29,30]. At the critical temperature there is no pressure difference between phases at the bubble walls, so the bubble expansion takes place almost in equilibrium. Assume at $T=T_c$ there are already regions with low-temperature phase. As the Universe expands, the fraction of space occupied by these regions increases, as the high-temperature phase converts to low-temperature phase. The loss of energy due to the expansion of the Universe is thus compensated by the latent heat released at the interfaces, and the temperature remains constant. In this scenario there is no supercooling.

Since at T_c the nucleation rate vanishes, such a first-order phase transition is only possible in the presence of impurities that induce the formation of bubbles. In this case inhomogeneous nucleation theory applies. In a phase transition mediated by impurities there will still be some supercooling, which we neglect in this section for simplicity. The role of impurities in the early Universe could be played for instance by topological [31,32] or nontopological solitons [33–35]. These may exist in the high-temperature phase, containing the low-temperature phase in their core. In this case their configurations become unstable or metastable below the critical temperature. When the system cools below T_c these objects begin to expand and convert the Universe into the true vacuum.

Another example of inhomogeneous nucleation is the case of the QCD phase transition in the presence of baryon number inhomogeneities [36]. These may arise as a natural consequence of electroweak baryogenesis [5], and can survive until the QCD scale [37]. Since the critical temperature is

[1] See for example Ref. [26].

different in regions with different chemical potential [38,39], bubbles will nucleate first in those regions with a higher T_c [22]. If such regions are relatively small, and if they achieve the necessary amount of supercooling while the surrounding background reaches the critical temperature, then the baryon inhomogeneities operate as impurities for inhomogeneous nucleation.

Even if the phase transition proceeds by homogeneous nucleation of bubbles, phase equilibrium will describe quite well a good part of the transition, whenever the latent heat is at least comparable to the energy density difference between the critical temperature and that at which nucleation effectively begins. In that case, the energy released will reheat the plasma back to a temperature very close to T_c. At that moment bubble nucleation virtually stops and the two phases remain close to equilibrium until the full latent heat of the transition is eliminated. We will analyze this case in the next section.

The customary equation for the adiabatic expansion, $\dot{\rho}=-3H(\rho+p)$, tells us how the Universe takes energy from the hot plasma. Here ρ is the energy density, p is the pressure, and H is the expansion rate. It can be equivalently written in terms of entropy density, $\dot{s}=-3Hs$, which is just the statement of entropy conservation, $S=$const. It will be convenient to use it in the form $s\propto a^{-3}$. At the beginning of the phase transition the whole Universe is in the symmetric phase, while at the end of the transition it is filled with the broken symmetry one, so we can write

$$s=s_u(T_c)a_i^3/a^3=s_b(T_c)a_f^3/a^3, \tag{2}$$

where $s_{u(b)}$ is the entropy density of the unbroken (broken) symmetry phase, a is the cosmic scale factor, and $a_{i(f)}\equiv a(t_{i(f)})$ its value at the beginning (end) of the transition. We assume that, since the phase transition occurs very slowly, the latent heat released at the interfaces is quickly distributed throughout space and the temperature is homogeneous. We also assume that pressure and temperature remain constant during the phase transition. Phase coexistence at $T=T_c$ means that there are regions of space with different equations of state. Thus the entropy density has different constant values $s_b(T_c)$ and $s_u(T_c)$ inside and outside the bubbles of broken symmetry phase respectively. The quantity s in Eq. (2) is the average entropy density of the whole system. The entropy in a comoving volume $V_U=V_b+V_u$ is the sum of two contributions, $S=s_bV_b+s_uV_u$. Thus,

$$s=s_u+(s_b-s_u)f_b, \tag{3}$$

where f_b is the fraction of space that is already in the broken-symmetry phase. The entropy, energy and pressure are derived from the free energy. At $T=T_c$ the pressure p_c is the same in the two phases, $\Delta p=-V(T_c)=0$. The latent heat of the phase transition is

$$L=\rho_u-\rho_b=T_c(s_u-s_b)=T_cV'(T_c). \tag{4}$$

From Eqs. (2) and (3), and using Eq. (4), we obtain the dependence of f_b on the scale factor

$$f_b=\frac{T_cs_u}{L}\left(1-\frac{a_i^3}{a^3}\right). \tag{5}$$

The dependence of the scale factor on time is given by the Friedman equation

$$H^2\equiv\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho, \tag{6}$$

where for simplicity we have neglected the term k/a^2. At constant temperature T_c and pressure p_c, the energy density is given by

$$\rho=T_cs-p_c=T_cs_ua_i^3/a^3-p_c, \tag{7}$$

where we have used Eq. (2) in the last equality. Inserting ρ in Eq. (6) and writing $H=\frac{1}{3}a^{-3}da^3/dt$ we can easily solve the equation for $a^3(t)$,

$$\left(\frac{a}{a_i}\right)^3=\frac{T_cs_u}{p_c}\sin^2[\omega(t-t_i)+\delta], \tag{8}$$

where $\omega=\sqrt{6\pi Gp_c}$ and the constant phase is determined by the initial condition $a(t_i)=a_i$, $\delta=\arcsin\sqrt{p_c/T_cs_u}$.

During the phase transition we have two coexisting phases in the radiation dominated era, and we may consider different possibilities for the equations of state. The simplest one is to assume that the Universe is radiation dominated before the phase transition, i.e.,

$$p_u=\rho_u/3, \tag{9}$$

with

$$\rho_u(T)=\pi^2g_*T^4/30, \tag{10}$$

where g_* is the number of effectively massless species.[2] In fact, this is not a realistic situation. In the symmetric phase the Higgs vacuum expectation value (VEV) does not correspond to the true vacuum, so we should add a constant energy density to account for the energy of this state and have a negligible cosmological constant after the phase transition. It is interesting, however, to consider first this simpler case. Hence, at $T=T_c$ Eqs. (9) and (10) give $p_c=\rho_u/3$ and $T_cs_u=4\rho_u/3$, so $T_cs_u/p_c=4$, $\delta=\pi/6$, and $\omega=\sqrt{3}H_i/2$.

Before the phase transition $H=\sqrt{8\pi G\rho_u/3}$, with $\rho_u\propto T^4\propto a^{-4}$, so Eq. (6) gives the familiar relation $H=(2t)^{-1}$. The temperature descends like $t^{-1/2}$, and at $T=T_c$ the phase transition begins, since we are assuming that no supercooling occurs. Hence we can use the relation $H_i=1/2t_i$ as an initial condition. The scale factor thus takes the very simple form

$$\left(\frac{a}{a_i}\right)^3=4\sin^2\left(\frac{\sqrt{3}}{4}\frac{t-t_i}{t_i}+\frac{\pi}{6}\right), \tag{11}$$

[2]In general g_* depends on T. We will assume for simplicity that g_* is constant during the phase transition. We discuss the effect of a variation of g_* in Sec. VI.

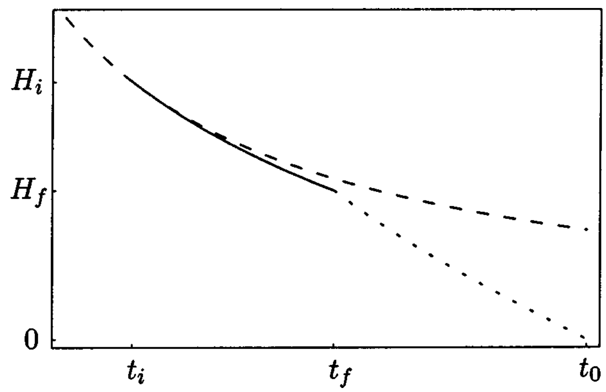

FIG. 1. The expansion rate of the Universe for a phase transition at constant $T=T_c$, in the case $\rho_u=\pi^2 g_* T^4/30$ (negative cosmological constant).

which does not depend on any parameter, i.e., during the transition the expansion of the Universe seems to be affected always in the same way, regardless the thermodynamical parameters of the theory. However, thermodynamics affects the duration of the phase transition and the subsequent evolution of the Universe, as we shall see immediately.

In the above equations it is apparent that the dynamics of the phase transition depends on thermodynamics only through the ratio

$$r \equiv \frac{s_u - s_b}{s_u} = \frac{L}{T_c s_u}. \tag{12}$$

The fraction of volume is thus $f_b = r^{-1}[1-(a/a_i)^{-3}]$, with a/a_i given by Eq. (11). The phase transition concludes when $f_b=1$ [equivalently [29], when s in Eq. (2) equals s_b], so its duration can be determined easily. Making use of some trigonometric algebra,

$$\frac{t_f - t_i}{t_i} = \frac{4}{\sqrt{3}} \arcsin \frac{\sqrt{3}(1-\sqrt{1-4r/3})}{4\sqrt{1-r}}. \tag{13}$$

Notice that Eq. (13) fails to give an answer for $r>3/4$. The problem is that the scale factor given by Eq. (11) reaches a maximum when the argument of the sinus is $\pi/2$. If r is small enough, that never occurs between t_i and t_f. For $r=3/4$ it occurs at $t=t_f$, and for larger r it happens before t_f, which means that the Universe begins to collapse before the phase transition has completed. This is not surprising. Indeed, if the energy density of the unbroken phase is given by Eq. (10), then the energy density of the broken symmetry phase is $\rho_b(T_c)=\pi^2 g_* T^4/30 - L$. So, there is a negative cosmological constant, which will begin to dominate sooner or later. If $L<\rho_u$ (i.e., $r<3/4$), this will not happen during the phase transition, but below the critical temperature the Universe will collapse.

In Fig. 1 we have plotted the expansion rate of the Universe from the beginning of the phase transition to the moment at which H becomes zero. The evolution of H without a phase transition is represented with a dashed line (long dashes). We have chosen a relatively large value $r=0.5$ in order to get a visible departure during the phase transition, which occurs between t_i and t_f (solid line). After the phase transition (short dashes) the expansion slows down, and the rate eventually vanishes at a time t_0.

To consider a more realistic situation we must add a constant term $\sim L$ to the initial energy density, so that we do not have a cosmological constant of the order of the scale of the phase transition. We take for simplicity

$$\rho_u(T) = \pi^2 g_* T^4/30 + L. \tag{14}$$

It will be more convenient to re-express the general solution (8) in terms of the conditions at $t=t_f$, $(a/a_f)^3 = (T_c s_b/p_c)\sin^2[\omega(t-t_f)+\delta']$, with $\delta' = \arcsin\sqrt{p_c/T_c s_b}$. From Eq. (14) it follows that $\rho_b(T_c)=\pi^2 g_* T_c^4/30$, $T_c s_u = 4\rho_b/3$, $T_c s_b = T_c s_u - L$, and $p_c = \rho_b/3 - L$. Therefore we can write

$$\left(\frac{a}{a_f}\right)^3 = \frac{4-4r}{1-4r}\sin^2\left[\frac{\sqrt{3}\sqrt{1-4r}}{4}\frac{t-t_f}{\tilde{t}} + \delta'\right] \tag{15}$$

with

$$\delta' = \arcsin\sqrt{\frac{1-4r}{4-4r}}, \tag{16}$$

where we have defined the time scale $\tilde{t}=(2H_f)^{-1}$. Before the phase transition, ρ is given by Eq. (14), so $a \propto \sinh^{1/2}(\sqrt{32\pi GL/3}t)$. This has the form $a\sim t^{1/2}$ for $t \ll (32\pi GL/3)^{-1/2}$, and departs from the radiation-domination behavior unless $\pi^2 g_* T^4/30 \gg L$.

The fraction of volume in the broken symmetry phase is

$$f_b = \frac{1}{r}\left[1-(1-r)\left(\frac{a}{a_f}\right)^{-3}\right], \tag{17}$$

and the duration of the phase transition is given by

$$\frac{t_f - t_i}{\tilde{t}} = \frac{4/\sqrt{3}}{\sqrt{1-4r}} \arcsin \frac{\sqrt{1-4r}(\sqrt{1+4r/3}-1)}{(4/\sqrt{3})\sqrt{1-r}}. \tag{18}$$

It can be easily checked that, for small r, this solution coincides with the previous one. Furthermore, for $r\to 0$, the duration of the phase transition vanishes, as expected. At first sight, there seems to be a problem if $L \geq \rho_b/3$ (i.e., for $r \geq 1/4$). However, all the previous expressions are still valid and real in the range $1/4 \leq r < 1$. They can be written in the form

$$\left(\frac{a}{a_f}\right)^3 = \frac{4-4r}{4r-1}\sinh^2\left[\frac{\sqrt{3}\sqrt{4r-1}}{4}\frac{t-t_f}{\tilde{t}} + \delta'\right], \tag{19}$$

with

$$\delta' = \operatorname{arcsinh}\sqrt{\frac{1-4r}{4-4r}}, \tag{20}$$

and

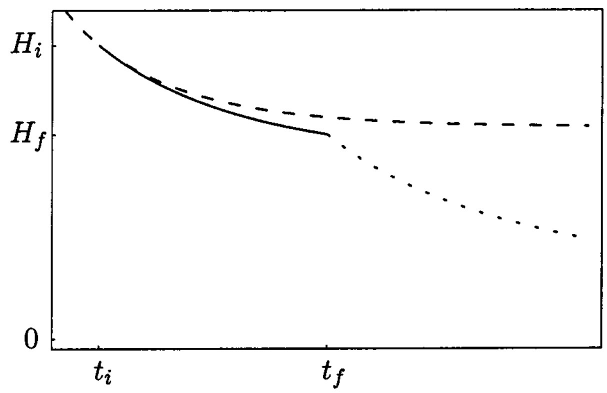

FIG. 2. The expansion rate of the Universe for a phase transition at constant $T=T_c$, in the case $\rho_u=\pi^2 g_* T^4/30+L$.

$$\frac{t_f-t_i}{\tilde{t}}=\frac{4/\sqrt{3}}{\sqrt{4r-1}}\text{arcsinh}\frac{\sqrt{4r-1}(\sqrt{1+4r/3}-1)}{(4/\sqrt{3})\sqrt{1-r}}. \quad (21)$$

For $r\to 1$ the duration of the phase transition becomes infinite because the constant energy density L is comparable to the energy density of the radiation, playing the role of a cosmological constant that starts dominating at $T\sim T_c$. Therefore the expansion of the Universe becomes too fast and the phase transition never ends. One expects that some of our initial assumptions will break down near this limit. For instance, the temperature and expansion rate will not be homogeneous, due to the significant energy density contrast between the two phases and the rapid expansion of the Universe.

In Fig. 2 we plot the expansion rate as a function of time for $r=0.8$. We see that without the phase transition the Universe would enter exponential expansion ($H\to$const). After the phase transition the evolution returns to the radiation-dominated relation $H=1/2t$. By the end of the transition the departure of the expansion rate from its previous evolution becomes appreciable, because in this case $t_f-t_i\sim\tilde{t}\sim t_i$. If the duration of the phase transition is short in comparison with the age of the Universe, the back reaction on H can be disregarded. According to Eq. (18), this happens when the energy released is small in comparison to the energy density of the plasma (small r).

IV. SUPERCOOLING

In the case of a phase transition mediated by homogeneous nucleation, bubbles start to nucleate at a temperature $T<T_c$, when the gain in free energy inside a bubble is enough to compensate the cost of gradient energy at the surface. We have seen in the previous section that, even if bubbles begin to grow at $T=T_c$, the phase transition may not come to an end if the parameter r is close to 1. The case of homogeneous nucleation is even worse due to the additional supercooling. As we shall see, a large latent heat is a general feature of strongly first-order phase transitions. In this case there may be extreme supercooling from which the Universe may never recover [32,40]. Considerable supercooling and latent heat release may occur for instance in the quark-hadron phase transition [41]. Notice that in the case of large supercooling there will be an important departure from equilibrium, at least at the beginning of the phase transition. For the rest of the paper we will be mostly interested in the case $r\ll 1$.

The nucleation and growth of bubbles in a first order phase transition has been extensively studied in the context of the QCD [4,41] and electroweak [4,5,30,42–44] phase transitions. After a bubble is formed, it grows due to the pressure difference at its surface. There is a very short acceleration stage until the wall reaches a terminal velocity due to the friction of the plasma. It can be seen that this initial period in the history of the bubble expansion is negligible. We will assume again that the system remains close to equilibrium, in accordance with the assumption that the velocity of the bubble wall is small. If the wall velocity is less than the speed of sound in the relativistic plasma, $v_w<c_s=\sqrt{1/3}$, the wall propagates as a deflagration front. This means that a shock front precedes the wall, with a velocity $v_{sh}>c_s$. For $v_w\ll c_s$, the latent heat is transmitted away from the wall and quickly distributed throughout space. We can take into account this effect by considering a homogeneous reheating of the plasma during the expansion of bubbles [5,13]. (For detailed treatments of hydrodynamics at different wall velocities see, e.g., Refs. [4,30].)

A. Bubble nucleation

The thermal tunnelling probability for bubble nucleation per unit volume and time is [45,46]

$$\Gamma\simeq A(T)e^{-S_3/T}. \quad (22)$$

The prefactor involves a ratio of determinants associated with the quantum fluctuations around the instanton. In general it must be evaluated numerically. It is usually assumed to be roughly of order T^4, since the nucleation rate is dominated by the exponential in Eq. (22). We will consequently assume $A(T)\simeq T_c^4$. $S_3(T)$ is the three-dimensional instanton action, which coincides with the free energy of a critical bubble (i.e., a bubble in unstable equilibrium between expansion and contraction),

$$S_3=4\pi\int_0^\infty r^2 dr\left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2+V(\phi(r),T)\right]. \quad (23)$$

The configuration of the nucleated bubble may be obtained by extremizing this action. Hence it obeys the equation

$$\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}=V'(\phi). \quad (24)$$

For temperatures very close to T_c, the width of the bubble wall at the moment of formation is much smaller than its radius, and a thin wall approximation can be used [1,46], in which S_3 is expressed as a function of the critical bubble radius R_c, the free energy difference V between the two minima of the potential, and the bubble wall surface energy

σ. The radius R_c can thus be obtained by finding the maximum of S_3. A similar approximation can be used to estimate the free energy and radius of a thick-walled bubble when temperature is not so close to T_c [43]. However, as pointed out in Ref. [44], due to the exponential dependence, the tunnelling probability may be strongly overestimated by using approximations to S_3, leading to a sooner completion of the phase transition.[3] In Ref. [48], different approaches (i.e., thin wall approximation, one- and two-loop perturbative estimates, etc.) have been compared to lattice results (which include calculation of the prefactor) for the case of the electroweak phase transition in the minimal standard model. There, it was found that the different approximations make errors that range from 20% to a factor of 2. Since we do not intend to do numerical calculations in the present work, we will use the thin wall approximation. This approximation may be reasonable or not, depending on the amount of supercooling (see the next section).

B. Phase transition dynamics

In the previous section we assumed constant temperature, and used Eqs. (2) and (3) to obtain the fraction of volume f_b in terms of the scale factor a. Then the Friedman equation determined $a(t)$. In the present case the temperature is not constant, so we need an extra equation to solve for the three quantities f_b, a and T. Such an equation arises by considering the nucleation and growth of bubbles [32],

$$f_b(t)=1-\exp\left\{-\frac{4\pi}{3}\int_{t_i}^{t}\left[\frac{a(t')}{a(t)}\right]^3\Gamma(T')R(t',t)^3dt'\right\}, \tag{25}$$

where T' is the temperature at $t=t'$, t_i is the time at which the Universe reaches the critical temperature,

$$t_i\simeq \xi M_P/T_c^2, \tag{26}$$

where $\xi=\sqrt{90/32\pi^3 g_*}$, and $R(t',t)$ is the radius of a bubble that nucleated at time t' and expanded until the time t,

$$R(t',t)=R_c(T')\frac{a(t)}{a(t')}+\int_{t'}^{t}v_w(T'')\frac{a(t)}{a(t'')}dt''. \tag{27}$$

The factors of a in Eqs. (25) and (27) take into account the fact that the number density of nucleated bubbles is diluted, and the radius of a bubble enlarged, due to the expansion of the Universe from t' to t (see e.g. Ref. [49]). We can assume that this effect is negligible if the duration of the phase transition is small in comparison with the age of the Universe. As we have seen in the previous section, this is true when $L/\rho_b \ll 1$, which is the case we will consider. (This assumption will not hold, in principle, for the QCD phase transition.) The wall velocity v_w is determined by the equilibrium between the pressure difference $V(T)$ and the friction force exerted by the plasma. The latter is proportional to the wall velocity. The constant of proportionality is the friction coefficient η (see Sec. VI), so

$$v_w=-V(T)/\eta. \tag{28}$$

The exponent in Eq. (25) is minus the fraction of volume occupied by bubbles that nucleated between t_i and t, if we do not take into account overlapping of bubbles. At the beginning of nucleation the formula $f_b(t) \simeq (4\pi/3)\int\Gamma(T')R(t',t)^3dt'$ is correct.

Using again Eqs. (2) and (3), we obtain the analogous of Eq. (5),

$$f_b=\frac{1}{s_u(T)-s_b(T)}\left(s_u(T)-s_u(T_c)\frac{a_i^3}{a^3}\right), \tag{29}$$

but since we already have an equation for f_b, namely, Eq. (25), we use Eq. (29) to express T in terms of f_b and a,

$$T^3=\frac{V'(T)}{2\pi^2 g_*/45}f_b+\frac{T_c^3a_i^3}{a^3}. \tag{30}$$

Equation (30) has come across within an approach that differs from previous works, so it is worthwhile spending a few words discussing its physical meaning. We may follow for instance Ref. [5], and use energy (non)conservation in the following way. On one hand, we may write the total energy in a volume $V_U=V_u+V_b$ as $E=[\rho_u+\Delta\rho f_b]V_U=\rho V_U$, where $\Delta\rho=\rho_b-\rho_u$. If the Universe were not expanding, energy conservation during the phase transition would give

$$\dot{\rho}=\dot{\rho}_u+\Delta\dot{\rho}f_b+\Delta\rho\dot{f}_b=0. \tag{31}$$

This gives the rate $\dot{\rho}_u$ at which the plasma takes energy from the change of phase. On the other hand, when it is not undergoing a phase transition, the Universe takes energy from the plasma at a rate

$$\dot{\rho}_u=\dot{\rho}=-4\rho_u H. \tag{32}$$

If we join the two equations, we obtain the total rate of change of energy as

$$\dot{\rho}_u=-\Delta\dot{\rho}f_b-\Delta\rho\dot{f}_b-4\rho_u H, \tag{33}$$

from where we get an equation for $\dot{T}$. However, if the phase transition and the expansion of the Universe are taken into account at the same time, additional terms appear both in Eqs. (31) and (32). On one hand, there is a term of the form $\rho\dot{V}_U$ in $\dot{E}$, which produces a new term $-3H\rho$. This just accounts for energy dilution. On the other hand, the expansion of the Universe takes energy not only from radiation, since it is not the only component in the equation of state. Bearing in mind the two coexisting phases during the phase

[3]In many cases the phase transition occurs in a tiny range of temperature about T_c, so it is a good approximation to replace almost every quantity by its value at $T\simeq T_c$. The important exception are quantities such as S_3, that depend directly on the free energy difference $V(\phi,T)$, which varies drastically with T at the critical temperature [47].

transition, the rate at which energy conservation is violated is given by $dE = -p_u dV_u - p_b dV_b$. Using this to obtain $\dot{\rho}$, we finally find

$$\dot{\rho}_u = -\Delta\dot{\rho} f_b - \Delta\rho \dot{f}_b - 3H(\rho_u + p_u + \Delta\rho f_b + \Delta p f_b) - \Delta p \dot{f}_b, \tag{34}$$

where $\Delta p = p_b - p_u$. Since $\rho + p = Ts$, it can be seen that the first terms of this equation reproduce Eq. (33), but there are additional terms. Using $s = dp/dT$ and rearranging Eq. (34), we see that it is just the equation for entropy conservation,

$$\dot{s}_u + \Delta\dot{s} f_b + \Delta s \dot{f}_b = -3H(s_u + \Delta s f_b), \tag{35}$$

from which we may re-obtain the result (30). The discrepancies between Eq. (33) and Eq. (34) will not be important as long as the latent heat L is not significant and T remains close to T_c. Indeed, we can neglect the last two terms inside the parenthesis in Eq. (34), provided that $\Delta\rho \ll \rho$. The last term in Eq. (34) is responsible for the appearance of the entropy difference $\Delta s = -V'(T)$ as the factor of $\dot{f}_b$ in Eq. (35), instead of the energy difference $\Delta\rho = V(T) - TV'(T)$. Since $V(T_c) = 0$, the two quantities are related by $\Delta\rho \simeq T\Delta s$ for $T \simeq T_c$. Therefore, Eq. (33) gives a good approximation in the case $r \ll 1$. Still, the fact that in Eq. (30) the temperature is already integrated may constitute an advantage for analytical calculations.

Finally, the evolution of the scale factor is given by the Friedman equation,

$$H^2 = \frac{8\pi G}{3}[\rho_u(T) + \Delta\rho f_b]. \tag{36}$$

If $\Delta\rho \ll \rho$, then one can use the customary formula $H \propto T^2$ for the expansion rate. In fact, the variation of ρ_u is of the same order of $\Delta\rho$, so if we neglect $\Delta\rho$, to be consistent we should also set $H = \text{const} \propto T_c^2$ in this approximation. Again, this is reasonable if the duration of the phase transition is short enough.

In each particular model, Eqs. (25), (30), and (36) can be used to calculate numerically the evolution of the phase transition. In this paper, though, we will make analytical estimates of the parameters that are relevant for the cosmological consequences, such as the temperature and wall velocity in the different stages of the transition.

C. Bubble coalescence

When bubbles occupy more than 30% of space, they meet and percolate. This occurs when bubbles have a characteristic radius $R_0 \sim (0.3/n_b)^{1/3}$, where n_b is the number density of bubbles. Bubble coalescence provides a different mechanism of bubble growth, in which the driving force is surface tension instead of pressure difference. When bubbles collide and percolate, they arrange themselves into a system of fewer, larger bubbles in order to minimize the surface area. We can estimate the characteristic time of this process as follows [14]. Assume that when two spherical bubbles of radius R meet, they form a single bubble of radius $2^{1/3}R$. In this process, a mass of fluid $m \sim \rho R^3$ is moved a distance of order R. If this is done in a time t, the kinetic energy involved in the process is $K \sim m(R/t)^2$. This energy is supplied by the surface of the bubbles. The surface energy released in the process is $\Delta E \sim \sigma R^2$, so this occurs in a time $t \sim (\rho R^3/\sigma)^{1/2}$.

Once bubble coalescence begins, it will be the fastest mechanism of bubble growth if the rate t^{-1} is larger than the rate v_w/R given by the wall velocity (28). Therefore this process will dominate until bubbles reach a characteristic size $R_1 \sim \sigma/v_w^2\rho$. If $R_1 < R_0$, then it never dominates. Otherwise, during the period in which the radius varies from R_0 to R_1 bubble coalescence is the fastest process and must be taken into account.

The process of bubble coalescence may end in two different ways. If v_w is large enough, the radius R_1 is very close to R_0, and after a short period bubbles continue to grow with velocity v_w. If, on the contrary, v_w is very small, then coalescence dominates for a larger period, until the low-temperature phase occupies more than 50% of the total volume. At this moment the regions of high-temperature phase detach into isolated bubbles and the process stops. The interface velocity is again determined by pressure difference, friction, and latent heat release, so these bubbles shrink with velocity v_w. This occurs at a bubble radius $R_2 \sim (0.5/n_b)^{1/3}$. Notice that, although this expression is similar to that for R_0, in fact R_2 may be very different from R_0 since n_b may decrease significantly during the process due to the expansion of the Universe.

Finally, when bubbles of the high temperature phase are so small that surface energy becomes dominant over volume energy, the shrinking is accelerated until the symmetric-phase bubbles disappear or, eventually, until a topological defect is formed.

V. BUBBLE NUMBER AND INTERFACE VELOCITY

In the case of homogeneous nucleation, the Universe cools down to a temperature $T_N < T_c$ before bubble nucleation becomes appreciable, so the phase transition effectively starts at a time $t_N > t_i$. Then, the phase transition proceeds in two main steps. At the beginning bubbles nucleate with a rate $\Gamma(T_N)$, and expand with a velocity given by the friction coefficient η and the pressure difference $V(T_N)$. After a short time the energy released by the change of phase reheats the Universe up to a temperature T_r close to T_c. Then, a longer period begins, in which the two phases are close to equilibrium. How long is this period, and how close is T_r to T_c, depends on the latent heat. The free energy difference in this stage is $V(T_r) \simeq 0$, so the velocity of bubble expansion decreases and the bubble nucleation rate becomes extremely suppressed. Therefore, this second stage may be very similar to the inhomogeneous nucleation phase transition discussed in Sec. III.

In order to avoid numerical calculations, we will need some approximations for the fraction of volume f_b, the temperature T, and the expansion rate of the Universe H [formulas (25), (30), and (36)]. We notice that the latter cannot be affected by the phase transition until the energy released becomes appreciable. Furthermore, when this occurs, the plasma has reheated up to $T_r \simeq T_c$, and enters the phase equilibrium stage. We have seen in Sec. III that H is not modified significantly if the parameter r defined in Eq. (12) is small. So, we do not need to consider back reaction on the scale factor and we will assume that the evolution of the Universe does not depart from the standard relations $H \simeq 1/2t$, $a \propto t^{1/2}$. We will also need an approximation for the nucleation rate and free energy difference.

A. Thin wall approximation and linearization of *V*

If the width of the wall is much less than the radius of the bubble, we can neglect the second term in Eq. (24) to obtain the wall profile (this approximation is exact at the critical temperature, at which $R_c \to \infty$, and gives the usual kink profile). If we then multiply by $d\phi/dr$ and integrate using the boundary conditions $d\phi/dr = 0$ and $V = 0$ outside the bubble, we find that

$$\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 = V(\phi), \tag{37}$$

so, $d\phi/dr = -\sqrt{2V}$ since at the wall ϕ falls from ϕ_m to 0. Inserting this in Eq. (23) we obtain the free energy of the critical bubble in the thin wall approximation,

$$S_3 = \frac{4\pi}{3} R_c^3 V(T) + 4\pi R_c^2 \sigma(T), \tag{38}$$

where the free energy difference $V(T)$ is defined in Eq. (1), and $\sigma(T)$ is the surface tension of the bubble wall,

$$\sigma = \int \left(\frac{d\phi}{dr}\right)^2 dr = \int_0^{\phi_m} \sqrt{2V}\, d\phi. \tag{39}$$

Maximizing with respect to R_c we get the values of the critical radius and action,

$$R_c = -2\sigma/V,$$

$$S_3 = 16\pi\sigma^3/3V^2.$$

Since σ does not change significantly during the phase transition, it can be approximated by $\sigma(T_c)$.

Since the thin wall approximation is valid when T does not depart significantly from T_c, we can also make a linear approximation for the free energy difference,

$$V(T) \simeq L(T - T_c)/T_c. \tag{40}$$

So, the exponent in the nucleation rate (22) becomes

$$\frac{S_3(T)}{T} \simeq \frac{16\pi\sigma^3 T_c}{3L^2(T_c - T)^2}, \tag{41}$$

and the critical radius is

$$R_c \simeq 2\sigma T_c / L(T_c - T). \tag{42}$$

With the linear approximation (40), the wall velocity is given by

$$v_w = L(T_c - T)/\eta T_c. \tag{43}$$

B. First stage: Nucleation and reheating

Neglecting the effects of latent heat, the phase transition may be assumed to occur roughly at $t = t_N$, since it goes on for a very short interval $\delta t \ll t_N$ [47]. With the inclusion of latent heat the phase transition evolves during a longer period, that begins at t_N. In this case, we expect that the first part of the evolution, in which the Universe is reheated up to a temperature T_r, has a time scale of the same order of the time interval δt of the phase transition without latent heat. This is confirmed by numerical calculations [13]. The nucleation rate Γ vanishes at $T = T_c$, but it changes very quickly with temperature and becomes of order T^4 at $T = T_0$, where the barrier between the two minima of the free energy disappears. This is an extremely large rate, so it is impossible that the Universe supercools close to T_0 [47]. We will assume that the temperature T_N is close enough to T_c that the approximations of the previous subsection can be used.

In Ref. [43] the onset of nucleation was assumed to occur when the probability that a bubble was nucleated inside each causal volume is 1,

$$\int_{t_i}^{t_N} V_H \Gamma\, dt \sim 1. \tag{44}$$

The causal volume is given by $V_H = d_H^3$, where the horizon size d_H scales like the age of the Universe, $d_H \sim 2t$. The cosmological scale t, however, is in general too large in comparison with the scale of phase transition dynamics, which at $t = t_N$ is given by $t_N - t_i$. The scale of phase transition dynamics is roughly determined by the temperature variation during the phase transition, which is bounded by the difference $T_c - T_0$.

Consequently, it may be more appropriate to consider a different causal distance d_c, related to the dynamics of the phase transition in the following way. We may say that bubbles begin to "see" each other at a time t_N when their mean separation is of the order of the distance travelled by a sound wave since time t_i. Then, the causal distance is given by $d_c \sim c_s(t_N - t_i)$, where $c_s = 1/\sqrt{3}$ is the velocity of sound in the relativistic fluid. This defines a causal volume in terms of T_N

$$V_c = [c_s \xi M_P (1/T_N^2 - 1/T_c^2)]^3. \tag{45}$$

We remark that the real improvement in using Eq. (45) instead of V_H does not come from considering the velocity of sound, which is ~ 1, but from the fact that in many cases the time elapsed from t_i to t_N will be much less than the age of the Universe $t_i \simeq \xi M_P / T_c^2$. The volume V_c is thus suppressed with respect to V_H by a factor $[(T_i - T_N)/T]^3$. The nucleation time t_N calculated in this way is larger, since more bubbles need to be nucleated before they are separated by a distance at which they are causally connected to each other.

There will be a bubble in each volume V_c when

$$\int_{t_i}^{t_N} V_c T^4 e^{-S_3(T)/T} dt \sim 1. \tag{46}$$

To evaluate the integral in Eq. (46) we make use of the following approximation [43] (see also [30]). The three-dimensional action in Eq. (41) can be expanded about any temperature T_* in the form

$$\frac{S_3(T)}{T} = \frac{S_3(T_*)}{T_*}\frac{1}{(1-x)^2} = \frac{S_3(T_*)}{T_*}(1+2x+\cdots), \tag{47}$$

where $x=(T-T_*)/(T_c-T_*)$. Since the integrand in Eq. (46) is sharply peaked at T_N, we choose $T_*=T_N$ and use the expansion (47) to evaluate the integral. We find

$$S_3(T_N)/T_N \simeq 4\log\frac{2\xi M_P}{T_N} + 4\log\frac{T_c-T_N}{T_c} - \log\frac{S_3(T_N)}{T_N}, \tag{48}$$

which will be in general dominated by the first term. For temperatures T several orders of magnitude below M_P, we have $S_3(T_N)/T_N \gtrsim 100$ (it is e.g. ~ 140 for the electroweak scale, and ~ 180 for the QCD scale). From Eq. (41) we obtain

$$\left(\frac{T_c}{T_c-T_N}\right)^2 \simeq \frac{3L^2T_c}{16\pi\sigma^3}\left(4\log\frac{2\xi M_P}{T_c} + \log\frac{3L^2T_c}{8\pi\sigma^3} + 6\log\frac{T_c-T_N}{T_c}\right), \tag{49}$$

where we have used that $T_N \simeq T_c$. In some of the following estimations we will replace T_c-T_N by T_c-T_0 inside the logs, since both differences are generally of the same order.

Immediately after $t=t_N$, the temperature increases at a rate which is given by Eq. (34). Under the current approximations we can write $\dot\rho = L\dot f_b - 4\rho H$. Therefore, the rate of change of energy of the plasma is $\dot\rho/\rho \simeq (3r/4)\dot f_b - 4H$. Thus, on one hand energy is increased at a rate $\sim r\,\delta f_1/\delta t_1$, where δf_1 is the fraction of volume converted to the low-T phase during the reheating stage, and δt_1 is the duration of this stage. The rate of energy decrease, on the other hand, is given by $H \sim 1/t$. Since δt_1 is much shorter than the age of the Universe, the increasing rate is much larger than the latter. The total change in energy density is thus $\delta\rho \sim L\,\delta f_1$.

The temperature cannot increase beyond T_c, so if $L \gtrsim \rho(T_c)-\rho(T_N)$, the fraction δf_1 will be less than 1. This means that before the phase transition completes, a final temperature T_r very close to T_c is reached, and the phase transition proceeds more slowly until $\delta f = 1$. For $L<\rho(T_c)-\rho(T_N)$ we have a variation $\delta\rho \sim L$ during the phase transition, which gives us an idea of the reheating that occurs. There may be significant reheating and a considerable variation of the wall velocity, but there will not be a long phase equilibrium stage, since in this case $\delta f_1 \simeq 1$. The case $L \ll \rho(T_c)-\rho(T_N)$ corresponds to weakly first-order phase transitions, which most likely occur by spinodal decomposition rather than by nucleation and expansion of bubbles [47,50]. We could also have $L \gg \rho(T_c)-\rho(T_N)$, although we are assuming $L \ll \rho(T_c)$. In this case we have $\delta f_1 \ll 1$, which means that most of the phase transition happens with the two phases near equilibrium, and we can apply the analysis of Sec. III. In this section we thus concentrate in the case in which the latent heat is comparable with the energy difference $\rho(T_c)-\rho(T_N)$.

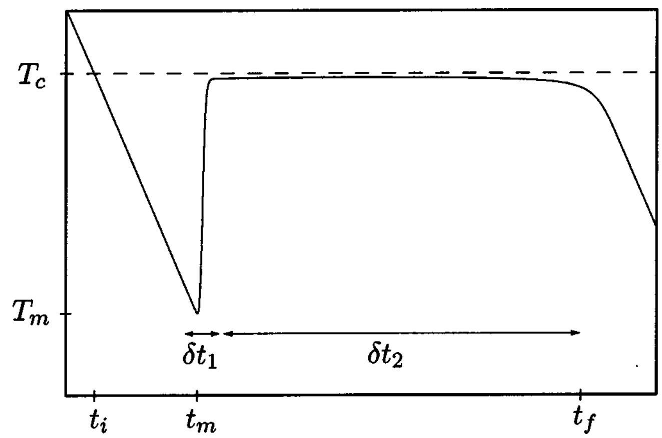

FIG. 3. Typical evolution of the temperature during a phase transition with supercooling.

The expansion of bubbles is governed by Eq. (27). At $T=T_N$ bubbles nucleate with a radius R_c given by Eq. (42), and after a time δt the radius has increased an amount $v_w\delta t$, with v_w given by Eq. (43). The ratio of the two distances is

$$\frac{v_w\delta t}{R_c} \simeq \frac{L^2(T_c-T_N)^2\delta t}{2\eta\sigma T_c^2}. \tag{50}$$

For the time scale of the phase transition dynamics, $\delta t \sim \xi M_P(T_c-T_0)/T^3$, we have

$$\frac{v_w\delta t}{R_c} \sim \frac{L^2\xi}{\eta\sigma T}\frac{M_P}{T}\left(\frac{T_c-T_0}{T}\right)^3. \tag{51}$$

Thus, if the phase transition takes place at a temperature sufficiently below the plank scale, the bubbles will grow so rapidly that we can safely neglect the initial radius R_c in Eq. (27).

At the beginning, bubbles expand with constant velocity $v_w(T_N)$. When reheating becomes important, the bubble expansion slows down and we enter the second stage. Consequently, during the first stage $\dot f_b$ can be roughly estimated without taking into account the liberation of latent heat. We can thus estimate the rate $\dot T/T \sim r\dot f_b$ without considering the back-reaction of the reheating on $\dot f_b$. It is only at the beginning and at the end of reheating that $r\dot f_b \sim H$ and $\dot T \simeq 0$. Indeed, soon after $t=t_N$, the temperature takes its minimum value $T_m \lesssim T_N$ (see Fig. 3), then it increases to $T_r \lesssim T_c$. The value of T_m is important because the nucleation rate turns on at $t \simeq t_N$, is maximal at $t=t_m$, and turns off again when the temperature has increased back to T_N. All the process occurs in a time $\delta t_\Gamma \sim (t_m - t_N)$, which is determined by the speed at which temperature changes. This interval δt_Γ in which Γ is not negligible is much less than δt_1, due to the exponential dependence of Γ on $T_c - T$.

Accordingly, the temperature T_m occurs when

$$r 4\pi v_w(T_m) \int_{t_i}^{t_m} \Gamma(T) v_w^2(T) (t_m - t)^2 dt \simeq H, \tag{52}$$

where the time-temperature relation $t = \xi M_P / T^2$ must be used to evaluate the integral. Using the expansion (47) about $T_* = T_m$, we find

$$\left[\frac{S_3(T_m)}{T_m}\right]^{-1} e^{-S_3(T_m)/T_m} \simeq \frac{2\pi\sigma^6 \eta^3 T_c^2}{9\xi^6 L^8} \left(\frac{T_c}{M_P}\right)^4 \left(\frac{T_c}{T_c - T_m}\right)^{10}. \tag{53}$$

The number density of bubbles is given by

$$n_b = \int_{t_c}^{t_f} \Gamma(t) dt. \tag{54}$$

Since $\Gamma(t)$ is sharply peaked at t_m, we can estimate n_b as $2\int_{t_c}^{t_m} \Gamma(t) dt$ and use again the dependence $t \propto T^{-2}$. The integral can be calculated again using the expansion (47), which gives

$$n_b \sim T_c^3 \frac{\xi M_P}{T_c} \frac{T_c - T_m}{T_c} \left[\frac{S_3(T_m)}{T_m}\right]^{-1} e^{-S_3(T_m)/T_m}. \tag{55}$$

We may define the interval δt_Γ by writing $n_b \simeq \Gamma(t_m) \delta t_\Gamma$. Then we can easily see in Eq. (55) that

$$\delta t_\Gamma \sim [S_3(T_m)/T_m]^{-1} (t_m - t_i). \tag{56}$$

Therefore, $t_m - t_N$ is much less than $t_m - t_i$ (most typically, about two orders of magnitude[4]). Using the result (53), with $T_m \simeq T_N$, we finally obtain the density of bubbles,

$$n_b \sim \frac{2\pi\sigma^6 \eta^3 T_c^2}{9\xi^5 L^8} \left(\frac{T_c}{M_P}\right)^3 \left(\frac{T}{T_c - T_N}\right)^9 T_c^3. \tag{57}$$

The bubbles thus nucleate during the short time δt_Γ about $t = t_m$, and expand with a velocity $v_1 \simeq v_w(t_N)$ for a time δt_1, until the temperature gets close to T_c. According to Eqs. (43) and (49),

$$v_1 \simeq \frac{1}{\eta} \left(\frac{16\pi\sigma^3}{3T_c}\right)^{1/2} K^{-1/2}, \tag{58}$$

where K is a shorthand for the sum of logs in Eq. (49). Notice that in Eq. (43), $v_w \propto L(T_c - T)$, but also $T_c - T_N \propto L^{-1}$, so v_w only depends on L through the logs in K.

[4]This shows that it is more accurate to estimate the difference $t_m - t_N \sim \delta t_\Gamma$ in this way, rather than subtracting the values given by Eqs. (49) and (53).

If a non-negligible fraction of the volume is taken up by bubbles during the first stage, the interval δt_1 can be estimated from $(4\pi/3) v_w^3(t_N) \delta t_1^3 n_b \sim 1$. This gives

$$\delta t_1 \sim \left(\frac{L^4}{\sigma^3 T_c^7}\right)^{1/6} \left(\frac{S_3(T_m)}{T_m}\right)^{-1/2} (t_N - t_i). \tag{59}$$

So, apart from a model-dependent factor, we find a general tendency to the relations $\delta t_\Gamma \ll \delta t_1 < t_N - t_i$. The value of $S_3(T_m)/T_m$ (and that of T_m) can be obtained similarly to the case of T_N. It is interesting to note that δt_1 has only a logarithmic dependence on the friction coefficient η. This is because the dependence on the wall velocity is twofold. On one hand, the lower the wall velocity, the longer the time δt_1 needed to reheat the plasma. But on the other hand, the lower the wall velocity, the longer will also be the time δt_Γ in which bubbles are formed, and the larger their number. This causes a shorter δt_1, since there are more bubbles to produce the reheating.

C. Second stage: Phase equilibrium

1. Inhomogeneous nucleation

If the formation of bubbles is associated with the presence of impurities, the phase transition occurs at $T \simeq T_c$, and the number density of bubbles n_b is an external parameter that depends on the density of impurities. According to Eqs. (15)–(17), for small r the evolution of the phase transition is given by

$$f_b \simeq 3Hr^{-1}(t - t_i), \tag{60}$$

and the rate at which the phase transition goes on is $\dot{f}_b = 3H/r$, i.e., a factor of $1/r$ larger than the rate of expansion $3H$ of a comoving volume. Our assumption $r \ll 1$ implies that $\dot{f}_b \gg H$ and $\delta t \ll t$.

To calculate the velocity of the interfaces, we assume that all the bubbles begin to expand at $t = t_i$. Thus, the fraction of volume occupied by bubbles is $f_b = n_b (4\pi/3) R_b^3$, where $R_b(t)$ is the bubble radius. At the midpoint of the transition we have $\dot{f}_b \sim 4\pi n_b \overline{R}^2 v_w$, where $\overline{R} = (4\pi n_b/3)^{-1/3}$ is the average radius. Therefore the mean velocity is given by $v_w = (4\pi n_b/3)^{-1/3} H/r$. We notice, however, that even in this case in which $\dot{f}_b$ is constant, the wall velocity may change significantly during the transition, since $v_w \propto R_b^{-2}$. The total variation of v_w depends on n_b.

2. Homogeneous nucleation

In the case of a phase transition with supercooling, the situation is very similar after the plasma has reheated up to a temperature $T_r \simeq T_c$. The transition proceeds at a rate

$$\dot{f}_b \sim 4\pi v_w(t) \int_{t_i}^{t} \Gamma(t') R^2(t', t) dt'. \tag{61}$$

In this stage the nucleation rate has turned off. We have seen that Γ peaks sharply at a certain time t_m in the previous stage, so we can write Eq. (61) as $\dot{f}_b \sim 4\pi v_2 n_b R^2(t_m,t)$, where $v_2 \equiv v_w(T_r)$, n_b is given by Eq. (57), and $R(t_m,t) \simeq v_2(t-t_m)$. In any case, the integral in Eq. (61) is an average of the squared radius of the bubbles, and for the present estimations we can set $\dot{f}_b \sim 4\pi v_2 n_b \overline{R^2}$, with $\overline{R}$ given by n_b, just as in the case of inhomogeneous nucleation. Since the temperature is almost constant, $L\dot{f}_b \simeq 4\rho H$, i.e., all the released latent heat is taken away by the expansion of the Universe. Thus, again $\dot{f}_b = 3H/r$ and the velocity coincides with that of the inhomogeneous nucleation case

$$v_2 \simeq \left(\frac{3}{4\pi n_b}\right)^{1/3} \frac{H}{r}. \tag{62}$$

Although in Eq. (62) it seems that the wall velocity during phase equilibrium does not depend on the friction, in fact it is proportional to η^{-1} due to the dependence of n_b.

Since $\dot{f}_b \sim H/r$, the duration of this stage is

$$\delta t_2 \sim r H^{-1}. \tag{63}$$

The reheating temperature T_r must be such that the pressure difference is adjusted so as to give the velocity (62). Using Eq. (43), we find

$$\frac{T_c - T_r}{T_c} = \frac{\eta H}{rL}\left(\frac{4\pi n_b}{3}\right)^{-1/3}. \tag{64}$$

As expected, the larger the latent heat, the closer will be T_r to T_c. Unlike δt_2, the values of T_r and v_2 are not easy to determine, since they depend on the number of bubbles that nucleated in the previous stage. Using Eq. (57) and taking into account that $HM_P/T^2 \sim 1$, we find that roughly

$$T_c - T_r \sim \left(\frac{T_c - T_N}{T_c}\right)^2 (T_c - T_N), \tag{65}$$

which confirms that generally, $T_c - T_r \ll T_c - T_N$. According to Eq. (43), the same relation holds for the velocities v_1 and v_2.

D. Coalescence

In the range $0.3 \lesssim f_b \lesssim 0.5$ bubble percolation takes place. We have seen that this process gives a contribution to the bubble expansion rate, of order $(\sigma/\rho R^3)^{1/2}$. For $f_b \sim 1$ this rate is

$$\dot{f}_{\mathrm{coalescence}} \sim (\sigma/\rho n_b)^{1/2}. \tag{66}$$

To establish the importance of this rate we should compare it with the rates δt_1^{-1} or δt_2^{-1}, corresponding to the two stages we have studied. Such a comparison is difficult to carry out without specifying a model, so we will ignore this effect in the subsequent discussions.

Although coalescence is bounded to occur in the above range of f_b, it could have important consequences if the associated bubble growth rate is significantly larger than those given by the time scales δt_1 and δt_2. In a specific model, comparison of Eqs. (59) and (63) with Eq. (66) should not be hard to do, once n_b has been evaluated.

VI. THERMODYNAMICAL PARAMETERS

We have seen that all the parameters that describe the dynamics of the phase transition (i.e. δt_1, δt_2, etc.) depend on a few thermodynamical parameters, such as the latent heat or the friction coefficient. The formation of the different cosmological products of a phase transition thus depends on these quantities, and also on other parameters, such as the conductivity of the plasma. These quantities are physically related, since all of them come from the equilibrium or nonequilibrium thermodynamics of the same underlying theory. This should be taken into account in phase transition calculations, when ranges of parameters are considered. Unfortunately, it is hard in practice to establish general relations between these physical parameters. Depending on the theory, it may even be impossible to compute some of these quantities.

One can gain some insight on the relations between thermodynamical quantities by conveniently modeling the free energy. The problem is further simplified by referring to the general form of the perturbative effective potential. In that case the thermodynamical quantities can be related to the parameters of the microscopic theory. We dedicate this section and the following to study the aforementioned relations. We will concentrate only in those parameters which influence directly the dynamics of the phase transition. An analysis of other parameters that affect the generation of cosmological remnants is considered in Ref. [7].

A. Free energy and viscosity

We assume the free energy density takes the form

$$\mathcal{F} = -\frac{\pi^2}{90} g_* T^4 + V(\phi,T) + L, \tag{67}$$

where the scalar field ϕ is the order parameter, and

$$V(\phi,T) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{\lambda}{4}\phi^4 \tag{68}$$

is the free energy density difference between the symmetric and the broken-symmetry phases. The parameter g_* is the number of light species of the plasma. In general, g_* depends on temperature, but it is usually approximated by

$$g_* = \sum_{\mathrm{bosons}} g_i + \frac{7}{8} \sum_{\mathrm{fermions}} g_i, \tag{69}$$

where the sums are on particles with masses $m_i < T$, and g_i is the number of degrees of freedom of species i (see Appendix A).

It depends on each particular case whether Eqs. (67) and (68) can be derived from the microscopic theory. At any rate, they can be regarded just as a simple model for studying the dynamics of the phase transition, being the latter first-order if the coefficient E is nonvanishing. The parameters of $V(\phi,T)$

can be chosen in such a way that the free energy carries the thermodynamical properties of the theory we wish to study [4,5,13]. For instance, these parameters determine the values of the critical temperature, latent heat, surface tension, and correlation length. The thermodynamical parameters could be obtained, e.g., with lattice simulations (see for example Ref. [51]). Then one can use those values to calculate the parameters T_0, D, E and λ. Furthermore, in general the order parameter ϕ is a Higgs field or a combination of Higgs fields, and $V(\phi,T)$ is the finite-temperature effective potential (see e.g. Ref. [6]). We will consider this case in the next section.

The effect of viscosity on the propagation of the bubble wall is calculated by considering its equation of motion in the hot plasma,

$$\Box\phi+V'(\phi)+\sum_i g_i\frac{dm_i^2}{d\phi}\int\frac{d^3p}{(2\pi)^3 2E_i}f_i(k,x)=0, \tag{70}$$

which can be derived by energy conservation considerations [2,10,44,52]. Here $V(\phi)$ is the zero temperature effective potential, the sum is over all particles that couple to ϕ, m_i are the ϕ-dependent masses (see the Appendixes), and f_i are the phase space population densities. This equation can be obtained by thermally averaging the operator equation for ϕ. If we separate f into the equilibrium population f_0 plus a small deviation δf, we obtain the equation

$$\Box\phi+V'(\phi,T)+\sum g_i\frac{dm_i^2}{d\phi}\int\frac{d^3p}{(2\pi)^3 2E_i}\delta f_i=0, \tag{71}$$

where $V(\phi,T)$ is the finite temperature effective potential, given by Eq. (68). Since the departure from equilibrium is proportional to the velocity of the bubble wall, it is the last term in Eq. (71) which gives the friction force of the plasma.

A simple approach to the calculation of the wall velocity [4,5] consists in replacing the last term in Eq. (71) with a typical damping term of the form $d\phi/dt$. Due to Lorentz invariance this term must be in fact of the form $u^\mu\partial_\mu\phi$, where u_μ is the four-velocity of the plasma. Equation (71) then may be written as

$$\Box\phi+V'+(\tilde{\eta}T)u^\mu\partial_\mu\phi=0, \tag{72}$$

where $\tilde{\eta}$ is a dimensionless damping coefficient that depends on the viscosity of the medium. Boosting to a frame that moves with the wall, and assuming stationary and nonrelativistic motion in the z direction, we have

$$\phi''=V'(\phi)-\tilde{\eta}Tv_w\phi', \tag{73}$$

where $\phi'\equiv d\phi/dz$. Multiplying both sides by ϕ' and integrating over $-\infty<x<\infty$ we obtain

$$\tilde{\eta}T\sigma v_w=V(T), \tag{74}$$

where $V(T)$ is the free energy difference between the two phases, defined in Eq. (1), and σ is the surface tension of the wall, given by Eq. (39). We have assumed here that temperature is constant across the wall. This is right if the wall velocity is small enough, so that the latent heat it releases has time to be uniformly distributed throughout space.

Hence, the pressure difference is equilibrated by a friction force proportional to the wall velocity. The constant of proportionality is the friction coefficient $\eta=\tilde{\eta}T\sigma$. Since the tension of the wall is related to the wall width L_w by $\sigma\simeq\phi_m^2/L_w$ [see the first integral in Eq. (39)], we can also write $\eta=\tilde{\eta}T\phi_m^2/L_w$.

A shortage of modeling the viscosity of the plasma in this way is that $\tilde{\eta}$ is a free parameter. The correct expression for η can be derived from Eq. (71). In Appendix B we show that particles with a thermal distribution give a friction coefficient

$$\eta_{\rm th}\simeq\tilde{\eta}_{\rm th}\frac{\phi_m^2}{T}\sigma, \tag{75}$$

while the contribution of infrared gauge bosons is

$$\eta_{\rm ir}\simeq\tilde{\eta}_{\rm ir}\frac{T^3}{L_w}. \tag{76}$$

Evidently, both formulas agree with the above result if $\phi_m\sim T$. This treatment also allows for the evaluation of the coefficients $\tilde{\eta}$, which depend only on the particle content of the plasma.

B. Thermodynamical quantities and phase transition dynamics

The free energy given by Eqs. (67) and (68) bears a first-order phase transition, with two minima separated by a barrier. The critical temperature is related to T_0 by

$$\frac{T_c^2-T_0^2}{T_c^2}=\frac{E^2}{\lambda D}. \tag{77}$$

At $T>T_c$ the global minimum of the potential is $\phi=0$. At the critical temperature the two minima become degenerate, and below this temperature the stable minimum is

$$\phi_m(T)=\frac{3ET}{2\lambda}\left[1+\sqrt{1-\frac{8}{9}\frac{\lambda D}{E^2}\left(1-\frac{T_0^2}{T^2}\right)}\right]. \tag{78}$$

At $T=T_0$ the barrier between minima disappears and $\phi=0$ becomes a maximum of the potential. Therefore the phase transition occurs at some stage in between T_c and T_0. The value $\phi=0$ corresponds to the symmetric, high-temperature phase, and $\phi\neq0$ corresponds to the broken-symmetry, low-temperature phase. The jump of the order parameter from the high temperature phase to the low temperature one is thus

$$\phi_m(T_c)=2ET_c/\lambda. \tag{79}$$

According to Eqs. (67) and (68), the free energy density of the symmetric phase is

$$\mathcal{F}_u = -\frac{\pi^2}{90} g_* T^4 + L. \tag{80}$$

This gives the equation of state of a hot relativistic plasma with a positive cosmological constant

$$\rho_u = \frac{\pi^2}{30} g_* T^4 + L, \quad p_u = \rho_u/3 - 4L/3. \tag{81}$$

The free energy of the broken-symmetry phase is $\mathcal{F} = \mathcal{F}_u + V(T)$. The energy density of the broken-symmetry phase is $\rho_b = \rho_u + \Delta\rho$, with

$$\Delta\rho = V(T) - TV'(T). \tag{82}$$

The entropy density of the symmetric phase is $s_u = 2\pi^2 g_* T^4/45$, and that of the broken-symmetry phase is $s_b = s_u - V'(T)$. The latent heat of the phase transition is given by $L = \Delta\rho(T_c) = T_c \Delta s(T_c)$; hence,

$$L = 8D\left(\frac{E}{\lambda}\right)^2 T_c^2 T_0^2. \tag{83}$$

Comparing with Eq. (79), we find the relation $L = 2D\phi_m^2 T_0^2$ between the discontinuity of the order parameter and that of the energy density. As expected, strongly first-order phase transitions (i.e., with large ϕ_m) have large latent heat.

The surface tension of the bubble wall is given by Eq. (39) in the thin wall approximation. At the critical temperature the effective potential is given by

$$V(\phi, T_c) = \frac{4(ET_c)^4}{\lambda^3} x^2 (1-x)^2, \tag{84}$$

where $x \equiv \lambda\phi/2ET_c = \phi/\phi_m$. Hence, Eq. (39) is easily integrated and

$$\sigma(T_c) = \frac{2\sqrt{2}E^3}{3\lambda^{5/2}} T_c^3. \tag{85}$$

Although we have not used it explicitly, in this approximation the field configuration near the wall can be solved analytically with the help of Eq. (84), and gives the kink profile,

$$\phi(z) = \frac{\phi_m}{2}\left(1 + \tanh\frac{z}{L_w}\right), \tag{86}$$

where

$$L_w = \phi_m^2/3\sigma \tag{87}$$

is the wall width.[5]

[5] L_w may change during the bubble expansion due to the friction with the plasma [2]. We shall neglect this effect.

Using Eqs. (79), (85), and (87) we find the values of the friction coefficients (75) and (76),

$$\eta_{\rm th} = \frac{8\sqrt{2}}{3} \frac{E^5 \tilde{\eta}_{\rm th}}{\lambda^{9/2}} T_c^4,$$

$$\eta_{\rm ir} = \frac{E\tilde{\eta}_{\rm ir}}{\sqrt{2\lambda}} T_c^4. \tag{88}$$

The two contributions have different parametrical dependence, so each will dominate in different regions of parameter space. For instance, if $E \ll \lambda$ the infrared boson contribution may be much larger than that of thermal particles. The maximum velocity of bubble walls occurs at $T \simeq T_N$. According to Eqs. (58) and (88), this velocity is the smallest among

$$v_{\rm th} \sim \frac{\lambda^{3/4}}{E^{1/2}\tilde{\eta}_{\rm th}} K^{-1/2}, \quad v_{\rm ir} \sim \frac{E^{7/2}}{\lambda^{13/4}\tilde{\eta}_{\rm ir}} K^{-1/2}. \tag{89}$$

To determine which one is the correct, it is necessary to know the relations between the coefficients E, λ, and $\tilde{\eta}$. We see that in the opposite limiting cases $E \ll \lambda$ and $E \gg \lambda$, one of the two velocities is $\ll 1$, unless $\tilde{\eta}$ is too small. In the case $E \sim \lambda$, the wall velocity will be small if one of the conditions $E^{1/4} \ll \tilde{\eta}_{\rm th}$ or $E^{1/4} \ll \tilde{\eta}_{\rm ir}$ is fulfilled, which does not seem unlikely in general (see the next section). This supports the assumption of nonrelativistic wall velocities.

It is evident that with the aid of the model (67), (68) we can get more information about the generalities of phase transition dynamics. For instance, if we write Eq. (49) as a function of E, D, and λ, and compare with Eq. (77), then we can locate the nucleation temperature in the interval $T_c - T_0$,

$$T_c - T_N \sim \frac{E^{1/2}}{\lambda^{3/4}} K^{-1/2} (T_c - T_0). \tag{90}$$

If E and λ are comparable, this gives a value of $T_c - T_N$ roughly an order of magnitude less than $T_c - T_0$.

The relations between the different quantities that determine the dynamics of the phase transition are apparent in the above expressions. Specific relations will be of interest for different cosmological consequences. As an example, let us consider the effect of modifying the theory in order to obtain a more strongly first-order phase transition. To do that, we have to enlarge the value of the order parameter. Assume we accomplish this by increasing the value of the parameter E and keeping the other parameters invariant [see Eq. (79)]. Then, there will be more supercooling, and one expects a larger departure from thermal equilibrium, since the pressure difference at $T = T_N$ will be larger. However, according to Eqs. (83) and (88), L and η also increase. This tends to decrease the wall velocity in the two stages of the phase transition, in opposition to the effect of supercooling.

In this work we assume for simplicity that g_* remains constant throughout the phase transition. In fact, the number of effectively massless degrees of freedom may change during the phase transition. It is conceivable that some particles acquire large masses and decouple from the thermal bath; then $\Delta g_* \equiv g_{*u} - g_{*b} > 0$. For instance, during the quark-hadron phase transition g_* changes substantially. It is interesting to note that such a change may affect considerably the dynamics of the phase transition, even in the case $\Delta g_* \ll g_*$. The effect of a decrease of g_* during the phase transition is twofold. To begin with, the free energy of the broken-symmetry phase is larger than in the case of constant g_*, so the critical temperature is lower [it is given by $V(T_c) = -\pi^2 \Delta g_* T_c^4/90$ [32,41]]. Therefore the phase transition is stronger,[6] and the latent heat $T_c V'(T_c)$ is larger. In addition, the entropy released by the decoupling species gives an extra contribution of $4\pi^2 \Delta g_* T_c^4/90$ to the latent heat. This contribution is comparable to the value of L as given by Eq. (83), if $\Delta g_* \gtrsim D(E/\lambda)^2(1-E^2/\lambda D)$. In the case of a perturbative effective potential, this condition may be easily fulfilled for $\Delta g_* \sim 1$.

VII. PHYSICAL QUANTITIES IN PERTURBATION THEORY

If perturbation theory is applicable, the one-loop effective potential at high temperature often has the form of Eq. (68), with parameters generally given by

$$D = \sum_{\text{bosons}} \frac{g_i h_i^2}{24} + \sum_{\text{fermions}} \frac{g_i h_i^2}{48},$$

$$T_0^2 = \frac{1}{D} \frac{m_h^2}{4},$$

$$E = \frac{2}{3} \sum_{\substack{\text{gauge}\\ \text{bosons}}} \frac{g_i h_i^3}{12\pi},$$

$$\lambda = m_h^2/2v^2. \tag{91}$$

Here, h_i are the couplings of the particles with ϕ, m_h is the Higgs mass, and v its zero temperature VEV. The coefficient E in general involves only gauge bosons. In Appendix A we review the derivation of these results and discuss on the general assumptions and approximations that lead to Eqs. (68), (69), and (91). In the discussions that follow we will sometimes take the electroweak theory as a reference point. The parameter T_0 gives the temperature scale of the phase transition. Its order of magnitude is determined by m_h, so it may be quite less than the scale v if λ is small. Anyway, for the dynamics of the phase transition, the difference $T_c - T_0$ is more important than the temperature scale T_0.

Regarding the viscosity of the plasma, we show in Appendix B that the contribution of thermal particles to the parameter $\tilde{\eta}$ is given by

$$\tilde{\eta}_{\text{th}} \simeq \sum 3 \left(\frac{\log \chi_i}{2\pi^2} \right)^2 g_i h_i^4 \tag{92}$$

where $\chi_i = 2$ for fermions and $\chi_i = h_i^{-1}$ for bosons. Therefore the contributions of bosons to η have an enhancement of $(\log h_i^{-1}/\log 2)^2$ with respect to fermions with the same Yukawa coupling. For instance, for $h \sim 0.1$ the boson enhancement is ~ 10. This means that friction may be much stronger in supersymmetric theories than in nonsupersymmetric ones. For instance, it was found in Ref. [3] that a light stop may slow down the electroweak bubble wall in the minimal supersymmetric standard model (MSSM) an order of magnitude with respect to the SM. The enhancement is larger for lighter particles, but these do not contribute to the friction due to the h_i^4 dependence.

The contribution of infrared gauge bosons is

$$\tilde{\eta}_{\text{ir}} \simeq \sum \frac{g_b \bar{g} h_b^2}{32\pi} \log[m_b(\phi_m) L_w]. \tag{93}$$

Here, the sum is only on gauge bosons, but the coefficient $\bar{g}$ also involves a sum over particle species (see Appendix B). Furthermore, the gauge coupling appears only squared, which means less suppression. The log enhancement in this case is $\simeq \log(h_b \phi_m^3/\sigma) \sim (\log h_b \lambda^{-1/2})$.

It is important to compare the value of the parameter E with the other parameters, since E is responsible for the first-order nature of the phase transition. We can see in the formulas of the previous section that all the thermodynamical quantities are proportional to some power of E, while the parameters D and λ usually appear in the denominators. In the perturbative approach (91), this parameter is generally smaller than the others. This is because E is a sum of gauge couplings to the third power, weighted with gauge boson degrees of freedom, while D involves squared couplings, and the sum is over all degrees of freedom. Regarding λ, it can be comparable to E, but this constrains the value of the Higgs mass.

The smallness of E indicates a tendency of perturbative effective potentials to give weakly first-order phase transitions.[7] This is apparent in the dependence of the order parameter, $\phi_m/T \sim E/\lambda$, or in the temperature interval in which the first-order phase transition can occur, $(T_c - T_0)/T_c \sim E^2/\lambda D$. For example, in the case of the electroweak phase transition, we have $E \sim 10^{-3}$ and $D \sim 10^{-1}$ for the minimal standard model. If we take a nonrealistic value for $\lambda \sim E$ to get an order parameter of order T, we find a temperature range $T_c - T_0 \sim 10^{-2} T_c$. In the specific case of the electroweak phase transition, a small value of the Higgs field $\phi_m(T)$ is undesirable for electroweak baryogenesis. In general, if this parameter is too small, the perturbative approximation breaks down (see Appendix A). In the electroweak case, the way out is to consider extensions of the SM which provide additional bosons that contribute to the parameter E [53,54].

[6]This could be important for baryogenesis [7].

[7]We are only discussing one-loop order here. Things may be different at two loops [42].

In the previous section we found that if we increase the parameter E while keeping the others constant, then we get a stronger phase transition and larger supercooling, but also larger values of η and L, which slow down the dynamics. It is evident that if E is augmented by adding particles to the theory, the value of D enlarges too, giving an additional increase of the latent heat L. If we add only a boson, the relative change will be more appreciable in E, because there are only a few terms in its expression, but if the boson comes together with several new species (as in the case of supersymmetry), then the change of D will be much more substantial. According to Eqs. (92) and (93), the friction coefficient will also increase significantly when adding bosons to the theory.

If $E^2/\lambda D \ll 1$, then $T_c \simeq T_0$, and $L/T_c^4 \simeq 8D(E/\lambda)^2$. It is interesting to compare the value of L with that of $\delta\rho \equiv \rho(T_c) - \rho(T_N)$, to assess the effect of reheating, as discussed in Sec. V. Using Eqs. (90) and (77), we may write

$$\delta\rho \simeq 4\frac{T_c - T_N}{T_c} \sim K^{-1/2}\frac{E^{5/2}}{D\lambda^{7/4}}. \tag{94}$$

Therefore,

$$\frac{L}{\delta\rho} \sim \left(\frac{30K^{1/2}}{\pi^2 g_*}\right)\left(\frac{D^2}{E^{1/2}\lambda^{1/4}}\right). \tag{95}$$

The first factor is likely of order 1 and depends essentially on the energy scale of the transition. The second factor is determined by the dynamics. It depends mainly on D, and may vary considerably if we change the particle content of the theory. Exemplifying again with the electroweak theory, D can vary from $\sim 10^{-1}$ in the SM to $D > 1$ in the MSSM, so we pass from little reheating in the first case to large reheating in the latter. We remark that things may be quite different if Eqs. (91) are not valid. For instance, in the case of the quark-hadron phase transition L and $\delta\rho$ are typically of the order of the energy density ρ.

VIII. CONCLUSIONS

In this paper we have performed an entirely analytical study of first-order phase transitions in the radiation-dominated era. We have seen that typically the high-temperature phase is supercooled to a temperature T_N, after which the transition proceeds in two steps, as sketched in Fig. 3. The first stage is complex, and some rough approximations must be made for an analytical treatment. Nevertheless, it can be checked with numerical results (e.g., Ref. [13]), that the orders of magnitude are correct. The second stage is much more simple, since bubble nucleation has effectively stopped and bubbles expand very slowly. This stage develops very close to the critical temperature, with almost zero pressure difference between the two phases. Therefore, this part of the evolution is similar to the case of inhomogeneous nucleation, in which the presence of impurities induces bubble nucleations without need of supercooling.

We have studied the case of a phase transition at phase equilibrium in some detail, taking advantage of the fact that it can be solved analytically for any value of the relevant parameter $r = L/T_c s(T_c) \sim L/\rho(T_c)$. This approach has allowed us to calculate the back reaction on the expansion rate H, which is important for large r. It is well known that supercooling may lead to exponential expansion of the Universe [32,40]. This possibility has been considered not only in the context of inflationary models, but also for the quark-hadron phase transition [49]. Although our approximations break down for $L \gtrsim \rho(T_c)$, we observe the manifestation of the energy of the false vacuum for large L. Even if the phase transition begins at $T = T_c$, when L is comparable to the energy density of the plasma the transition may take a long time to complete due to vacuum energy dominance.

For the more probable case of a phase transition with variation of temperature, we have given a derivation of the integro-differential equations that govern the dynamics. In particular, we have found a simple algebraic relation between the temperature, the fraction of volume occupied by the low-temperature phase, and the scale factor of the Universe, which holds under the usual assumption of adiabatic expansion. Using the thin wall approximation and the linearized form of the free energy difference, we have found analytical formulas for all the quantities that characterize the dynamics of the phase transition and may be relevant for the determination of its cosmological consequences. These parameters are the durations δt_1 and δt_2 of the two stages of the transition, the wall velocities v_1 and v_2 at each stage, the total number density of bubbles, the time interval δt_Γ in which bubble nucleation is active, etc. We have expressed these quantities in terms of those that determine the dynamics, namely, thermodynamical parameters like the latent heat L, the wall tension σ, or the friction coefficient η. As expected, for the phase equilibrium stage we have simple expressions, of the sort $\delta t_2 \simeq r H^{-1}$, with obvious interpretation. More complex formulas arise instead for the reheating stage. Although these parameters must be calculated in each particular case, some relations can be established, that hold quite broadly. They allowed us to confirm some natural premises for the dynamics, e.g., that $\delta t_\Gamma \ll \delta t_1 \ll \delta t_2$ and $v_2 \ll v_1$.

We have studied also the interrelations between the thermodynamical parameters. When necessary, we made use of a simple model for the free energy. Aside from reproducing the desired features of the phase transition, it is well known that this model corresponds to the simplest high-temperature effective potential that arises in perturbation theory. We have derived general expressions for the parameters of this potential, which is useful in establishing further relations between the thermodynamical parameters. We have also derived general expressions for the friction on the bubble walls. This is caused by the perturbation from equilibrium of the particles of the plasma due to the motion of the interfaces. We have compared the case of thermal particles to that of coherent infrared bosons. We have found a different parametric dependence of each contribution, which indicates that each of them will dominate in different parameter ranges. We have argued that probably one of these contributions will cause the wall to move nonrelativistically. This justifies the near-equilibrium approximations that simplify the analysis of the phase transition.

We have thus been able to find some general relations between the parameters that determine the dynamics. For instance, we have seen that if the first-order phase transition is strengthened, then the supercooling is intensified, but also the friction and latent heat are generically enlarged, giving a slower evolution. This general feature is easily detected with *ad hoc* variations of the parameters of the free energy, and further confirmed by the relations that arise for perturbation-theory values of these parameters. The amount of supercooling is characterized by the difference between the energy density of the plasma at the critical temperature, and that at which nucleation begins, $\delta\rho=\rho(T_c)-\rho(T_N)$. The specific relation between $\delta\rho$ and the latent heat is decisive for the phase transition dynamics. As we have seen, the ratio $L/\delta\rho$ can either be small or large, depending on the theory. Of course, it is larger for stronger phase transitions. We have argued that the case of interest corresponds to a latent heat comparable to $\delta\rho$. On one hand, a small L is related to too weakly first-order phase transitions. On the other hand, for large L most of the phase transition occurs close to phase equilibrium, and can be described as a phase transition at constant $T\simeq T_c$. This includes the case $r\sim 1$ if $\delta\rho\ll\rho$, thus justifying the approximation $r\ll 1$ in the case of supercooling.

For the study of the different cosmological consequences, additional specific relations will be relevant in each case [7], which can be obtained from the present analysis. Our analytical approximations will thus prove useful to include details of phase transition dynamics in the calculation of cosmic remnants, particularly with regard to the variation of the pertinent parameters throughout the phase transition. For example, the importance of the phase equilibrium stage has been already investigated in Refs. [5,13] in the context of electroweak baryogenesis. Other cosmological consequences are affected as well. For instance, the fact that the nucleation rate is turned off due to reheating evidently modifies the number of nucleated bubbles, and thus the density of topological defects.

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APPENDIX A: PERTURBATIVE FREE ENERGY

Following Ref. [6], we will obtain the high-temperature effective potential (or free energy) in the one-loop approximation, including leading-order plasma effects. Additional terms appear at higher-loop order. For instance, potentially important terms of the form $\phi^2\log\phi$ arise at two loops [42]. However, inclusion of two-loop corrections makes the situation more complicated and lies out of the scope of our general analysis.

We consider a gauge theory which is spontaneously broken by a VEV of a scalar field ϕ. The tree-level potential is

$$V_0(\phi)=-\frac{m^2}{2}\phi^2+\frac{\lambda}{4}\phi^4, \tag{A1}$$

so the VEV is given by $v^2=m^2/\lambda$ and the Higgs mass by $m_h^2=2\lambda v^2$.

At one-loop the effective potential picks up zero-temperature and finite-temperature corrections. With a cutoff regularization and tree level values for v and m_h, the zero-temperature contribution of a particle species is [43]

$$\pm\frac{g}{64\pi^2}\left\{m^4(\phi)\left[\log\frac{m^2(\phi)}{m^2(v)}-\frac{3}{2}\right]+2m^2(v)m^2(\phi)\right\}, \tag{A2}$$

where the $\pm$ is for bosons (fermions), g is the number of degrees of freedom of the species, and $m(\phi)$ is the mass of the particle in the presence of the background scalar field. It is in general of the form

$$m^2(\phi)=\mu^2+h^2\phi^2, \tag{A3}$$

where h is the coupling of the particle with ϕ (i.e., the Yukawa coupling, gauge coupling, etc.). The finite-temperature corrections are of the form

$$\pm\frac{g}{2\pi^2\beta^4}J_{B,F}[m^2(\phi)\beta^2], \tag{A4}$$

where the functions J_B and J_F can be expanded in powers of m/T for $m\ll T$, and fall off exponentially for large m/T. Therefore species with $m\gg T$ decouple from the plasma and we make a high-temperature approximation in which we consider only particles with $m<T$. Expanding up to $\mathcal{O}(m/T)^4$ we have (see e.g. Ref. [6])

$$\begin{aligned}\mathcal{F}(\phi,T)={}&\text{const}-\frac{\pi^2}{90}g_*T^4-\frac{m_h^2}{4}\phi^2+\sum_b g_b\left(\frac{m_b^2}{32\pi^2}+\frac{T^2}{24}\right)m_b^2(\phi)+\sum_f g_f\left(-\frac{m_f^2}{32\pi^2}+\frac{T^2}{48}\right)m_f^2(\phi)\\&-\frac{T}{12\pi}\sum_b g_b m_b^3(\phi)+\frac{\lambda}{4}\phi^4\\&-\sum_b\frac{g_b}{64\pi^2}\log\left(\frac{m_b^2}{T^2A_b}\right)m_b^4(\phi)\\&+\sum_f\frac{g_f}{64\pi^2}\log\left(\frac{m_f^2}{T^2A_f}\right)m_f^4(\phi),\end{aligned} \tag{A5}$$

where g_* is the effective number of relativistic degrees of freedom, given in Eq. (69) and $m_i\equiv m_i(\phi=v)$ are the physical masses. We assume that particles contributing to $\mathcal{F}$ do not decouple during the phase transition, i.e., that the condition $m(\phi)\ll T$ is preserved in the range of temperatures of interest. This is a reasonable assumption provided that $\phi\ll v$ for temperatures close to T_c, which is consistent with the one-loop approximation. If some particles decouple from the plasma during the transition, the main effect is a change in g_*.

The m^3 term is the contribution of the bosonic zero modes to the one-loop effective potential. This term is the most important to us; without it the phase transition would be of second order. However, for the zero modes the loop expansion has an infrared problem. The perturbative expansion breaks down at higher-loop order, since higher loops contribute powers of $\alpha = h^2T^2/m^2(\phi)$ and of $\beta = h^2T/m(\phi)$ [6]. The way out is to dress the zero modes with daisy and superdaisy diagrams. The result of this resummation is a contribution of the form [to all order in α and to $\mathcal{O}(\beta)$]

$$-\frac{T}{12\pi}\sum_b g_b \mathcal{M}_b^3(\phi) \tag{A6}$$

plus contributions proportional to ϕ^2 which are unimportant within the present approximations. Thus the mass m_b gets replaced with its Debye mass

$$\mathcal{M}_b^2 = m_b^2(\phi) + \Pi_b(\phi,T), \tag{A7}$$

where Π_b is the self-energy of the boson particle. In general it is a combination of squared coupling constants times T^2. The exception are the transverse components of the gauge bosons, for which $\Pi = 0$.

If we replace the Debye mass (A7) in the cubic term of Eq. (A5), and the masses (A3) everywhere else, the resulting terms can be grouped as follows:

a. Constant terms. These are contributions to the cosmological constant. The total cosmological constant must be set by hand, so that it is almost zero after the phase transition.

b. T-dependent, ϕ-independent terms. Apart from the first term in Eq. (A5), there are also T^2 and logarithmic terms, but these are of order $(\mu_i/T)^2$ and $(\mu_i/T)^4$ with respect to the T^4 term, so we can neglect them within the approximation $m_i \ll T$. We notice, however, that we could have a large negative μ_i of order T, such that m_i is small (see below). In any case, these corrections modify the equations of state of both phases in the same way, and we do not expect them to affect significantly the dynamics of the phase transition.

c. ϕ^2 terms. The coefficient of ϕ^2 is the sum of a term proportional to T^2, a constant term $m_h^2/4$, and other constant and logarithmic terms, which are $\sim h_i^2 m_i^2/32\pi^2$. The latter are suppressed unless the Higgs mass is too small, so we will disregard them. In any case, these corrections are inconsequential for our purposes. They contribute to the value of the characteristic temperature T_0 of the phase transition. However, for the dynamics of the phase transition the precise value of T_0 is not relevant; the important parameter is the relative difference between this temperature and the critical one, $(T_c - T_0)/T_0$, which is independent of T_0.

d. ϕ^3 terms. The $\mathcal{M}^3$ term has contributions from all the bosons, proportional to

$$(h_b^2\phi^2 + \mu_b^2 + \Pi_b)^{3/2}. \tag{A8}$$

These terms may strongly affect the nature of the phase transition, depending on the value of $\mu_b^2 + \Pi_b(T_c)$. There are two limiting cases:

(i) if $\mu_b^2 + \Pi_b(T_c) \simeq 0$, Eq. (A8) contributes a term of the form $T\phi^3$ to the free energy, which favors a strongly first-order phase transition, and

(ii) if $\mu_b^2 + \Pi_b(T_c) \gg h_b^2\phi^2$, we can expand Eq. (A8) in powers of ϕ. This gives higher-order corrections to the coefficients of ϕ^2 and ϕ^4, and no contribution to ϕ^3.

Almost all particles fall in the second case, since Π is of order h^2T^2, and $\phi \ll T$ near the phase transition. The transverse components of the vector bosons, on the contrary, are protected against this thermal screening. Another exception may be a scalar with a negative $\mu \sim T$. Such a particle would fall in the first case or in an intermediate case, and may play a role in determining the character of the phase transition [53]. However, such a tuning may induce unwanted minima in the scalar potential [53,54]. We are not going to consider this possibility here. Accordingly, the cubic term in the effective potential is

$$-\frac{T}{12\pi}\sum \frac{2}{3} g_b h_b^3 \phi^3 \tag{A9}$$

where the sum is only over gauge bosons, and the factor 2/3 is due to the fact that only two degrees of freedom of the massive vector contribute [44].

e. ϕ^4 terms. The corrections to λ depend logarithmically on T, so the effective value of λ may be regarded to be constant during the phase transition. Furthermore, these corrections are of order $h_i^4/64\pi^2$, so they can be neglected provided that $\lambda \gtrsim h_i^2$. For simplicity we will assume that this is the case.

Putting all these terms together we see that, under the above assumptions and approximations, the free energy density takes the form displayed in Eqs. (67), (68), with coefficients given by Eqs. (91).

APPENDIX B: FRICTION COEFFICIENT

In this appendix we make a derivation of the friction exerted by the hot plasma on the bubble walls. For that, we must calculate the departure from equilibrium of the phase space population functions, δf in Eq. (71). The friction on the wall has been extensively studied in the case of the electroweak phase transition [2,3,10,44,52,55,56]. Our aim here is to discuss the general dependence of the friction on the particle content of a theory, so we will need to use some approximations in order to keep the description as general as possible.

1. Fluid approximation

We begin by considering the contribution of particles with $p \gg L_w^{-1}$ (L_w = wall width), for which the background field varies slowly and the semiclassical (WKB) approximation is valid. Since in general $L_w^{-1} \gg T$, this condition is satisfied for all but the most infrared particles [2], which we study below. We follow Refs. [2,3], but we use a simpler ansatz for the deviations from equilibrium distributions. This will suffice for our purposes. We assume that the population density of a particle species in the background of the domain wall (that moves along the z direction) is governed by the Boltzmann equation

$$[\partial_t+(\partial_{p_z}E)\partial_z-(\partial_z E)\partial_{p_z}]f=-C[f], \tag{B1}$$

where $E=\sqrt{p^2+m(z,t)^2}$ is the particle energy, $\partial_{p_z}E=p_z/E$ is the particle velocity, $-\partial_z E=-\partial_z(m^2)/2E$ is the force on the particle, and $C[f]$ is the collision integral.

We use the ansatz $f=f_0(E/T-\delta)$, where

$$f_0(x)=\frac{1}{e^x\pm 1}, \tag{B2}$$

so the deviation from $f_0(E/T)$ is $\delta f=-f_0'(E/T)\delta$. Thus we obtain an equation for δ by linearizing the Boltzmann equation. Keeping only terms of order $(m/T)^2$ we have

$$\left(\frac{1}{2ET}\partial_t m^2-\partial_t\delta+\frac{p_z}{E}\partial_z\delta\right)f_0'+C[f]=0. \tag{B3}$$

The mass of the particle is a function of $z-v_w t$, and so is the perturbation δ if we assume a stationary state. Thus we make the replacements $\partial_t m^2=-v_w(m^2)'$ and $\partial_t\delta=-v_w\delta'$, where the prime means derivative with respect to z. We further simplify Eq. (B3) by making the integration $\int d^3p/(2\pi)^3$. We obtain

$$c_2 v_w\delta'-\Gamma\delta=c_1 v_w m^{2'}/2T^2, \tag{B4}$$

where c_1 and c_2 are defined by the integrals

$$c_1\equiv-\frac{1}{T^2}\int\frac{d^3p}{(2\pi)^3E}f_0',\quad c_2\equiv-\frac{1}{T^3}\int\frac{d^3p}{(2\pi)^3}f_0',$$

and we have written the collision integral in the form [2]

$$\int\frac{d^3p}{(2\pi)^3}\frac{C[f]}{T^2}=T\Gamma\delta. \tag{B5}$$

To lowest order in m/T we have $c_{1f}=\log 2/2\pi^2$ and $c_{2f}=1/12$ for fermions, and $c_{1b}=\log(T/m)/2\pi^2$ and $c_{2b}=1/6$ for bosons.

For each particle species i we have a fluid equation of the form (B4). These equations are coupled through the collision term (B5), and Γ is in principle a matrix with indices running over all particle species.[8] However, only particles with large Yukawa couplings are relevant for the friction force, since they have stronger interactions with the bubble wall. In accordance with Refs. [2,3], in this appendix we call "heavy" these particles with large h_i. Notice however that heavy particles with large mass μ_i in the unbroken phase may be thermally decoupled and not contribute to the friction at all. The remaining "light" particles can be treated as a common background perturbation δ_{bg}. The fluid equation for the background is simpler and can be solved to eliminate δ_{bg}. Moreover, the heavy particles primarily collide with the light particles, so direct coupling between heavy species can be neglected. The effect of the light background, though, once eliminated from the equations, is to introduce a weak coupling between heavy particles. As a consequence, heavy particles are only weakly coupled through the background, and the nondiagonal terms of Γ are suppressed with respect to the diagonal terms by a factor $1/g_{*\mathrm{light}}$, where $g_{*\mathrm{light}}$ is the number of light species of the background (it is proportional to the heat capacity of the plasma). We will therefore neglect nondiagonal terms of Γ in our analysis. Calculating the rates Γ is well beyond the scope of this work. They are of the form $\alpha^2\log(1/\alpha)T$, where α is a gauge coupling [2]. We will assume that in general, $\Gamma\lesssim 10^{-1}T$.

The right-hand side of Eq. (B4) is the source term of the equation. It is localized at the bubble wall, so we expect the same for δ. Therefore we have $\delta'/\delta\sim 1/L_w$. Normally, $L_w\gtrsim 10T^{-1}$, so if the wall velocity is small, the first term on the left-hand side of Eq. (B4) is much less than the second one and can be neglected. With this approximation the equation has a simple solution,

$$\delta=-v_w\frac{c_1 m^{2'}}{2T^2}. \tag{B6}$$

If we now insert $\delta f_i=-f_0'(E_i/T)\delta_i$ in Eq. (71) we obtain

$$(v_w^2-1)\phi''+V'(\phi,T)+\frac{T^2}{2}\sum g_i c_{1i}\frac{dm_i^2}{d\phi}\delta_i. \tag{B7}$$

Replacing the value of δ_i given by Eq. (B6), multiplying times ϕ', then integrating with respect to z, we get

$$-V(\phi_m,T)=v_w\sum g_i\int\frac{c_{1i}^2(m_i^{2'})^2}{4\Gamma}dz. \tag{B8}$$

The left-hand side is the pressure difference between the two phases. It is equilibrated by a friction force of the form ηv_w. The friction coefficient is thus

$$\eta=\sum_{i=\text{“heavy”}}\frac{g_i h_i^4}{\Gamma}\int c_{1i}^2\phi^2\phi'^2dz. \tag{B9}$$

The coefficient c_1 for bosons depends on m_i, but it is easy to see that its variation with z can be neglected, and we can make the approximation

$$c_{1b}=\log h_b^{-1}/2\pi^2. \tag{B10}$$

To evaluate the integral in Eq. (B9) we use the thin wall approximation (37),

[8] In Refs. [2,3] a more complex approximation was made, where the perturbation δ is split up into three different perturbations, $\delta=\mu/T+E\delta T/T^2+p_z v/T$. In that case there are three fluid equations for each particle species, with different rates Γ_μ, Γ_T and Γ_v, and there arise additional constants c_3 and c_4. Our approximation corresponds to considering only the term μ/T.

$$\int \phi^2 \phi'^2 dz = \int_0^{\phi_m} \phi^2 \sqrt{2V} d\phi.$$

It is clear that this integral goes like $\phi_m^2 \sigma$. For the model (68), the integral is easily calculated using Eq. (84). It gives $(3/10)\phi_m^2\sigma$, so the friction coefficient is given by

$$\eta = \sum 3 \left(\frac{\log \chi_i}{2\pi^2} \right)^2 \frac{g_i h_i^4}{(\Gamma_i/10^{-1}T)} \frac{\phi_m^2 \sigma}{T}, \tag{B11}$$

where $\chi_i = 2$ for fermions and $\chi_i = h_i^{-1}$ for bosons. According to the arguments above, we will make the assumption that the parentheses in the denominator of the last equation is roughly ~ 1. This gives Eqs. (75) and (92).

2. Infrared bosons

It has been shown [55] that coherent gauge fields can have important contributions to the friction. Following Ref. [55], we will estimate the contribution of a gauge boson to η. Infrared boson excitations must be treated classically [56]; furthermore, the dynamics of the soft fields is overdamped by hard particles [57]. As a consequence, the equation for the population function is given by [55]

$$\frac{\pi m_D^2}{8p} \frac{df}{dt} = -E^2 f + \text{noise}, \tag{B12}$$

which comes from a similar equation for the amplitude of the field. Here, m_D is the Debye mass, given by $m_D^2 \sim \overline{g} h^2 T^2$, where, according to our previous notation, h is the gauge coupling, and $\overline{g}$ is roughly proportional to the number of particles that couple to the gauge field. Averaging over the noise, we get the restoring term $-E^2 \delta f$ in the right-hand side of Eq. (B12). Since $f = f_0(E/T) + \delta f$, and $\delta f \ll f_0$ for small v_w, we can write

$$\delta f = -\frac{\pi m_D^2}{16 p T E^3} f_0' \frac{dm^2}{d\phi} \phi' v_w. \tag{B13}$$

Inserting in Eq. (71), multiplying by ϕ', and integrating as before, we obtain

$$V(T) = -\sum_{\text{gauge}} g_b \frac{v_w \pi m_D^2}{8} \int dz (m_b^{2\prime})^2 \int \frac{d^3p}{(2\pi)^3} \frac{f_0'}{4pTE^4}. \tag{B14}$$

Since the momentum integral is infrared dominated, we can approximate $f_0'(x) \simeq -1/x^2$, so the momentum integral yields $T/32\pi^2 m_b^4$. With $m_b^2 = h_b^2 \phi^2$, we have

$$V(T) = v_w \sum_{\text{gauge}} \frac{g_b m_D^2 T}{64\pi} \int dz \frac{\phi'^2}{\phi^2}. \tag{B15}$$

The last integral can be calculated using again the thin wall approximation, Eqs. (37) and (84),

$$\int_0^{\phi_m} \frac{d\phi}{\phi^2} \sqrt{2V} = \frac{2}{L_w} \int_0^1 dx \frac{1-x}{x}. \tag{B16}$$

There is a logarithmic divergence that must be cut off where the approximations used in this derivation break down [55]. Perturbation theory breaks down when $m_b(\phi) \sim h_b^2 T$, i.e., at $\phi/\phi_m \sim h_b \lambda/E$. The kinetic theory description that leads to Eq. (71) breaks down when $m_b(\phi) \sim L_w^{-1}$, i.e., at $\phi/\phi_m \sim \sqrt{\lambda/h_b^2}$. The latter occurs first, so the log is cut off at $m_b(\phi) L_w \sim 1$. In Ref. [55] it is argued that the contribution of very infrared degrees of freedom is subdominant, since their wavelength cannot resolve the thickness of the wall. Hence, the integral in Eq. (B16) gives to leading log, $\log \phi_m/\phi = \log[m_b(\phi_m) L_w]$, and the friction coefficient is

$$\eta = \sum_{\text{gauge}} \frac{g_b m_D^2 T}{32\pi L_w} \log[m_b(\phi_m) L_w], \tag{B17}$$

which gives Eqs. (76) and (93).

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