Renormalization in reparametrization invariance

Yu-Qi Chen
Institute of Theoretical Physics, Academia Sinica, Beijing 100080, People's Republic of China
(Received 27 July 2002; published 18 May 2004)

The renormalization issue in the reparametrization invariance in heavy quark effective theory and NRQCD is investigated. I argue that the renormalization of the transformation of the heavy quark field under the variation of the velocity parameter V is attributed to the renormalization of the small component field in the proposed transformation. I derive the matching conditions for determining the renormalized small component field by imposing an infinitesimal transformation of V on the relations between the Green's functions in QCD full theory and those in the effective theory. As an application, I determine the renormalized transformation to order $1/m^2$ using the matching conditions. The obtained result is in disagreement with that determined by the indirect method.

DOI: 10.1103/PhysRevD.69.096001 PACS number(s): 11.10.Gh, 11.10.Ef, 11.30.Ly

I. INTRODUCTION

Heavy quark effective theory (HQET) [1] and nonrelativistic QCD (NRQCD) [2] are powerful tools in dealing with the dynamics of heavy-light and heavy-heavy systems, respectively. In those systems, the off-shell momentum of the heavy quark is much smaller than its mass. The effective theories are designed to reproduce the results of the QCD full theory at the low energy scale in a simpler way by integrating out the effects at the energy scale of the heavy quark mass. In the past decade both effective theories and their applications have been intensively studied.

A. What is the reparametrization invariance

One interesting theoretical issue in those effective theories is reparametrization invariance (RPI). It arises from the fact that the effective theory explicitly depends on the four velocity parameter V. In constructing the effective Lagrangian, one needs to divide the heavy quark momentum P into a large and small part as $P=mV+k$, where m is the heavy quark mass and k is a small residual momentum. One also needs to decompose the Dirac 4-fermion field as large and small two-component fields with respect to V and use the large one to describe the heavy quark or antiquark. These procedures lead to the effective Lagrangian being V dependent. The choice of V which satisfies $V^2=1$ is not unique. But the physical prediction should be unchanged against the variation of the velocity parameter V. This is the RPI. It is required by the consistency of effective theory and also conducts interesting applications. It was first proposed in HQET. However, the same invariance also holds for NRQCD effective theory.

B. A brief review of previous studies

To implement RPI in the effective theory, it is essential to find out an appropriate transformation of the heavy quark field under the variation of V. It was first studied by Manohar and Luke [3] in HQET. They used the Lorentz boost of the four component spinor field as the transformation of the heavy quark field from finite velocity V to V'. Their transformation suffers from operator ordering ambiguities when it is expanded to higher orders of $1/m$. Later on, Manohar [4] discussed its higher order expansion. Chen [5] proposed an infinitesimal transformation of the heavy quark field under the velocity variation from $V\to V+\Delta V$. Chen's transformation keeps the tree level effective Lagrangian invariant to all orders of $1/m$. Finkemeier, Georgi, and McIrvin [6] showed that to order $1/m^2$ the effective Lagrangian constrained by Manohar and Luke's transformation and Chen's transformation may be related to each other by a field redefinition.

Chen's transformation can be expanded as inverse power series of the heavy quark mass. Each term contains the product of some covariant derivatives and the heavy quark field. It can be thought of as a composite operator. Since the heavy quark expansion changes the ultraviolet behavior of the original transformation, beyond the next leading order, it turns out that it needs to be renormalized. Its renormalization is different from that of the effective Lagrangian since each term in the Lagrangian is a bilinear function of the heavy quark field. The renormalization of the operators in the Lagrangian is familiar to us and we have appropriate matching conditions to determine the coefficients of them. However, the renormalization of the composite operators in the transformation is a new case. How to carry out its renormalization is not obvious.

Kilian and Ohl [7] proposed a renormalized transformation. The form is exactly the same as Chen's transformation except that the covariant derivative D^μ in Chen's transformation is substituted by another operator which they called the general covariant derivative. Sundrum [8] discussed this issue using the auxiliary field method and obtained a similar result as that of Kilian and Ohl. The results presented in these papers are formal. They did not show how to determine the general covariant derivative by certain matching conditions. Actually, in the literature no specific calculation for determining the transformation has been made with this method. The only calculation to determine the renormalized transformation was given by Balzereit in an unpublished paper [9]. He first calculated the effective Lagrangian to order $1/m^3$ at one loop level in the leading logarithmic approximation. By requiring that the effective Lagrangian be invariant, he could then determine the renormalized transformation to order $1/m^2$ indirectly. There are some drawbacks to this kind of calculation. First, since it is an indirect determination, one is not able to get too many insights into the renormalization issue in the RPI. Second, it makes the RPI less practically useful. An interesting application of the RPI is that once we know the transformation, we can use it to constraint the higher order effective Lagrangian and make the calculations simpler [3,5,6]. Since Balzereit did it in an inverse order, RPI cannot be used to constraint the effective Lagrangian with that method. Third, the calculations are quite complicated since the determination of the higher order effective Lagrangian is usually tedious work.

C. New method to renormalize the transformation

In this paper, I propose a new method to study the renormalization issue in the RPI. In this method, I derive explicit matching conditions to determine the coefficients of the new composite operators in the transformation so that the renormalized transformation can be directly calculated to all orders of $1/m$ and α_s without knowing the higher order effective Lagrangian. The obtained transformation can be used to constrain the effective Lagrangian in $1/m$ expansion.

Recall that there are some general relations between the Green's functions in the QCD full theory and those in the effective theory [1]. These relations ensure that the effective theory reproduces the same physical predictions as the full theory and can be used as the matching conditions to determine the renormalized effective Lagrangian. Since these relations are valid for arbitrary velocity parameter V, we may impose an infinitesimal transformation of V on both sides of the relations and gain some new relations. We will show that the composite operators in the heavy quark transformation are inserted on the effective theory side of these relations and the new relations obtained can be used as the matching conditions to determine the renormalized transformation.

As a specific example, I will use these matching conditions to determine the renormalized transformation to order $1/m^2$ at one-loop level. The obtained result is in disagreement with that obtained by Balzereit [9]. Since the same Lagrangian (MRR Lagrangian) is used, the disagreement between these two different results cannot be accounted for by a field redefinition.

I will also show that the renormalized transformation determined by these matching conditions can be written in the form of Chen's transformation with the covariant derivative substituted by an operator which may be called a general covariant derivative. Thus the renormalized transformation determined by the matching conditions presented in this paper is consistent with the result of Kilian and Ohl [7].

The remainder of the paper is organized as follows. In Sec. II, after a brief review of the tree level transformation, I argue that the renormalized transformation of the heavy quark field can be attributed to the renormalization of the small component field. I then show that the matching conditions for determining it can be obtained by imposing an infinitesimal transformation on the general relations between the Green's functions in the QCD full theory and those in the effective theory. As an example, in Sec. III, I determine the renormalized transformation to order $1/m^2$ by matching the two-point and three-point functions. In Sec. IV, I show that previous results can be understood more clearly by constructing the effective Lagrangian in an alternative way, where a four-component effective Lagrangian is constructed first, followed by its reduction to the effective Lagrangian in the two-component field. I then show that the remormalized small component field determined by the matching conditions is consistent with the result of Kilian and Ohl [7]. Then I show that the renormalized effective Lagrangian is reparametrization invariant under the renormalized transformation. Conclusions are given in Sec. V. Finally, in the Appendix, I derive the general relations between the Green's functions with the generating functional method.

II. RENORMALIZED TRANSFORMATION OF THE HEAVY QUARK FIELD

In the heavy quark effective theory, the heavy quark is described by a two component field while in QCD full theory it is described by the Dirac four-component field. Thus, to construct the effective theory, one first needs to decompose the Dirac four-component field as the two-component fields. A simple way to realize this decomposition is

$$h_{V\pm}(x) \equiv \exp(imV\cdot x)P_{\pm}\Psi(x), \tag{1}$$

where

$$P_{\pm} \equiv \frac{1\pm \not{V}}{2} \tag{2}$$

are the projection operators. The introduced phase factor just removes the large part mV from the heavy quark momentum p when it is written as $p=mV+k$, with k the residue momentum. This definition of the field was first introduced by Georgi [1] and has been used by most people [1] in the literature. Nevertheless, it is not unique. Different definitions lead to different forms of the effective Lagrangian. However, they can be related to each other by a field redefinition and produce the same physical predictions [6]. Throughout this paper, we use the definition (1).

A. The tree level transformation

With the definition in Eq. (1), the effective Lagrangian reads [5,13]

$$\mathcal{L}^0_{\rm eff} = \bar{h}_{V+}(x) iD\cdot V h_{V+}(x) - \bar{h}_{V+}(x)\not{D}\frac{1}{2m+iD\cdot V}P_-\not{D}h_{V+}(x), \tag{3}$$

where $D^\mu \equiv \partial^\mu - ig_s A^\mu$ is the covariant derivative. This is the nonlocal form of the effective theory. The effective theory in this form is equivalent to that of the full theory in the sense that they produce the same S-matrix elements. Without expansion, the effective Lagrangian does not receive renormalization.

Obviously, this effective Lagrangian depends on the velocity parameter V. The choice of this parameter is not unique. The RPI implies that the physical predictions by the effective theory are independent of the choice of V. In Ref. [5], it was shown that the effective Lagrangian (3) is invariant under an infinitesimal transformation $V \to V+\Delta V$

$$\Delta h_{V+}(x) = \frac{\Delta\!\!\!/ V}{2}[h_{V+}(x)+h_{V-}(x)], \tag{4}$$

with $h_{V-}(x)$ being the small component field given by

$$h_{V-}(x) = \frac{1}{2m+iD\cdot V}P_{-}iD\!\!\!/ h_{V+}(x). \tag{5}$$

ΔV is constrained by $\Delta V\cdot V=0$ due to $V^2=1$.

Both the effective Lagrangian (3) and the transformation given by Eqs. (4) and (5) can be expanded as a power series of $1/m$. The RPI is then valid order by order in $1/m$. This implies that the tree level transformation makes the tree level effective Lagrangian valid at any order of $1/m$.

B. Matching conditions for renormalizing the transformation

When the effective theory is expanded in terms of $1/m$, the ultraviolet behavior of the theory is changed. Both the effective Lagrangian and the transformation receive renormalization.

The renormalization procedure of the effective Lagrangian is well known. The counterterms are bilinear operators of the heavy quark field. In Feynman diagrams, contributions from these operators can be expressed as insertions of corresponding vertices on the heavy quark lines. In order that the effective theory reproduces the results of the QCD full theory, the Green's functions in the effective theory with the heavy quark field defined in Eq. (1) and those in QCD full theory are required to satisfy certain relations. The coefficients of those operators can be fixed by virtue of these relations. Thus those relations can be used as the matching conditions to determine the renormalized effective Lagrangian.

The renormalization of the transformation is a new case since each composite operator is a linear function of the heavy quark. It can only be inserted at the endpoints of the quark line as tadpole diagrams. Thus, to renormalize the transformation, one needs to seek matching conditions with insertions of such operators at the endpoints of the quark line on the side of the effective theory. Fortunately, it is found that these matching conditions can be obtained by imposing an infinitesimal transformation of V on both sides of the matching conditions for renormalizing the effective Lagrangian.

Now let us first look at the relations between the Green's functions in the full QCD and those in the effective theory. I denote the Green's functions in the full theory by $G(x,y;B)$ and those in the effective theory by $G_V(x,y;B)$, respectively, where B is an arbitrary background field. They are defined by

$$G(x,y;B) \equiv \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle^B \tag{6}$$

and

$$G_V(x,y;B) \equiv \langle 0|Th_{V+}(x)\bar{h}_{V+}(y)|0\rangle^B. \tag{7}$$

Any insertion of interaction vertex with gluons can be obtained by functional differentiating over the gluon field $B(x)$. When the quark field $h_{V+}(x)$ is related to the field in the full theory by Eq. (1), the Green's functions satisfy the following simple relation (a derivation of this relation using the generating functional method is given in Appendix):

$$G_V(x,y;B) \doteq P_+G(x,y;B)P_+, \tag{8}$$

where $\doteq$ means that we omit the phase factor $\exp[imV\cdot(x-y)]$ and the renormalization constant $Z[m/\mu,\alpha_s(\mu)]$ which arises from the renormalization of the heavy quark field. Both sides are valid to all orders in $1/m$ and α_s expansion. This relation ensures that the S-matrix elements in the effective theory are identical to those in the full theory.

The relation (8) is just the matching conditions to determine the coefficients of the operators in the effective Lagrangian. Below starting from this relation we derive the matching conditions to determine the renormalized transformation of the heavy quark field.

Obviously, the relation (8) is valid for arbitrary V. It implies that we may impose an infinitesimal transformation $V \to V+\Delta V$ on both sides. It follows that

$$\Delta G_V(x,y;B) \doteq \frac{\Delta\!\!\!/ V}{2}G(x,y;B)P_+ + P_+G(x,y;B)\frac{\Delta\!\!\!/ V}{2}. \tag{9}$$

Again the symbol $\doteq$ means that we omit the renormalization constant, the phase factor, and a term arising from its infinitesimal shift which is trivial under the transformation.

Given the definitions of the Green's functions in Eqs. (6), (7), we have the following unique solution to Eq. (9):

$$\langle 0|T\Delta h_{V+}(x)\bar{h}_{V+}(y)|0\rangle^B \doteq \frac{\Delta\!\!\!/ V}{2}\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle^B P_+. \tag{10}$$

$\Delta h_{V+}(x)$ in this equation defines the infinitesimal transformation of the heavy quark field. Equation (10) implies that $\Delta h_{V+}(x)$ is proportional to $\Delta\!\!\!/ V$. Thus we may write it in the following generic form as

$$\Delta h_{V+}(x) = \frac{\Delta\!\!\!/ V}{2}[P_+h'_{V+}(x)+P_-h'_{V-}(x)]. \tag{11}$$

Substituting it into Eq. (10), it can then be decomposed as two equations by the projection operators

$$\langle 0|Th'_{V+}(x)\bar{h}_{V+}(y)|0\rangle^B \doteq P_+\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle^B P_+, \tag{12}$$

$$\langle 0|Th'_{V-}(x)\bar{h}_{V+}(y)|0\rangle^B \doteq P_-\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle^B P_+. \tag{13}$$

We see that Eq. (12) is nothing but Eq. (8) if $h'_{V+}(x)$ is identical to $h_{V+}(x)$. Equation (13) is a new one which can be regarded as the definition of the $h'_{V-}(x)$. In this new equation, the right-hand side is still the projected Green's functions in the full theory while the left-hand side is the

Green's functions in the effective theory with the operator h'_{V-} inserted at the endpoint x. Both sides can be expanded in terms of $1/m$. With the effective Lagrangian, it is calculable order by order in $1/m$ and α_s expansion.

Equation (11) together with Eqs. (12) and (13) implies that the renormalized transformation keeps the same form as the tree-level transformation (4). But only the small component field needs to be renormalized.

Now let us illustrate how Eq. (13) determines the renormalized $h_{V-}(x)$ field as the matching conditions in the hard cutoff regularization. Similar arguments are applicable to the dimensional regularization.

Suppose one takes different hard cutoff regularization energy scales Λ_e in the effective theory and Λ_f in the full theory, respectively. The Λ_f should be much larger than the heavy quark mass m for including both quark and antiquark contributions. The Λ_e should be much smaller than m for making the $1/m$ expansion eligible. So they satisfy a hierarchy relation $\Lambda_e \ll m \ll \Lambda_f$. In calculating the tree level diagrams, the relation (13) is valid for arbitrary momenta of external gluons and heavy quarks with the tree level effective Lagrangian and the transformation. However, in calculating loop diagrams the integration bounds are different on both sides. (On the effective theory side, it is integrated out from 0 to Λ_e while on the full theory side it is integrated from 0 to Λ_f.) Thus, to make the relation (13) valid, one has to add the contributions of the loop integrals with the loop momenta from Λ_e to Λ_f to the effective theory side. Since the loop momentum in this region is larger than the external momenta of gluons and residue momentum of quarks, their contributions can be expressed as the insertions of the local operators on the heavy quark line either at a point between x to y or at the endpoint x. Those local operators not inserted at the endpoint x correspond to the counterterms in the renormalized effective Lagrangian. Those local operators inserted at the endpoint x correspond to the counter terms in the renormalized small component field $h'_{V-}(x)$. Since the effective Lagrangian can be fixed by the matching conditions (8), the renormalized small component field $h'_{V-}(x)$ can uniquely be determined by these matching conditions after subtracting the insertion of the high-dimensional operators in the renormalized effective Lagrangian. Therefore, Eq. (11) can be used as the matching conditions for determining the renormalized small component field $h_{V-}(x)$. Equation (13) allows one to determine the renormalized $h_{V-}(x)$ to any order of $1/m$ and α_s.

III. RENORMALIZED TRANSFORMATION TO ORDER $1/m^2$

As a specific example, in this section, we determine the renormalized effective Lagrangian and the $h_{V-}(x)$ field up to the next leading order corrections of $1/m$ using the matching conditions (8) and (13), respectively. For simplicity, we carry our calculations in the dimensional regularization and Feynman gauges.

Up to next leading order corrections of $1/m$, the most general form of the renormalized effective Lagrangian can be expressed as

$$\begin{aligned} L_V^1(x) = {} & Z\bar{h}_{V+}(x) iD\cdot V h_{V+}(x) + \frac{Z}{2m}\bar{h}_{V+}(x)\mathbf{D}^2 h_{V+}(x) \\ & + \frac{ZZ_e}{2m}\bar{h}_{V+}(x)(iD\cdot V)^2 h_{V+}(x) \\ & + \frac{ZZ_m}{4m} h_{V+}(x)\sigma^{\mu\nu}G_{\mu\nu}(x) h_{V+}(x), \end{aligned} \tag{14}$$

$\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$, and the most general form of the renormalized $h_{V-}(x)$ field can be written as

$$\begin{aligned} h_{V-}(x) = P_-\Bigg(& \frac{d_0(\mu)}{2m} i\not{D} + \frac{d_1(\mu)}{4m^2} D\cdot V\not{D} \\ & + \frac{d_2(\mu)}{4m^2}\not{D} D\cdot V\Bigg) h_{V+}(x), \end{aligned} \tag{15}$$

where $Z, Z_e, Z_m, d_0(\mu), d_1(\mu), d_2(\mu)$ are the short distance coefficients to be determined.

In the following matching procedures, all the short-distance coefficients are assumed to be calculated to all orders in α_s, which makes the matching procedure applicable for higher order calculations. But in this paper we only present the one-loop result in the final expression.

Since the off-shell momenta of the heavy quark and the momenta of the gluons are much smaller than the quark mass in the heavy quark limit, in the matching procedure, the integrand can be expanded as power series of these momenta over the heavy quark mass. This leads to the remainder part of the loop momentum integrals no longer depending on the heavy quark mass. We are free to choose the infrared regulator since the infrared divergences cancel on both sides. If we use a limitation order in which the external off-shell momenta of heavy quark and the momenta of the gluons go to zero first, followed by $\epsilon = 2 - D/2$ going to zero, as used by Eichten and Hill in Ref. [1], then all terms such as k^ϵ arising from the loop momentum integrals vanish. In the effective theory, this implies that all contributions from the loop diagrams vanish since all loop momentum integrals are proportional to k^ϵ while in the full theory it implies that there is no logarithmic nonlocal term of these external momenta. This simplifies the matching calculations significantly. It is easy to see that to determine these coefficients, we need to match both the two-point and three-point functions.

A. Matching the two-point function

Let us first match the two-point function and see what we can learn from it. For external momentum p of the heavy quark near mass shell, the generic form of the QCD heavy quark self-energy, the inverse of the two-point function, with above infrared regulator can be written as

$$\Sigma(p) = A(\not{p} - m) + 2Bm\Delta, \tag{16}$$

where Δ is defined as

$$\Delta \equiv \frac{p^2 - m^2}{4m^2}. \tag{17}$$

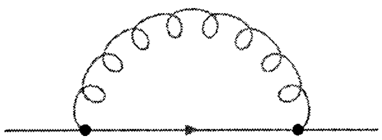

FIG. 1. Self-energy diagram on QCD side.

The heavy quark expansion implies that $\Delta \ll 1$. Thus A,B can be expanded as power series of Δ. Up to next leading order, they can be written as

$$A = 1 + c_0(\mu) + c_2(\mu)\Delta,$$
$$B = c_1(\mu) + c_4(\mu)\Delta. \tag{18}$$

At one-loop level in QCD, there is only one 1PI diagram contributing to the self-energy, as shown in Fig. 1. Carrying out a specific calculation with the above infrared regulator, we obtain that

$$c_0(\mu) = \frac{C_F}{4\pi}\alpha_s(\mu)\left(\ln\frac{\mu^2}{m^2} + 2\right),$$
$$c_1(\mu) = \frac{C_F}{2\pi}\alpha_s(\mu)\left(\ln\frac{\mu^2}{m^2} + 1\right),$$
$$c_2(\mu) = -\frac{C_F}{\pi}\alpha_s(\mu)\left(\ln\frac{\mu^2}{m^2} + 2\right),$$
$$c_4(\mu) = -\frac{C_F}{\pi}\alpha_s(\mu)\left(\ln\frac{\mu^2}{m^2} - 1\right), \tag{19}$$

with $C_F = 4/3$.

The two-point Green's function reads

$$G(p) = \frac{i}{\Sigma(p)} = i\frac{A(\not{p} + m) - 2Bm\Delta}{4m^2\Delta(A^2 + AB - B^2\Delta)}. \tag{20}$$

Now we first determine the effective Lagrangian up to order $1/m$ corrections using the matching condition (8). With Eq. (20), the QCD side of Eq. (8) reads

$$P_+G(p)P_+ = i\frac{A(2m + k\cdot V) - 2Bm\Delta}{4m^2\Delta(A^2 + AB - B^2\Delta)}P_+ = \frac{i}{ck\cdot V}\left(1 + \frac{\mathbf{k}^2}{2mk\cdot V} - \frac{c_1 + c_2 + c_4}{c}\delta\right)P_+, \tag{21}$$

where $\delta \equiv k\cdot V/2m$ and $c = 1 + c_0 + c_1$.

As argued above, on the side of the effective theory, contributions from the loop diagrams vanish. Thus all contributions to the right-hand side of the matching condition (8) arise from the tree diagrams. With all possible insertions of higher order terms, the right-hand side of the matching condition (8) reads

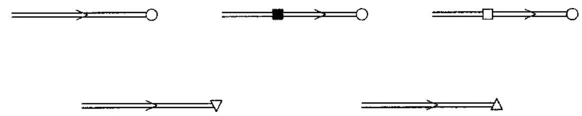

FIG. 2. Diagrams contributing to the matching conditions on the effective theory side. The circle, up and down triangles represent the operators with coefficient d_0, d_1, and d_2, respectively, while the solid and the blank boxes represent the insertion of the kinetic and $(D\cdot V)^2/2m$ operators in the effective Lagrangian.

$$\frac{i}{Zk\cdot V}\left(1 + \frac{\mathbf{k}^2}{2mk\cdot V} - Z_e\delta\right)P_+. \tag{22}$$

Comparing Eq. (22) to Eq. (21), we see that $Z = c = 1 + c_0 + c_1$, and $Z_e = (c_1 + c_2 + c_4)/c$. With the one-loop values given in Eq. (19), we have

$$Z = 1 + \frac{C_F}{4\pi}\alpha_s(\mu)\left(3\ln\frac{\mu^2}{m^2} + 4\right),$$
$$Z_e = -\frac{C_F}{2\pi}\alpha_s(\mu)\left(3\ln\frac{\mu^2}{m^2} + 1\right). \tag{23}$$

These results are in agreement with those presented in the literature [1]. The coefficient Z_m can only be determined by matching the 3-point function in the next subsection.

We then use the matching condition (13) to determine the renormalized $h_{V-}(x)$ field up to order $1/m$ corrections. With the two-point function given in Eq. (20), the QCD side of Eq. (13) reads

$$P_-G(p)P_+ = i\frac{AP_-\not{k}P_+}{4m^2\Delta(A^2 + AB - B^2\Delta)} = i\frac{P_-\not{k}P_+}{2mck\cdot V}\left[1 + \frac{\mathbf{k}^2}{2mk\cdot V} - \left(1 + \frac{c_2 + c_4}{c} - \frac{c_1^2}{c(1 + c_0)}\right)\delta\right]. \tag{24}$$

On the effective theory side, again contributions from loop diagrams vanish while only tree diagrams survive. Up to next leading order correction terms five diagrams give nonzero contributions, as shown in Fig. 2. The Feynman rules for the operator insertions in these diagrams can easily be obtained from Eqs. (14) and (15). Their contributions to the right-side of Eq. (13) reads

$$i\frac{P_-\not{k}P_+}{2m}\left[\frac{d_0}{Zk\cdot V} + \frac{\mathbf{k}^2}{2mZ(k\cdot V)^2} - \left(\frac{Z_e}{2mZ} + \frac{d_1 + d_2}{2mZ}\right)\delta\right]. \tag{25}$$

Comparing Eqs. (24) to (25), we obtain that

$$d_0=1,$$

$$d_1+d_2=1-\frac{c_1}{1+c_0}. \tag{26}$$

We see that only the combination of the d_1 and d_2 can be determined by matching the 2-point function. To determine each of them separately, we need to match the 3-point function.

B. Matching the three-point function

In this subsection, we determine the short-distance coefficients Z_m in Eq. (14) and d_1, d_2 in Eq. (15) by matching the three-point function. Here the Feynamn diagrams with the 3-gluon vertex are involved. We use the background field method [14], in which the calculations can be simplified significantly. In this method, the QCD Ward identity for the self-energy and the 1PI quark-gluon vertex takes a QED-like form

$$k_\mu\Gamma^\mu(p_1,p_2)=\Sigma(p_2)-\Sigma(p_1), \tag{27}$$

where $\Gamma^\mu(p_1,p_2)$ is the 1PI 3-vertex with external quark momenta p_1, p_2, and gluon momentum $k=p_2-p_1$.

Up to next leading order correction terms, the general form of the QCD 1PI vertex satisfying the Ward identity (27) with quark near threshold can be written as

$$\Gamma^\mu(p_1,p_2)=\bar{A}\gamma^\mu+\frac{\bar{B}}{m}\bar{p}^\mu+\frac{c_2}{2m^2}\bar{p}^\mu(\bar{\not p}-m)+\frac{c_3}{4m}[\not k,\gamma^\mu], \tag{28}$$

where

$$\bar{A}=1+c_0+c_2\bar{\Delta},$$

$$\bar{B}=c_1+2c_4\bar{\Delta},$$

$$\bar{p}\equiv\frac{1}{2}(p_1+p_2),$$

$$\bar{\Delta}\equiv\frac{p_1^2+p_2^2-2m^2}{8m^2}. \tag{29}$$

The one-loop coefficients c_0, c_1, c_2, and c_4 have been given in Eq. (19). Thus we only need to evaluate the c_3. For simplicity, we take the gluon polarization vector e satisfying

$$e\cdot p_1=e\cdot p_2=e\cdot k=0. \tag{30}$$

Then we have

$$\Gamma^e(p_1,p_2)\equiv\Gamma^\mu(p_1,p_2)e_\mu=\bar{A}\not e+\frac{c_3}{4m}[\not k,\not e]. \tag{31}$$

At one-loop level in QCD, there are two Feynman diagrams contributing to $\Gamma^e(p_1,p_2)$ as shown in Fig. 3. A straightforward calculation gives

FIG. 3. Vertex diagrams on QCD side.

$$c_3(\mu)=(2C_F+C_A)\frac{\alpha_s(\mu)}{4\pi}\left(\ln\frac{\mu^2}{m^2}+2\right), \tag{32}$$

with $C_A=3$.

In the matching procedure, the calculations can be significantly simplified by taking the polarization vector e. In QCD, the general form of the 3-point Green's function with this vertex contributing to the matching conditions (8) and (13) is given by

$$G^e(p_1,p_2)=G(p_1)\Gamma^e(p_1,p_2)G(p_2). \tag{33}$$

Now we first determine Z_m by using Eq. (8). The QCD side of Eq. (8) reads

$$P_+G(p_1)\Gamma^e(p_1,p_2)G(p_2)P_+$$
$$=-P_+\frac{A_1(\not k_1+2m)-2B_1m\Delta}{4m^2\Delta_1(A_1^2+A_1B_1-B_1^2\Delta_1)}\left(\bar{A}\not e+\frac{c_3}{4m}[\not k,\not e]\right)$$
$$\times\frac{A_2(\not k_2+2m)-2B_2m\Delta_2}{4m^2\Delta_2(A_2^2+A_2B_2-B_2^2\Delta_2)}P_+, \tag{34}$$

where the subscript 1 and 2 denote the momentum being p_1 and p_2, respectively. Expanding it to leading order of k, we have

$$-\frac{1}{4mc^2k_1\cdot Vk_2\cdot V}(1+c_0+c_3)P_+[\not k,\not e]P_+. \tag{35}$$

On the effective theory side, only the insertion of the color-magnetic dipole term gives nonzero contribution for the gluon polarization vector satisfying Eq. (30). It reads

$$-\frac{Z_m}{4mZk_1\cdot Vk_2\cdot V}P_+[\not k,\not e]P_+. \tag{36}$$

Comparing Eqs. (35) to (36), we determine that

$$Z_m=\frac{1+c_0+c_3}{c}=1+\frac{c_3-c_1}{c}. \tag{37}$$

With the coefficients given in Eqs. (19) and (32), Z_m at one-loop level reads

$$Z_m=1+\frac{\alpha_s(\mu)}{4\pi}\left(C_A\ln\frac{\mu^2}{m^2}+2C_A+2C_F\right). \tag{38}$$

This is in agreement with that obtained in the literature [1].

We then determine d_1 and d_2 in the renormalized h_{V-} field (15) by using the matching condition (13). The QCD side of Eq. (13) reads

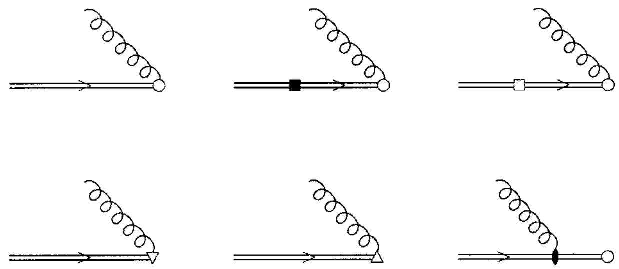

FIG. 4. Diagrams contributing to the matching conditions on the effective theory side. The notations are the same as in Fig. 2. The solid oval represents the insertion of the color-magnetic dipole operator.

$$P_- G(p_1)\Gamma^e(p_1,p_2)G(p_2)P_+ = -P_- \frac{A_1(\not k_1+2m)-2B_1 m\Delta}{4m^2\Delta_1(A_1^2+A_1B_1-B_1^2\Delta_1)}\left(\overline{A}\not e+\frac{c_3}{4m}[\not k,\not e]\right) \times\frac{A_2(\not k_2+2m)-2B_2 m\Delta_2}{4m^2\Delta_2(A_2^2+A_2B_2-B_2^2\Delta_2)}P_+ . \quad (39)$$

The expression can be expanded as power series of k's. Keeping only the leading corrections, Eq. (39) reads

$$P_-\not e P_+\frac{1}{2mck_2\cdot V}\left[1+\frac{\mathbf{k}_2^2}{2mk_2\cdot V}-\left(1-\frac{2c_1-c_2-2c_3}{2(1+c_0)}\right)\delta_1 -\frac{c_1+c_2+c_4}{c}\delta_2+\frac{c_2+2c_3}{2(1+c_0)}\delta_2\right] -\frac{1+c_0+c_3}{4mc^2}\frac{1}{k_1\cdot Vk_2\cdot V}P_-\not k P_+[\not k,\not e]P_+ . \quad (40)$$

On the effective theory side, contributions may arise from the insertions of the operators both in $h_{V-}(x)$ given in Eq. (15) and in the effective Lagrangian given in Eq. (14). With the polarization vector e, there are only 6 Feynman diagrams contributing to it as shown in Fig. 4. With appropriate Feynman rules, they read

$$P_-\not e P_+\frac{1}{2mZk_2\cdot V}\left[1+\frac{\mathbf{k}_2^2}{2mk_2\cdot V}-d_1\delta_1-\frac{Z_e}{Z}\delta_2-d_2\delta_2\right] -\frac{Z_m}{4mZ}\frac{1}{k_1\cdot Vk_2\cdot V}P_-\not k P_+[\not k,\not e]P_+ . \quad (41)$$

Comparing Eq. (39) to Eq. (41), we determine that

$$d_1=1-\frac{2c_1-c_2-2c_3}{2(1+c_0)},$$

$$d_2=-\frac{c_2+2c_3}{2(1+c_0)}. \quad (42)$$

These values are consistent with Eq. (26) which is obtained by matching the 2-point function.

Substituting the short-distance coefficients in Eqs. (19) and (32) into Eq. (42), we obtain the one-loop renormalized coefficients for d_1 and d_2:

$$d_1(\mu)=1+\frac{\alpha_s(\mu)}{4\pi}\left[(C_A-2C_F)\ln\frac{\mu^2}{m^2}-2C_F+2C_A\right], \quad (43)$$

$$d_2(\mu)=-\frac{\alpha_s(\mu)}{4\pi}C_A\left(\ln\frac{\mu^2}{m^2}+2\right). \quad (44)$$

These are the central results of this section. We find that they are in disagreement with those obtained by Balzereit in Ref. [9]. Since we use the same definition of the heavy quark field, both results should be equal. The result presented in this paper is derived rigorously using the matching conditions while Balzereit obtained it indirectly from the requirement of the invariance of the effective Lagrangian which is much more complicated.

IV. RPI OF THE RENORMALIZED EFFECTIVE LAGRANGIAN

In this section, I compare the renormalized transformation determined by the matching conditions (13) with those given in Ref. [7]. I show that their results can easily be understood by constructing the effective Lagrangian in an alternative way, in which an effective theory in four-component field is constructed first, followed by its reduction to the effective theory in the two-component field. I then prove that the renormalized transformation determined by the matching condition (13) can be written as the same form with the transformation given by Eqs. (4) and (5) with the covariant derivative substituted by the operator which may be called the generalized covariant derivative. It means that the result presented in this paper is consistent with that given in Ref. [7]. Finally, I will show that the renormalized effective Lagrangian is reparametrization invariant under the renormalized transformation.

A. Effective Lagrangian in four-component field

In the conventional method, a renormalized effective Lagrangian is constructed by the following steps. First a proper field to describe the low energy particles is chosen. In HQET and NRQCD, this effective field for describing the heavy quark is just the two-component field. Then the effective Lagrangian in this field is expanded as a sum of local operators in terms of appropriate counting rules. Finally the renormalized short distance coefficients of these local operators are determined by matching the full theory and the effective theory. We call this method ‘‘matching after expansion.’’

Here we introduce an alternative way to determine the renormalized effective Lagrangian. In this method, renormalized local operators expressed in the field of the full theory are added to the Lagrangian of the full theory by matching conditions. Then it is expanded in terms of the two-component field. We call this method ‘‘matching before expansion.’’

Let us illustrate how this works in a hard cutoff regularization. As in the last section, we take different hard cutoff regularization energy scales Λ_e in the effective theory and Λ_f in the full theory, respectively. They satisfy $\Lambda_e \ll m \ll \Lambda_f$. In calculating the one-loop 1PI diagrams in full QCD theory, we need to calculate the loop momentum integrals from zero to Λ_f. They can be separated into integrals from 0 to Λ_e and integrals from Λ_e to Λ_f. The first part is just the same with that in the effective theory while the second part gives extra contributions. As argued above, the contributions from this region can be written as local terms of external momenta and can be expressed as contributions from local operators. Therefore, once those local operators are added to the Lagrangian, the effective theory with hard cutoff Λ_e can produce the same result of the full theory with cutoff Λ_f. This argument can easily be generalized to the case of multiloops.

At this stage, those local operators are written in terms of Dirac four-component field. A general form of the renormalized effective Lagrangian density with hard cutoff Λ_e for heavy quark field can formally be expressed as

$$\mathcal{L}_{\mathrm{eff}} = \bar{\Psi}(x)(i\not{D} - m)\Psi(x) + \bar{\Psi}(x)O_1(x)\Psi(x), \quad (45)$$

where $D^\mu = \partial^\mu - igA^a_\mu T^a$ is the covariant derivative. It may be denoted as

$$\mathcal{L}_{\mathrm{eff}} = \bar{\Psi}(x)O(x)\Psi(x) \quad (46)$$

in a compact form by defining $O(x) \equiv i\not{D} - m + O_1(x)$.

The first term in Eq. (45) is just the tree-level Lagrangian while the second term arises from the renormalization with the cutoff $\Lambda_e \ll m$. The operators in this term are generally the function of the covariant derivative and the heavy quark mass. It may contain terms such as $D^2 + m^2$ and $g_s G^{\mu\nu} = i[D^\mu, D^\nu]$, which are suppressed by the off-shell momentum of the heavy quark or the momenta of the external gluons. They can be organized via appropriate power counting rules. In perturbative calculations, the loop momentum integral is from zero to Λ_e. In this region, both the external and the loop momenta are smaller than m, hence the $1/m$ expansion is allowed and the quark mass dependence is extracted explicitly. Thus the energy scale m is no longer involved in the effective theory. I emphasis here that the effective Lagrangian density in this form is independent of the velocity parameter V. Thus it automatically satisfies the RPI.

Higher-dimensional operators appear in $O_1(x)$. It implies that power divergences arise in the loop momentum integrals. In the full theory the power divergences cancel when both the contributions from quark and antiquark are included. However, in the effective theory when we impose a hard cutoff $\Lambda_e \ll m$ on the loop momentum integrals, the contributions from the antiquark are excluded so that the power divergences do not cancel. Nevertheless, those power divergences are artificial since they cancel between those from diagram calculations and those from the short distance coefficients.

At one loop and the leading order of $1/m$, the most general form of the four-component effective Lagrangian is

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = {} & (1+c_0)\bar{\Psi}(x)(i\not{D} - m)\Psi(x) \\ & - \frac{c_1}{2m}\bar{\Psi}(x)(D^2+m^2)\Psi(x) - \frac{ic_2}{8m^2}\bar{\Psi}(x) \\ & \times[(i\not{D}-m)(D^2+m^2)+(D^2+m^2)(i\not{D}-m)]\Psi(x) \\ & + \frac{c_3}{4m}\bar{\Psi}(x)g_s\sigma^{\mu\nu}G_{\mu\nu}\Psi(x) \\ & + \frac{c_4}{8m^3}\bar{\Psi}(x)(D^2+m^2)^2\Psi(x). \end{aligned} \quad (47)$$

Calculating the 1PI diagrams shown in Figs. 1 and 3 using this effective Lagrangian and full QCD, we see that these coefficients are the same $c_1 - c_4$'s given in Eqs. (19) and (32).

B. Effective Lagrangian in two-component field

Now let us reduce Eq. (47) to the effective Lagrangian in the two-component field. The equation of motion now reads

$$P_-\bar{O}(x)[h_{V+}(x) + h_{V-}(x)] = 0, \quad (48)$$

where $\bar{O}(x)$ is the $O(x)$ in which the covariant derivative iD is replaced by $iD + mV$ due to the phase factor in the field redefinition. It can be regarded as the renormalized equation of motion.

From Eq. (48), we can express $h_{V-}(x)$ as a function of $h_{V+}(x)$ formally as

$$\begin{aligned} h_{V-}(x) = {} & \frac{1}{2m + iD(x)\cdot V - P_-\bar{O}_1(x)P_-} \\ & \times P_-[i\not{D} + \bar{O}_1(x)]h_{V+}(x), \end{aligned} \quad (49)$$

where $\bar{O}_1(x)$ is the $O_1(x)$ in which the covariant derivative iD is substituted by $iD + mV$. This modifies the tree level expression (5). Once the form of $O_1(x)$ is given, the right-hand side of Eq. (49) can be expanded as power series of $1/m$. With $O_1(x)$ given in Eq. (47), up to order α_s and $1/m^2$, $h_{V-}(x)$ reads

$$\begin{aligned} h_{V-}(x) = P_-\Bigg[& \frac{1}{2m}i\not{D} + \frac{1}{4m^2}\left(1 - \frac{2c_1 - c_2 - 2c_3}{2(1+c_0)}\right)D\cdot V\not{D} \\ & - \frac{c_2 + 2c_3}{8(1+c_0)m^2}\not{D}D\cdot V\Bigg]h_{V+}(x). \end{aligned} \quad (50)$$

Comparing this with Eqs. (15),(42), we see that they are in agreement.

Finally, with equation of motion (48), the effective Lagrangian (46) is reduced to

$$\begin{aligned} \mathcal{L}_{\mathrm{eff}} &= \bar{h}_{V+}(x)\bar{O}(x)[h_{V+}(x) + h_{V-}(x)] \\ &= [\bar{h}_{V+}(x) + \bar{h}_{V-}(x)]\bar{O}(x)h_{V+}(x). \end{aligned} \quad (51)$$

This is just the two-component effective Lagrangian. It can be expanded as power series of $1/m$. In this way, the four-component effective Lagrangian is reduced to the two-component effective Lagrangian. Up to order α_s and $1/m$ correction, it is reduced to the effective Lagrangian (14). Therefore, the effective Lagrangian obtained by these two different approaches are equal.

C. Comparison with previous studies

In Sec. II, we derive the matching conditions for determining the renormalized $h'_{V-}(x)$ field as given in Eq. (13). In the last subsection, the $h_{V-}(x)$ field was obtained using the equation of motion. Its expression is given by Eq. (49). In this subsection, I will show that the small component fields obtained by these two different methods are identical. They uniquely determine the renormalized transformation of the heavy quark field against the infinitesimal variation of the velocity parameter V. Adding both sides of Eqs. (12) and (13) together, we have

$$\langle 0|T[h_{V+}(x)+h'_{V-}(x)]\overline{h}_{V+}(y)|0\rangle^B \doteq \langle 0|T\Psi(x)\overline{\Psi}(y)|0\rangle^B P_+, \tag{52}$$

where $\langle 0|T\Psi(x)\overline{\Psi}(y)|0\rangle^B$ is a full propagator under arbitrary external field $B^\mu(x)$. It is satisfied order by order in α_s. Suppose we calculate the left-hand side at tree level with the renormalized effective Lagrangian. To validate this equation, the right-hand side then should also be calculated to the tree level with the renormalized four-component effective Lagrangian (45). Thus it satisfies the following equation:

$$O(x)G(x,y;B)=i\delta^4(x-y). \tag{53}$$

Acting an operator $P_-\overline{O}(x)$ on the left-hand side and $P_-O(x)$ on the right-hand side of Eq. (52), the right-hand side vanishes immediately due to $P_-\cdot P_+=0$. Since we only calculate them at tree level, the operator $\overline{O}(x)$ can be moved within the bracket

$$P_-\langle 0|T[\overline{O}(x)[h_{V+}(x)+h'_{V-}(x)]\overline{h}_{V+}(y)]|0\rangle^B=0. \tag{54}$$

Since the argument y in $\overline{h}_{V+}(y)$ is arbitrary and this correlation function contains interaction with arbitrary background gluon field, the unique solution of this equation is

$$P_-\overline{O}(x)[h_{V+}(x)+h'_{V-}(x)]=0. \tag{55}$$

This is identical to Eq. (48) if $h'_{V-}(x)$ is the same as $h_{V-}(x)$. This implies that the renormalized $h'_{V-}(x)$ determined by the matching conditions (13) is identical to that from the equation of motion (48).

D. RPI of the renormalized effective Lagrangian

In this subsection, we prove that the renormalized effective Lagrangian (51) is invariant under the transformation (4) or (11) with the renormalized small component field. It follows that from an infinitesimal transformation of the effective Lagrangian (51)

$$\begin{aligned}\Delta\mathcal{L}_{\rm eff} &\doteq \Delta\overline{h}_{V+}(x)\overline{O}(x)h_V(x)+\overline{h}_{V+}(x)\overline{O}(x)\Delta h_V(x)\\ &=\overline{h}_V(x)\frac{\Delta\not{V}}{2}\overline{O}(x)h_V(x)+\overline{h}_{V+}(x)\overline{O}(x)\Delta h_V(x).\end{aligned} \tag{56}$$

We have used a shorthand notation $h_V(x)=h_{V+}(x)+h_{V-}(x)$. It is emphasized here that the operator $O(x)$ introduced in the four component effective field theory is invariant against the variation of the velocity V. Any change arising from the phase factor in the definition of the effective field has been omitted simply because it is trivial under the transformation.

Imposing an infinitesimal transformation on the equation of motion (48), we obtain that

$$-\frac{\Delta\not{V}}{2}\overline{O}(x)h_V(x)+P_-\overline{O}(x)\Delta h_V(x)\doteq 0. \tag{57}$$

With it, Eq. (56) can be rewritten as

$$\Delta\mathcal{L}_{\rm eff}\doteq\overline{h}_V(x)\overline{O}(x)\Delta h_V(x). \tag{58}$$

Notice that $P_+h_{V-}(x)=0$. Imposing an infinitesimal transformation on it, we immediately have

$$P_+\Delta h_{V-}(x)=-\frac{\Delta\not{V}}{2}h_{V-}(x). \tag{59}$$

Adding it together with $P_+\Delta h_{V+}(x)=\Delta\not{V}/2h_{V-}(x)$, we have

$$P_+\Delta h_V(x)=0. \tag{60}$$

With it, Eq. (56) is reduced to

$$\Delta\mathcal{L}_{\rm eff}\doteq\overline{h}_V(x)\overline{O}(x)P_-\Delta h_V(x). \tag{61}$$

It follows that $\Delta\mathcal{L}_{\rm eff}=0$ from the equation of motion $\overline{h}_V(x)O(x)P_-=0$. Thus we have shown that the renormalized effective Lagrangian (51) is invariant against the variation of the velocity parameter V under the infinitesimal transformation (4) or (11) with the renormalized small component field.

V. CONCLUSION

The RPI is an important theoretical issue in the heavy quark effective theory and the NRQCD effective theory. It is required by the consistency of the effective theory. It also leads to interesting applications [10–12]. The transformation of heavy quark field under the variation of the velocity parameter V proposed by Chen [5] with the tree level expression of the small-component field keeps the tree level effective theory invariant. However, at loop level, the transformation needs to be renormalized. In this paper, I have shown that the renormalized transformation of the heavy quark keeps the same form as Chen's transform while the small component field needs to be renormalized. I derived the matching conditions for determining the renormalized transformation by imposing an infinitesimal transformation on the relations between the Green's functions in the full QCD and those in the effective theory. These matching conditions are essential for studying the renormalization issue in RPI. As an application of these matching conditions, I determined the renormalized transformation up to order $1/m^2$. I also showed that the previous result in Ref. [7] can be understood clearly by building the effective theory in an alternative way, in which the renormalized effective Lagrangian in Dirac four-component field is constructed first, followed by its reduction to the two-component effective Lagrangian. The renormalized small component field is then obtained by the equation of motion. The four-component effective Lagrangian automatically satisfies RPI. Thus RPI cannot give any constraints on any operators in it. When it is reduced to the two-component effective theory, the same operator with certain coefficients may appear in different terms. The RPI can be used to connect those terms. I also showed that the renormalized small component fields obtained by these two methods turn out to be the same while the matching conditions provide a systematic way to determine the renormalized transformation to any desired order in $1/m$ and α_s expansions.

ACKNOWLEDGMENTS

The author would like to thank Professor Tao Han for reading this manuscript. This work was supported in part by the National Science Foundation of China.

APPENDIX: GENERATING FUNCTIONAL OF GREEN'S FUNCTIONS

In this appendix, we derive the relations between the Green's functions in QCD full theory and those in the effective theory using generating functional method. It is similar to that given in Refs. [13] and [5]. We use the background field method [14,15] for gluon field interactions, which explicitly preserves the gauge covariant.

In QCD full theory, the generating functional reads

$$Z[\eta,\bar{\eta},J,B]=\int d[\psi,\bar{\psi},A]\exp i\int d^4x[I_Q(x)+I_g(x)], \tag{A1}$$

where $\eta,\bar{\eta},J$ are the external sources for heavy quark, antiquark, and gluon field, B is the background gluon field, I_g is given by

$$I_g(x)=-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}-\frac{1}{2\xi}(G^a)^2+\ln\det\left[\frac{\delta G^a}{\delta\omega^b}\right]+J^a_\mu A^{a\mu}, \tag{A2}$$

with

$$F^a_{\mu\nu}=\partial_\mu(A+B)^a_\nu-\partial_\nu(A+B)^a_\mu+gf^{abc}(A+B)^b_\mu(A+B)^c_\nu, \tag{A3}$$

$$G^a=\partial_\mu A^a_\mu+gf^{abc}B^b_\mu A^{c\mu}, \tag{A4}$$

being the gauge-fixing term. If J_μ satisfies the following relation:

$$\frac{\delta W}{\delta B^a_\mu}+\int d^4y\left[\frac{\delta W}{\delta J^b_\nu}\frac{\delta J^b_\nu(y)}{\delta B^a_\mu}\right]=-J^a_\mu, \tag{A5}$$

with $W[\eta,\bar{\eta},J,B]=-i\ln Z[\eta,\bar{\eta},J,B]$, $W[\eta,\bar{\eta},J,B]$ is just the effective action regarding to the gluon field B with gauge-fixing term

$$G^a=\partial_\nu(A-B)^a_\nu+gf^{abc}B^b_\mu A^c_\nu \tag{A6}$$

and I_Q reads

$$I_Q(x)=\bar{\Psi}(x)(i\not{D}-m)\Psi(x)+\bar{\eta}(x)\Psi(x)+\bar{\Psi}(x)\eta(x). \tag{A7}$$

The quark field can be integrated out formally and then we have

$$Z[\eta,\bar{\eta},J,B]=\int d[A]\det[i\not{D}-m]\exp i\int d^4x(I'_Q+I_g), \tag{A8}$$

where I'_Q remains the same form as I_Q. But the quark field now is related to the external source $\eta(x)$ by the following equation of motions:

$$(i\not{D}-m)\Psi(x)=-\eta(x). \tag{A9}$$

The generating functional of the effective theory is similar to that of the full theory except the heavy quark action. The effective Lagrangian is substituted by Eq. (46). In the external source term of the heavy quark only the large component effective field defined in Eq. (1) couples to the external source. The action of the heavy quark is given by

$$I_Q^{V+}(x)=\bar{\Psi}(x)O(x)\Psi(x)+\bar{\eta}(x)P_+h_{V+}(x)+\bar{h}_{V+}(x)P_+\eta(x). \tag{A10}$$

Similarly, integrating out the heavy quark field, the generating functional takes the same form as Eq. (A8) with the effective action of the heavy quark section is substituted by

$$I'^{V+}_Q(x)=\bar{h}_V(x)\bar{O}(x)h_V(x)+\bar{\eta}(x)P_+h_{V+}(x)+\bar{h}_{V+}(x)P_+\eta(x), \tag{A11}$$

with $h_V(x)=h_{V+}(x)+h_{V-}(x)$.

The quark field now is related to the external source $\eta(x)$ by the following equation of motion:

$$\bar{O}(x)h_{V+}(x)=-P_+\eta(x). \tag{A12}$$

Multiplying P_- on both sides, the right-hand side vanishes and we obtain the renormalized equation of motion

$$P_-\overline{O}(x)[h_{V+}(x)+h_{V-}(x)]=0. \tag{A13}$$

This is Eq. (48). The renormalized $h_{V-}(x)$ can be related to $h_{V+}(x)$ by Eq. (49). With the equation of motion (A13), Eq. (A11) can be simplified as

$$I'^{V}_{Q}=\overline{h}_{V+}(x)\overline{O}(x)h_V(x)+\overline{\eta}(x)P_+h_{V+}(x)$$
$$+\overline{h}_{V+}(x)P_+\eta(x). \tag{A14}$$

This gives the effective Lagrangian density (51).

The quark determinant in Eq. (A8) is responsible for the contributions of the heavy quark loop. It is the same in the full theory and in the effective theory and is suppressed by $1/m^2$ at least. Thus we may ignore it.

The full quark propagator with background field $B^\mu(x)$ is derived by differentiating over the external sources

$$G(x,y;B)=\frac{\delta^2}{\delta\eta(x)\delta\overline{\eta}(y)}W(\eta,\overline{\eta},J,B). \tag{A15}$$

If the hard cutoff energy scale is set to Λ_f, the same as that in the QCD full theory, the $O(x)$ is then to be $i\not{D}-m$, just as in the full theory. The effective Lagrangian is just the nonlocal form (3). In this case the only difference of the effective theory and the full theory is the external source term. One immediately gains the relations between the Green's functions of the full theory and those in the effective theory (8). This relation ensures that the nonlocal effective theory is equivalent to the QCD full theory. The local effective theory with the hard cutoff regularization scale Λ_e is equivalent to the nonlocal effective theory with a hard cutoff Λ_f. This ensures the validity of relation (8).

[1] M.A. Shifman and M.B. Voloshin, Yad. Fiz. **45**, 463 (1987) [Sov. J. Nucl. Phys. **45**, 292 (1987)]; **47**, 801 (1988) [**47**, 511 (1988)]; N. Isgur and M.B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990); H.D. Politzer and M.B. Wise, *ibid.* **206**, 681 (1988); **208**, 504 (1988); E. Eichten and B. Hill, *ibid.* **234**, 511 (1990); **240**, 447 (1990); B. Grinstein, Nucl. Phys. **B339**, 253 (1990); A.F. Falk, H. Georgi, B. Grinstein, and M.B. Wise, *ibid.* **B343**, 1 (1990); H. Georgi and M.B. Wise, Phys. Lett. B **243**, 279 (1990); M. Neubert, Phys. Rep. **245**, 259 (1994).

[2] W.E. Caswell and G.P. Lepage, Phys. Lett. **167B**, 437 (1986); G.P. Lepage and B.A. Thacker, Nucl. Phys. B (Proc. Suppl.) **4**, 199 (1988); G.T. Bodwin, E. Braaten and G.P. Lepage, Phys. Rev. D **51**, 1125 (1995); **55**, 5853 (1997).

[3] M.E. Luke and A.V. Manohar, Phys. Lett. B **286**, 348 (1992).

[4] A.V. Manohar, Phys. Rev. D **56**, 230 (1997).

[5] Y.Q. Chen, Phys. Lett. B **317**, 421 (1993).

[6] M. Finkemeier, H. Georgi, and M. McIrvin, Phys. Rev. D **55**, 6933 (1997).

[7] W. Kilian and T. Ohl, Phys. Rev. D **50**, 4649 (1994).

[8] R. Sundrum, Phys. Rev. D **57**, 331 (1998).

[9] C. Balzereit, hep-ph/9809226.

[10] M. Neubert, Phys. Lett. B **306**, 357 (1993).

[11] Y.Q. Chen and Y.P. Kuang, Z. Phys. C **67**, 627 (1995).

[12] Y.Q. Chen, Y.P. Kuang, and R.J. Oakes, Phys. Rev. D **52**, 264 (1995).

[13] T. Mannel, W. Roberts, and Z. Ryzak, Nucl. Phys. **B368**, 204 (1992).

[14] B.S. Dewitt, Phys. Rev. **162**, 1195 (1967).

[15] L.F. Abbott, Nucl. Phys. **B185**, 189 (1981).