

**Conditions for generating scale-invariant density perturbations**Steven Gratton,<sup>1,\*</sup> Justin Khoury,<sup>1,2</sup> Paul J. Steinhardt,<sup>1</sup> and Neil Turok<sup>3</sup><sup>1</sup>*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA*<sup>2</sup>*Institute for Strings, Cosmology, and Astroparticle Physics, Columbia University, New York, New York 10027, USA*<sup>3</sup>*DAMTP, CMS, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

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We analyze the general conditions on the equation of state  $w$  required for quantum fluctuations of a scalar field to produce a scale-invariant spectrum of density perturbations, including the possibility of models which (in the four dimensional effective description) bounce from a contracting to an expanding phase. We show that there are only two stable cases:  $w \approx -1$  and  $w \gg 1$ . The first is utilized in inflationary models and the second in the ekpyrotic/cyclic scenario, assuming the perturbations propagate through the bounce. All other cases, including the  $w \approx 0$  case considered by some authors, require extreme fine-tuning of initial conditions and/or the effective potential. For the ekpyrotic/cyclic ( $w \gg 1$ ) case, we also analyze the small deviations from scale invariance.

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**INTRODUCTION**

Until recently, the only known mechanism for generating superhorizon, adiabatic, scale-invariant density perturbations was inflation [1], a period of rapid, accelerated expansion. In the last few years, the ekpyrotic [2–4] and cyclic models [5,6] have been proposed which suggest a second, alternative mechanism based on a period of slow contraction. The models assume that the scale-invariant perturbations produced during the contraction phase pass through the bounce with the same spectral shape into the expanding phase [3,4,7,8]. In this paper, we introduce a gauge invariant, systematic analysis (including gravitational back reaction) that identifies the most general conditions for obtaining a nearly scale-invariant spectrum for cases where the density perturbations arise as a result of quantum fluctuations of a single scalar field  $\phi$  with potential  $V(\phi)$ . We include the possibility that the universe may bounce from a contracting to an expanding phase, and that the perturbations can be matched across such a bounce in an unambiguous way [3,4,7,8].

First, we must obtain scale-invariant fluctuations during a period in which the scalar field dominates the energy density of the universe. The conditions for this to occur can be characterized by the equation of state  $w$  during this epoch. We find three interesting cases, for each of which  $w$  is nearly constant: (i) an expanding universe with  $w \approx -1$ , corresponding to slow-roll inflation; (ii) a *contracting* universe with  $w \gg 1$ , corresponding to the ekpyrotic/cyclic models; and (iii) a *contracting* universe with  $w \approx 0$ , as discussed by Wands [9] and by Finelli and Brandenberger [10]. Although the last case does generate a scale invariant spectrum of curvature perturbations, we shall show that the corresponding Newtonian potential has a very red power spectrum. This points to a serious instability of the background, which we explicitly identify in the infinite wavelength limit. Unlike the ekpyrotic/cyclic cases, the  $w=0$  background solution is un-

stable and hence not a dynamical attractor. The only additional cases are ones in which  $w$  is rapidly time-varying, but these require extreme fine-tuning of  $V(\phi)$ .

Hence, we find that the  $w \gg 1$  case, the one considered in ekpyrotic/cyclic models, is the only conceivable alternative to inflation. Of course, this assumes that a bounce from contraction to expansion is possible and that the  $k$ -dependence of the perturbation spectrum is preserved through the bounce (at least in the long-wavelength limit). Whether this concept is physically plausible remains controversial. There have been criticisms of the models [11] generally (but see [12] for responses), and arguments have been raised suggesting that a bounce is impossible [13], or that scale-invariant perturbations produced in the contracting phase cannot pass into the expanding phase [14] (but see also counterarguments in [15]). However, very recent work shows that, for  $w \gg 1$ , the contracting phase satisfies ultralocal conditions [16] that make the evolution for long wavelengths simple to track up until a few Planck times prior to the bounce. If the time remaining before the bounce is so small and the spatial geometry is flat, causality suggests that the bounce should not disturb correlations between distant points (separated by much more than a Planck length) which cannot communicate with one another in this finite time interval. Under these conditions, works by several independent groups [8,17,18] suggest that perturbations generated during the contracting phase may pass into the expanding phase with the spectral shape unperturbed. At present, the issue is unsettled and the subject continues to be an area of active research. If the recent results are found to be rigorous, then this paper shows that there is only one viable alternative to inflation (under the assumptions stated below). If the recent results are negated, then this paper shows that inflation is unique.

Our analysis is restricted to the case of a single scalar field, which includes the simplest inflationary scenarios as well as the ekpyrotic/cyclic models. In Newtonian gauge, the perturbed metric for a spatially flat background can be expressed in terms of a single gauge invariant variable  $\Phi$ , the Newtonian potential, as

$$ds^2 = a^2(\tau) \cdot \{ -[1 + 2\Phi(\vec{x}, \tau)]d\tau^2 + [1 - 2\Phi(\vec{x}, \tau)]d\vec{x}^2 \},$$

\*Present address: DAMTP, CMS, Wilberforce Road, Cambridge CB3 0WA, United Kingdom.

where  $\tau$  is conformal time, and where we have used the fact that fluctuations of a scalar field do not generate anisotropic stress.

While knowledge of  $\Phi$  is sufficient to determine the perturbed metric, it is useful to introduce a second variable,  $\zeta$ , which is the curvature perturbation on comoving hypersurfaces [19,20].  $\Phi$  and  $\zeta$  are related by

$$\zeta = \frac{2}{3a^2(1+w)} \left( \frac{\Phi}{a'/a^3} \right)', \quad (1)$$

where a prime denotes differentiation with respect to  $\tau$ . The variable  $\zeta$  has the virtue of remaining nearly constant at superhorizon wavelengths during epochs of *expansion*. In inflationary models, it allows one to easily match the Newtonian potential at the horizon reentry in the matter dominated phase to that calculated at the horizon exit during the inflationary phase. In the ekpyrotic/cyclic scenarios, once its spectrum has been determined after the bounce to an expanding phase,  $\zeta$  also gives the perturbation amplitude at horizon reentry.

It is tempting to suppose that we should only be interested in tracking the evolution of  $\zeta$ . However, even though  $\zeta$  is continuous throughout a contracting phase or an expanding phase, it can undergo a rapid jump during the transition between the two [4,8]. It is necessary to match the incoming  $\Phi_{in}$  and  $\zeta_{in}$  to the outgoing  $\Phi_{out}$  and  $\zeta_{out}$ . Generically there is some mixing and  $\zeta_{out}$  depends on a combination of the  $\Phi_{in}$  and  $\zeta_{in}$  [4]. Hence, it is important to know both  $\Phi_{in}$  and  $\zeta_{in}$  at the bounce.

For the ekpyrotic/cyclic case,  $\Phi_{in}$  is scale-invariant and  $\zeta_{in}$  is blue (decreasing at long wavelengths), and, hence,  $\zeta_{out}$  is dominated by the scale-invariant contribution (due to  $\Phi_{in}$ ) at long wavelengths, leading to a scale-invariant spectrum as modes reenter the horizon during the expanding phase. For the  $w \approx 0$  case,  $\zeta_{in}$  is scale-invariant before the bounce and  $\Phi_{in}$  turns out to be red (increasing at long wavelengths and exponentially larger than  $\zeta_{in}$ ). This suggests that the background possesses a serious long wavelength instability; we confirm this by showing that the background solution is not a dynamical attractor.

### THE $\Phi$ SPECTRUM

The differential equation for the  $k$ -mode  $u_k$  of the gauge invariant variable  $u$ , related to the Newtonian potential by  $u = a\Phi/\phi'$  [where we drop the (*in*) subscript, henceforth], is

$$u_k'' + \left( k^2 - \frac{\beta(\tau)}{\tau^2} \right) u_k = 0; \quad (2)$$

$$\beta(\tau) \equiv \tau^2 H^2 a^2 \left\{ \bar{\epsilon} + \frac{(1+\bar{\epsilon})}{2} \left( \frac{d \ln \bar{\epsilon}}{dN} \right) + \frac{1}{4} \left( \frac{d \ln \bar{\epsilon}}{dN} \right)^2 - \frac{1}{2} \frac{d^2 \ln \bar{\epsilon}}{dN^2} \right\}, \quad (3)$$

where  $H = a'/a^2$  is the Hubble parameter,  $N \equiv \ln a$ , and  $\bar{\epsilon} \equiv 3(1+w)/2$ . Note that in the case of inflation,  $\bar{\epsilon}$  reduces to the usual ‘‘slow-roll’’ parameter, while  $N$  is the number of e-folds of expansion.

In the regime  $k^2 \tau^2 \gg |\beta|$ , Eq. (2) reduces to the equation for a simple harmonic oscillator, and  $u_k$  is stable. When  $k^2 \tau^2 \ll |\beta|$ , however, the amplitude of the mode is unstable. In order to have a situation where successive modes with increasing  $k$  are becoming unstable and growing, we need  $\beta$  positive, and assuming that  $\beta$  is slowly varying, we require that  $\tau$  be negative and increasing. This applies to expanding models, such as inflation, or contracting models, such as the ekpyrotic/cyclic scenarios.

For general time-varying  $w$ ,  $\beta(\tau)$  will be a complicated function of time, and one can use numerical methods to solve Eq. (2). In the most plausible cases, however, it is reasonable to approximate  $w$  as constant, at least for the observationally relevant range of modes.

It is well-known [4,21] that solutions with constant  $w$  correspond to potentials of the exponential form,  $V(\phi) = -V_0 e^{-c\phi}$ , where  $c$  and  $V_0$  are constants. In this case, the equation of state is related to the slope of the potential by

$$\bar{\epsilon} \equiv \frac{3}{2}(1+w) = \frac{1}{2} \left( \frac{V_{,\phi}}{V} \right)^2 \quad (4)$$

(we use units where  $8\pi G = 1$ ) and the solution for the background, assuming homogeneity, isotropy and spatial flatness, is given by

$$a(\tau) \sim (-\tau)^{1/(\bar{\epsilon}-1)}, \quad H = \frac{1}{(\bar{\epsilon}-1)a\tau}$$

$$\phi' = \sqrt{\frac{2}{\bar{\epsilon}}} \left( \frac{\bar{\epsilon}}{\bar{\epsilon}-1} \right) \tau^{-1}, \quad V(\phi) = -\frac{(\bar{\epsilon}-3)}{(\bar{\epsilon}-1)^2 a^2 \tau^2}. \quad (5)$$

Substituting the above into Eq. (3), we obtain  $\beta = \bar{\epsilon}/(\bar{\epsilon}-1)^2$ . Since  $\beta$  is constant in this case, Eq. (2) can be solved analytically, with the general solution

$$u_k = \sqrt{-k\tau} [C_1(k)J_n(-k\tau) + C_2(k)J_{-n}(-k\tau)], \quad (6)$$

where  $n \equiv \sqrt{\beta+1/4}$ ,  $J_n$  is the Bessel function of the first kind of order  $n$ , and  $C_i(k), i=1,2$  are arbitrary functions of  $k$ .

The functions  $C_i(k)$  are determined by specifying initial conditions when the mode is stable, i.e., when  $k^2 \tau^2 \gg \beta$ . In this limit, we make the usual assumption that the fluctuations in  $\phi$  are in their Minkowski vacuum, which corresponds to  $u_k \approx i e^{-ik\tau}/(2k)^{3/2}$ . Using the relation  $u_k = a\Phi_k/\phi'$  and the asymptotic properties of Bessel functions, this gives

$$u_k = \frac{\sqrt{-\pi k\tau}}{4k^{3/2} \sin(\pi n)} [J_{-n}(-k\tau) - e^{-i\pi n} J_n(-k\tau)], \quad (7)$$

where we have neglected an irrelevant phase factor.

We are interested in the amplitude of the mode in the long-wavelength regime,  $k^2 \tau^2 \ll \beta$ . In this limit, we can expand the Bessel functions to obtain

$$k^{3/2} \Phi_k \approx \frac{\sqrt{\pi}}{2^{3/2} \sin(\pi n) \Gamma(1-n)} \left( \frac{\phi'}{a} \right) \left( \frac{-k\tau}{2} \right)^{-n+1/2} \times \left\{ 1 - e^{-i\pi n} \frac{\Gamma(1-n)}{\Gamma(1+n)} \left( \frac{-k\tau}{2} \right)^{2n} - \frac{\Gamma(1-n)}{\Gamma(2-n)} \left( \frac{-k\tau}{2} \right)^2 \right\}. \quad (8)$$

The Newtonian potential has a scale-invariant spectrum if the rms amplitude of  $\Phi_k$  varies with  $k$  as  $k^{-3/2}$ . Hence, we conclude that this will be the case if  $n \approx 1/2$ . Recalling that  $n = \sqrt{\beta+1}/4$ , this can be expressed as a constraint on  $\beta$  or, equivalently,  $\bar{\epsilon}$ :

$$\beta = \frac{\bar{\epsilon}}{(\bar{\epsilon}-1)^2} \ll 1. \quad (9)$$

We have thus translated the requirement of scale invariance for  $\Phi$  into a condition on the background equation of state. Therefore, we may now determine what choice of  $w$  will satisfy Eq. (9) and lead to a Harrison-Zel'dovich spectrum.

First, this condition is clearly satisfied when  $\bar{\epsilon} \ll 1$ , that is, when  $w \approx -1$ . This corresponds to the case of slow-roll inflation [1]. Note, however, that there is a second regime in which condition (9) holds, namely when  $\bar{\epsilon} \gg 1$ , corresponding to  $w \gg 1$ . This is the limit relevant to the production of fluctuations in the ekpyrotic and cyclic scenarios [2–6]. These two regimes are in some sense at opposite ends of parameter space. In the inflationary case,  $\bar{\epsilon}$  plays the role of a slow-roll parameter and is therefore small. In the ekpyrotic and cyclic scenarios, however,  $\bar{\epsilon}$  is large compared to unity. Also, from Eq. (5), we see that the universe is expanding in the first case and contracting in the second.

This analysis assumed a nearly constant  $w$  so that  $\beta(\tau)$  in Eq. (3) and, consequently, the spectral index is nearly constant. Note that this assumption is not necessary. It is possible, in principle, to build models for which the time-variation of  $w$  is non-negligible, and yet the derivative terms in Eq. (3) conspire to cancel for a significant range of e-folds,  $N$ . This has been discussed for inflation by Wang *et al.* [24] who showed that maintaining the cancellation for many e-folds requires extreme fine-tuning of  $V(\phi)$  compared to the constant  $w$  cases. Hence, these models seem highly unlikely.

### THE $\zeta$ SPECTRUM

To calculate the spectrum of the second gauge invariant variable of interest,  $\zeta$ , we substitute the expression for  $\Phi$  obtained in Eq. (8) into Eq. (1). The leading term in the expansion for  $\Phi$  is

$$k^{3/2} \Phi_k \sim \left( \frac{\phi'}{a} \right) (-k\tau)^{-n+1/2}, \quad (10)$$

where we have omitted the numerical coefficient. As before, we approximate  $w$  as constant, and, thus, so is  $n$ . Using Eqs. (5), we find

$$k^{3/2} \Phi_k \sim k^{-n+1/2} (-\tau)^\lambda; \quad \lambda \equiv -\frac{1+\bar{\epsilon}}{2(\bar{\epsilon}-1)} \left\{ 1 + \frac{\bar{\epsilon}-1}{|\bar{\epsilon}-1|} \right\}, \quad (11)$$

as well as

$$\frac{a'}{a^3} \sim (-\tau)^{-(1+\bar{\epsilon})/(\bar{\epsilon}-1)}. \quad (12)$$

Since the expression for  $\lambda$  involves a factor of  $|\bar{\epsilon}-1|$ , we must consider the cases  $\bar{\epsilon} > 1$  and  $\bar{\epsilon} < 1$  separately. Starting with the latter (which includes slow-roll inflation), we find  $\lambda = 0$ , and therefore

$$k^{3/2} \left( \frac{\Phi_k}{a'/a^3} \right) \sim k^{-\bar{\epsilon}/(1-\bar{\epsilon})} (-\tau)^{(1+\bar{\epsilon})/(\bar{\epsilon}-1)}. \quad (13)$$

Using the relation between  $\zeta$  and  $\Phi$  given in Eq. (1), we see that  $\zeta$  will have a scale-invariant spectrum if  $\bar{\epsilon} \ll 1$ . Recall that this limit corresponds to slow-roll inflation and that it also leads to a scale-invariant spectrum for  $\Phi$ .

For the case  $\bar{\epsilon} > 1$ , which includes the ekpyrotic/cyclic scenarios, we find  $\lambda = -(1+\bar{\epsilon})/(\bar{\epsilon}-1)$ , and thus

$$k^{3/2} \left( \frac{\Phi_k}{a'/a^3} \right) \sim k^{-1/(\bar{\epsilon}-1)}. \quad (14)$$

Since the right hand side is independent of time, Eq. (1) implies that this leading term for  $\Phi$  does not contribute to  $\zeta$ . In order to determine the long-wavelength piece of  $\zeta$ , we must therefore keep the higher-order terms in the expansion for  $\Phi$  given in Eq. (8). It is straightforward to show that the result is of the form

$$k^{3/2} \zeta \sim f_1(\tau) k^{\bar{\epsilon}/(\bar{\epsilon}-1)} + f_2(\tau) k^{(2\bar{\epsilon}-3)/(\bar{\epsilon}-1)}, \quad (15)$$

where  $f_1(\tau)$  and  $f_2(\tau)$  are time-dependent factors.

For the ekpyrotic/cyclic scenarios, corresponding to the regime  $\bar{\epsilon} \gg 1$ , the first-term in Eq. (15) gives the dominant contribution at long-wavelengths, and thus  $k^{3/2} \zeta$  goes like  $k$ . Hence, while the condition  $\bar{\epsilon} \gg 1$  led to a scale-invariant spectrum for  $\Phi$  in the pre-big-bang phase, it yields a blue spectrum for  $\zeta$ . As we have seen above, this is a consequence of the fact that the growing mode of  $\Phi$ , which is scale-invariant in this limit, is projected out of  $\zeta$ . Thus,  $\zeta$  is determined by the next-order correction in the expansion for  $\Phi$ , which is down by a factor of  $k^{2n} \approx k$ . Nevertheless, as mentioned earlier,  $\zeta$  jumps at the bounce (mixes with the

scale-invariant  $\Phi$  mode) and is dominated by the scale-invariant contribution at long wavelengths after the universe begins to expand.

Note that, if we choose  $\bar{\epsilon} \approx 3/2$ , then the second term in Eq. (15) dominates at large wavelengths, and the resulting spectrum for  $\zeta$  is nearly scale-invariant before the bounce. This case, identified by Wands [9,22] and recently studied by Finelli and Brandenberger [10], describes a contracting universe with a dustlike equation of state,  $w \approx 0$ . However, using our results above, we see that the corresponding spectrum of  $\Phi$  is strongly red ( $k^{3/2}\Phi_k \sim k^{-2}\tau^{-5}$ ), indicating a severe long wavelength instability. Then if  $\Phi$  and  $\zeta$  mix at all at the bounce, the red contribution would dominate at long wavelengths rendering the resulting universe phenomenologically unacceptable.

### STABILITY OF THE BACKGROUND SOLUTION

Before considering evolution of quantum fluctuations, we should first consider whether the constant  $w$  solutions we have assumed as background solutions are stable attractors to the equations of motion. If they are not, then they could not arise in a cosmological solution without extraordinary fine-tuning of initial conditions. The expanding inflationary ( $w = -1$ ) is known to be a stable attractor. Here we show that the contracting ekpyrotic/cyclic ( $w \gg 1$ ) phase is also a stable attractor, but the contracting  $w = 0$  phase is not.

The stability of the contracting background solutions may be studied in the infinite wavelength (i.e., homogeneous) limit simply by considering the scalar field equation in a homogeneous universe,

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -V_{,\phi}, \quad (16)$$

together with the corresponding Friedmann equation. Here dots denote derivative with respect to proper time  $t$ . Now we consider linear perturbations, with  $\phi \rightarrow \phi + \delta\phi$ ,  $a \rightarrow a + \delta a$ , and spatial curvature  $ka^{-2} \rightarrow \delta ka^{-2}$ , since we consider perturbations about a flat background. The Friedmann equation allows us to express  $\delta(\dot{a}/a)$  in terms of  $\delta\phi$ ,  $\delta\dot{\phi}$ , and  $\delta k$ . Inserting the background scaling solution

$$a(t) \propto (-t)^p, \quad (17)$$

$$V(\phi_B) = -V_0 e^{-c\phi_B(t)} = p(3p-1)/t^2,$$

where  $p = 2/c^2$ , and using the Friedmann equation, Eq. (16) becomes

$$\delta\ddot{\phi} + \frac{1+3p}{t}\delta\dot{\phi} - \frac{1-3p}{t^2}\delta\phi = \frac{3}{\sqrt{2pa^2}}\delta k, \quad (18)$$

with the generic solution (that is for  $p \neq \frac{2}{3}$  or  $\frac{3}{2}$ )

$$\delta\phi = D\frac{\delta kt^2}{a^2} + A_0 t^{-1} + A_1 (-t)^{1-3p}, \quad (19)$$

where  $D$  is a number dependent on  $p$ , and  $A_0$  and  $A_1$  arbitrary constants. For  $p = \frac{3}{2}$ ,  $Dt^2$  is replaced by  $ET\ln(-t)$ , with  $E$  a number. For  $p = \frac{2}{3}$ , the second homogeneous solution becomes,  $A_1 t^{-1}\ln(-t)$ . The  $t^{-1}$  solution is just an infinitesimal shift in the time to the big crunch,  $\delta\phi \propto \dot{\phi}$ , and is present for all  $p$ . Such a shift provides a solution to the Einstein-scalar equations because they are time translation invariant, but is physically irrelevant since it can be removed by a redefinition of time. In contrast, the second homogeneous solution represents a physical perturbation of the background solution.

For a contracting universe with  $p > \frac{1}{3}$ , or  $w < 1$ , the physical perturbation grows as  $t$  approaches zero, indicating an instability of the background scaling solution. (Note that this still holds in the special  $p = \frac{2}{3}$  case.) For  $p < \frac{1}{3}$ , or  $w > 1$ , as in the ekpyrotic or cyclic case, the physical perturbation goes to zero in this limit, indicating stability.

This interpretation of the two homogeneous solutions to Eq. (18) can be confirmed (for  $p \neq \frac{1}{3}$ ) by calculating the ratio of kinetic energy to potential energy in the scalar field, which is constant and equal to

$$\frac{K}{V} = \frac{1+w}{1-w} = \frac{1}{3p-1} \quad (20)$$

(even for  $p = \frac{2}{3}$ ) in the background solution. This ratio is unaltered by the  $t^{-1}$  solution (since it is a time shift) but the  $t^{1-3p}$  solution yields

$$\delta\left(\frac{K}{V}\right) \propto (-t)^{1-3p} \quad (21)$$

which for  $p > \frac{1}{3}$  diverges as  $t=0$  is approached.

Our analysis shows in particular that the  $w=0$  ( $p = \frac{2}{3}$ ) background scaling solution is unstable and hence not an attractor. Conversely, the ekpyrotic/cyclic cases, which correspond to  $p \ll 1$ , possess scaling solutions which are stable attractors in the infinite wavelength limit, since the only growing mode is, as we have discussed, just a time translation. Similar conclusions have been reached in Ref. [22].

Another general and perhaps more heuristic argument can be obtained by comparing the Friedmann equation for the three cases,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2}, \quad (22)$$

where  $H$  is the Hubble parameter and  $\rho$  is the energy density. For the expanding  $w = -1$  case, the scalar field energy density is nearly constant, whereas the curvature, radiation ( $\propto 1/a^4$ ), matter ( $\propto 1/a^3$ ), and other forms of energy density all decrease as  $a$  expands. Hence, the constant energy density  $w = -1$  state is an attractor. For the contracting  $w \gg 1$  case ( $a \sim (-t)^p$  with  $p \ll 1$ ), the scalar field energy density increases as  $t^{-2} \propto a^{-(2/p)}$  as  $a$  decreases, whereas the curvature, matter, radiation, or other forms of energy density increase at a slower rate. Hence, the  $w \gg 1$  contracting solution is also an attractor of the Friedmann equation. However, for the  $w \approx 0$  solution, the scalar field energy density increases as  $1/a^3$ , but radiation density increases at a more rapid rate. Furthermore,

as shown above, a small perturbation drives the universe away from  $w \approx 0$  ( $K \approx V$ ) towards  $w = 1$  ( $K \gg V$ ), a state in which the energy density ( $\propto 1/a^6$ ) increases even more rapidly. Hence, the  $w \approx 0$  background is not a stable attractor.

We can state the conclusion more generally. For an exponential potential proportional to  $\exp(-c\phi)$ , we have a scaling solution when  $V$  is positive (negative) for  $c < \sqrt{6}$  ( $c > \sqrt{6}$ ). In the scaling solution, we have  $w = (2-3p)/(3p) = (c^2/3) - 1$ . It follows from the analysis above that *in a contracting universe, the scaling solution is only a stable attractor if  $w > 1$  (or  $c > \sqrt{6}$ ) and the scalar potential is negative*. In contrast, we infer that *in an expanding universe the scaling solution is only a stable attractor if  $w < 1$  ( $c < \sqrt{6}$ ) and the scalar potential is positive*. Inflationary and quintessence-type scaling solutions are both included in this latter case.

Thus, we have completed our classification of the possibilities. In particular, we have seen that when the universe is contracting with  $w \gg 1$  the Newtonian potential  $\Phi$  develops a scale-invariant spectrum while that of  $\zeta$  is blue. In the ekpyrotic/cyclic models, it is proposed that  $\Phi$  and  $\zeta$  mix at the bounce (which is to say that the growing mode of the contracting phase does not match to a pure decaying mode in the expanding phase) [4]. Whether this is possible or not remains controversial [14]. However, recent analyses of the bounce by Tolley *et al.* [8], Craps and Ovrut [17] and Battefield *et al.* [18] support this notion. In this case, one would obtain scale-invariant density perturbations in the expanding phase. When the universe is contracting with a dustlike equation of state ( $w \approx 0$ ),  $\zeta$  acquires a scale-invariant spectrum, while  $\Phi$  acquires a red spectrum. With mixing at the bounce, one obtains an unacceptable red spectrum. Moreover, as shown above, the  $w = 0$  ( $p = 2/3$ ) scaling solution is not an attractor. Therefore, it is highly unlikely to find the universe in this scaling solution in the first place.

More generally, we have shown that the only way to obtain a stable attractor scaling background solution in a contracting universe is to have a *negative* scalar field potential (as occurs in the ekpyrotic and cyclic models).

### SPECTRAL INDEX IN EKPYROTIC /CYCLIC MODELS

In the remaining part of this paper, we shall focus on the ekpyrotic/cyclic generation of perturbations and calculate the spectral index, giving a treatment analogous to that given for inflation by Wang *et al.* [23]. Here, as throughout the paper (see the Introduction), we assume that the bounce from a contracting to an expanding phase is possible and that the scale-invariant perturbations produced during the contracting phase pass through the bounce to become scale-invariant perturbations in the expanding phase.

Recall that approximate scale invariance of the power spectrum in the ekpyrotic/cyclic scenario requires that  $\bar{\epsilon}$  be large and nearly constant as modes become unstable. Since  $\bar{\epsilon} \gg 1$ , it is convenient to introduce a small, “fast-roll” parameter  $\epsilon$  as

$$\epsilon \equiv \frac{1}{2\bar{\epsilon}} = \left( \frac{V}{V_{,\phi}} \right)^2. \quad (23)$$

The condition  $\bar{\epsilon} \gg 1$  implies  $\epsilon \ll 1$ , which translates into the requirement that the potential be steep.

Since  $\bar{\epsilon}$  is large and nearly constant, the parameter  $\beta$  defined in Eq. (3) reduces to

$$\beta \approx \tau^2 H^2 a^2 \bar{\epsilon} \left\{ 1 + \frac{1}{2} \left( \frac{d \ln \bar{\epsilon}}{dN} \right) \right\}, \quad (24)$$

where we have assumed that  $d^2 \ln \bar{\epsilon} / dN^2$  and  $(d \ln \bar{\epsilon} / dN)^2$  are much smaller than  $d \ln \bar{\epsilon} / dN$ .

Recalling from Eq. (4) that  $\bar{\epsilon} \approx V_{,\phi}^2 / 2V^2$  for nearly constant  $\bar{\epsilon}$ , we obtain

$$\frac{d \ln \bar{\epsilon}}{dN} = \left( \frac{\phi'}{aH} \right) \frac{d \ln \bar{\epsilon}}{d\phi} = -2 \left( \frac{V_{,\phi}}{V} \right) \left( \frac{\phi'}{aH} \right) \eta, \quad (25)$$

where we have introduced a second fast-roll parameter  $\eta$ , defined by

$$\eta \equiv 1 - \frac{VV_{,\phi\phi}}{V_{,\phi}^2}. \quad (26)$$

Note that  $\eta = 0$  corresponds to pure exponential potentials.

Substituting for  $\phi'/(aH)$  using Eqs. (5), Eq. (25) reduces to

$$\frac{d \ln \bar{\epsilon}}{dN} \approx 4\bar{\epsilon}\eta. \quad (27)$$

Since  $\bar{\epsilon}$  is assumed to be nearly constant and large, Eq. (27) implies  $|\eta| \ll 1$ ; that is, the potential must be nearly exponential.

From the background solution given in Eqs. (5), it is easily seen that

$$\tau Ha \approx \left( \frac{1}{\bar{\epsilon}} \right) \cdot \{1 + \mathcal{O}(\epsilon, \eta)\}. \quad (28)$$

Therefore, substituting Eqs. (23), (27) and (28) into Eq. (24), we find

$$\beta \approx 2(\epsilon + \eta). \quad (29)$$

We may now proceed to calculate the spectral index of density perturbations. As seen from Eq. (8), the long-wavelength limit of the Newtonian potential is given by

$$k^{3/2} \Phi_k \sim k^{-n+1/2} \approx k^{-\beta}, \quad (30)$$

where we have used the fact that  $n = \sqrt{\beta+1/4}$  and  $\beta \ll 1$  in this case.

Equation (30) describes the spectrum of  $\Phi$  for modes that went unstable during the contracting phase. In Ref. [4], it was argued that the pre-big-bang spectrum of  $\Phi$  gets im-

printed on the long-wavelength part of  $\zeta$  as the universe undergoes reversal from contraction to expansion. (See also Refs. [6,8], and [24].) Then, the post-big-bang spectrum of energy density perturbations is given by  $\delta_k \sim k^{-\beta}$ , corresponding to a spectral index

$$n_s - 1 \equiv \frac{d \ln |\delta_k|^2}{d \ln k} = -2\beta, \quad (31)$$

where  $n_s = 1$  corresponds to an exactly scale-invariant (Harrison-Zel'dovich) spectrum.

Substituting Eq. (29) into Eq. (31), we find that the spectral index of density perturbations in the cyclic and ekpyrotic scenarios is given by

$$n_s - 1 = -4(\epsilon + \eta) = -4 \left\{ \left( \frac{V}{V_{,\phi}} \right)^2 + 1 - \frac{VV_{,\phi\phi}}{V_{,\phi}^2} \right\}. \quad (32)$$

In the case of pure exponential potentials,  $\eta$  vanishes identically, and therefore the spectrum is red (since  $\epsilon > 0$ ). For potentials of larger curvature than an exponential, such as  $-e^{-c\phi^2}$ , one has  $\eta > 0$  and the spectrum is also red. However, for potentials of smaller curvature than an exponential, such as  $e^{-c\sqrt{\phi}}$ , one has  $\eta < 0$ , and the spectrum will be blue if  $\epsilon + \eta$  is also less than zero. For instance, the string-inspired potential of Ref. [2] led to a blue spectrum. We now see that both red and blue spectra can be achieved, as anticipated by Linde *et al.* [25].

It is instructive to compare Eq. (32) with its counterpart in slow-roll inflation [23,26]

$$n_s - 1 = -6\bar{\epsilon} + 2\bar{\eta}, \quad (33)$$

where  $\bar{\epsilon} = V_{,\phi}^2/2V^2$  and  $\bar{\eta} = V_{,\phi\phi}/V$  are the usual slow-roll parameters of inflation. It is easily seen that pure exponential potentials also yield a red spectrum in this case. Once again, it is possible to find potentials for which the spectrum can be either red or blue.

In summary, our work shows that an expanding universe with  $w \approx -1$  (inflation) and a contracting universe with  $w \gg 1$  are the only approaches for generating a nearly adiabatic, scale-invariant spectrum of fluctuations from a single scale field without extreme fine-tuning. In order for the contracting case to be relevant for our universe, it must be possible for the universe to bounce from a contracting to an expanding phase and for the spectrum of fluctuations to remain scale-invariant after the bounce, as proposed in the ekpyrotic and cyclic models. Whether this is physically plausible is controversial [13,14], but recent developments [8,16–18] have suggested reasons to be optimistic. Taking the optimistic point-of-view, we have shown for the ekpyrotic case that the spectral index is related to fast-roll parameters that characterize the slope and curvature of the scalar field potential scale. In Ref. [6], we show that this constraint requires essentially the same amount of fine tuning as the slow-roll conditions for inflation. In Ref. [27], we consider the spectrum in a mixed case where the scalar field rolls from an expanding inflationary regime to a contracting ekpyrotic regime.

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- [1] A.H. Guth, Phys. Rev. D **23**, 347 (1981); A.D. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
- [2] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001).
- [3] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt, and N. Turok, Phys. Rev. D **65**, 086007 (2002).
- [4] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **66**, 046005 (2002).
- [5] P.J. Steinhardt and N. Turok, Science **296**, 1436 (2002); Phys. Rev. D **65**, 126003 (2002).
- [6] J. Khoury, P.J. Steinhardt, and N. Turok, Phys. Rev. Lett. **92**, 031302 (2004).
- [7] A. Tolley and N. Turok, Phys. Rev. D **66**, 106005 (2002).
- [8] A. Tolley, N. Turok, and P.J. Steinhardt, Phys. Rev. D (to be published), hep-th/0306109.
- [9] D. Wands, Phys. Rev. D **60**, 023507 (1999).
- [10] F. Finelli and R.H. Brandenberger, Phys. Rev. D **65**, 103522 (2002).
- [11] R. Kallosh, L. Kofman, and A. Linde, Phys. Rev. D **64**, 123523 (2001); L. Kofman, A. Linde, and A. Tseytlin, *ibid.* **64**, 123524 (2001).
- [12] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, hep-th/0105212; see also <http://feynman.princeton.edu/~steinh/cyclicFAQS>
- [13] H. Liu, G. Moore, and N. Seiberg, J. High Energy Phys. **06**, 045 (2002); **10**, 031 (2002); O. Aharony, M. Fabinger, G. Horowitz, and E. Silverstein, *ibid.* **07**, 007 (2002); M. Fabinger and J. McGreevy, *ibid.* **06**, 042 (2003); M. Fabinger and S. Hellenman, hep-th/0212223; G.T. Horowitz and J. Polchinski, Phys. Rev. D **66**, 103512 (2002); D. Lyth, Phys. Lett. B **524**, 1 (2002).
- [14] R. Brandenberger and F. Finelli, J. High Energy Phys. **11**, 056 (2001); J.-c. Hwang, Phys. Rev. D **65**, 063514 (2002); P. Peter and N. Pinto-Neto, *ibid.* **66**, 063509 (2002); J. Martin, P. Peter, N. Pinto-Neto, and D.J. Schwarz, *ibid.* **65**, 123513 (2002); **67**, 028301 (2003); J. Martin and P. Peter, Phys. Rev. Lett. **92**, 061301 (2004).
- [15] R. Durrer and F. Vernizzi, Phys. Rev. D **66**, 083503 (2002); C. Cartier, R. Durrer, and E.J. Copeland, *ibid.* **67**, 103517 (2003).
- [16] J.K. Erickson, D.H. Wesley, P.J. Steinhardt, and N. Turok, Phys. Rev. D **69**, 063514 (2004).
- [17] B. Craps and B.A. Ovrut, Phys. Rev. D **69**, 066001 (2004).
- [18] T.J. Battefield, S.P. Patil, and R. Brandenberger, hep-th/0401010.
- [19] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

- [20] J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D **28**, 679 (1983).
- [21] D.H. Lyth and E.D. Stewart, Phys. Lett. B **274**, 168 (1992); D.H. Lyth, *ibid.* **524**, 1 (2002).
- [22] I.P.C. Heard and D. Wands, Class. Quantum Grav. **19**, 5435 (2002).
- [23] L. Wang, V.F. Mukhanov, and P.J. Steinhardt, Phys. Lett. B **414**, 18 (1997).
- [24] R. Durrer, hep-th/0112026; R. Durrer and F. Vernizzi, Phys. Rev. D **66**, 083503 (2002).
- [25] Kallosh, Kofman, and Linde [11].
- [26] E.D. Stewart and D.H. Lyth, Phys. Lett. B **302**, 171 (1993); A.R. Liddle and M.S. Turner, Phys. Rev. D **50**, 758 (1994); J.E. Lidsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys. **69**, 373 (1997).
- [27] J. Erickson, S. Gratton, P.J. Steinhardt, and N. Turok (in preparation).