Meson-meson bound states in a (2+1)-dimensional strongly coupled lattice QCD model

Paulo A. Faria da Veiga* and Michael O'Carroll
Departamento de Matemática, ICMC-USP, C.P. 668, 13560-970 São Carlos, SP, Brazil

Antônio Francisco Neto
Departamento de Física e Informática, IFSC-USP, C.P. 369, 13560-970 São Carlos, SP, Brazil

(Received 21 October 2003; published 13 May 2004)

We consider bound states of two mesons (antimesons) in lattice quantum chromodynamics in an Euclidean formulation. For simplicity, we analyze an SU(3) theory with a single flavor in $2+1$ dimensions and two-dimensional Dirac matrices. For a small hopping parameter κ and small plaquette coupling g_0^{-2}, such that $0<g_0^{-2}\ll\kappa\ll 1$, recently we showed the existence of a (anti)mesonlike particle, with an asymptotic mass of the order of $-2\ln\kappa$ and with an isolated dispersion curve—i.e., an upper gap property persisting up to near the meson-meson threshold which is of the order of $-4\ln\kappa$. Here, in a ladder approximation, we show that there is no meson-meson (or antimeson-antimeson) bound state solution to the Bethe-Salpeter equation up to the two-meson threshold. Remarkably the absence of such a bound state is an effect of a potential which is nonlocal in space at order κ^2, i.e., the leading order in the hopping parameter κ. A local potential appears only at order κ^4 and is repulsive. The relevant spectral properties for our model are unveiled by considering the correspondence between the lattice Bethe-Salpeter equation and a lattice Schrödinger resolvent equation with a nonlocal potential.

DOI: 10.1103/PhysRevD.69.097501 PACS number(s): 12.38.Gc, 11.10.St, 11.15.Ha, 24.85.+p

In quantum chromodynamics (QCD), it is a long-standing problem to establish rigorously the low energy-momentum (EM) spectrum of particles and their bound states, in particular to show the existence of mesons and baryons, their bound states, and scattering. One way to attack this question is to use a lattice regularization.

Recently, in Ref. [1], we showed the existence of baryons in a $2+1$ imaginary-time formulation of lattice SU(3) QCD, with 2×2 Pauli spin matrices, one quark flavor, and a strong coupling regime corresponding to a small plaquette coupling g_0^{-2}, and a small hopping parameter κ, such as $0<g_0^{-2}\ll\kappa\ll 1$. In Refs. [2,3] we showed the existence of baryons and mesons for $(2+1)$- and $(3+1)$-dimensional one-flavor lattice QCD, using 4×4 Dirac spin matrices, and for the same region of parameters. The baryon (meson) particle asymptotic mass is $-3\ln\kappa$ $(-2\ln\kappa)$ and they are associated with isolated dispersion curves in the EM spectrum. Mass splitting for these particles is also obtained. The existence of mesons for $2+1$ dimensions and 2×2 Pauli spin matrices, as we consider here, follows by an adaptation of Ref. [3].

Going up in the spectrum, in Ref. [4], we showed that there is no baryon-baryon bound state in the EM spectrum for the $(2+1)$-dimensional one-flavor case up to the two-baryon threshold ($\approx -6\ln\kappa$). This is done using a lattice version of the Bethe-Salpeter (BS) equation, where we find that the leading interaction is a space-range-1 local repulsive energy-independent potential at order κ^2. A key step for this analysis is a spectral representation for both the two- and four-point baryon correlations. Here we consider the existence of meson-meson bound states below the two-meson threshold ($\approx -4\ln\kappa$). Again, we employ the BS equation and establish spectral representations for the two- and four-point meson functions, and find that the dominant interaction between two mesons occurs at κ^2 and is a range-1 energy-independent and, surprisingly, nonlocal potential. A local potential, which is repulsive, appears only at order κ^4. We show the correspondence between the relative coordinate BS equation, for zero system momentum, and a one-particle lattice Schrödinger resolvent equation with a range-1 nonlocal potential. This Schrödinger operator exhibits bound states for all large enough coupling, regardless of its sign. However, for small hopping parameter and in the strong coupling regime, which can be analytically treated with the method given here, there are no bound states.

We recall that there are many attempts to analyze the hadron-hadron interactions using numerical simulations (see, e.g., Refs. [5,6] and Ref. [7] and references therein). It is hard to compute correlations which rapidly vanish into noise because of the relatively massive particles involved. An attractive potential between the two hadron quark clusters is found for some versions of the model, and standard mass computations are carried out. However, the lack of spectral representations for the correlations makes it difficult to establish the connection between the obtained results and the spectrum.

The model we consider here is the same lattice QCD model as in Ref. [1], but our analysis is now restricted to the meson sector of the underlying physical Hilbert space $\mathcal{H}$. To show how our results are obtained, we recall that the partition function is formally given by

$$Z=\int e^{-S(\psi,\bar{\psi},g)}d\psi\, d\bar{\psi}\, d\mu(g),$$

and, for $F(\bar{\psi},\psi,g)$, the normalized expectations are denoted by $\langle F\rangle$. The model action $S(\psi,\bar{\psi},g)$ is

*Electronic address: veiga@icmc.usp.br

$$S(\psi,\bar{\psi},g)=\frac{\kappa}{2}\sum \bar{\psi}_{\alpha,a}(u)\Gamma^{\epsilon e^{\mu}}_{\alpha\beta}(g_{u,u+\epsilon e^{\mu}})_{ab}\psi_{\beta,b}(u+\epsilon e^{\mu}) + \sum_{u\in\mathbb{Z}^3_0}\bar{\psi}_{\alpha,a}(u)M_{\alpha\beta}\psi_{\beta,a}(u)-\frac{1}{g_0^2}\sum_p \chi(g_p),$$

where here Σ runs over $u\in\mathbb{Z}^3_0$, $\epsilon=\pm 1$, and $\mu=0,1,2$.

We use the same notation and conventions as in Ref. [1] and adapt the treatment of symmetries given in Refs. [2,3]. We recall that the Fermi (one-flavor quark and antiquark) fields $\psi_{\alpha,a}(u)$ and $\bar{\psi}_{\alpha,a}(u)$, where $a=1,2,3$ is the color index, $\alpha=1,2\equiv +,-$ is the *spin* index, $u=(u^0,\vec{u})=(u^0,u^1,u^2)$, are defined on the lattice with half-integer time coordinates $u\in\mathbb{Z}^3_0\equiv\mathbb{Z}_{1/2}\times\mathbb{Z}^2$, where $\mathbb{Z}_{1/2}=\{\pm 1/2,\pm 3/2,...\}$. Letting e^{μ}, $\mu=0,1,2$, denote the unit lattice vectors, there is a gauge group matrix $U(g_{u+e^{\mu},u})=U(g_{u,u+e^{\mu}})^{-1}$ associated with the directed bond u, $u+e^{\mu}$, and we drop U from the notation.

The space $\mathcal{H}$ and the EM operators are obtained starting from gauge-invariant correlation functions, with support restricted to $u^0\geqslant 1/2$, by a standard construction. Letting $T_0^{x^0}$, $T_i^{x^i}$, $i=1,2$, denote translation of the functions of Grassmann and gauge variables by $x^0\geqslant 0$, $x\in\mathbb{Z}^3$, and for F and G only depending on coordinates with $u^0\geqslant 1/2$, we have the Feynman-Kac (FK) formula

$$(G,T_0^{x^0}T_1^{x^1}T_2^{x^2}F)_{\mathcal{H}}=\langle[T_0^{x^0}T_1^{x^1}T_2^{x^2}F]\Theta G\rangle,$$

where Θ is an antilinear operator which involves time reflection (see Ref. [2]). We do not distinguish between Grassmann, gauge variables, and their associated Hilbert space vectors in our notation. As linear operators in $\mathcal{H}$, T_{μ}, $\mu=0,1,2$, are mutually commuting; T_0 is self-adjoint, with $-1\leqslant T_0\leqslant 1$, and $T_{j=1,2}$ are unitary, so that we write $T_j=e^{iP^j}$ and $\vec{P}=(P^1,P^2)$ is the self-adjoint momentum operator, with spectral points $\vec{p}\in\mathbf{T}^2\equiv(-\pi,\pi]^2$. Since $T_0^2\geqslant 0$, we define the energy operator $H\geqslant 0$ by $T_0^2=e^{-2H}$ and refer to each point in the EM spectrum associated with zero momentum as mass. We work in the subspace $\mathcal{H}_e\subset\mathcal{H}$ generated by an even number of $\bar{\psi}$ or ψ.

To determine the meson bound-state spectrum, we first give spectral results for the meson and antimeson particles, which are the same using the symmetry results of Ref. [2] for charge conjugation. We introduce the meson fields (see Ref. [3]) $\Pi(u)=(1/\sqrt{3})\bar{\psi}_{a,-}(u)\psi_{a,+}(u)$ and the associated field $\mu(u)=(1/\sqrt{3})\psi_{a,-}(u)\bar{\psi}_{a,+}(u)$. Considering the FK formula for $(\Pi(1/2,\vec{x}_1),T_0^{|x^0|-1}\Pi(1/2,\vec{x}_2))_{\mathcal{H}}$, $x^0\neq 0$, we are led to define the associated two-point correlation function (χ is the characteristic function, $*$ is complex conjugation, and T means truncation)

$$G(u^0,\vec{x}_1;v^0,\vec{x}_2)=\chi_{u^0\leqslant v^0}\langle\mu(u^0,\vec{x}_1)\Pi(v^0,\vec{x}_2)\rangle^T + \chi_{u^0>v^0}[\langle\Pi(u^0,\vec{x}_1)\mu(v^0,\vec{x}_2)\rangle^T]^*,$$

where $x^0=v^0-u^0\in\mathbb{Z}$. By translation invariance and with an abuse of notation, $G(u,v)=G(v-u)$. The function $G(x)$, $x^0\neq 0$, admits the spectral representation

$$G(x)=\int_{-1}^{1}\int_{\mathbf{T}^2}(\lambda^0)^{|x^0|-1}e^{i\vec{\lambda}\cdot\vec{x}}d_{\lambda^0}\alpha_{\vec{\lambda}}(\lambda^0)d\vec{\lambda},$$

where $d_{\lambda^0}\alpha_{\vec{\lambda}}(\lambda^0)d\vec{\lambda}=d_{\lambda^0}d_{\vec{\lambda}}(\Pi,\mathcal{E}(\lambda^0,\vec{\lambda})\Pi)_{\mathcal{H}}$ and $\mathcal{E}$ is the product of the spectral families for the energy and momentum component operators. For its Fourier transform $\tilde{G}(p)=\Sigma_{x\in\mathbb{Z}^3}e^{-ip\cdot x}G(x)$, $p=(p^0,\vec{p})\in\mathbf{T}^3$, we get $\tilde{G}(p)=\tilde{G}(\vec{p})+(2\pi)^2\int_{-1}^{1}\int_{\mathbf{T}^2}f(p^0,\lambda^0)\ \delta(\vec{p}-\vec{\lambda})\ d_{\lambda^0}\alpha_{\vec{\lambda}}(\lambda^0)d\vec{\lambda}$, where $f(x,y)\equiv(e^{ix}-y)^{-1}+(e^{-ix}-y)^{-1}$, $\tilde{G}(\vec{p})=\Sigma_{\vec{x}\in\mathbb{Z}^2}\times e^{-i\vec{p}\cdot\vec{x}}G(x^0=0,\vec{x})$, $\Pi=\Pi(1/2,\vec{0})$. The associated dispersion relation is

$$w(\vec{p})=-2\ln\kappa+r(\kappa,\vec{p}) =-2\ln\kappa+\ln\left[1-2\frac{\kappa^2}{4}(\cos p^1+\cos p^2)\right]+\mathcal{O}(\kappa^3),$$

with $r(\kappa,\vec{p})$ real analytic in κ and each component p^j ($j=1,2$). Clearly, $w(\vec{p})\approx m_{\kappa}+(\kappa^2/4)|\vec{p}|^2$, $|\vec{p}|\leqslant 1$, where $m_{\kappa}\equiv w(\vec{0})$ is the meson mass. Furthermore, separating the one-particle contribution, the spectral measure has the decomposition $d_{\lambda^0}\alpha_{\vec{\lambda}}(\lambda^0)=Z(\vec{\lambda})\delta(\lambda^0-e^{-w(\vec{\lambda})})d\lambda^0+d\nu(\lambda^0,\vec{\lambda})$, where, for $\tilde{\Gamma}(p)=\tilde{G}(p)^{-1}$, we have $Z(\vec{p})^{-1}=-(2\pi)^2e^{w(\vec{p})}(\partial\tilde{\Gamma}/\partial\chi)(p^0=i\chi,\vec{p})|_{\chi=w(\vec{p})}$, such that $Z(\vec{p})=(2\pi)^{-2}e^{-w(\vec{p})}+\mathcal{O}(\kappa^3)$, with $Z(\vec{p})$ also real analytic in κ and p^j, $j=1,2$. The λ^0 support of $d\nu(\lambda^0,\vec{\lambda})$ is contained in $|\lambda^0|\leqslant|\kappa|^{4-\epsilon}$, $\epsilon>0$, and $\int_{-1}^{1}d\nu(\lambda^0,\vec{\lambda})\leqslant\mathcal{O}(\kappa^3)$. Points in the spectrum occur as p^0 singularities of $\tilde{G}(p)$, for fixed $\vec{p}$, and the meson mass points occur as singularities for $p^0=\pm iw(\vec{p})$. Our analysis shows that points of the form $p^0=\pi+i\chi$, $|\chi|<-(4-\epsilon)\ln\kappa$, are regular. Notice that the above measure decomposition shows the dispersion curve is isolated up to $-(4-\epsilon)\ln\kappa$ (upper gap property), making possible the particle identification. The isolated dispersion curve in the EM spectrum associated with the Π field is the *only* spectrum in $\mathcal{H}_e$, up to mass $-(4-\epsilon)\ln\kappa$. This can be shown by adapting the subtraction method of Ref. [2].

Concerning the symmetries, we follow Ref. [2]. For a spatial rotation of $\pi/2$ in the x^1, x^2 plane, given by $\mathcal{R}e^1=e^2$, $\mathcal{R}e^2=-e^1$, the fields $\Pi(u)$ and $\mu(u)$ are transformed to $i\Pi(u^0,\mathcal{R}\vec{u})$ and $-i\mu(u^0,\mathcal{R}\vec{u})$, respectively. The improper zero-momentum state $\Sigma_{\vec{u}}\Pi(u^0,\vec{u})$ transforms under the irreducible representation of the group Z^4 generated by i. Under the local charge conjugation symmetry, Π and μ are multiplied by (-1) as is also the case for parity and the expectations are invariant. Thus, using either charge conjugation or parity, we have $\langle\Pi(u)\rangle=\langle\mu(u)\rangle=0$, and the truncation in the definition of $G(x)$ can be dropped. As remarked before, Π is its own antiparticle.

To determine the existence of meson-meson Π-Π bound states, we consider the states generated by $\Pi(1/2,\vec{x}_1)\Pi(1/2,\vec{x}_2)$. From the FK formula, we have

$(\Pi(1/2,\vec{u}_1)\Pi(1/2,\vec{u}_2), \quad (T^0)^{|x^0|-1}\vec{T}^{\vec{x}}\Pi(1/2,\vec{u}_3)\Pi(1/2,\vec{u}_4))_{\mathcal{H}}$ for $x^0\neq 0$, where $\mathcal{G}(x)=\mathcal{G}(u_1,u_2,u_3+\vec{x},u_4+\vec{x})$, with $x=(x^0=v^0-u^0,\vec{x})\in\mathbb{Z}^3$, and, for $u_1^0=u_2^0=u^0$ and $u_3^0=u_4^0=v^0$,

$$\mathcal{G}(u_1,u_2,u_3,u_4)=\langle\mu(u_1)\mu(u_2)\Pi(u_3)\Pi(u_4)\rangle\chi_{u^0\leqslant v^0} + \langle\Pi(u_1)\Pi(u_2)\mu(u_3)\mu(u_4)\rangle^*\chi_{u^0>v^0}.$$

We now give a rough description of our method before going into detail. We first obtain a spectral representation for $\mathcal{G}(x)$ and its Fourier transform $\tilde{\mathcal{G}}(k)$. In this way, we can relate k singularities in $\tilde{\mathcal{G}}(k)$ to the EM spectrum. Next, using a lattice BS equation in a ladder approximation (see below), we look for the singularities of $\tilde{\mathcal{G}}(p)$ below the two-meson threshold.

Taking the Fourier transform and inserting the spectral representations for T^0, T^1, and T^2, we have

$$\tilde{\mathcal{G}}(k)=\tilde{\mathcal{G}}(\vec{k})+(2\pi)^2\int_{-1}^{1}\int_{\mathbf{T}^2} f(k^0,\lambda^0)\,\delta(\vec{k}-\vec{\lambda}) \times d_\lambda d_{\vec{\lambda}}(\Pi(1/2,\vec{u}_1)\Pi(1/2,\vec{u}_2), \times\mathcal{E}(\lambda^0,\vec{\lambda})\Pi(1/2,\vec{u}_3)\Pi(1/2,\vec{u}_4))_{\mathcal{H}},$$

where $\tilde{\mathcal{G}}(\vec{k})=\Sigma_{\vec{x}\in\mathbf{T}^2}e^{-i\vec{k}\cdot\vec{x}}\mathcal{G}(x^0=0,\vec{x})$. The singularities in $\tilde{\mathcal{G}}(k)$, for $k=(k^0=i\chi,k=0)$ and $e^{\pm\chi}\leqslant 1$, are points in the mass spectrum—i.e., the EM spectrum at system momentum zero.

To analyze $\tilde{\mathcal{G}}(k)$, we follow the analysis for spin models as in Ref. [8]. We relabel the time direction coordinates in $\mathcal{G}(x)$ by integer labels, with $u_i^0-1/2=x_i^0$, $\vec{u}_i=\vec{x}_i$, $i=1,...4$, and write $D(x_1,x_2,x_3+\vec{x},x_4+\vec{x})$, $x_1^0=x_2^0$ and $x_3^0=x_4^0$, $x^0=x_3^0-x_2^0$, where x_i and x are points on the $\mathbb{Z}^3$ lattice. Now we pass to difference coordinates and then to lattice relative coordinates $\xi=x_2-x_1$, $\eta=x_4-x_3$, and $\tau=x_3-x_2$ to obtain $D(x_1,x_2,x_3+\vec{x},x_4+\vec{x})=D(0,x_2-x_1,x_3-x_1+\vec{x},x_4-x_1+\vec{x})\equiv D(\vec{\xi},\vec{\eta},\tau+\vec{x})$ and $\tilde{\mathcal{G}}(k)=e^{ik\cdot\vec{\tau}}\hat{D}(\vec{\xi},\vec{\eta},k)$, where $\hat{D}(\vec{\xi},\vec{\eta},k)=\Sigma_{\tau\in\mathbb{Z}^3}D(\vec{\xi},\vec{\eta},\tau)e^{-ik\cdot\tau}$. Explicitly, we have

$$\begin{aligned}D(x_1,x_2,x_3,x_4) &=\langle\mu(x_1^0+1/2,\vec{x}_1)\mu(x_2^0+1/2,\vec{x}_2)\\ &\quad\times\Pi(x_3^0+1/2,\vec{x}_3)\Pi(x_4^0+1/2,\vec{x}_4)\rangle\chi_{x_2^0\leqslant x_3^0}\\ &\quad+\langle\Pi(x_1^0+1/2,\vec{x}_1)\Pi(x_2^0+1/2,\vec{x}_2)\\ &\quad\times\mu(x_3^0+1/2,\vec{x}_3)\mu(x_4^0+1/2,\vec{x}_4)\rangle^*\chi_{x_2^0>x_3^0}.\end{aligned}$$

The point of all this is that the singularities of $\tilde{\mathcal{G}}(k)$ are the same as those of $\hat{D}(\vec{\xi},\vec{\eta},k)$ and the BS equation for $\hat{D}(\vec{\xi},\vec{\eta},k)$ and its analysis are familiar (see Refs. [8,9]).

The BS equation in operator form and in what we call the equal time representation is $D=D_0+D_0KD$. In terms of kernels, with $x_1^0=x_2^0$ and $x_3^0=x_4^0$,

$$\begin{aligned}D(x_1,x_2,x_3,x_4)&=D_0(x_1,x_2,x_3,x_4)\\ &\quad+\int D_0(x_1,x_2,y_1,y_2)K(y_1,y_2,y_3,y_4)\\ &\quad\times D(y_3,y_4,x_3,x_4)\,\delta(y_1^0-y_2^0)\\ &\quad\times\delta(y_3^0-y_4^0)dy_1dy_2dy_3dy_4,\end{aligned}$$

where $D_0(x_1,x_2,x_3,x_4)=G(x_1,x_3)G(x_2,x_4)+G(x_1,x_4)\times G(x_2,x_3)$, and we use a continuum notation for sums over lattice points. D, D_0, and $K=D_0^{-1}-D^{-1}$ are to be taken as matrix operators acting on $\ell_2^s(\mathcal{A})$, the symmetric subspace of $\ell_2(\mathcal{A})$, where $\mathcal{A}=\{(x_1,x_2)\in\mathbb{Z}^3\times\mathbb{Z}^3/x_1^0=x_2^0\}$. In terms of the $(\vec{\xi},\vec{\eta},\tau)$ relative coordinates and taking the Fourier transform in τ, the BS equation becomes (see Ref. [8]) $\hat{D}(\vec{\xi},\vec{\eta},k)=\hat{D}_0(\vec{\xi},\vec{\eta},k)+\int\hat{D}_0(\vec{\xi},\vec{\xi}',k)\hat{K}(-\vec{\xi}',-\vec{\eta}',k)\times\hat{D}(\vec{\eta}',\vec{\eta},k)d\vec{\xi}'d\vec{\eta}'$. With k fixed, $\hat{D}(\vec{\xi},\vec{\eta},k)$, etc., is taken as a matrix operator on $\ell_2(\mathbb{Z}^2)$, for $k=(k^0,\vec{k}=\vec{0})$ on the even subspace of $\ell_2(\mathbb{Z}^2)$. The kernel $\hat{K}(-\vec{\xi}',-\vec{\eta}',k)$, in general, acts as an energy-dependent nonlocal potential in the nonrelativistic lattice Schrödinger operator analogy.

The key to successfully solve the BS equation is to obtain appropriate decay properties for the kernel of K. In particular, we want a temporal decay faster than the two-particle decay, here $\kappa^{4|x_3^0-x_1^0|}$. The typical tool employed to obtain the decays is the hyperplane decoupling method (see Refs. [8,9] and references therein). Next, we look for a solution to the approximate equation where K is replaced by its first nonvanishing order in κ, which is commonly called a *ladder* approximation L to K. These ingredients together with the control of perturbations to the ladder approximation lead to a rigorous solution of the BS equation and two-particle spectral results for the complete model as in Ref. [9].

For our case and as is well known, D does not have temporal decay, due to vacuum contributions. Even after subtracting out these contributions and using the hyperplane decoupling method, we obtain only a κ^4 falloff of K for temporal distance one. This would force us to deal with an energy-dependent potential in the ladder approximation. To avoid these complications, replace the kernel $D(x_1,x_2,x_3,x_4)$ by $h(x_1,x_2)D'(x_1,x_2,x_3,x_4)h(x_3,x_4)$ in *all* expressions above, where $h(x_i,x_j)=c\,\delta(x_i-x_j)+[1-\delta(x_i-x_j)]$, $c=\sqrt{3/2}$, and D' is the partially truncated D, obtained by subtracting, such as for $x_2^0\leqslant x_3^0$, $\langle\mu(x_1^0+1/2,\vec{x}_1)\mu(x_2^0+1/2,\vec{x}_2)\rangle\langle\Pi(x_3^0+1/2,\vec{x}_3)\Pi(x_4^0+1/2,\vec{x}_4)\rangle$ from $D(x_1,x_2,x_3,x_4)$. From now on, without changing the notation, we assume these replacements have been made. Note that when, e.g., $x_1=x_2$ in the expression for $\Pi(x_1)\Pi(x_2)$, the color-diagonal quark terms are zero by Pauli exclusion, essentially producing a change in normalization. For the case of a noncomposite boson field, this does not occur, and for the case of the composite baryon field ϕ, with one-flavor quarks as in Ref. [1], we have $[\phi(x)]^2=0$ by Pauli exclusion. Our case is intermediate, and with this modified D, D_0 behaves as a good unperturbed part of it. Finally, with this new D, K has the appropriate decay

$\kappa^{6|x_3^0-x_1^0|}$ and the coincident point contribution is zero until $\mathcal{O}(\kappa^2)$.

We now obtain our ladder approximation L to K (see Ref. [4]). L is given by the κ^2 contribution to $D_0^{-1}D^sD_0^{-1}$, the first nonvanishing term in the Neumann series for $K=D_0^{-1}-[D_0+D^s]^{-1}$, where $D^s=D-D_0$. We obtain, with $c''(\kappa)=(\sqrt{2/3}-1)\kappa^2/8$, $L(x_1,x_2,x_3,x_4)=c''(\kappa)\Sigma_{j=1,2;\epsilon=\pm1}[\delta(x_2-x_1+\epsilon e^j)\delta(x_3-x_2)\delta(x_4-x_3)+\delta(x_3-x_1)\delta(x_4-x_1)\delta(x_2-x_1-\epsilon e^j)+\delta(x_2-x_1)\delta(x_3-x_1-\epsilon e^j)\delta(x_4-x_1)+\delta(x_2-x_1)\delta(x_3-x_1)\delta(x_4-x_1-\epsilon e^j)]$, and, in relative coordinates and for $c'(\kappa)=2c''(\kappa)$, $\hat{L}(\vec{\xi},\vec{\eta},k^0)=c'(\kappa)\Sigma_{j,\epsilon}[\delta(\vec{\xi})\delta(\vec{\eta}-\epsilon e^j)+\delta(\vec{\xi}-\epsilon e^j)\delta(\vec{\eta})]$, with the abbreviated notation k^0 for $k=(k^0,\vec{k}=\vec{0})$, which we omit below. In the lattice Schrödinger operator analogy, $\hat{L}$ corresponds to a nonlocal energy-independent potential, where we recall that a local potential is given by contributions to $L(x_1,x_2,x_3,x_4)$ with $x_1=x_3$ and $x_2=x_4$. The nonlocal potential, which occurs here, results from the κ^2 contribution with one (three) of the fields in D and D_0 at a point y, for example, and the other three (one) at the nearest-neighbor points $y+\epsilon e^j$.

Next, we derive the solution of the BS equation in the ladder approximation which reads

$$\hat{D}(\vec{\xi},\vec{\eta})=\hat{D}_0(\vec{\xi},\vec{\eta})+c'(\kappa)\sum_{j,\epsilon}[\hat{D}_0(\vec{\xi},\vec{0})\hat{D}(\epsilon e^j,\vec{\eta})+\hat{D}_0(\vec{\xi},\epsilon e^j)\hat{D}(\vec{0},\vec{\eta})].$$

Using $\hat{D}_0(\pm\epsilon e^i,\vec{0})=\hat{D}_0(\vec{0},\mp\epsilon e^i)$ and $\hat{D}_0(\epsilon e^i,\epsilon' e^j)=\hat{D}_0(\epsilon e^j,\epsilon' e^i)$, $i,j=1,2$ and ϵ, $\epsilon'=\pm1$, we only need to determine $\hat{D}(\vec{0},\vec{\eta})$ and $\hat{D}(e^i,\vec{\eta})$, $i=1,2$, in $\hat{D}(\vec{0},\vec{\eta})=\hat{D}_0(\vec{0},\vec{\eta})+2c'(\kappa)\{\hat{D}_0(\vec{0},\vec{0})\Sigma_j\hat{D}(e^j,\vec{\eta})+2\hat{D}_0(e^1,\vec{0})\hat{D}(\vec{0},\vec{\eta})\}$ and $\hat{D}(e^i,\vec{\eta})=\hat{D}_0(e^i,\vec{\eta})+2c'(\kappa)\Sigma_j[\hat{D}(e^j,\vec{\eta})\hat{D}_0(e^i,\vec{0})+\hat{D}(e^i,e^j)\hat{D}(\vec{0},\vec{\eta})]$. Letting $\mathcal{M}$ denote the 2×2 matrix with entries $\mathcal{M}_{11}=\mathcal{M}_{22}=4c'(\kappa)\hat{D}_0(\vec{0},e^1)$, $\mathcal{M}_{12}=4c'(\kappa)\hat{D}_0(\vec{0},\vec{0})$, and $\mathcal{M}_{21}=2c'(\kappa)[\hat{D}_0(e^1,e^1)+\hat{D}_0(e^1,e^2)]$, the only singularities of $\hat{D}(\vec{0},\vec{\eta})$ and $\hat{D}(e^{j=1,2},\vec{\eta})$, on the imaginary k^0 axis, below the two-meson threshold, occur as zeros of $\det[I-\mathcal{M}]$. The same holds for $\hat{D}(\vec{\xi},\vec{\eta})$.

To analyze the BS equation, we develop an approximate correspondence with a one-particle lattice Schrödinger operator resolvent equation $(H-z)^{-1}=(H_0-z)^{-1}-\lambda(H_0-z)^{-1}V(H-z)^{-1}$, where $H=H_0+\lambda V$, $H_0=-a\Delta/2$ [Δ is the lattice Laplacian on $\ell_2(\mathbb{Z}^2)$], and V is a nonlocal potential. To obtain this correspondence, we keep only the product of one-meson contributions to $\hat{D}_0(\vec{\xi},\vec{\eta},k^0)$ to get

$$\hat{D}_0(\vec{\xi},\vec{\eta},k^0)\simeq2(2\pi)^2\int_{\mathbf{T}^2}\frac{Z(\vec{p})^2\cos\vec{p}\cdot\vec{\xi}\cos\vec{p}\cdot\vec{\eta}}{e^{ik^0}-e^{-2w(\vec{p})}}d\vec{p}.$$

Using the small distance behavior of the two-point function G, we find $Z(\vec{p})\simeq(2\pi)^{-2}c_{20}^{-1}e^{-w(\vec{p})}$, $c_{20}=1$, where $c_{20}\kappa^2$ is the κ^2 contribution to $\langle\mu(0)\Pi(e^0)\rangle$. Writing $w(\vec{p})\simeq m_\kappa+c_m\kappa^2(-\tilde{\Delta}(\vec{p}))$, $c_m=1/4$, where $-\tilde{\Delta}(\vec{p})=2\Sigma_{j=1,2}(1-\cos p^j)$, and letting $k^0=i(2m_\kappa-\epsilon')$ so that $\epsilon'>0$ is the meson-meson binding energy,

$$\hat{D}_0(\vec{\xi},\vec{\eta},k^0)\simeq\frac{2}{(2\pi)^2}\int_{\mathbf{T}^2}\frac{\cos\vec{p}\cdot\vec{\xi}\cos\vec{p}\cdot\vec{\eta}}{2c_m\kappa^2[-\Delta(\vec{p})]+\epsilon'}d\vec{p}.$$

With these approximations in the ladder BS equation, we make the identifications $a=\kappa^2/2$, $\lambda=[1-\sqrt{2/3}]\kappa^2/4$, $V=\Sigma_{j,\epsilon}[\delta(\vec{\xi})\delta(\vec{\eta}-\epsilon e^j)+\delta(\vec{\xi}-\epsilon e^j)\delta(\vec{\eta})]$, and $z=-\epsilon'/2$ in the lattice Schrödinger resolvent equation, acting on the even subspace of $\ell_2(\mathbb{Z}^2)$.

We now turn to the determination of the negative energy bound-state spectrum of $H'=-\Delta/2+\alpha V=a^{-1}H$, $\alpha=\lambda/a=(1-\sqrt{2/3})/2$, first for any α real. As a self-adjoint operator V has the discrete spectrum 2, -2, and 0 with multiplicities 1, 1, and ∞, respectively. As $-\Delta/2$ is bounded, with absolutely continuous spectrum [0, 4], H' has a negative energy bound state with binding energy $\approx2|\alpha|$, for large $|\alpha|$, regardless of the sign of α. The resolvent equation for H' is solved in the same way as the ladder BS equation and leads to the determinant zero condition $(1+4\alpha R_{01})^2-8\alpha^2R_{00}(R_{11}+R_{12})=0$, for a bound state of binding energy b, where, for $R_0(\vec{\xi},\vec{\eta})=(2\pi)^{-2}\int_{\mathbf{T}^2}[\cos\vec{p}\cdot\vec{\xi}\cos\vec{p}\cdot\vec{\eta}]/[\Sigma_{j=1,2}(1-\cos p^j)+b]d\vec{p}$, we set $R_{00}=R_0(\vec{0},\vec{0})$, $R_{0j}=R_0(\vec{0},e^j)$, and $R_{ij}=R_0(e^i,e^j)$. Using the identities $R_{01}=-1/2+(1/2)(2+b)R_{00}$ and $R_{11}+R_{12}=(2+b)R_{01}$, obtained by multiplying the integrand of R_{00} by the denominator and also the denominator squared of R_{00}, the bound-state equation becomes $(1-2\alpha)^2+4\alpha(1-\alpha)(2+b)R_{00}=0$. As R_{00} is positive, there is no solution for $0<\alpha<1$; there is a solution for $\alpha<0$ and $\alpha>1$. For our meson-meson problem, $0<\alpha=[1-\sqrt{2/3}]/2<1$, so there is *no* meson-meson bound state in our model.

This work was supported by CNPq and FAPESP. A.F.N. thanks the CNPq for a doctoral scholarship.

[1] P. A. Faria da Veiga, M. O'Carroll, and R. Schor, Phys. Rev. D **67**, 017501 (2003).
[2] P. A. Faria da Veiga, M. O'Carroll, and R. Schor, Commun. Math. Phys. **245**, 383 (2004).
[3] A. Francisco Neto, P. A. Faria da Veiga, and M. O'Carroll, J. Math. Phys. **45**, 628 (2004).
[4] P. A. Faria da Veiga, M. O'Carroll, and R. Schor, Phys. Rev. D **68**, 037501 (2003).
[5] H. R. Fiebig, H. Markum, A. Mihály, and K. Rabitsch, Nucl. Phys. B (Proc. Suppl.) **53**, 804 (1997).
[6] C. Stewart and R. Koniuk, Phys. Rev. D **57**, 5581 (1998).
[7] H. R. Fiebig and H. Markum, in *International Review of Nuclear Physics*, edited by A. M. Green (World Scientific, Singapore, 2003), Vol. 9.
[8] R. S. Schor and M. O'Carroll, Phys. Rev. E **62**, 1521 (2000); J. Stat. Phys. **99**, 1207 (2000); **99**, 1265 (2000); **109**, 279 (2002).
[9] P. A. Faria da Veiga, M. O'Carroll, E. Pereira, and R. Schor, Commun. Math. Phys. **220**, 377 (2001).