Lower bounds on the curvature of the Isgur-Wise function

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Using the operator product expansion, we obtain new sum rules in the heavy quark limit of QCD, in addition to those previously formulated. Key elements in their derivation are the consideration of the nonforward amplitude, plus the systematic use of boundary conditions that ensure that only a finite number of j^P intermediate states (with their tower of radial excitations) contribute. A study of these sum rules shows that it is possible to bound the curvature $\sigma^2=\xi''(1)$ of the elastic Isgur-Wise function $\xi(w)$ in terms of its slope $\rho^2=-\xi'(1)$. In addition to the bound $\sigma^2\geqslant\frac{5}{4}\rho^2$, previously demonstrated, we find the better bound $\sigma^2\geqslant\frac{1}{5}[4\rho^2+3(\rho^2)^2]$. We show that the quadratic term $\frac{3}{5}(\rho^2)^2$ has a transparent physical interpretation, as it is leading in a nonrelativistic expansion in the mass of the light quark. At the lowest possible value for the slope $\rho^2=\frac{3}{4}$, both bounds imply the same bound for the curvature $\sigma^2\geqslant\frac{15}{16}$. We point out that these results are consistent with the dispersive bounds and, furthermore, that they strongly reduce the allowed region by the latter for $\xi(w)$.

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I. INTRODUCTION

In a recent paper [1] we set a systematic method to obtain sum rules (SRs) in the heavy quark limit of QCD that relate the derivatives of the elastic Isgur-Wise (IW) function $\xi(w)$ to sums over IW functions of excited states. The method is based on the operator product expansion (OPE) [2] applied to heavy quark transitions [3] and its key element is the consideration, following Uraltsev [4], of the nonforward amplitude, i.e., $B(v_i)\to D^{(n)}(v')\to B(v_f)$ with in general $v_i\neq v_f$. Then, the OPE side of the SR contains the elastic IW function $\xi(w_{if})$ and therefore the SR depend in general on three variables w_i, w_f, and w_{if} that lie within a certain domain. By differentiation relatively to these variables within the domain and taking the limit to its boundary, one finds a very general class of SR that have interesting consequences on the shape of $\xi(w)$.

More precisely, as shown in Ref. [1], using the OPE—as formulated, for example, in Ref. [5] and generalized to $v_i\neq v_f$ [4]—the trace formalism [6] and arbitrary heavy quark currents

$$J_1=\overline{h_{v'}^{(c)}}\ \Gamma_1\ h_{v_i}^{(b)},\quad J_2=\overline{h_{v_f}^{(b)}}\ \Gamma_2\ h_{v'}^{(c)} \tag{1}$$

the following sum rule can be written in the heavy quark limit [1]:

$$\left\{\sum_{D=P,V}\sum_n \mathrm{Tr}[\overline{\mathcal{B}_f}(v_f)\Gamma_2\mathcal{D}^{(n)}(v')]\mathrm{Tr}[\overline{\mathcal{D}}^{(n)}(v')\Gamma_1\mathcal{B}_i(v_i)]\right.$$
$$\left.\times\xi^{(n)}(w_i)\xi^{(n)}(w_f)+\text{other excited states}\right\}$$
$$=-2\xi(w_{if})\mathrm{Tr}[\overline{\mathcal{B}_f}(v_f)\Gamma_2P'_+\Gamma_1\mathcal{B}_i(v_i)]. \tag{2}$$

In this formula v' is the intermediate meson four-velocity, the projector

$$P'_+=\frac{1}{2}(1+\not{v}') \tag{3}$$

comes from the residue of the positive energy part of the c-quark propagator and $\xi(w_{if})$ is the elastic Isgur-Wise function that appears because one assumes $v_i\neq v_f$. $\mathcal{B}_i$ and $\mathcal{B}_f$ are the 4×4 matrices of the ground state B or B^* meson and $\mathcal{D}^{(n)}$ those of all possible ground state or excited state D mesons coupled to B_i and B_f through the currents. In formula (2) we have made explicit the $j=\frac{1}{2}^-$ D and D^* mesons and their radial excitations.

The variables w_i, w_f, and w_{if} are defined as

$$w_i=v_i\cdot v',\quad w_f=v_f\cdot v',\quad w_{if}=v_i\cdot v_f. \tag{4}$$

The domain of (w_i, w_f, w_{if}) is [1]

$$w_i,w_f\geqslant1,$$
$$w_iw_f-\sqrt{(w_i^2-1)(w_f^2-1)}$$
$$\leqslant w_{if}\leqslant w_iw_f+\sqrt{(w_i^2-1)(w_f^2-1)}. \tag{5}$$

There is a subdomain for $w_i=w_f=w$:

$$w\geqslant1,\ \ 1\leqslant w_{if}\leqslant 2w^2-1. \tag{6}$$

Calling now $L(w_i,w_f,w_{if})$ the left-hand side (LHS) and $R(w_i,w_f,w_{if})$ the right-hand side (RHS) of Eq. (2), this SR writes

$$L(w_i,w_f,w_{if})=R(w_i,w_f,w_{if}), \tag{7}$$

where $L(w_i,w_f,w_{if})$ is the sum over the intermediate D states and $R(w_i,w_f,w_{if})$ is the OPE side. Within the domain (5) one can differentiate relatively to any of the variables w_i, w_f, and w_{if}

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$$\frac{\partial^{p+q+r}L}{\partial w_i^p \partial w_f^q \partial w_{if}^r} = \frac{\partial^{p+q+r}R}{\partial w_i^p \partial w_f^q \partial w_{if}^r} \tag{8}$$

and obtain different SRs taking different limits to the frontiers of the domain. One must take care in taking these limits, as we point out below.

Let us parametrize the elastic Isgur-Wise function $\xi(w)$ near zero recoil,

$$\xi(w) = 1 - \rho^2(w-1) + \frac{\sigma^2}{2}(w-1)^2 - \cdots . \tag{9}$$

From the SR (2), we gave in Ref. [1] a simple and straightforward demonstration of Bjorken [7,8] and of another SR, that combined with the former, implied Uraltsev SR [4]. The Bjorken and Uraltsev SRs imply the lower bound on the elastic slope

$$\rho^2 = -\xi'(1) \geqslant \frac{3}{4}. \tag{10}$$

A crucial simplifying feature of the calculation was to consider, for the currents (1), vector or axial currents aligned along the initial and final velocities v_i and v_f. In Ref. [1] we also obtained, modulo a very natural phenomenological hypothesis, a new bound on the curvature:

$$\sigma^2 = \xi''(1) \geqslant \frac{5}{4}\rho^2 \geqslant \frac{15}{16}. \tag{11}$$

This bound was obtained from the consideration in the SR of the whole tower of j^P intermediate states [9]. A crucial feature of the calculation was the needed derivation of the projector on the polarization tensors of particles of arbitrary integer spin [10].

Using the SR involving the whole sum over all j^P intermediate states, we pursued our study in Ref. [11] and demonstrated that the IW function $\xi(w)$ is an alternate series in powers of $(w-1)$. Moreover, we obtained the bound for the nth derivative at zero recoil $(-1)^n\xi^{(n)}(1)$

$$(-1)^n\xi^{(n)}(1) \geqslant \frac{2n+1}{4}(-1)^{n-1}\xi^{(n-1)} \geqslant \frac{(2n+1)!!}{2^{2n}} \tag{12}$$

rigorously demonstrating the bound (11) and generalizing Eqs. (10) and (11) to any derivative.

The aim of this paper is to investigate whether the systematic use of the sum rules can allow us to obtain better bounds on the curvature. As we will see below, the answer is positive. The reason is that only a finite number of j^P states, with their radial excitations, contribute to the relevant sum rules and one is left with a relatively simple set of algebraic linear equations. As we will see, this is due to the crucial fact that we adopt particular conditions at the boundary of the domain (5).

The paper is organized as follows. Section II is a self-contained reminder of the formalism exposed in Refs. [1] and [11]. Sections III, IV, and V are devoted to the deduction of the new SR involving the curvature of the IW function. Section VI gives the bounds on the curvature and Sec. VII exposes new implications on the P-wave IW functions. Section VIII demonstrates that the quadratic term in the new bound on the curvature has a clean physical interpretation in the nonrelativistic limit for the light quark. In Sec. IX we give an example of the fit to the data for $B \to D^* \ell \nu$ with a phenomenological ansatz for the IW function that satisfies the demonstrated bounds. In Sec. X we compare our bounds with the dispersive contraints on the IW function, and in Sec. XI our phenomenological formula for the IW function to the dispersive approach. In Sec. XII we conclude. Our motivation in studying the relation of our approach in the heavy quark limit of QCD to the dispersive approach is in part a pragmatic one, since this is the most widely used approach to fit the data (see, for example, Ref. [12]). Of course, this is not the unique approach to study the IW function. Other results in the heavy quark limit of QCD are relevant, namely, the upper bounds on the IW slope coming from Voloshin SR [13] or from the experimental limits on the kinetic energy expectation value [14]. Also, one should quote the lattice approaches to the IW function [15].

II. VECTOR AND AXIAL SUM RULES

We choose as initial and final states the B meson,

$$\mathcal{B}_i(v_i) = P_{i+}(-\gamma_5), \quad \mathcal{B}_f(v_f) = P_{f+}(-\gamma_5), \tag{13}$$

where the projectors P_{i+}, P_{f+} are defined as in Eq. (3). Moreover, we take vector or axial currents projected along the v_i and v_f four-velocities. Considering the vector currents

$$J_1 = \overline{h}_{v'}^{(c)} \not{v}_i h_{v_i}^{(b)}, \quad J_2 = \overline{h}_{v_f}^{(b)} \not{v}_f h_{v'}^{(c)} \tag{14}$$

and gathering the formulas (48) and (89)–(91) of Ref. [1] we obtain for the SR (2) with the sum of all excited states j^P, as written down in Ref. [11]:

$$\begin{aligned}(w_i+1)(w_f+1)\sum_{\ell\geqslant 0}\frac{\ell+1}{2\ell+1}S_\ell(w_i,w_f,w_{if})\\ \times\sum_n \tau_{\ell+1/2}^{(\ell)(n)}(w_i)\tau_{\ell+1/2}^{(\ell)(n)}(w_f)\\ +\sum_{\ell\geqslant 1}S_\ell(w_i,w_f,w_{if})\sum_n \tau_{\ell-1/2}^{(\ell)(n)}(w_i)\tau_{\ell-1/2}^{(\ell)(n)}(w_f)\\ =(1+w_i+w_f+w_{if})\xi(w_{if}).\end{aligned} \tag{15}$$

Choosing instead the axial currents

$$J_1 = \overline{h}_{v'}^{(c)} \not{v}_i\gamma_5 \, h_{v_i}^{(b)}, \quad J_2 = \overline{h}_{v_f}^{(b)} \not{v}_f\gamma_5 \, h_{v'}^{(c)} \tag{16}$$

the SR (2) is written as, from the formulas (48) and (92)–(94) of Ref. [1], obtained in Ref. [11]:

$$\sum_{\ell \geqslant 0} S_{\ell+1}(w_i,w_f,w_{if}) \sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w_i)\tau^{(\ell)(n)}_{\ell+1/2}(w_f)$$
$$+(w_i-1)(w_f-1)\sum_{\ell \geqslant 1} \frac{\ell}{2\ell-1} S_{\ell-1}(w_i,w_f,w_{if})$$
$$\times \sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i)\tau^{(\ell)(n)}_{\ell-1/2}(w_f)$$
$$=-(1-w_i-w_f+w_{if})\xi(w_{if}). \quad (17)$$

Following the formulation of heavy-light states for arbitrary j^P given by Falk [9], we have defined in Ref. [1] the IW functions $\tau^{(\ell)(n)}_{\ell+1/2}(w)$ and $\tau^{(\ell)(n)}_{\ell-1/2}(w)$, that correspond to the orbital angular momentum ℓ of the light quark relative to the heavy quark, $j=\ell \pm \frac{1}{2}$ being the total angular momentum of the light cloud. For the lower values of ℓ, one has the identities with the traditional notation of Isgur and Wise [8]:

$$\tau^{(0)}_{1/2}(w) \equiv \xi(w), \quad \tau^{(1)}_{1/2}(w) \equiv 2\tau_{1/2}(w), \quad \tau^{(1)}_{3/2}(w) \equiv \sqrt{3}\tau_{3/2}(w), \quad (18)$$

where a radial quantum number is implicit. Therefore, the functions $\tau^{(1)}_{1/2}(w)$ and $\tau^{(1)}_{3/2}(w)$ correspond, respectively, to the functions $\zeta(w)$ and $\tau(w)$ defined by Leibovich, Ligeti, Steward, and Wise [16].

In Eqs. (14) and (16) the quantity S_n is defined by

$$S_n = v_{f\nu_1}\cdots v_{f\nu_n} T^{\nu_1 \cdots \nu_n, \mu_1 \cdots \mu_n} v_{i\mu_1}\cdots v_{i\mu_n} \quad (19)$$

and the polarization projector $T^{\nu_1 \cdots \nu_k, \mu_1 \cdots \mu_n}$, given by

$$T^{\nu_1 \cdots \nu_n, \mu_1 \cdots \mu_n} = \sum_\lambda \varepsilon'^{(\lambda)*\nu_1 \cdots \nu_n} \varepsilon'^{(\lambda)\mu_1 \cdots \mu_n} \quad (20)$$

depends only on the four-velocity v'. The tensor $\varepsilon'^{(\lambda)\mu_1 \cdots \mu_n}$ is the polarization tensor of a particle of integer spin $J=n$, symmetric, traceless, i.e., $\varepsilon'^{(\lambda)\mu_1 \cdots \mu_n} g_{\mu_i \mu_j}=0$ $(i \neq j \leqslant n)$, and transverse to v', $v'_{\mu_i}\varepsilon'^{(\lambda)\mu_1 \cdots \mu_n}=0$ $(i \leqslant n)$ [1,10].

Moreover, as demonstrated in the Appendix A of Ref. [1], S_n is given by the following expression:

$$S_n(w_i,w_f,w_{if}) = \sum_{0 \leqslant k \leqslant n/2} C_{n,k}(w_i^2-1)^k(w_f^2-1)^k$$
$$\times (w_i w_f - w_{if})^{n-2k} \quad (21)$$

with

$$C_{n,k} = (-1)^k \frac{(n!)^2}{(2n)!} \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}. \quad (22)$$

The relation

$$L^V(w_i,w_f,w_{if})|_{w_{if}=1, w_i=w_f=w}$$
$$= R^V(w_i,w_f,w_{if})|_{w_{if}=1, w_i=w_f=w} \quad (23)$$

gives, dividing by $2(w+1)$, the Bjorken SR [7,8], now including the whole sum of intermediate states:

$$\frac{w+1}{2}\sum_{\ell \geqslant 0} \frac{\ell+1}{2\ell+1} C_\ell (w^2-1)^\ell \sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w)\tau^{(\ell)(n)}_{\ell+1/2}(w)$$
$$+\frac{w-1}{2}\sum_{\ell \geqslant 1} C_\ell (w^2-1)^{\ell-1} \sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w)\tau^{(\ell)(n)}_{\ell-1/2}(w)$$
$$=1, \quad (24)$$

where, from Eqs. (21) and (22)

$$S_n(w,w,1) = C_n(w^2-1)^n, \quad C_n = \sum_{0 \leqslant k \leqslant n/2} C_{n,k} = 2^n \frac{(n!)^2}{(2n)!}. \quad (25)$$

Remember that, usually, the first terms in the sum (24) are written in the notation (18) of Isgur and Wise [8]:

$$\frac{w+1}{2}\sum_n [\xi^{(n)}(w)]^2 + (w-1)$$
$$\times \sum_n \{2[\tau^{(n)}_{1/2}(w)]^2 + (w+1)^2[\tau^{(n)}_{3/2}(w)]^2\} + \cdots = 1. \quad (26)$$

Going now to the axial current SR (17), the condition

$$L^A(w_i,w_f,w_{if})|_{w_{if}=1, w_i=w_f=w}$$
$$= R^A(w_i,w_f,w_{if})|_{w_{if}=1, w_i=w_f=w} \quad (27)$$

gives again, dividing this time by $2(w-1)$, the complete Bjorken SR (24). Notice that one obtains the same SR from the vector (14) and the axial current (16) because, from Eq. (25), one has

$$(2n+1)C_{n+1} = (n+1)C_n. \quad (28)$$

III. EQUATIONS FROM THE VECTOR SUM RULE

In what follows, to look for independent relations, we make use of the fact that the SR (15) and (17) are symmetric in the exchange $w_i \leftrightarrow w_f$. Let us first consider the derivatives of the SR for vector currents (15) relatively to w_{if} with the boundary condition $w_{if}=1$. For $w_{if}=1$, the domain (5) implies

$$w_i = w_f = w. \quad (29)$$

We define, therefore,

$$L_V(w_{if},w) \equiv L_V(w_{if},w_i,w_f)|_{w_i=w_f=w},$$
$$R_V(w_{if},w) \equiv R_V(w_{if},w_i,w_f)|_{w_i=w_f=w}. \quad (30)$$

We then take the $p+q$ derivatives

$$\left(\frac{\partial^{p+q} L_V}{\partial w_{if}^p \partial w^q}\right)_{w_{if}=w=1} = \left(\frac{\partial^{p+q} R_V}{\partial w_{if}^p \partial w^q}\right)_{w_{if}=w=1} \quad (31)$$

and systematically exploit the obtained relations. To get information on the curvature σ^2 of the elastic IW function (9) we need to go up to the second order derivatives. Notice that we could have differentiated first relative to w_i and taken the limit $w_i=1$, and then differentiated with respect to $w=w_{if}=w_f$. We do not obtain, however, new information from these sum rules than with the former boundary conditions.

Let us proceed with care and begin with the first order derivatives. From Eqs. (15) and (31), we obtain the following results.

For $p=q=0$ we obtain a trivial result, while for the derivatives $p=1$, $q=0$ we obtain the Bjorken SR for the slope ρ^2

$$\rho^2=\frac{1}{4}+\frac{2}{3}\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2+\frac{1}{4}\sum_n [\tau_{1/2}^{(1)(n)}(1)]^2. \quad (32)$$

The relation to the Isgur-Wise notation is given by Eq. (18).

For $p=0$, $q=1$ we get $\xi(1)=\xi(1)$. For a purpose that will be clarified below, we make explicit the IW functions between $j^P=\frac{1}{2}^-$ states using the notation of Isgur and Wise $\xi^{(n)}(w)$ (18).

For $p=2$, $q=0$:

$$\rho^2-2\sigma^2+\frac{12}{5}\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2+\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2=0. \quad (33)$$

For $p=1$, $q=1$:

$$\rho^2-\frac{4}{3}\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2-\frac{8}{3}\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1) -\sum_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)\prime}(1)-2\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2 -\frac{24}{5}\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2=0. \quad (34)$$

For $p=0$, $q=2$:

$$1-8\rho^2+4\sigma^2+4\sum_n [\xi^{(n)\prime}(1)]^2+8\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2 +\sum_n [\tau_{1/2}^{(1)(n)}(1)]^2+\frac{32}{3}\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1) +4\sum_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)\prime}(1)+\frac{8}{3}\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2 +\frac{32}{5}\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2=0. \quad (35)$$

Equations (33)–(35) are a set of linear equations in the elastic slope ρ^2 and the curvature σ^2, and the following quantities, that are series on the radial excitations, indicated by the sums over n:

$$\sum_n [\xi^{(n)\prime}(1)]^2, \quad (36)$$

$$\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2, \quad (37)$$

$$\sum_n [\tau_{1/2}^{(1)(n)}(1)]^2, \quad (38)$$

$$-\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1), \quad (39)$$

$$-\sum_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)\prime}(1), \quad (40)$$

$$\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2, \quad (41)$$

$$\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2. \quad (42)$$

We realize that due to the fact that we compute the second order derivatives in Eq. (31) $(p+q=2)$ and use the boundary conditions $w_{if}=w=1$, the series in j^P states is truncated and includes at most the $\ell=2$ states $j^P=\frac{3}{2}^-,\frac{5}{2}^-$ corresponding to the unknowns (41) and (42). On the other hand (36) is the square of the derivatives at zero recoil of the lowest $j^P=\frac{1}{2}^-$, and Eqs. (37) and (38) depend on the IW functions of the transitions to the *P*-wave states $j^P=\frac{1}{2}^+,\frac{3}{2}^+$, that are simply related to the slope ρ^2 through Bjorken and Uraltsev SR, as we write down below again. Finally, we have two other unknowns (39) and (40) that involve the derivatives of the *P*-wave IW functions $\tau_{3/2}^{(1)(n)}(w)$, $\tau_{1/2}^{(1)(n)}(w)$ at zero recoil. These quantities were already introduced in Ref. [1].

IV. EQUATIONS FROM THE AXIAL SUM RULE

Let us now consider the derivatives of the SR for axial currents (17) with the boundary condition $w_{if}=1$, $w_i=w_f=w\rightarrow 1$:

$$\left(\frac{\partial^{p+q}L_A}{\partial w_{if}^p \partial w^q}\right)_{w_{if}=w=1}=\left(\frac{\partial^{p+q}R_A}{\partial w_{if}^p \partial w^q}\right)_{w_{if}=w=1}. \quad (43)$$

Since all terms in Eq. (17) vanish for $w_i=w_f=w_{if}=1$, to obtain information on the curvature σ^2 we will need to go up to the third order derivatives.

For $p=q=0$ and for $p=1$, $q=0$, and $p=0$, $q=1$ the results are trivial.

For $p=2$, $q=0$, and $p=q=1$ we get

$$\rho^2=\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2 \quad (44)$$

while for $p=0$, $q=2$, we get the Bjorken SR

$$\rho^2=\frac{1}{4}+\frac{2}{3}\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2+\frac{1}{4}\sum_n [\tau_{1/2}^{(1)(n)}(1)]^2. \quad (45)$$

Both Eqs. (44) and (45) imply the Uraltsev SR

$$\frac{1}{3}\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2-\frac{1}{4}\sum_n [\tau_{1/2}^{(1)(n)}(1)]^2=\frac{1}{4}. \quad (46)$$

Going now to the third order derivatives, we obtain the following results.

For $p=3,\ q=0$:

$$\sigma^2=2\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2. \quad (47)$$

For $p=2,\ q=1$:

$$\sigma^2=2\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1)+6\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2. \quad (48)$$

For $p=1,\ q=2$:

$$\sigma^2+\sum_n [\xi^{(n)\prime}(1)]^2+2\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2 +8\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1)+\frac{2}{3}\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2 +\frac{48}{5}\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2=0. \quad (49)$$

For $p=0,\ q=3$:

$$-3\rho^2+3\sigma^2+3\sum_n [\xi^{(n)\prime}(1)]^2+4\sum_n [\tau_{3/2}^{(1)(n)}(1)]^2 +8\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1) +3\sum_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)\prime}(1)+2\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2 +\frac{24}{5}\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2=0. \quad (50)$$

Equations (47)–(50) depend on ρ^2, σ^2 and the same set of unknowns listed in Eqs. (36)–(42).

V. LINEARLY INDEPENDENT RELATIONS

Let us concentrate on Eqs. (33)–(35) and (47)–(50) obtained, respectively, from the vector and axial sum rules. Using the Bjorken SR (45), the relation

$$\frac{4}{3}\rho^2-1=\sum_n [\tau_{1/2}^{(1)(n)}(1)]^2 \quad (51)$$

obtained from Eqs. (32) and (44) and Eqs. (44) and (47) we finally obtain the following set of linearly independent relations:

$$\rho^2=-\frac{4}{5}\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1) +\frac{3}{5}\sum_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)\prime}(1), \quad (52)$$

$$\sigma^2=-\sum_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)\prime}(1), \quad (53)$$

$$\sigma^2=2\sum_n [\tau_{5/2}^{(2)(n)}(1)]^2, \quad (54)$$

$$\rho^2-\frac{4}{5}\sigma^2+\sum_n [\tau_{3/2}^{(2)(n)}(1)]^2=0, \quad (55)$$

$$\frac{4}{3}\rho^2-\frac{5}{3}\sigma^2+\sum_n [\xi^{(n)\prime}(1)]^2=0. \quad (56)$$

Relations (52) and (53) were obtained in Ref. [1], and relations (54) and (55) in Ref. [11]. The systematic study of the present paper using all possibilities (31) and (43) involving the curvature gives the new equation (56).

VI. BOUNDS ON THE CURVATURE

The last two equations (55) and (56) involve the curvature with a negative sign and positive definite quantities. Making explicit in the sum $\Sigma_n[\xi^{(n)\prime}(1)]^2$ the ground state IW function slope $\xi^{(0)\prime}(1)=-\rho^2$, one obtains the two equations

$$\rho^2-\frac{4}{5}\sigma^2+\sum_n |\tau_{3/2}^{(2)(n)}(1)|^2=0, \quad (57)$$

$$\frac{4}{3}\rho^2+(\rho^2)^2-\frac{5}{3}\sigma^2+\sum_{n\neq 0} |\xi^{(n)\prime}(1)|^2=0 \quad (58)$$

which imply, respectively, the bounds

$$\sigma^2\geqslant\frac{5}{4}\rho^2, \quad (59)$$

$$\sigma^2\geqslant\frac{1}{5}[4\rho^2+3(\rho^2)^2]. \quad (60)$$

The bound (59) was obtained in Ref. [1] using the relations (52) and (53) and making the assumption

$$-\sum_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)\prime}(1)\geqslant 0. \quad (61)$$

Later, Eq. (59) was demonstrated rigorously in Ref. [11] and generalized to the nth derivative. However, in this latter paper, only derivatives relatively to w_{if} were taken, while in the present work a systematic use of Eqs. (31) and (43) is carried out.

The inequality (60) is the best of the bounds that we have obtained for σ^2 for any value of ρ^2, and is the main result of this paper. Interestingly enough, both bounds (59) and (60) coincide at the lower bound $\rho^2 \geqslant \frac{3}{4}$ implied by the Bjorken and Uraltsev SRs (32) and (46). At the value $\rho^2 = \frac{3}{4}$ one then gets the same absolute bound (i.e., independent of ρ^2) for σ^2, namely (11), $\sigma^2 \geqslant \frac{15}{16}$.

VII. IMPLICATION ON THE *P*-WAVE IW FUNCTIONS AT ZERO RECOIL

Let us now express the sums of products of the *P*-wave Isgur-Wise functions $\frac{1}{2}^- \to \frac{1}{2}^+$ and their derivatives $\Sigma_n \tau_{3/2}^{(1)(n)}(1)\tau_{3/2}^{(1)(n)'}(1)$ and $\Sigma_n \tau_{1/2}^{(1)(n)}(1)\tau_{1/2}^{(1)(n)'}(1)$ in terms of ρ^2 and σ^2. From Eqs. (52) and (53) we obtain, using now the notation of Isgur and Wise (18),

$$-\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)'}(1) = \frac{1}{3}\sigma^2, \tag{62}$$

$$-\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)'}(1) = -\frac{5}{12}\rho^2 + \frac{1}{3}\sigma^2. \tag{63}$$

Using the bounds (10) and (11) for ρ^2 and σ^2 one finds

$$-\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)'}(1) \geqslant \frac{5}{16}, \tag{64}$$

$$-\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)'}(1) \geqslant 0. \tag{65}$$

Strictly speaking, these relations do not give information on the slope of the lowest $n=0$ IW functions $\tau_{3/2}^{(0)'}(1)$ and $\tau_{1/2}^{(0)'}(1)$. However, if the $n=0$ state dominates the sum, the inequalities (64) and (65) imply that the slopes $\tau_{3/2}^{(0)'}(1)$ and $\tau_{1/2}^{(0)'}(1)$ are negative, as is plausible on physical grounds for form factors that do not involve radially excited states.

This is indeed the case for the Bakamjian-Thomas type of quark models, that satisfy IW scaling [17] and Bjorken and Uraltsev sum rules [18]. We have conjectured in Ref. [1] that this class of models presumably satisfy all the SRs of the heavy quark limit of QCD that follow from zero order moments.

In the Bakamjian-Thomas model one finds for the phenomenologically successful spectroscopic model of Godfrey and Isgur [19], the numbers

$$-\tau_{3/2}^{(0)}(1)\tau_{3/2}^{(0)'}(1) = 0.43, \tag{66}$$

$$-\tau_{1/2}^{(0)}(1)\tau_{1/2}^{(0)'}(1) = 0.04 \tag{67}$$

that by themselves satisfy the preceding bounds, so that the $n=0$ state seems to give a dominant contribution to the LHS of Eqs. (64) and (65).

VIII. NONRELATIVISTIC LIMIT OF THE BOUNDS

There is a simple intuitive argument to understand the term $\frac{3}{5}(\rho^2)^2$ in the best bound (60). Let us consider the nonrelativistic quark model, i.e., a nonrelativistic light quark q interacting with a heavy quark Q through a potential. The form factor—to be identified with the IW function—then has the simple form

$$F(\mathbf{k}^2) = \int d\mathbf{r}\varphi_0^+(r)\exp\left(i\frac{m_q}{m_q+m_Q}\mathbf{k}\cdot\mathbf{r}\right)\varphi_0(r), \tag{68}$$

where $\varphi_0(r)$ is the ground state radial wave function. In the small momentum transfer limit, the IW variable w is written, in the initial heavy hadron rest frame

$$w \cong 1 + \frac{\mathbf{v}'^2}{2} = 1 + \frac{\mathbf{k}^2}{2m_Q^2}. \tag{69}$$

Identifying the nonrelativistic IW function $\xi_{\rm NR}(w)$ with the form factor $F(\mathbf{k}^2)$ (68), one finds, because of rotational invariance

$$\xi_{\rm NR}(w) \cong 1 - m_q^2\langle 0|z^2|0\rangle(w-1) + \frac{1}{2}\frac{1}{3}m_q^4\langle 0|z^4|0\rangle(w-1)^2 + \cdots, \tag{70}$$

where $|0\rangle$ stands for the ground state wave function, and we have neglected in the $(w-1)^2$ coefficient subleading terms in powers of $1/(m_q z)$ (internal velocity). Therefore, one has the following expressions for the slope and the curvature in the nonrelativistic limit:

$$\rho_{\rm NR}^2 = m_q^2\langle 0|z^2|0\rangle, \quad \sigma_{\rm NR}^2 \cong \frac{1}{3}m_q^4\langle 0|z^4|0\rangle. \tag{71}$$

From spherical symmetry one has

$$\langle 0|z^4|0\rangle = \frac{1}{5}\langle 0|r^4|0\rangle. \tag{72}$$

Using now completeness $\Sigma_n|n\rangle\langle n| = 1$,

$$\langle 0|r^4|0\rangle = |\langle 0|r^2|0\rangle|^2 + \sum_{n\neq 0}|\langle n|r^2|0\rangle|^2 \tag{73}$$

we again use spherical symmetry

$$\langle 0|r^4|0\rangle = 9|\langle 0|z^2|0\rangle|^2 + 9\sum_{n\neq 0,\rm rad}|\langle n|z^2|0\rangle|^2, \tag{74}$$

where the latter sum runs only over radial excitations.

Therefore, from Eqs. (71)–(74) we can rewrite $\sigma_{\rm NR}^2$ as

$$\sigma_{\rm NR}^2 = \frac{3}{5}\left\{[m_q^2|\langle 0|z^2|0\rangle|]^2 + m_q^4\sum_{n\neq 0,\rm rad}|\langle n|z^2|0\rangle|^2\right\} \tag{75}$$

or

$$\sigma_{\rm NR}^2=\frac{3}{5}[\rho_{\rm NR}^2]^2+\frac{3}{5}m_q^4\sum_{n\neq 0,{\rm rad}}|\langle n|z^2|0\rangle|^2 \tag{76}$$

and therefore

$$\sigma_{\rm NR}^2\geqslant\frac{3}{5}[\rho_{\rm NR}^2]^2. \tag{77}$$

Notice that, denoting the bound state radius by R and the light quark mass by m_q, in the nonrelativistic limit, just from expressions (71), one can see that $\rho_{\rm NR}^2$ scales as $m_q^2R^2$, while $\sigma_{\rm NR}^2$ scales as $m_q^4R^4$ and both the LHS and the RHS in Eq. (77) scale in the same way.

Going back to the relativistic bounds (59), (60), we observe that the terms proportional to ρ^2 are subleading in the nonrelativistic expansion and correspond to relativistic corrections specific to QCD in the heavy quark limit. In the nonrelativistic limit $\rho^2\sim m_q^2R^2\gg 1$, and the power $(\rho^2)^2$ is leading. We can understand therefore the appearance of the term $\frac{3}{5}(\rho^2)^2$ in the RHS of the inequality (60).

IX. AN EXAMPLE OF FIT TO THE DATA

An interesting phenomenological remark is that the simple parametrization for the IW function [19]

$$\xi(w)=\left(\frac{2}{w+1}\right)^{2\rho^2} \tag{78}$$

gives

$$\sigma^2=\frac{\rho^2}{2}+(\rho^2)^2 \tag{79}$$

that satisfies the inequalities (59), (60) if $\rho^2\geqslant\frac{3}{4}$, i.e., for all values allowed for ρ^2. Moreover, interestingly, at the lowest bound of the slope $\rho^2=\frac{3}{4}$, Eq. (79) implies precisely the lowest bound of the curvature $\sigma^2=\frac{15}{16}$, as pointed out in Ref. [11].

Notice that in Ref. [19], within the class of Bakamjian-Thomas quark models, the approximate form (78) was found with $\rho^2=1.02$ in the particular case of the spectroscopic model of Godfrey and Isgur. This gives a curvature (79) $\sigma^2=1.55$, close to the bound (60), that gives $\sigma^2\geqslant 1.44$, stronger than the bound (59), which implies $\sigma^2\geqslant 1.27$.

As a simple example of a fit with the simple function (78), we can use BELLE data on $\overline{B}^0\to D^{*+}e^-\overline{\nu}$ for the product $|V_{cb}|\mathcal{F}^*(w)$ [12], as shown in Fig. 1. The function $\mathcal{F}^*(w)$ is equal to the Isgur-Wise function $\xi(w)$ in the heavy quark limit. Assuming only departures of this limit at $w=1$, i.e., fitting $\xi(w)$ from the data with

$$|V_{cb}|\mathcal{F}^*(w)=|V_{cb}|\mathcal{F}^*(1)\xi(w) \tag{80}$$

we obtain the following results for the normalization and the slope:

$$\mathcal{F}^*(1)|V_{cb}|=0.036\pm 0.002, \quad \rho_{\mathcal{F}^*}^2=1.15\pm 0.18 \tag{81}$$

with the other derivatives of $\xi(w)$ fixed by Eq. (78) (Fig. 1).

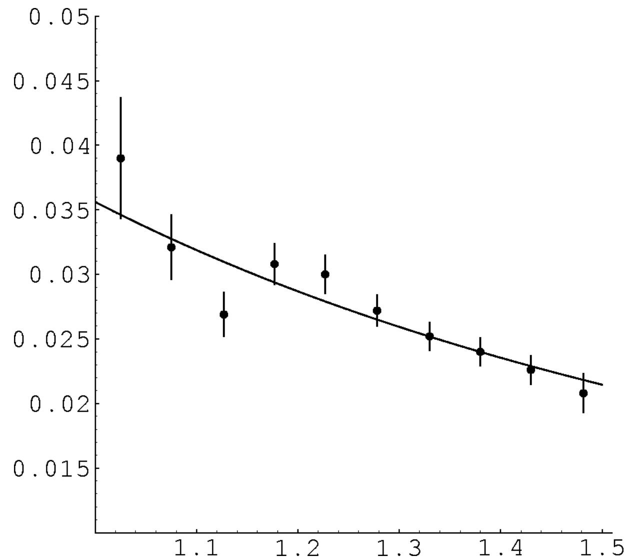

FIG. 1. Fit to $\mathcal{F}^*(w)|V_{cb}|$ using the phenomenological formula (78) and the BELLE data for $\overline{B}\to D^*\ell\nu$ [16], assuming only violations to the heavy quark limit at $w=1$. The fit gives the results (81).

As we can see, the determination of $\mathcal{F}^*(1)|V_{cb}|$ is rather precise, while $\rho_{\mathcal{F}^*}^2$ has a larger error. However, the values obtained for $|V_{cb}|$ and $\rho_{\mathcal{F}^*}^2$ are strongly correlated. It is important to point out that the most precise data points are the ones at large w, so that higher derivatives contribute in this region. Due to the alternate character of $\xi(w)$ as a series of $(w-1)$, one does not clearly see the curvature of $\xi(w)$ in Fig. 1, but the curve is definitely not close to a straight line. Linear fits, as are commonly used, should be ruled out at the view of the bounds that we have found.

We must emphasize that the fit that we present is a simple exercise in the heavy quark limit. Radiative corrrections and $1/m_Q$ corrections that enter in the relation between the actual function $\mathcal{F}^*(w)$ and its heavy quark limit $\xi(w)$ should be taken into account, although this does not seem to be an easy task [20]. The slope $\rho_{\mathcal{F}^*}^2$ has to be distinguished from $\rho_{A_1}^2$ that is usually tabulated [16].

X. COMPARISON WITH THE DISPERSIVE BOUNDS

Considerable effort has gone into formulating dispersive constraints on the shape of the form factors in $\overline{B}\to D^*\ell\nu$ [21–25]. Dispersion relations relate the hadronic spectral functions to the QCD two-point functions in the deep Euclidean region, and positivity allows to bound the contribution of the relevant states, leading to constraints on the semileptonic form factors.

We will now compare our method that gives information on the derivatives of the Isgur-Wise function with the dispersive approach. A first remark to be made is that our approach, based on Bjorken-like SRs, holds *in the physical region* of the semileptonic decays $\overline{B}\to D^{(*)}\ell\nu$ and *in the heavy quark limit*. However, concerning this last simplifying feature, we should underline that there is no objection to the inclusion in the calculation of radiative corrections and subleading corrections in powers of $1/m_Q$.

The dispersive approach starts from bounds *in the crossed channel* by comparison of the OPE and the sum over hadrons coupled to the corresponding current $\overline{BD}$, $\overline{BD^*}$, ... Then, one analytically continues to the physical region of the semileptonic decays. This is done for a single reference form factor, for example, the combination

$$V_1(w)=h_+(w)-\frac{m_B-m_D}{m_B+m_D}h_-(w) \tag{82}$$

that enters the $\overline{B}\to D\ell\nu$ rate. In the heavy quark limit $h_-(w)=0$, $V_1(w)=h_+(w)=\xi(w)$. The ratios of the remaining form factors to $V_1(w)$ are computed *in the physical region* by introducing $1/m_Q$ and α_s corrections to the heavy quark limit. The dispersive approach considers *physical quark masses*, in contrast with the heavy quark limit of our method.

The two approaches are quite different in spirit and in their results. However, it can be interesting to numerically compare our bounds with the ones of the dispersive approach, as they happen to be complementary. We must, however, keep in mind the differences between the two methods. We have demonstrated in Ref. [11] that the IW function $\xi(w)$ is an alternating series in powers of $(w-1)$, with the moduli of the derivatives satisfying the bounds (12) and (60).

A. Comparison with the work of Caprini, Lellouch, and Neubert

Let us consider the main results of Ref. [24], that are summarized by the one-parameter formula

$$\frac{V_1(w)}{V_1(1)}\cong 1-8\rho^2 z+(51\rho^2-10)z^2-(252\rho^2-84)z^3 \tag{83}$$

with the variable $z(w)$ defined by

$$z=\frac{\sqrt{w+1}-\sqrt{2}}{\sqrt{w+1}+\sqrt{2}} \tag{84}$$

and the allowed range for ρ^2 being

$$-0.17<\rho^2<1.51. \tag{85}$$

Of course, the function $V_1(w)/V_1(1)$ contains finite mass corrections that are absent at present in our method. Nevertheless, let us first compare these results with our lower bounds (12), assuming the rough approximation

$$\frac{V_1(w)}{V_1(1)}\cong \xi(w). \tag{86}$$

Of course, since the expansion (83) stops at third order in z, it would only make sense in the comparison to go up to the third derivative of $\xi(w)$. Using our notation, the results of Sec. IV of Ref. [24] for the first derivatives write, from the expansion (83), in terms of the slope ρ^2:

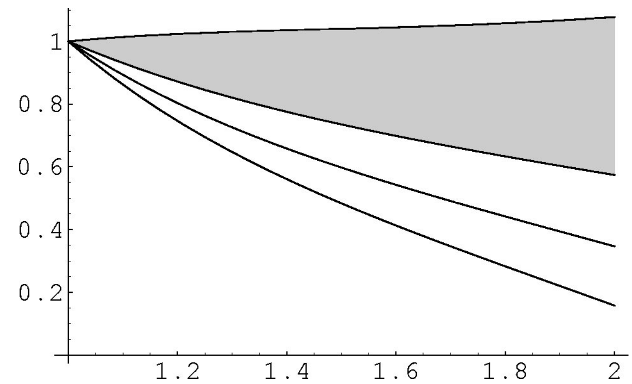

FIG. 2. The upper (lower) curves are the representations of $\xi(w)$ according to the dispersive approach of Caprini *et al.* [21] (83)–(86). The upper (lower) curve correspond to $\rho^2=-0.17$ ($\rho^2=1.51$). The shadowed region is the region forbidden by the Uraltsev bound $\rho^2\geqslant\frac{3}{4}$. The remaining allowed region corresponds to Eq. (89). The curve within this allowed region is our fit, according to (83), (86), to BELLE data [11] for $\mathcal{F}^*(w)$, normalized to $w=1$, that gives $\mathcal{F}^*(1)|V_{cb}|=0.036\pm0.002$, $\rho^2_{\mathcal{F}^*}=1.16\pm0.15$, in practice the same fit as Eq. (81) with the phenomenological formula (78).

$$\xi''(1)=\frac{1}{32}(67\rho^2-10), \tag{87}$$

$$\xi'''(1)=-\frac{1}{256}(1487\rho^2-372) \tag{88}$$

with ρ^2 in the range (85). From Eqs. (87) and (88), using the notation $\xi(w)=1-\rho^2(w-1)+c(w-1)^2+d(w-1)^3+\cdots$ one gets the numerical relations [24] $c\cong 1.05\rho^2-0.15$, $d\cong -0.97\rho^2+0.24$.

Let us now comment on the implications of our bounds (12). The first important remark is that, within the simplifying hypothesis (86), the range (85) is considerably tightened by the lower bound on $\rho^2\geqslant\frac{3}{4}$ implied by Bjorken and Uraltsev sum rules. Therefore, we will consider hereafter, instead of Eq. (85), the improved range

$$\frac{3}{4}\leqslant\rho^2<1.51 \tag{89}$$

that shows that our type of lower bounds are complementary to the upper bounds obtained from dispersive methods. Within the hypothesis of the heavy quark limit, the region allowed by the dispersive bounds for $\xi(w)$ with ρ^2 within the range (85) is obviously much reduced by the bounds (89) (Fig. 2).

Finally, let us look for the implications of our improved bound on the curvature, Eq. (60). Combining the linear dependence obtained from dispersive methods (87) with the inequality (60) one obtains the condition

$$\frac{1}{32}(67\rho^2-10)\geqslant\frac{1}{5}[4\rho^2+3(\rho^2)^2] \tag{90}$$

that gives the range

$$0.28 \lesssim \rho^2 \lesssim 1.88. \tag{91}$$

Interestingly, the condition (90) gives *by itself* an upper bound for ρ^2 that is of the same order than the upper bound (85). Moreover, the range (91) contains the improved range (89), and things appear to be coherent.

B. Comparison with the work of Boyd, Grinstein, and Lebed

Let us now compare the results of the work of Boyd, Grinstein, and Lebed with the dispersive method [25]. In this work, the QCD part of the calculation includes α_s and non-perturbative (condensate) corrections, and new poles below the annihilation threshold ignored in Ref. [23], but not in Ref. [24]. In a form that allows us to make the comparison with our results, the authors of Ref. [25] obtain the following expansion for the scalar form factor:

$$\tilde{f}_0(w) = \tilde{f}_0(1) + [1.72a_1 - 0.77\tilde{f}_0(1)](w-1) + [-1.74a_1 + 0.21a_2 + 0.55\tilde{f}_0(1)](w-1)^2 + \cdots, \tag{92}$$

where $\tilde{f}_0(w)$ has been defined by Caprini and Neubert [23],

$$\tilde{f}_0[w(q^2)] = \frac{f_0(q^2)}{(M_B - M_D)\sqrt{M_B M_D}(w+1)} \tag{93}$$

with

$$f_0(q^2) = (M_B^2 - M_D^2) f_+(q^2) + q^2 f_-(q^2). \tag{94}$$

$f_\pm(q^2)$ are the form factors governing the rate $\overline{B} \to D \ell \nu$ $(q = p - p')$:

$$\langle D(p')|V_\mu|B(p)\rangle = f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu. \tag{95}$$

Heavy quark symmetry implies

$$\tilde{f}_0[w(q^2)] \cong \xi(w). \tag{96}$$

The coefficients a_n in Eq. (92) are defined by the expression for a generic form factor [24,25]

$$F(z) = \frac{1}{P(z)\varphi(z)} \sum_{n=0}^{\infty} a_n z^n, \tag{97}$$

where z is defined by Eq. (84). The functions $P(z)$ and $\varphi(z)$—respectively, the Blaschke factor and the outer function—contain the subthreshold singularities in the annihilation channel, respectively, the B_c poles and the kinematic singularities. The basic result of the dispersive approach is that the coefficients a_n of the series obey

$$\sum_{n=0}^{\infty} a_n^2 \leqslant 1. \tag{98}$$

To compare with our bounds we proceed as we did above. Since Eq. (96) holds in the heavy quark limit, we set $\tilde{f}_0(1) = 1$ and write the Isgur-Wise function in terms of the coefficients a_n:

$$\xi(w) \cong 1 + (1.72a_1 - 0.77)(w-1) + (-1.74a_1 + 0.21a_2 + 0.55)(w-1)^2 + \cdots. \tag{99}$$

Notice that in Eqs. (92) and (99) it does not make much sense to consider higher powers $(w-1)^n$ $(n \geqslant 3)$ unless the corresponding a_n $(n \geqslant 3)$ are introduced. Then, our lower bounds (12) are written

$$-1.72a_1 + 0.77 \geqslant \frac{3}{4} 2(-1.74a_1 + 0.21a_2 + 0.55) \geqslant \frac{5}{4}(-1.72a_1 + 0.77) \tag{100}$$

implying, respectively,

$$a_1 \leqslant 0.01,$$

$$a_2 \geqslant 3.17a_1 - 0.33. \tag{101}$$

Since, from Eqs. (101) and (98) we have

$$-1 \leqslant a_1 \leqslant 0.01 \tag{102}$$

and the coefficient of a_1 in Eq. (101) is large, the whole range

$$-1 \leqslant a_2 \leqslant 1 \tag{103}$$

is allowed. This seems to support the statement of Ref. [25] that a_2 cannot always be neglected.

Moreover, using the quadratic bound (60), one obtains

$$3(\rho^2)^2 - 6\rho^2 + 2(1 - a_2) \lesssim 0 \tag{104}$$

and therefore

$$-0.5 \lesssim a_2 \leqslant 1 \tag{105}$$

giving the range for ρ^2 in terms of a_2

$$1 - \sqrt{\frac{1+2a_2}{3}} \lesssim \rho^2 \lesssim 1 + \sqrt{\frac{1+2a_2}{3}} \tag{106}$$

and therefore the wide range

$$0 \lesssim \rho^2 \lesssim 2. \tag{107}$$

For $a_2 = 0$, implicitly assumed in Ref. [24], one finds the range

$$0.42 \lesssim \rho^2 \lesssim 1.58, \tag{108}$$

a domain qualitatively consistent with but somewhat narrower than the corresponding one (91) obtained from the linear relation between the curvature and the slope given by Ref. [24].

In conclusion, there is no contradiction between the dispersive bounds and the type of bounds that we have obtained using Bjorken-like sum rules in the heavy quark limit. The latter appear rather as lower bounds that are complementary to the upper bounds of the dispersive approach, considerably tightening the allowed range for ρ^2 and for the higher derivatives of $\xi(w)$ as well.

XI. A PHENOMENOLOGICAL ANSATZ FOR THE ISGUR-WISE FUNCTION AND THE DISPERSIVE CONSTRAINTS

In light of the preceding discussion, we are now going to address the question of whether the phenomenological ansatz for the IW function proposed in Sec. IX

$$\xi(w)=\left(\frac{2}{w+1}\right)^{2\rho^2} \tag{109}$$

satisfies, assuming the heavy quark limit (86) or (96), the constraints of the dispersive approach.

We will follow here the formulation of Boyd *et al.* [25] and consider the form factors $f_+(q^2)$ and $f_0(q^2)$ defined by Eqs. (94) and (95). In the heavy quark limit, one has the relations

$$f_+[q^2(w)]\cong\frac{M_B+M_D}{2\sqrt{M_BM_D}}\xi(w), \tag{110}$$

$$f_0[q^2(w)]\cong(M_B-M_D)\sqrt{M_BM_D}(w+1)\xi(w). \tag{111}$$

We now denote generically any of these form factors by $F[q^2(w)]$, or through the transformation (84), $F[q^2(z)]$.

We adopt the phenomenological formula (109) for $\xi(w)$ and define the corresponding series (97)

$$\sum_{n=0}^{\infty} a_n z^n=P(z)\varphi(z)F(z), \tag{112}$$

where $P(z)$ and $\varphi(z)$ are the Blaschke factor and the outer function of the corresponding form factors.

We now want to compare the coefficients a_n obtained from Eqs. (110), (111), assuming $F(z)=\xi[w(z)]$ given by Eq. (109), to the condition (98):

$$\sum_{n=0}^{\infty} a_n^2\leqslant 1. \tag{113}$$

The outer functions $\varphi(z)$ and the Blaschke factors $P(z)$ for $f_+(q^2)$ and $f_0(q^2)$ are given in Ref. [25], respectively, by formula (4.23) and Table 1 and by formula (4.25) and Table 3. We have singled out $f_+(q^2)$ and $f_0(q^2)$ as given by Eqs. (110) and (111) but we could have taken any other form factor related, up to kinematic factors, to $\xi(w)$. Of course, the results for the coefficients a_n would differ according to the considered form factor.

We use the numerical parameters of this paper, and two choices for ρ^2 in formula (109), namely, $\rho^2=1.023$, which corresponds to the Isgur-Wise function obtained within the Bakamjian-Thomas scheme from the Godfrey-Isgur spectroscopic model, as found in Ref. [19], and $\rho^2=1.15$ obtained from the fit of Sec. IX.

Denoting the Blaschke factor and outer function for each form factor by the corresponding subindices, we find, for $\rho^2=1.023$, the series (112) for $f_+[q^2(z)]$

$$P_+(z)\varphi_+(z)f_+[q^2(z)]=0.0143-0.0179z-0.1164z^2 +0.3277z^3-0.1995z^4-0.4497z^5 +1.2347z^6+\cdots \tag{114}$$

and for $f_0[q^2(z)]$

$$P_0(z)\varphi_0(z)f_0[q^2(z)]=0.0834-0.1750z-0.1725z^2 +0.8673z^3-1.1600z^4-0.8943z^5 -0.4346z^6+\cdots. \tag{115}$$

For $\rho^2=1.15$ we find, respectively,

$$P_+(z)\varphi_+(z)f_+[q^2(z)]=0.0143-0.0326z-0.0907z^2 +0.4294z^3-0.5779z^4-0.0306z^5 +1.3868z^6+\cdots \tag{116}$$

and

$$P_0(z)\varphi_0(z)f_0[q^2(z)]=0.0834-0.2599z+0.0484z^2 +0.9094z^3-2.0079z^4+2.5089z^5 -2.3596z^6+\cdots. \tag{117}$$

Compared to the condition (113), we observe two points. First, the first coefficients have squares well below 1, especially for $f_+(q^2)$. That this happens to be the case for this form factor that has three Blaschke factors, while $f_0(q^2)$ has only two, reinforces the idea that one should be closer to the IW function: when the number of subthreshold poles increases, the form factor should become closer to the Isgur-Wise function [25]. Second, high powers of z have coefficients that can be of $O(1)$. Therefore, the phenomenological formula (109) is ruled out on strict theoretical grounds. However, since the variable z defined by Eq. (84) is small in the whole physical region ($z_{\max}\simeq 0.056$), high powers z^n($n\geqslant 4$) are completely irrelevant in the actual phase space. Our conclusion is that, owing to the fact that the coefficients, up to order z^3 included, satisfy the condition (113), the "dipole" formula (109) gives, *on phenomenological grounds*, a fair enough representation of the form factors (110), (111).

XII. CONCLUSIONS

In conclusion, using sum rules in the heavy quark limit of QCD, as formulated in Ref. [1], we have found an improved bound for the curvature of the Isgur-Wise function $\sigma^2 = \xi''(1) \geqslant \frac{1}{5}[4\rho^2 + 3(\rho^2)^2]$ that implies the already demonstrated [1,11] absolute bound $\sigma^2 \geqslant \frac{15}{16}$.

Beyond the simple ansatz (78) introduced above, any phenomenological parametrization of $\xi(w)$ intending to fit the CKM matrix element $|V_{cb}|$ in $B \to D^{(*)}\ell\nu$ should have, for a given slope ρ^2 satisfying the bound (10), a curvature σ^2 satisfying the new bound (60). Moreover, we discuss these bounds in comparison with the dispersive approach. We show that there is no contradiction as our bounds restrain the region for $\xi(w)$ allowed by this latter method.

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