Splitting strong and electromagnetic interactions in $K_{\ell 4}$ decays

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We recently considered $K_{\ell 4}$ decays in the framework of chiral perturbation theory based on the effective Lagrangian including mesons, photons, and leptons. There, we published analytic one-loop-level expressions for form factors f and g corresponding to the mixed process, $K^0 \to \pi^0 \pi^- \ell^+ \nu_\ell$. We propose here a possible splitting between strong and electromagnetic parts allowing analytic (and numerical) evaluation of isospin breaking corrections. The latter are sensitive to the infrared divergence subtraction scheme and are sizable near the $\pi\pi$ production threshold. Our results should be used for the extraction of the P-wave isovector $\pi\pi$ phase shift from the outgoing data of the currently running KTeV experiment at Fermilab.

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I. INTRODUCTION

Every time that a kaon decays into a couple of pions and a lepton-neutrino pair, $\pi\pi$ scattering occurs in the final state. Whenever a pion scatters on its twin, it offers us an additional opportunity to scrutinize the fundamental state of strong interactions (see Ref. [1] for references). Let δ_l^I be the phase of a two-pion state of angular momentum l and isospin I and consider the $K_{\ell 4}$ decay process

$$K(p) \to \pi(p_1)\pi(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \tag{1}$$

where the lepton ℓ is either a muon μ or an electron e, and ν stands for the corresponding neutrino. In the isospin limit, the decay amplitude $\mathcal{A}$ for process (1) can be parametrized in terms of three vectorial (F, G, and R) and one anomalous (H) form factors:

$$\begin{aligned}\mathcal{A} \doteq i\frac{G_F}{\sqrt{2}}V_{us}^*\overline{u}(\mathbf{p}_\nu)\gamma_\mu(1-\gamma^5)v(\mathbf{p}_\ell)\Bigg\{&\frac{i}{M_{K^\pm}}[(p_1+p_2)^\mu F \\ &+(p_1-p_2)^\mu G+(p_\ell+p_\nu)^\mu R] \\ &-\frac{1}{M_{K^\pm}^3}\epsilon^{\mu\nu\rho\sigma}(p_\ell+p_\nu)_\nu(p_1+p_2)_\rho(p_1-p_2)_\sigma H\Bigg\},\end{aligned} \tag{2}$$

where V_{us} denotes the Cabibbo-Kobayashi-Maskawa flavor-mixing matrix element and G_F is the so-called Fermi coupling constant. Note that form factors are made dimensionless by inserting the normalizations $M_{K^\pm}^{-1}$ and $M_{K^\pm}^{-3}$. The fact that we have used the *charged* kaon mass is a purely conventional matter and corresponds to the choice of defining the isospin limit in terms of charged masses. In the following, we will be interested only in two form factors F and G and denote by $(F,G)^{+-}$ and $(F,G)^{0-}$ those corresponding to the physical processes

$$K^+(p) \to \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu) \tag{3}$$

and

$$K^0(p) \to \pi^0(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \tag{4}$$

respectively.

Form factors are analytic functions of three independent Lorentz invariants,

$$s_\pi \doteq (p_1+p_2)^2, \quad s_\ell \doteq (p_\ell+p_\nu)^2, \tag{5}$$

and the angle θ_π formed by $\mathbf{p}_1$, in the dipion rest frame, and the line of flight of the dipion as defined in the kaon rest frame [2,3]. It has been shown in Ref. [4] that, in the experimentally relevant region, the partial wave expansion,

$$F^{+-} = [f_S(s_\pi)+f_\ell s_\ell]e^{i\delta_0^0(s_\pi)} + \tilde{f}_P XY\cos\theta_\pi e^{i\delta_1^1(s_\pi)}, \tag{6}$$

$$G^{+-} = (g_P + g_P' s_\pi + g_\ell s_\ell)e^{i\delta_1^1(s_\pi)} + \tilde{g}_D XY\cos\theta_\pi e^{i\delta_2^0(s_\pi)}, \tag{7}$$

is proving sufficient to parametrize form factors. In the preceding,

$$X \doteq \frac{1}{2}\lambda^{1/2}(s_\pi, s_\ell, M_{K^\pm}^2), \quad Y \doteq \frac{1}{s_\pi}\lambda^{1/2}(s_\pi, M_{\pi^\pm}^2, M_{\pi^\pm}^2), \tag{8}$$

with

$$\lambda(x,y,z) \doteq x^2+y^2+z^2-2xy-2xz-2yz \tag{9}$$

the usual Källén function. Note the linear dependence of the first term in the partial wave expansion of form factors on s_ℓ. Isospin symmetry, Bose symmetry, and the $\Delta I = 1/2$ rule lead to

$$F^{0-} = \sqrt{2}\tilde{f}_P XY\cos\theta_\pi e^{i\delta_1^1(s_\pi)}, \tag{10}$$

$$G^{0-} = \sqrt{2}(g_P + g_P' s_\pi + g_\ell s_\ell)e^{i\delta_1^1(s_\pi)}. \tag{11}$$

It follows that $K_{\ell 4}$ decay of the neutral kaon is dominated by P waves. Therefore, a precise measurement of form factors

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for the decay in question would allow an accurate determination of the P-wave isovector $\pi\pi$ phase shift.

The currently running KTeV experiment [5] aims at measuring form factors for $K_{\ell 4}$ decay of the neutral kaon with an accuracy 3 times better than the one offered by previous measurement [6,7]. The outgoing data on form factors contain, besides a strong interaction contribution, a contribution coming from the electroweak interaction. The latter breaks isospin symmetry and is expected to be sizable near the $\pi\pi$ production threshold [8]. In order to extract $\pi\pi$ scattering parameters from the KTeV measurement, the isospin breaking correction to form factors should therefore be under control. In this direction, we recently published analytic expressions for F^{0-} and G^{0-} form factors calculated at one-loop level in the framework of chiral perturbation theory based on the effective Lagrangian including mesons, photons, and leptons [1]. In the present work, we will split analytically the isospin limit and isospin breaking part in form factors, allowing a first evaluation of isospin breaking effects in $K_{\ell 4}$ decays.

II. KINEMATICAL VARIABLES

In the following, we shall consider process (4) and use, unless mentioned, notations of Ref. [1]. In the presence of isospin breaking, the decay amplitude for process (4) can be written as follows by Lorentz covariance,

$$\mathcal{A}^{0-} \doteq \frac{G_F V_{us}^*}{\sqrt{2}} \overline{u}(\mathbf{p}_\nu)(1+\gamma^5)\Bigg\{ \frac{1}{M_{K^\pm}}[(p_1+p_2)^\mu f^{0-} + (p_1-p_2)^\mu g^{0-} + (p_\ell+p_\nu)^\mu r^{0-}]\gamma_\mu + \frac{i}{M_{K^\pm}^3}\epsilon^{\mu\nu\rho\sigma}(p_\ell+p_\nu)_\nu(p_1+p_2)_\rho(p_1-p_2)_\sigma h^{0-} + \frac{1}{2M_{K^\pm}^2}[\gamma_\mu,\gamma_\nu]p_1^\mu p_2^\nu T\Bigg\} v(\mathbf{p}_l).$$

The quantities f, g, r, and h will be called the *corrected* $K_{\ell 4}$ form factors since their isospin limits are nothing else than the $K_{\ell 4}$ form factors, F, G, R, and H, respectively. The tensorial form factor T is purely isospin breaking and does not contribute to the mixed process at leading chiral order. The corrected form factors as well as the tensorial one are analytic functions of five independent Lorentz invariants s_π, s_ℓ, θ_π, θ_ℓ, and ϕ. θ_ℓ is the angle formed by $\mathbf{p}_\ell$, in the dilepton rest frame, and the line of flight of the dilepton as defined in the kaon rest frame. ϕ is the angle between the normals to the planes defined in the kaon rest frame by the pion pair and the lepton pair, respectively. Let us denote by δF and δG the next-to-leading order corrections to the F^{0-} and G^{0-} form factors, respectively,

$$f^{0-} = \frac{M_{K^\pm}}{F_0}(0+\delta F),$$

$$g^{0-} = \frac{M_{K^\pm}}{F_0}(1+\delta G).$$

The analytic expressions for δF and δG were given in Ref. [1]. We shall distinguish between *photonic* and *nonphotonic* contributions to δF and δG. The photonic contribution comes from those Feynman diagrams with a virtual photon exchanged between two meson legs or one meson leg and a pure strong vertex. Obviously, this contribution is proportional to e^2, where e is the electric charge, and depends in general on the five independent kinematical variables, s_π, s_ℓ, θ_π, θ_ℓ, and ϕ through Lorentz invariants such as $(p_2+p_\ell)^2$, say. The nonphotonic contribution comes from diagrams having similar topology as those in the pure strong theory with isospin breaking allowed in propagators and vertices. This contribution generates isospin breaking terms proportional to the rate of $SU(2)$ to $SU(3)$ breaking,

$$\epsilon \doteq \frac{\sqrt{3}}{4}\frac{m_d-m_u}{m_s-\hat{m}}, \quad \hat{m}\doteq\frac{1}{2}(m_u+m_d), \tag{12}$$

and to mass square difference between charged and neutral mesons,

$$\Delta_\pi \doteq M_{\pi^\pm}^2 - M_{\pi^0}^2 = 2Z_0 e^2 F_0^2 + \mathcal{O}(p^4), \tag{13}$$

$$\Delta_K \doteq M_{K^\pm}^2 - M_{K^0}^2 = 2Z_0 e^2 F_0^2 - B_0(m_d-m_u) + \mathcal{O}(p^4), \tag{14}$$

or equivalently, $(m_d-m_u)/(m_s-\hat{m})$, $Z_0 e^2$, and m_d-m_u. The kinematical dependence is on three Lorentz invariants, $(p_1+p_2)^2$, $(p-p_1)^2$, and $(p-p_2)^2$, which represent, respectively, the dipion mass square, the exchange energy between the kaon and the neutral pion, and that between the kaon and the charged pion. In terms of independent kinematical variables, the preceding scalars are functions of s_π, s_ℓ, and $\cos\theta_\pi$.

A. The photonic contribution

A generic term in the photonic contribution can be

$$\text{photonic contribution} = e^2\sum_i \xi_i((p_2+p_\ell)^2,\ldots), \tag{15}$$

where ξ_i is an arbitrary loop integral function of $(p_2+p_\ell)^2$. To the order we are working, that is, to leading order in isospin breaking, the power counting scheme we use dictates the following on-shell conditions to be used in the argument of ξ_i,

$$p^2 = M_K^2 \doteq B_0(m_s+\hat{m}), \quad p_1^2 = p_2^2 = M_\pi^2 \doteq 2B_0\hat{m}. \tag{16}$$

Therefore, $(p_2+p_\ell)^2$ in (15) should be replaced by the following expression [1],

$$M_\pi^2+m_\ell^2+\frac{1}{4}\left(1+\frac{m_\ell^2}{s_\ell}\right)(M_K^2-s_\ell-s_\pi)$$
$$-\frac{1}{4}\left(1+\frac{m_\ell^2}{s_\ell}\right)\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2}\lambda^{1/2}(s_\pi,s_\ell,M_K^2)\cos\theta_\pi$$
$$+\frac{1}{4}\left(1-\frac{m_\ell^2}{s_\ell}\right)\lambda^{1/2}(s_\pi,s_\ell,M_K^2)\cos\theta_\ell$$
$$-\frac{1}{4}\left(1-\frac{m_\ell^2}{s_\ell}\right)\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2}(M_K^2-s_\ell-s_\pi)\cos\theta_\pi\cos\theta_\ell$$
$$+\frac{1}{2}\left(1-\frac{m_\ell^2}{s_\ell}\right)\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2}(s_\pi s_\ell)^{1/2}$$
$$\times\sin\theta_\pi\sin\theta_\ell\cos\phi.$$

From the foregoing, it is clear that for $s_\ell=m_\ell^2$ the photonic contribution neither depends on θ_ℓ nor on ϕ. In order to reduce the complexity of the study and allow the treatment of photonic and nonphotonic contributions to $\mathcal{A}^{0-}$ on an equal footing, we will assume that

$$s_\ell=m_\ell^2 \tag{17}$$

and use for $(p_2+p_\ell)^2$ in (15) the following expression,

$$(p_2+p_\ell)^2=\frac{1}{2}(M_K^2+2M_\pi^2+m_\ell^2-s_\pi)-\frac{1}{2}\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_K^2)\cos\theta_\pi. \tag{18}$$

It follows that (15) can be written as

$$\text{photonic contribution}=e^2\varsigma(s_\pi)+e^2\vartheta(s_\pi)\cos\theta_\pi, \tag{19}$$

where ς and ϑ are analytic functions of s_π.

B. The nonphotonic contribution

In order to split strong and electromagnetic terms in the nonphotonic contribution, one has to expand the exchange energies $(p-p_1)^2$ and $(p-p_2)^2$ in powers of the fine structure constant α and m_d-m_u. To this end, we shall first express these scalars in terms of s_π and $\cos\theta_\pi$ for $s_\ell=m_\ell^2$ and in the presence of isospin breaking. From Ref. [1],

$$(p-p_1)^2=M_{K^0}^2+M_{\pi^0}^2-\frac{1}{2s_\pi}[(M_{K^0}^2-m_\ell^2+s_\pi)(s_\pi+M_{\pi^0}^2-M_{\pi^\pm}^2)+\lambda^{1/2}(s_\pi,m_\ell^2,M_{K^0}^2)\lambda^{1/2}(s_\pi,M_{\pi^0}^2,M_{\pi^\pm}^2)\cos\theta_\pi], \tag{20}$$

$$(p-p_2)^2=M_{K^0}^2+M_{\pi^\pm}^2-\frac{1}{2s_\pi}[(M_{K^0}^2-m_\ell^2+s_\pi)(s_\pi-M_{\pi^0}^2+M_{\pi^\pm}^2)-\lambda^{1/2}(s_\pi,m_\ell^2,M_{K^0}^2)\lambda^{1/2}(s_\pi,M_{\pi^0}^2,M_{\pi^\pm}^2)\cos\theta_\pi]. \tag{21}$$

Let us denote by t_π and u_π the isospin limits of the preceding Lorentz scalars,

$$t_\pi=\frac{1}{2}(M_{K^\pm}^2+2M_{\pi^\pm}^2+m_\ell^2-s_\pi)-\frac{1}{2}\left(1-\frac{4M_{\pi^\pm}^2}{s_\pi}\right)^{1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_{K^\pm}^2)\cos\theta_\pi, \tag{22}$$

$$u_\pi=\frac{1}{2}(M_{K^\pm}^2+2M_{\pi^\pm}^2+m_\ell^2-s_\pi)+\frac{1}{2}\left(1-\frac{4M_{\pi^\pm}^2}{s_\pi}\right)^{1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_{K^\pm}^2)\cos\theta_\pi. \tag{23}$$

For completeness, it is convenient to note the following proposition,

$$\cos\theta_\pi=0\Rightarrow t_\pi=u_\pi=\frac{1}{2}(M_{K^\pm}^2+2M_{\pi^\pm}^2+m_\ell^2-s_\pi). \tag{24}$$

Using the replacements

$$M_{\pi^0}^2\to M_{\pi^\pm}^2-\Delta_\pi,\quad M_{K^0}^2\to M_{K^\pm}^2-\Delta_K, \tag{25}$$

and expanding (20) and (21) to first order in Δ_π and Δ_K, we obtain

$$(p-p_1)^2=\frac{1}{2}(M_{K^\pm}^2+2M_{\pi^\pm}^2+m_\ell^2-s_\pi)+\frac{1}{2s_\pi}(M_K^2-m_\ell^2-s_\pi)\Delta_\pi-\frac{1}{2}\Delta_K$$
$$+\left[-\frac{1}{2}\left(1-\frac{4M_{\pi^\pm}^2}{s_\pi}\right)^{1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_{K^\pm}^2)-\frac{1}{2s_\pi}\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{-1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_K^2)\Delta_\pi\right.$$
$$\left.+\frac{1}{2}\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2}(M_K^2-m_\ell^2-s_\pi)\lambda^{-1/2}(s_\pi,m_\ell^2,M_K^2)\Delta_K\right]\cos\theta_\pi, \tag{26}$$

$$(p-p_2)^2=\frac{1}{2}(M_{K^\pm}^2+2M_{\pi^\pm}^2+m_\ell^2-s_\pi)-\frac{1}{2s_\pi}(M_K^2-m_\ell^2+s_\pi)\Delta_\pi-\frac{1}{2}\Delta_K$$
$$+\left[\frac{1}{2}\left(1-\frac{4M_{\pi^\pm}^2}{s_\pi}\right)^{1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_{K^\pm}^2)+\frac{1}{2s_\pi}\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{-1/2}\lambda^{1/2}(s_\pi,m_\ell^2,M_K^2)\Delta_\pi\right.$$
$$\left.-\frac{1}{2}\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2}(M_K^2-m_\ell^2-s_\pi)\lambda^{-1/2}(s_\pi,m_\ell^2,M_K^2)\Delta_K\right]\cos\theta_\pi. \tag{27}$$

Note that terms of order $\mathcal{O}(\Delta_\pi\Delta_K)$ are forbidden by our power counting scheme since they are first order in isospin breaking. Although Eqs. (26) and (27) are simple to derive, their utility is of great importance to the present study. In fact, the involved expansion could be generalized to any $K_{\ell 4}$ observable as we will see below.

C. Splitting strong and electromagnetic interactions

The first step in our program consists on injecting Eqs. (26) and (27) in the nonphotonic contribution to the decay amplitude $\mathcal{A}^{0-}$. Then, we expand once more to first order in Δ_π and Δ_K dropping out terms of order $\mathcal{O}(\Delta_\pi\Delta_K)$. As a result, form factors for $K_{\ell 4}$ decay of the neutral kaon can be written in the following compact form, which shows explicitly the splitting between strong and electromagnetic interactions,

$$x^{0-}(s_\pi,(p-p_1)^2,(p-p_2)^2,(p_2+p_\ell)^2,\ldots)$$
$$=\frac{M_{K^\pm}}{F_0}[\delta_{xg}+U^x(s_\pi)+V^x(s_\pi)\cos\theta_\pi],\quad x=f,g, \tag{28}$$

where

$$W^x=W_s^x+W_\pi^x\Delta_\pi+W_K^x\Delta_K+W_{e^2}^x e^2+W_\epsilon^x\frac{\epsilon}{\sqrt{3}},\qquad W=U,V, \tag{29}$$

are analytic functions of s_π. If one makes the following substitutions,

$$\Delta_\pi\to 2Z_0e^2F_0^2, \tag{30}$$

$$\Delta_K\to 2Z_0e^2F_0^2-\frac{4\epsilon}{\sqrt{3}}(M_K^2-M_\pi^2), \tag{31}$$

then Eqs. (28) and (29) read

$$W^x=W_s^x+W_\alpha^x e^2+W_{m_d-m_u}^x\frac{\epsilon}{\sqrt{3}}, \tag{32}$$

$$W_\alpha^x=W_{e^2}^x+2Z_0F_0^2(W_\pi^x+W_K^x), \tag{33}$$

$$W_{m_d-m_u}^x=W_\epsilon^x-4(M_K^2-M_\pi^2)W_K^x. \tag{34}$$

The aim of the present work is to determine the U functions corresponding to f and g form factors for $K_{\ell 4}$ decay of the neutral kaon.

III. THE PHOTONIC CONTRIBUTION

From now on, we will work under proposition (24) keeping in mind that, in the isospin breaking contribution, the power counting dictates the following,

$$\text{Isospin breaking}\to t_\pi=u_\pi=\frac{1}{2}(M_K^2+2M_\pi^2+m_\ell^2-s_\pi). \tag{35}$$

Taking the photonic contribution from Ref. [1], applying assumption (17), and performing the preceding expansion, it is easy at a first sight to derive U_{e^2}. The problem is that, in practice, one encounters loop integrals with a vanishing Gramm determinant when reducing vector and tensor integrals to scalar ones [9]. After a long and tedious calculation, one obtains

$$U_{e^2}^f=\frac{1}{3}(-6K_3+3K_4+2K_5+2K_6-6X_1)-\frac{2}{3}\frac{M_\pi^2}{M_\pi^2-M_\eta^2}(6K_3-3K_4-2K_5-2K_6+2K_9+2K_{10})$$
$$+B(M_\pi^2,0,M_\pi^2)\left\{1-\frac{1}{4}[(M_K^2-m_\ell^2-s_\pi)^2-4m_\ell^2M_\pi^2]\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)+\frac{3}{4}m_\ell^2M_\pi^2(s_\pi-4M_\pi^2)\right.$$
$$\left.\times(M_K^2-m_\ell^2-s_\pi)\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}+B(m_\ell^2,0,m_\ell^2)\left\{-\frac{3}{4}m_\ell^2(M_K^2-m_\ell^2-s_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(0,m_\ell^2,M_K^2)\left\{\frac{m_\ell^2}{4t_\pi}+\frac{m_\ell^2}{4t_\pi}(s_\pi-4M_\pi^2)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$

$$+B(m_\ell^2,0,M_K^2)\left\{2m_\ell^4M_\pi^2(s_\pi-4M_\pi^2)\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)+\frac{1}{16}m_\ell^2(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2-s_\pi)^2\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(t_\pi,m_\ell^2,M_\pi^2)\left\{-1+\frac{1}{2}[t_\pi(M_K^2-2m_\ell^2-s_\pi)-(M_\pi^2-m_\ell^2)(M_K^2-s_\pi)]\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(t_\pi,M_\pi^2,M_K^2)\left\{-\frac{m_\ell^2}{4t_\pi}-\frac{m_\ell^2}{4t_\pi}(s_\pi-4M_\pi^2)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$\left.+\frac{1}{4}m_\ell^2(s_\pi-4M_\pi^2)[-t_\pi(3M_\pi^2-m_\ell^2+t_\pi)+(M_\pi^2-m_\ell^2)(5M_\pi^2+m_\ell^2-t_\pi)]\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+C(M_\pi^2,t_\pi,m_\ell^2,0,M_\pi^2,M_K^2)\left\{m_\ell^2M_\pi^2(M_K^2-m_\ell^2)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$\left.+\frac{3}{4}m_\ell^2M_\pi^2(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2)(M_K^2-m_\ell^2-s_\pi)\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+C(m_\ell^2,0,m_\ell^2,0,m_\ell^2,M_K^2)\left\{-\frac{1}{2}m_\ell^2(M_K^2-m_\ell^2)(M_\pi^2+m_\ell^2-t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+C(t_\pi,t_\pi,0,m_\ell^2,M_\pi^2,M_K^2)\left\{-\frac{1}{2}m_\ell^2(M_K^2-m_\ell^2)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$+\frac{m_\ell^2}{8t_\pi}(s_\pi-4M_\pi^2)(5M_\pi^2+m_\ell^2-t_\pi)(M_\pi^2-m_\ell^2-t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)$$
$$\left.+\frac{m_\ell^2}{16t_\pi}(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2-s_\pi)(3M_\pi^2-3m_\ell^2-t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)+\frac{m_\ell^2}{4t_\pi}(M_\pi^2-m_\ell^2+t_\pi)\right\}, \tag{36}$$

$$U_{e^2}^g=-\frac{1}{18}(24K_1+24K_2+8K_5+8K_6-36K_{12}+12X_1+9X_6)+\frac{1}{M_\pi^2}A(M_\pi^2)-\frac{1}{2m_\ell^2}A(m_\ell^2)$$
$$-\frac{1}{32\pi^2}\left(5+2\ln\frac{m_\gamma^2}{M_\pi^2}+2\ln\frac{m_\gamma^2}{m_\ell^2}\right)+B(M_\pi^2,0,M_\pi^2)\left\{-m_\ell^2M_\pi^2\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$\left.-(M_\pi^2+m_\ell^2-t_\pi)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)-\frac{3}{4}m_\ell^2M_\pi^2(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2-s_\pi)\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(m_\ell^2,0,m_\ell^2)\left\{1+\frac{1}{4}(M_K^2-m_\ell^2-s_\pi)(M_K^2+2m_\ell^2-s_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(0,m_\ell^2,M_K^2)\left\{\frac{1}{2}-\frac{m_\ell^2}{4t_\pi}-\frac{m_\ell^2}{4t_\pi}(s_\pi-4M_\pi^2)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(m_\ell^2,0,M_K^2)\left\{-\frac{1}{2}-2m_\ell^4M_\pi^2(s_\pi-4M_\pi^2)\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)-\frac{1}{16}m_\ell^2(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2-s_\pi)^2\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(t_\pi,m_\ell^2,M_\pi^2)\left\{\frac{1}{2}m_\ell^2(M_\pi^2-m_\ell^2)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)-\frac{1}{2}t_\pi(2M_K^2-3m_\ell^2-2s_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+B(t_\pi,M_\pi^2,M_K^2)\left\{\frac{m_\ell^2}{4t_\pi}+\frac{m_\ell^2}{4t_\pi}(s_\pi-4M_\pi^2)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$\left.+\frac{1}{4}m_\ell^2(s_\pi-4M_\pi^2)[t_\pi(3M_\pi^2-m_\ell^2+t_\pi)-(M_\pi^2-m_\ell^2)(5M_\pi^2+m_\ell^2-t_\pi)]\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$

$$+C(M_\pi^2,t_\pi,m_\ell^2,0,M_\pi^2,M_K^2)\left\{-m_\ell^2M_\pi^2(M_K^2-m_\ell^2)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$\left.-\frac{3}{4}m_\ell^2M_\pi^2(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2)(M_K^2-m_\ell^2-s_\pi)\lambda^{-2}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+C(m_\ell^2,0,m_\ell^2,0,m_\ell^2,M_K^2)\left\{\frac{1}{2}m_\ell^2(M_K^2-m_\ell^2)(M_\pi^2+m_\ell^2-t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}$$
$$+C(t_\pi,t_\pi,0,m_\ell^2,M_\pi^2,M_K^2)\left\{-\frac{m_\ell^2}{4t_\pi}(M_\pi^2-m_\ell^2+t_\pi)+\frac{1}{2}m_\ell^2(M_K^2-m_\ell^2)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right.$$
$$-\frac{m_\ell^2}{8t_\pi}(s_\pi-4M_\pi^2)(5M_\pi^2+m_\ell^2-t_\pi)(M_\pi^2-m_\ell^2-t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)+\frac{1}{4}m_\ell^2(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2-s_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)$$
$$\left.-\frac{3m_\ell^2}{16t_\pi}(s_\pi-4M_\pi^2)(M_K^2-m_\ell^2-s_\pi)(M_\pi^2-m_\ell^2+t_\pi)\lambda^{-1}(t_\pi,m_\ell^2,M_\pi^2)\right\}-(M_K^2-m_\ell^2-s_\pi)C(m_\ell^2,t_\pi,M_\pi^2,m_\gamma^2,m_\ell^2,M_\pi^2). \tag{37}$$

IV. THE NONPHOTONIC CONTRIBUTION

A. One-point functions

Let P denotes a pion π or a kaon K, and Δ_P the difference

$$\Delta_P \doteq M_{P^\pm}^2 - M_{P^0}^2. \tag{38}$$

We shall expand the one-point function,

$$A(M_{P^0}^2) \doteq -i\mu^{4-D}\int \frac{d^Dl}{(2\pi)^D}\frac{1}{l^2-M_{P^0}^2}, \tag{39}$$

to leading order in isospin breaking.

In dimensional regularization, the preceding integral reads,

$$A(M_{P^0}^2)=M_{P^0}^2\left[-2\bar{\lambda}-\frac{1}{16\pi^2}\ln\left(\frac{M_{P^0}^2}{\mu^2}\right)\right].$$

By (38), this is equivalent to

$$A(M_{P^0}^2)=-2\bar{\lambda}(M_{P^\pm}^2-\Delta_P)-\frac{1}{16\pi^2}(M_{P^\pm}^2-\Delta_P)\times\ln\left[\left(\frac{M_{P^\pm}^2}{\mu^2}\right)\left(1-\frac{\Delta_P}{M_P^2}\right)\right].$$

Expanding to first order in Δ_P, we obtain the splitting between strong and electromagnetic interactions in one-point functions,

$$A(M_{P^0}^2)=A(M_{P^\pm}^2)+\left[\frac{1}{16\pi^2}-\frac{1}{M_P^2}A(M_P^2)\right]\Delta_P. \tag{40}$$

B. Two-point functions

The loop integral

$$B(p_1,m_0,m_1) \doteq -i\mu^{4-D}\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l^2-m_0^2)[(p_1+l)^2-m_1^2]} \tag{41}$$

is function of three scalars p_1^2, m_0^2, and m_1^2. In order to obtain isospin breaking corrections generated from (41) we shall expand $B(p_1^2+\delta,m_0^2+\delta_0,m_1^2+\delta_1)$ to first order in δ, δ_0, and δ_1, where these quantities are leading order in isospin breaking,

$$\delta,\delta_0,\delta_1=\mathcal{O}(\alpha,m_d-m_u). \tag{42}$$

In dimensional regularization,

$$B(p_1^2,m_0^2,m_1^2)=\frac{1}{16\pi^2}\left[\frac{2}{4-D}+\ln(4\pi\mu^2)+\Gamma'(1)\right]-\frac{1}{16\pi^2}\int_0^1dx\,\ln[xm_0^2+(1-x)m_1^2-x(1-x)p_1^2].$$

One then has

$$B(p_1^2+\delta,m_0^2+\delta_0,m_1^2+\delta_1)$$
$$=\frac{1}{16\pi^2}\left[\frac{2}{4-D}+\ln(4\pi\mu^2)+\Gamma'(1)\right]$$
$$-\frac{1}{16\pi^2}\int_0^1 dx\,\ln[x(m_0^2+\delta_0)+(1-x)(m_1^2+\delta_1)$$
$$-x(1-x)(p_1^2+\delta)].$$

Expanding to first order in δ, δ_0, and δ_1, the preceding equation takes the form

$$B(p_1^2+\delta,m_0^2+\delta_0,m_1^2+\delta_1)$$
$$=B(p_1^2,m_0^2,m_1^2)$$
$$-\frac{1}{16\pi^2}\int_0^1 dx\,\frac{x}{xm_0^2+(1-x)m_1^2-x(1-x)p_1^2}\,\delta_0$$
$$-\frac{1}{16\pi^2}\int_0^1 dx\,\frac{1-x}{xm_0^2+(1-x)m_1^2-x(1-x)p_1^2}\,\delta_1$$
$$-\frac{1}{16\pi^2}\int_0^1 dx\,\frac{-x(1-x)}{xm_0^2+(1-x)m_1^2-x(1-x)p_1^2}\,\delta.$$

If we denote by τ the generic integral

$$\tau(p_1^2,m_0^2,m_1^2)\doteq\int_0^1 dx\,\frac{1}{xm_0^2+(1-x)m_1^2-x(1-x)p_1^2}, \tag{43}$$

then the splitting between strong and electromagnetic interactions in two-point functions is easily obtained from the following compact formula,

$$B(p_1^2+\delta,m_0^2+\delta_0,m_1^2+\delta_1)=B(p_1^2,m_0^2,m_1^2)-\frac{1}{32\pi^2p_1^2}\left[\ln\left(\frac{m_0^2}{m_1^2}\right)+(p_1^2+m_1^2-m_0^2)\tau(p_1^2,m_0^2,m_1^2)\right]\delta_0$$
$$+\frac{1}{32\pi^2p_1^2}\left[\ln\left(\frac{m_0^2}{m_1^2}\right)-(p_1^2-m_1^2+m_0^2)\tau(p_1^2,m_0^2,m_1^2)\right]\delta_1$$
$$-\frac{1}{32\pi^2p_1^4}\left\{2p_1^2+(m_1^2-m_0^2)\ln\left(\frac{m_0^2}{m_1^2}\right)+[(m_1^2-m_0^2)^2-p_1^2(m_1^2+m_0^2)]\tau(p_1^2,m_0^2,m_1^2)\right\}\delta. \tag{44}$$

As an application, consider the two-point function, $B(p_1-p,M_{\pi^0},M_{K^0})$, selected from the t-channel contribution to $\mathcal{A}^{0-}$. The following replacements in (44),

$$\delta\to-\frac{1}{s_\pi}(M_\pi^2+m_\ell^2-t_\pi)\Delta_\pi-\frac{1}{2}\Delta_K,\quad \delta_0\to-\Delta_\pi,\quad \delta_1\to-\Delta_K,$$

lead to the expression

$$B(p_1-p,M_{\pi^0},M_{K^0})$$
$$=B(t_\pi,M_{\pi^\pm}^2,M_{K^\pm}^2)+\frac{\Delta_\pi}{32\pi^2t_\pi}\left\{\frac{2}{s_\pi}(M_\pi^2+m_\ell^2-t_\pi)+\left[1+\frac{1}{s_\pi t_\pi}(M_\pi^2+m_\ell^2-t_\pi)(M_K^2-M_\pi^2)\right]\ln\left(\frac{M_\pi^2}{M_K^2}\right)\right.$$
$$\left.+\left[M_K^2-M_\pi^2+t_\pi-\frac{1}{s_\pi}(M_K^2+M_\pi^2)+\frac{1}{s_\pi t_\pi}(M_\pi^2+m_\ell^2-t_\pi)(M_K^2-M_\pi^2)^2\right]\tau(t_\pi,M_\pi^2,M_K^2)\right\}$$
$$+\frac{\Delta_K}{32\pi^2t_\pi}\left\{1+\frac{1}{2t_\pi}(M_K^2-M_\pi^2-2t_\pi)\ln\left(\frac{M_\pi^2}{M_K^2}\right)-\left[\frac{1}{2}(3M_K^2-M_\pi^2-2t_\pi)-\frac{1}{2t_\pi}(M_K^2-M_\pi^2)^2\right]\tau(t_\pi,M_\pi^2,M_K^2)\right\}. \tag{45}$$

C. Isospin limit

We have

$$U_s^f=0, \tag{46}$$

$$\begin{aligned}U_s^g=&-\frac{1}{24F_0^2}\Bigg\{\frac{1}{16\pi^2}\Bigg[4(2M_{K^\pm}^2+4M_{\pi^\pm}^2-s_\pi)+(2M_{K^\pm}^2+4M_{\pi^\pm}^2-t_\pi)\\&-\frac{3}{t_\pi}(M_{\pi^\pm}^2-M_{K^\pm}^2)(M_{\pi^\pm}^2+M_{K^\pm}^2)+(6M_{K^\pm}^2-t_\pi)+\frac{9}{t_\pi}(M_\eta^2-M_{K^\pm}^2)(M_\eta^2+M_{K^\pm}^2)\Bigg]\\&+48[2(M_{\pi^\pm}^2+M_{K^\pm}^2-t_\pi)L_3+2(M_{\pi^\pm}^2+2M_{K^\pm}^2)L_4-m_\ell^2L_9]+A(M_{\pi^\pm}^2)\Bigg[5-\frac{6}{t_\pi}M_{\pi^\pm}^2-\frac{6}{t_\pi^2}(M_{\pi^\pm}^2-M_{K^\pm}^2)^2\Bigg]\\&+A(M_\eta^2)\Bigg[-3-\frac{6}{t_\pi}(5M_\eta^2-6M_{K^\pm}^2)+\frac{18}{t_\pi^2}(M_\eta^2-M_{K^\pm}^2)^2\Bigg]\\&+A(M_{K^\pm}^2)\Bigg[-2+\frac{12}{t_\pi}(M_{\pi^\pm}^2-M_{K^\pm}^2)+\frac{12}{t_\pi}(M_\eta^2-M_{K^\pm}^2)+\frac{6}{t_\pi^2}(M_{\pi^\pm}^2-M_{K^\pm}^2)^2-\frac{18}{t_\pi^2}(M_\eta^2-M_{K^\pm}^2)^2\Bigg]\\&+4(4M_{\pi^\pm}^2-s_\pi)B(s_\pi,M_{\pi^\pm}^2,M_{\pi^\pm}^2)+2(4M_{K^\pm}^2-s_\pi)B(s_\pi,M_{K^\pm}^2,M_{K^\pm}^2)\\&+B(t_\pi,M_{\pi^\pm}^2,M_{K^\pm}^2)\Bigg[-6(M_{\pi^\pm}^2-M_{K^\pm}^2+t_\pi)+\frac{6}{t_\pi}(M_{\pi^\pm}^2-M_{K^\pm}^2)^2+\frac{6}{t_\pi^2}(M_{\pi^\pm}^2-M_{K^\pm}^2)^3\Bigg]\\&+B(t_\pi,M_\eta^2,M_{K^\pm}^2)\Bigg[-6(M_\eta^2-3M_{K^\pm}^2+t_\pi)+\frac{6}{t_\pi}(M_\eta^2-M_{K^\pm}^2)(5M_\eta^2-3M_{K^\pm}^2)-\frac{18}{t_\pi^2}(M_\eta^2-M_{K^\pm}^2)^3\Bigg]\Bigg\}.\end{aligned} \tag{47}$$

D. The ϵ terms

We have

$$\begin{aligned}U_\epsilon^f=&-3+\frac{1}{24F_0^2}\frac{1}{16\pi^2}\Bigg[2(6M_\eta^2+28M_K^2+20M_\pi^2-9t_\pi)+\frac{15}{t_\pi}(M_\pi^2-M_K^2)(M_\pi^2+M_K^2)-\frac{9}{t_\pi}(M_\eta^2-M_K^2)(M_\eta^2+M_K^2)\Bigg]\\&+\frac{2}{F_0^2}[2(M_K^2+5M_\pi^2-2s_\pi-t_\pi)L_3+6(M_\pi^2+2M_K^2)L_4+48(M_K^2-M_\pi^2)(3L_7+L_8)-3m_\ell^2L_9]\\&+\frac{1}{8F_0^2}\Bigg\{-A(M_\pi^2)\Bigg[\frac{32M_\pi^2}{M_\pi^2-M_\eta^2}+15+\frac{2}{t_\pi}M_\pi^2-\frac{10}{t_\pi^2}(M_\pi^2-M_K^2)^2\Bigg]\\&+3A(M_\eta^2)\Bigg[3+\frac{2}{t_\pi}(3M_\eta^2-4M_K^2)-\frac{2}{t_\pi^2}(M_\eta^2-M_K^2)^2\Bigg]\\&+2A(M_K^2)\Bigg[\frac{16M_\pi^2}{M_\pi^2-M_\eta^2}+3+\frac{1}{t_\pi}(5M_K^2-4M_\pi^2)-\frac{5}{t_\pi^2}(M_\pi^2-M_K^2)^2-\frac{3}{t_\pi}(2M_\eta^2-3M_K^2)+\frac{3}{t_\pi^2}(M_\eta^2-M_K^2)^2\Bigg]\\&-12(2M_\pi^2-s_\pi)B(s_\pi,M_\pi^2,M_\pi^2)+2(4M_K^2-3s_\pi)B(s_\pi,M_K^2,M_K^2)+4(3M_\eta^2+M_\pi^2-3s_\pi)B(s_\pi,M_\eta^2,M_\pi^2)\\&+2B(t_\pi,M_\pi^2,M_K^2)\Bigg[7M_K^2+13M_\pi^2-9t_\pi+\frac{1}{t_\pi}(M_\pi^2-M_K^2)(5M_K^2+M_\pi^2)-\frac{5}{t_\pi^2}(M_\pi^2-M_K^2)^3\Bigg]\\&+2B(t_\pi,M_\eta^2,M_K^2)\Bigg[3M_\eta^2-5M_K^2+3t_\pi-\frac{9}{t_\pi}(M_\eta^2-M_K^2)^2+\frac{3}{t_\pi^2}(M_\eta^2-M_K^2)^3\Bigg]\Bigg\},\end{aligned} \tag{48}$$

$$U_\epsilon^g = \frac{1}{4F_0^2}\Bigg\{3\left(1+\frac{2}{t_\pi}M_K^2\right)A(M_\pi^2)-\left[3-\frac{2}{t_\pi}(3M_\eta^2-4M_K^2)\right]A(M_\eta^2)$$
$$-2\left[\frac{3}{t_\pi}M_K^2+\frac{1}{t_\pi}(3M_\eta^2-4M_K^2)\right]A(M_K^2)-\frac{6}{t_\pi}M_K^2(M_\pi^2-M_K^2)B(t_\pi,M_\pi^2,M_K^2)+2B(t_\pi,M_\eta^2,M_K^2)$$
$$\times\left[3(M_\eta^2-M_K^2)-\frac{1}{t_\pi}(M_\eta^2-M_K^2)(3M_\eta^2-4M_K^2)\right]\Bigg\}. \tag{49}$$

E. The Δ_π terms

We have

$$U_\pi^f = -\frac{2}{F_0^2}\left(1-\frac{\Delta_{\ell K}}{s_\pi}\right)L_3-\frac{1}{192\pi^2F_0^2}\frac{1}{s_\pi}(4M_K^2+20M_\pi^2-3s_\pi)-\frac{1}{256\pi^2F_0^2}\frac{1}{s_\pi}\Bigg\{4(M_\pi^2+m_\ell^2-t_\pi)+\frac{6}{t_\pi}(M_\eta^2+M_\pi^2)\Delta_{\ell K}$$
$$-\frac{1}{t_\pi}(M_\eta^2+6M_K^2+M_\pi^2)s_\pi+\frac{1}{t_\pi^2}[(5M_K^2-3M_\pi^2)\Delta_{\pi K}-5(3M_\eta^2-M_K^2)\Delta_{\eta K}]\Delta_{\ell K}-\frac{2}{t_\pi^3}(\Delta_{\pi K}^2+\Delta_{\eta K}^2)\Delta_{\ell K}\Delta_{\pi K}\Bigg\}$$
$$+\frac{1}{24F_0^2}\frac{1}{s_\pi}A(M_\pi^2)\left[40+\frac{6}{t_\pi}\frac{s_\pi}{M_\pi^2}\Delta_{\pi K}+\frac{3}{t_\pi^2}\Delta_{\pi K}s_\pi+\frac{3}{t_\pi^2}(M_K^2-2M_\pi^2)\Delta_{\ell K}-\frac{6}{t_\pi^3}\Delta_{\ell K}\Delta_{\pi K}^2\right]$$
$$-\frac{1}{8F_0^2}\frac{1}{s_\pi}A(M_\eta^2)\left[\frac{1}{t_\pi}s_\pi-\frac{1}{t_\pi^2}\Delta_{\eta K}s_\pi+\frac{1}{t_\pi^2}(6M_\eta^2-7M_K^2)\Delta_{\ell K}+\frac{2}{t_\pi^3}\Delta_{\ell K}\Delta_{\pi K}\Delta_{\eta K}\right]$$
$$+\frac{1}{24F_0^2}\frac{1}{s_\pi}A(M_K^2)\left[8-\frac{9}{t_\pi}s_\pi-\frac{3}{t_\pi^2}(\Delta_{\pi K}+\Delta_{\eta K})s_\pi+\frac{9}{t_\pi^2}(M_\eta^2-2M_K^2+M_\pi^2)\Delta_{\ell K}+\frac{6}{t_\pi^3}(\Delta_{\pi K}+\Delta_{\eta K})\Delta_{\ell K}\Delta_{\pi K}\right]$$
$$-\frac{1}{256\pi^2F_0^2}\frac{1}{s_\pi}\ln\left(\frac{M_\pi^2}{M_K^2}\right)\left[\frac{2}{t_\pi}M_K^2s_\pi+\frac{2}{t_\pi^2}M_K^2\Delta_{\pi K}s_\pi-\frac{1}{t_\pi^2}(M_K^2+3M_\pi^2)\Delta_{\ell K}\Delta_{\pi K}+\frac{2}{t_\pi^3}\Delta_{\ell K}\Delta_{\pi K}^3+\frac{1}{t_\pi^4}\Delta_{\ell K}\Delta_{\pi K}^4\right]$$
$$+\frac{1}{512\pi^2F_0^2}\frac{1}{s_\pi}\ln\left(\frac{M_\eta^2}{M_K^2}\right)\left[\frac{2}{t_\pi^2}(3M_\eta^2-M_K^2)\Delta_{\ell K}\Delta_{\eta K}-\frac{4}{t_\pi^3}(3M_\eta^2-2M_K^2)\Delta_{\ell K}\Delta_{\eta K}^2-\frac{2}{t_\pi^4}\Delta_{\ell K}\Delta_{\pi K}\Delta_{\eta K}^3\right]$$
$$-\frac{1}{256\pi^2F_0^2}\frac{1}{s_\pi}\tau(t_\pi,M_\pi^2,M_K^2)\Bigg[2M_K^2s_\pi-\frac{1}{t_\pi}(M_K^2+3M_\pi^2)\Sigma_{\pi K}\Delta_{\ell K}+\frac{1}{t_\pi^2}(3M_K^2+5M_\pi^2)\Delta_{\ell K}\Delta_{\pi K}^2$$
$$-\frac{2}{t_\pi^2}M_K^2\Delta_{\pi K}^2s_\pi+\frac{1}{t_\pi^3}(3M_K^2-M_\pi^2)\Delta_{\ell K}\Delta_{\pi K}^3-\frac{1}{t_\pi^4}\Delta_{\ell K}\Delta_{\pi K}^5\Bigg]$$
$$-\frac{1}{512\pi^2F_0^2}\frac{1}{s_\pi}\tau(t_\pi,M_\eta^2,M_K^2)\Delta_{\ell K}(\Delta_{\eta K}^2-\Sigma_{\eta K}t_\pi)\left[\frac{2}{t_\pi^2}(3M_\eta^2-M_K^2)-\frac{4}{t_\pi^3}(3M_\eta^2-2M_K^2)\Delta_{\eta K}-\frac{2}{t_\pi^4}\Delta_{\pi K}\Delta_{\eta K}^2\right]$$
$$-\frac{1}{3F_0^2}\frac{1}{s_\pi}(5M_\pi^2-2s_\pi)B(s_\pi,M_\pi^2,M_\pi^2)-\frac{1}{3F_0^2}\frac{1}{s_\pi}(M_K^2-s_\pi)B(s_\pi,M_K^2,M_K^2)$$
$$-\frac{1}{8F_0^2}\frac{1}{s_\pi}B(t_\pi,M_\pi^2,M_K^2)\left[s_\pi-\frac{1}{t_\pi}(5M_K^2-2M_\pi^2)s_\pi+\frac{1}{t_\pi^2}(2M_K^2-2m_\ell^2+s_\pi)\Delta_{\pi K}^2-\frac{2}{t_\pi^3}\Delta_{\ell K}\Delta_{\pi K}^3\right]$$
$$+\frac{1}{16F_0^2}\frac{1}{s_\pi}B(t_\pi,M_\eta^2,M_K^2)\left[\frac{2}{t_\pi}M_\eta^2s_\pi-\frac{2}{t_\pi^2}\Delta_{\eta K}^2s_\pi+\frac{4}{t_\pi^2}(3M_\eta^2-2M_K^2)\Delta_{\ell K}\Delta_{\eta K}+\frac{4}{t_\pi^3}\Delta_{\ell K}\Delta_{\pi K}\Delta_{\eta K}^2\right], \tag{50}$$

$$U_\pi^g=\frac{8}{F_0^2}L_4+\frac{1}{768\pi^2F_0^2}\frac{1}{s_\pi}\bigg\{-12(M_\eta^2-4M_K^2+M_\pi^2-3s_\pi)-\frac{1}{t_\pi}[6(2M_K^2-3M_\pi^2-m_\ell^2)\Delta_{\pi K}+(M_K^2+5M_\pi^2)s_\pi$$
$$-18M_K^2(3M_\eta^2-8M_K^2-m_\ell^2)+3(M_\eta^2+M_K^2)s_\pi-6M_\eta^2(7M_\eta^2+2M_K^2+m_\ell^2)]$$
$$+\frac{1}{t_\pi^2}[4\Delta_{\pi K}^2s_\pi+3M_K^2(7M_K^2-M_\pi^2+3m_\ell^2)\Delta_{\pi K}-3M_\pi^2(2M_K^2-2M_\pi^2+m_\ell^2)\Delta_{\pi K}-9M_K^2(10M_\eta^2-5M_K^2-m_\ell^2)\Delta_{\eta K}$$
$$+3M_\eta^2(66M_\eta^2-81M_K^2-13m_\ell^2)\Delta_{\eta K}]-\frac{6}{t_\pi^3}[(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\pi K}^3+3(6M_\eta^2-9M_K^2-m_\ell^2)\Delta_{\eta K}^3]\bigg\}$$
$$+\frac{1}{24F_0^2}\frac{1}{s_\pi}A(M_\pi^2)\bigg[-\frac{3}{M_\pi^2}s_\pi-\frac{2}{t_\pi}(3M_\pi^2+2s_\pi)-\frac{6}{t_\pi^2}(2M_K^2-3M_\pi^2)M_K^2+\frac{3}{t_\pi^2}(3M_K^2-2M_\pi^2+m_\ell^2)M_\pi^2$$
$$-\frac{1}{t_\pi^2}\left(7-\frac{2M_K^2}{M_\pi^2}\right)\Delta_{\pi K}s_\pi+\frac{6}{t_\pi^3}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\pi K}^2\bigg]$$
$$+\frac{1}{16F_0^2}\frac{1}{s_\pi}A(M_\eta^2)\bigg[-\frac{4}{t_\pi}(5M_\eta^2-6M_K^2-s_\pi)-\frac{2}{t_\pi^2}\Delta_{\eta K}s_\pi-\frac{2}{t_\pi^2}(18M_\eta^2-17M_K^2-5m_\ell^2)M_\eta^2$$
$$+\frac{4}{t_\pi^2}(20M_\eta^2-21M_K^2-3m_\ell^2)M_K^2-\frac{12}{t_\pi^3}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\eta K}^2\bigg]$$
$$+\frac{1}{48F_0^2}\frac{1}{s_\pi}A(M_K^2)\bigg[-\frac{2}{t_\pi}(6M_K^2-12M_\pi^2-7s_\pi)+\frac{6}{t_\pi}(4M_\eta^2-6M_K^2-s_\pi)$$
$$+\frac{2}{t_\pi^2}(5\Delta_{\pi K}s_\pi-6\Sigma_{\ell K}M_\pi^2-15\Delta_{\pi K}M_K^2+3\Delta_{\ell\pi}M_K^2)+\frac{6}{t_\pi^2}\Delta_{\eta K}s_\pi-\frac{6}{t_\pi^2}(3M_K^2+2m_\ell^2)M_\eta^2$$
$$+\frac{18}{t_\pi^2}(5M_K^2+m_\ell^2-3M_\eta^2)M_K^2-\frac{12}{t_\pi^3}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\pi K}^2-\frac{36}{t_\pi^3}(6M_\eta^2-9M_K^2-m_\ell^2)\Delta_{\eta K}^2\bigg]$$
$$+\frac{1}{768\pi^2F_0^2}\frac{1}{s_\pi}\ln\left(\frac{M_\pi^2}{M_K^2}\right)\bigg\{-2(3M_K^2-3M_\pi^2+2s_\pi)+\frac{1}{t_\pi}[2(2M_K^2+3M_\pi^2)s_\pi-3(3M_K^2+m_\ell^2)\Delta_{\pi K}]$$
$$+\frac{1}{t_\pi^2}[2M_K^2s_\pi+3(M_K^2-4M_\pi^2-m_\ell^2)\Delta_{\pi K}]\Delta_{\pi K}+\frac{1}{t_\pi^3}(9M_K^2+3m_\ell^2-2s_\pi)\Delta_{\pi K}^3+\frac{3}{t_\pi^4}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\pi K}^4\bigg\}$$
$$-\frac{1}{512\pi^2F_0^2}\ln\left(\frac{M_\eta^2}{M_K^2}\right)\Delta_{\eta K}\bigg[\frac{2}{t_\pi}+\frac{2}{t_\pi^2}(M_\eta^2-3M_K^2)-\frac{2}{t_\pi^3}(5M_\eta^2-3M_K^2)\Delta_{\eta K}+\frac{6}{t_\pi^4}\Delta_{\eta K}^3\bigg]$$
$$-\frac{1}{192\pi^2F_0^2}(4M_\pi^2-s_\pi)\tau(s_\pi,M_\pi^2,M_\pi^2)+\frac{1}{768\pi^2F_0^2}\frac{1}{s_\pi}\tau(t_\pi,M_\pi^2,M_K^2)(-\Delta_{\pi K}+t_\pi)$$
$$\times\bigg\{2(3\Sigma_{\pi K}-2s_\pi)+\frac{3}{t_\pi^4}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\pi K}^4-\frac{1}{t_\pi^3}[2\Delta_{\pi K}s_\pi-3M_\pi^2\Delta_{\ell K}+3M_K^2(5M_K^2+3m_\ell^2)]\Delta_{\pi K}^2$$
$$-\frac{1}{t_\pi^2}[3M_\pi^2(\Delta_{\ell K}+4M_\pi^2)+M_K^2(27M_K^2+9m_\ell^2-2s_\pi)]\Delta_{\pi K}-\frac{1}{t_\pi}[3M_\pi^2(\Delta_{\ell K}-2s_\pi)+M_K^2(21M_K^2+3m_\ell^2-4s_\pi)]\bigg\}$$
$$+\frac{1}{512\pi^2F_0^2}\tau(t_\pi,M_\eta^2,M_K^2)(\Delta_{\eta K}^2-\Sigma_{\eta K}t_\pi)\bigg[\frac{2}{t_\pi}+\frac{2}{t_\pi^2}(M_\eta^2-3M_K^2)-\frac{2}{t_\pi^3}(5M_\eta^2-3M_K^2)\Delta_{\eta K}+\frac{6}{t_\pi^4}\Delta_{\eta K}^3\bigg]$$

$$+\frac{1}{3F_0^2}B(s_\pi,M_\pi^2,M_\pi^2)+\frac{1}{24F_0^2}\frac{1}{s_\pi}B(t_\pi,M_\pi^2,M_K^2)\Bigg\{-2s_\pi+\frac{1}{t_\pi}[6\Delta_{\pi K}^2-(7M_K^2-4M_\pi^2)s_\pi]$$
$$-\frac{1}{t_\pi^2}(15M_K^2-6M_\pi^2+3m_\ell^2-7s_\pi)\Delta_{\pi K}^2-\frac{6}{t_\pi^3}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\pi K}^3\Bigg\}$$
$$+\frac{1}{16F_0^2}\frac{1}{s_\pi}B(t_\pi,M_\eta^2,M_K^2)\Bigg\{\frac{2}{t_\pi}[2(5M_\eta^2-3M_K^2)\Delta_{\eta K}-(2M_\eta^2-M_K^2)s_\pi]$$
$$-\frac{2}{t_\pi^2}[-M_\eta^2(18M_\eta^2-9M_K^2-5m_\ell^2)-\Delta_{\eta K}s_\pi+3M_K^2(10M_\eta^2-5M_K^2-m_\ell^2)]\Delta_{\eta K}+\frac{12}{t_\pi^3}(M_K^2+2M_\pi^2+m_\ell^2)\Delta_{\eta K}^3\Bigg\}. \quad (51)$$

In the preceding expressions,

$$\Delta_{\ell P}\doteq m_\ell^2-M_P^2,\quad \Sigma_{\ell P}\doteq m_\ell^2+M_P^2, \quad (52)$$

$$\Delta_{PQ}\doteq M_P^2-M_Q^2,\quad \Sigma_{PQ}\doteq M_P^2+M_Q^2. \quad (53)$$

F. The Δ_K terms

We have

$$U_K^f=\frac{1}{768\pi^2F_0^2}\Bigg\{-16\frac{M_\pi^2+2M_K^2}{M_\pi^2-M_\eta^2}+12+\frac{3}{t_\pi}(3M_\eta^2-6M_K^2-M_\pi^2)+\frac{2}{t_\pi^2}[2(M_K^2-3M_\pi^2)\Delta_{\pi K}-9\Delta_{\eta K}^2]-\frac{4}{t_\pi^3}\Delta_{\pi K}^3\Bigg\}$$
$$+\frac{1}{24F_0^2}\frac{1}{t_\pi^2}\Delta_{\pi K}A(M_\pi^2)-\frac{1}{24F_0^2}\frac{1}{M_K^2}A(M_K^2)\Bigg\{-8\frac{M_\pi^2+2M_K^2}{M_\pi^2-M_\eta^2}+\frac{1}{t_\pi}(9M_\eta^2-13M_K^2-5M_\pi^2)$$
$$+\frac{1}{t_\pi^2}[(2M_K^2-M_\pi^2)\Delta_{\pi K}+9(2M_K^2-M_\eta^2)\Delta_{\eta K}]\Bigg\}+\frac{1}{8F_0^2}\left(-\frac{1}{t_\pi}+\frac{3}{t_\pi^2}\Delta_{\eta K}\right)A(M_\eta^2)$$
$$+\frac{1}{384\pi^2F_0^2}\ln\left(\frac{M_\pi^2}{M_K^2}\right)\left[-3+\frac{1}{t_\pi}(4M_K^2-M_\pi^2)+\frac{1}{t_\pi^2}M_K^2\Delta_{\pi K}-\frac{1}{t_\pi^3}(M_K^2-3M_\pi^2)\Delta_{\pi K}^2+\frac{1}{t_\pi^4}\Delta_{\pi K}^4\right]$$
$$+\frac{1}{512\pi^2F_0^2}\ln\left(\frac{M_\eta^2}{M_K^2}\right)\left[\frac{2}{t_\pi}(3M_\eta^2-M_K^2)-\frac{4}{t_\pi^2}(3M_\eta^2-2M_K^2)\Delta_{\eta K}+\frac{6}{t_\pi^3}\Delta_{\eta K}^3\right]$$
$$+\frac{1}{384\pi^2F_0^2}\tau(t_\pi,M_\pi^2,M_K^2)\left[-(7M_K^2-M_\pi^2-3t_\pi)-\frac{1}{t_\pi}(5M_K^2+3M_\pi^2)\Delta_{\pi K}+\frac{2}{t_\pi^2}M_\pi^2\Sigma_{\pi K}\Delta_{\pi K}-\frac{2}{t_\pi^3}\Delta_{\pi K}^4-\frac{1}{t_\pi^4}\Delta_{\pi K}^5\right]$$
$$-\frac{1}{512\pi^2F_0^2}\tau(t_\pi,M_\eta^2,M_K^2)(\Delta_{\eta K}+t_\pi)\left[\frac{2}{t_\pi}(3M_\eta^2-M_K^2)-\frac{4}{t_\pi^2}(3M_\eta^2-2M_K^2)\Delta_{\eta K}+\frac{6}{t_\pi^3}\Delta_{\eta K}^3\right]$$
$$+\frac{1}{24F_0^2}B(t_\pi,M_\pi^2,M_K^2)\left[3+\frac{1}{t_\pi}(2M_K^2-5M_\pi^2)-\frac{2}{t_\pi^2}\Delta_{\pi K}^2\right]+\frac{1}{16F_0^2}B(t_\pi,M_\eta^2,M_K^2)\left[4+\frac{2}{t_\pi}(4M_\eta^2-5M_K^2)-\frac{12}{t_\pi^2}\Delta_{\eta K}^2\right], \quad (54)$$

$$U_K^g=\frac{2}{F_0^2}(L_3+2L_4)+\frac{1}{768\pi^2F_0^2}\Bigg\{6+\frac{1}{t_\pi}(34M_K^2+9M_\eta^2-3M_\pi^2)+\frac{1}{t_\pi^2}[(17M_K^2-3M_\pi^2)\Delta_{\pi K}-3(7M_\eta^2+3M_K^2)\Delta_{\eta K}]$$
$$-\frac{2}{t_\pi^3}(\Delta_{\pi K}^3-9\Delta_{\eta K}^3)\Bigg\}+\frac{1}{8F_0^2}A(M_\pi^2)\left[\frac{1}{t_\pi}-\frac{1}{t_\pi^2}(3M_K^2-4M_\pi^2)+\frac{2}{t_\pi^3}\Delta_{\pi K}^2\right]+\frac{1}{16F_0^2}A(M_\eta^2)\left[\frac{2}{t_\pi^2}(4M_\eta^2-5M_K^2)-\frac{12}{t_\pi^3}\Delta_{\eta K}^2\right]$$
$$-\frac{1}{48F_0^2}A(M_K^2)\Bigg\{-\frac{4}{M_K^2}+\frac{3}{t_\pi}\left(22-\frac{4M_\pi^2}{M_K^2}-\frac{4M_\eta^2}{M_K^2}\right)-\frac{3}{t_\pi^2}\left[8M_K^2-2M_\pi^2\left(7-\frac{M_\pi^2}{M_K^2}\right)+2M_\eta^2\left(5-\frac{3M_\eta^2}{M_K^2}\right)\right]$$

$$
\begin{aligned}
&+\frac{12}{t_\pi^3}(\Delta_{\pi K}^2-3\Delta_{\eta K}^2)\Bigg\}-\frac{1}{768\pi^2F_0^2}\ln\left(\frac{M_\pi^2}{M_K^2}\right)(\Delta_{\pi K}+3t_\pi)\left[\frac{1}{t_\pi}+\frac{2}{t_\pi^2}M_K^2+\frac{4}{t_\pi^3}M_K^2\Delta_{\pi K}-\frac{1}{t_\pi^4}\Delta_{\pi K}^3\right]\\
&-\frac{1}{512\pi^2F_0^2}\ln\left(\frac{M_\eta^2}{M_K^2}\right)(\Delta_{\eta K}+t_\pi)\left[\frac{2}{t_\pi}+\frac{2}{t_\pi^2}(M_\eta^2-3M_K^2)-\frac{2}{t_\pi^3}(5M_\eta^2-3M_K^2)\Delta_{\eta K}+\frac{6}{t_\pi^4}\Delta_{\eta K}^3\right]\\
&-\frac{1}{384\pi^2F_0^2}(4M_K^2-s_\pi)\tau(s_\pi,M_K^2,M_K^2)+\frac{1}{768\pi^2F_0^2}\tau(t_\pi,M_\pi^2,M_K^2)\Bigg[2M_K^2+8M_\pi^2+3t_\pi\\
&-\frac{1}{t_\pi}(19M_K^4-8M_\pi^2M_K^2+5M_\pi^4)-\frac{1}{t_\pi^2}(21M_K^4-16M_\pi^2M_K^2+3M_\pi^4)\Delta_{\pi K}+\frac{2}{t_\pi^3}(4M_K^2-M_\pi^2)\Delta_{\pi K}^3-\frac{1}{t_\pi^4}\Delta_{\pi K}^5\Bigg]\\
&+\frac{1}{512\pi^2F_0^2}\tau(t_\pi,M_\eta^2,M_K^2)\left(-2M_K^2+t_\pi+\frac{1}{t_\pi}\Delta_{\eta K}^2\right)\left[2+\frac{2}{t_\pi}(M_\eta^2-3M_K^2)-\frac{2}{t_\pi^2}(5M_\eta^2-3M_K^2)\Delta_{\eta K}+\frac{6}{t_\pi^3}\Delta_{\eta K}^3\right]\\
&+\frac{1}{6F_0^2}B(s_\pi,M_K^2,M_K^2)+\frac{1}{8F_0^2}B(t_\pi,M_\pi^2,M_K^2)\left[-2+\frac{3}{t_\pi}(2M_K^2-M_\pi^2)-\frac{5}{t_\pi^2}\Delta_{\pi K}^2-\frac{2}{t_\pi^3}\Delta_{\pi K}^3\right]\\
&+\frac{1}{16F_0^2}B(t_\pi,M_\eta^2,M_K^2)\left[2-\frac{2}{t_\pi}(2M_\eta^2-5M_K^2)-\frac{2}{t_\pi^2}\Sigma_{\eta K}\Delta_{\eta K}+\frac{12}{t_\pi^3}\Delta_{\eta K}^3\right].
\end{aligned}
\tag{55}
$$

V. RESULTS

A. Input

The numerical values of the physical parameters must be fixed through experimental input. However, this input may not necessarily consist of direct measurements of the renormalized parameters; it may be obtained from any suitable set of experimental results. In practice one uses those experiments that have the highest experimental accuracy and theoretical reliability. This criterion is certainly fulfilled for the following set of parameters whose numerical values are taken from Ref. [10]:

(1) the fine structure constant,

$$\alpha=1/137.03599976(50),$$

corresponding to the classical electron charge $e=\sqrt{4\pi\alpha}$;

(2) the masses of the charged leptons,

$$m_e=0.510998902(21)\ \text{MeV},\quad m_\mu=105.658357(5)\ \text{MeV};$$

(3) the Fermi constant,

$$G_F=1.16639(1)\times10^{-5}\ \text{GeV}^{-2},$$

which is directly related to the muon lifetime;

(4) the Cabibbo-Kobayashi-Maskawa quark-mixing matrix element,

$$|V_{us}|=0.2196\pm0.0026,$$

coming from the analysis of K_{e3} decays;

(5) the masses of the light mesons,

$$M_{\pi^\pm}=139.57018(35)\ \text{MeV},\quad M_{\pi^0}=134.9766(6)\ \text{MeV},$$

$$M_{K^\pm}=493.677\pm0.016\ \text{MeV},$$

$$M_{K^0}=497.672\pm0.031\ \text{MeV},$$

$$M_\eta=547.30\pm0.12\ \text{MeV},\quad M_\rho=771.1\pm0.9\ \text{MeV};$$

(6) the charged light meson decay constants,

$$F_{\pi^\pm}=92.419\pm0.325\ \text{MeV},$$

$$F_{K^\pm}=112.996\pm1.301\ \text{MeV},$$

coming from the analysis of $\pi_{\mu2}$ and $K_{\mu2}$ decays, respectively.

Let us consider the parameters M_π, M_K, and ϵ, related to light quark masses. Since M_π and M_K figure in our expressions only at next-to-leading order, it is completely safe to replace them by their leading order expressions. In fact, the quantity M_π will be identified by the neutral pion mass,

$$M_\pi\to M_{\pi^0}=134.9766(6)\ \text{MeV},$$

while M_K^2 will be replaced by

$$M_K^2\to\frac{1}{2}(M_{K^\pm}^2+M_{K^0}^2+M_{\pi^0}^2-M_{\pi^\pm}^2)$$

to get

$$M_K=495.042\pm0.034\ \text{MeV}.$$

For ϵ, we will use the value [11],

$$\epsilon=(1.061\pm0.083)\times10^{-2},$$

extracted from the mass splitting in the baryon octet.

We will turn now to the determination of low-energy constants in the strong sector. Following Ref. [12], these constants will be evaluated at one-loop accuracy, that is, by fitting experimental measurements of the concerned observables to their ChPT expressions at next-to-leading order. Note that all of our expressions will be evaluated at the scale μ equal to the rho mass. The $K_{\ell 4}$ form factors are sensitive to variations of the low-energy constants L_1, L_2 and L_3. By fitting experimental results on $K_{\ell 4}$ form factors [13] to their ChPT expressions at next-to-leading chiral order we obtain [14]

$$L_1^r=(0.46\pm 0.24)\times 10^{-3},$$

$$L_2^r=(1.49\pm 0.23)\times 10^{-3},$$

$$L_3^r=(-3.18\pm 0.85)\times 10^{-3}.$$

The constant L_5 can be extracted from the ratio of the kaon to the pion decay constant in the isospin limit [12],

$$\frac{F_{K^\pm}}{F_{\pi^\pm}}=1+\frac{4}{F_{\pi^\pm}^2}(M_{K^\pm}^2-M_{\pi^\pm}^2)L_5^r+\frac{5M_{\pi^\pm}^2}{128\pi^2F_{\pi^\pm}^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}-\frac{M_{K^\pm}^2}{64\pi^2F_{\pi^\pm}^2}\ln\frac{M_{K^\pm}^2}{\mu^2}-\frac{3M_\eta^2}{128\pi^2F_{\pi^\pm}^2}\ln\frac{M_\eta^2}{\mu^2},$$

and reads

$$L_5^r=(1.49\pm 0.14)\times 10^{-3}.$$

Having L_5, it is easy to determine L_8 from the quantity Δ_M accounting for $SU(3)$ breaking [12],

$$\Delta_M=\frac{8}{F_{\pi^\pm}^2}(M_{K^\pm}^2-M_{\pi^\pm}^2)(2L_8^r-L_5^r)-\frac{M_{\pi^\pm}^2}{32\pi^2F_{\pi^\pm}^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}+\frac{M_\eta^2}{32\pi^2F_{\pi^\pm}^2}\ln\frac{M_\eta^2}{\mu^2},$$

the value of which reads [11]

$$\Delta_M=0.065\pm 0.065.$$

The result is

$$L_8^r=(1.02\pm 0.22)\times 10^{-3}.$$

The constant L_7 is obtained from L_5 and L_8 with the help of the isospin limit quantity,

$$\Delta_{\rm GMO}\doteq(4M_{K^\pm}^2-M_{\pi^\pm}^2-3M_\eta^2)/(M_\eta^2-M_{\pi^\pm}^2)=0.2027(15), \tag{56}$$

by matching its value to the ChPT expression at next-to-leading order [12],

$$\Delta_{\rm GMO}=-\frac{6}{F_{\pi^\pm}^2}(M_\eta^2-M_{\pi^\pm}^2)(12L_7+6L_8^r-L_5^r)+\frac{2}{M_\eta^2-M_{\pi^\pm}^2}\left(\frac{M_{\pi^\pm}^4}{32\pi^2F_{\pi^\pm}^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}-\frac{M_{K^\pm}^4}{8\pi^2F_{\pi^\pm}^2}\ln\frac{M_{K^\pm}^2}{\mu^2}+\frac{3M_\eta^4}{32\pi^2F_{\pi^\pm}^2}\ln\frac{M_\eta^2}{\mu^2}\right). \tag{57}$$

We obtain for L_7 the value

$$L_7=(-0.44\pm 0.12)\times 10^{-3}.$$

The constant L_9 is fixed from the electromagnetic charge radius of the pion [15],

$$L_9^r=(5.5\pm 0.2)\times 10^{-3}.$$

Finally, it is difficult to fix the constants L_4 and L_6 by direct experimental determination. These constants are suppressed by the Okubo-Zweig-Iizuka (OZI) rule and measure the amount by which m_s affects the values of the order parameters F and $\langle\bar{q}q\rangle$. The constant L_4 was derived from Roy and Steiner equations for S and P waves of π-K scattering amplitude [16],

$$L_4^r=(0.53\pm 0.39)\times 10^{-3}.$$

The constant L_6 has been obtained from a chiral sum rule [17],

$$L_6^r=(0.4\pm 0.2)\times 10^{-3}.$$

To end the discussion about the strong sector we have to fix the parameter F_0. At leading chiral order this parameter is given by the pseudoscalar decay constants, F_π, F_K, or F_η. One can then see the latter as the "renormalized" quantities corresponding to the "bare" quantity F_0 and thus replace it by one of them after accounting for next-to-leading order contributions. But the main question is which expression for the decay constants should be used especially since the difference between their numerical values is relatively big. For instance, the expressions for the pion and kaon decay constants at next-to-leading order are given in the isospin limit by [12]

$$F_{\pi^\pm}=F_0\left[1+\frac{4}{F_{\pi^\pm}^2}(M_{\pi^\pm}^2+2M_{K^\pm}^2)L_4^r+\frac{4M_{\pi^\pm}^2}{F_{\pi^\pm}^2}L_5^r-\frac{M_{\pi^\pm}^2}{16\pi^2F_{\pi^\pm}^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}-\frac{M_{K^\pm}^2}{32\pi^2F_{\pi^\pm}^2}\ln\frac{M_{K^\pm}^2}{\mu^2}\right],$$

$$F_{K^\pm}=F_0\left[1+\frac{4}{F_{\pi^\pm}^2}(M_{\pi^\pm}^2+2M_{K^\pm}^2)L_4^r+\frac{4M_{K^\pm}^2}{F_{\pi^\pm}^2}L_5^r -\frac{3M_{\pi^\pm}^2}{128\pi^2F_{\pi^\pm}^2}\ln\frac{M_{\pi^\pm}^2}{\mu^2}-\frac{3M_{K^\pm}^2}{64\pi^2F_{\pi^\pm}^2}\ln\frac{M_{K^\pm}^2}{\mu^2} -\frac{3M_\eta^2}{128\pi^2F_{\pi^\pm}^2}\ln\frac{M_\eta^2}{\mu^2}\right].$$

Taking as input the aforecited values for $M_{\pi^\pm}$, $M_{K^\pm}$, $F_{\pi^\pm}$, $F_{K^\pm}$, L_4^r, and L_5^r, we obtain for F_0 the central values $F_0=67.53$ MeV and $F_0=57.40$ MeV from $F_{\pi^\pm}$ and $F_{K^\pm}$, respectively. If, for comparison, we take for L_4 its large-N_c estimate, the central values modify to $F_0=79.16$ MeV and $F_0=71.62$ MeV from $F_{\pi^\pm}$ and $F_{K^\pm}$, respectively. This amounts for a 15% to 20% deviation for the value of F_0. In our calculation we will use for F_0 the two values given by the bounds of the following inequality,

$$57.40\leqslant F_0\leqslant 67.53,$$

and give the difference between the two obtained results as an error on the final result.

In the electroweak sector it is quasi-impossible to have an experimental determination of the low-energy constants due to the relatively big number of constants from one side and to the relatively small magnitude of the electroweak effects from the other side. We will use for the constants K_i in the mesonic sector the following central values obtained by means of resonance saturation [18],

$$K_1^r=-6.4\times10^{-3},\quad K_2^r=-3.1\times10^{-3},\quad K_3^r=6.4\times10^{-3},$$

$$K_4^r=-6.4\times10^{-3},\quad K_5^r=19.9\times10^{-3},\quad K_6^r=8.6\times10^{-3},$$

$$K_9^r=0,\quad K_{10}^r=0,\quad K_{12}^r=-9.2\times10^{-3},$$

with an error of $\pm6.3\times10^{-3}$ assigned to each of them coming from naive dimensional analysis. The latter will also be used to fix the bounds on low-energy constants in the electroweak leptonic sector,

$$|X_i|\leqslant 6.3\times10^{-3},$$

since these constants have not been yet determined.

B. The *f* form factor

In what follows we will refer to the inequality

$$4M_{\pi^\pm}^2\leqslant s_\pi\leqslant(M_{K^\pm}-m_\ell)^2, \tag{58}$$

as the *allowed kinematical region*. The first term in the partial wave expansion for *f* form factor is infrared finite. It contains, however, singular (Coulomb) terms for

$$s_\pi=(M_K-m_\ell)^2+2m_\ell(M_K\mp2M_\pi-m_\ell). \tag{59}$$

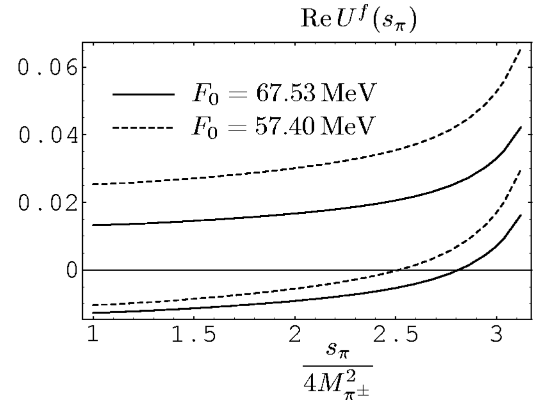

FIG. 1. The real part of the first term in the partial wave expansion for *f* form factor under the assumption $s_\ell=m_\ell^2=m_e^2$. The error band comes exclusively from the uncertainty in the determination of low-energy constants and has been developed in quadrature.

As can be easily seen, the singularity is outside the allowed kinematical region for $m_\ell\neq0$ and approaches the upper bound from the right when m_ℓ tends to zero. Therefore, there is no apparent reason for subtracting Coulomb terms in the case of the nonvanishing lepton mass. In order to see the impact of such terms on the whole correction, let us consider the following imaginary part,

$$\mathrm{Im}\,U^f(s_\pi)=\frac{3}{32\pi F_0^2}\frac{\epsilon}{\sqrt{3}}(s_\pi-2M_\pi^2)\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2} +\frac{\Delta_\pi}{48\pi F_0^2}\left(2-\frac{5M_\pi^2}{s_\pi}\right)\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2} +\frac{3e^2}{32\pi}\frac{m_\ell^2}{t_\pi}(t_\pi+M_\pi^2-m_\ell^2)\lambda^{-1/2}(t_\pi,m_\ell^2,M_\pi^2). \tag{60}$$

The plot of the preceding expression as a function of s_π is given by Fig. 3. It is easy to see that e^2 (singular) terms are almost negligible with respect to Δ_π or ϵ terms.

C. The *g* form factor

Unlike the *f* form factor, the *g* form factor is infrared divergent. We have shown in Ref. [1] that this divergence is canceled at the level of differential decay rate by the one coming from real soft photon emission. In $K_{\ell4}$ experiments, one has to measure modules and phases for form factors. Therefore, a subtraction of the infrared divergence should be applied at the level of form factors. The trouble is that the subtraction is not unique. A possible choice corresponds to a *minimal subtraction* and consists of dropping out the $\ln m_\gamma$ term. Another possible choice that we qualify by a *reasonable subtraction* consists of treating *f* and *g* form factors on an equal footing. While the *f* form factor is infrared finite, the infrared divergence in the *g* form factor comes from wave function renormalization of external charged particles and from virtual photon exchange. The latter contribution is generated from the C_0 function,

$$C_0(-p_l,p_2,m_\gamma,m_l,M_\pi), \tag{61}$$

expressed by formula (A.71) in the Appendix of Ref. [1]. In the reasonable subtraction scheme, one drops out the $\ln m_\gamma$ term coming from wave function renormalization and the *full* contribution of the C_0 function. Formally, one introduces a subtraction parameter ξ, which equals 1 in the minimal subtraction scheme and vanishes in the reasonable one. Having this, we define the subtracted real part,

$$\begin{aligned} g_P(s_\pi,\xi) &= 1+\mathrm{Re}\,U^g(s_\pi)+\frac{e^2}{8\pi^2}\ln m_\gamma^2 \\ &\quad -\frac{e^2}{8\pi^2}\frac{t_\pi-M_\pi^2-m_\ell^2}{\sqrt{t_\pi-(m_\ell+M_\pi)^2}\sqrt{t_\pi-(m_\ell-M_\pi)^2}}\xi \\ &\quad \times\ln\frac{\sqrt{t_\pi-(m_\ell-M_\pi)^2}+\sqrt{t_\pi-(m_\ell+M_\pi)^2}}{\sqrt{t_\pi-(m_\ell-M_\pi)^2}-\sqrt{t_\pi-(m_\ell+M_\pi)^2}}\ln m_\gamma^2 \\ &\quad +2e^2(t_\pi-M_\pi^2-m_\ell^2)(1-\xi) \\ &\quad \times\mathrm{Re}\,C(m_\ell^2,t_\pi,M_\pi^2,m_\gamma^2,m_\ell^2,M_\pi^2). \end{aligned} \tag{62}$$

Finally, from the imaginary part,

$$\begin{aligned} &\mathrm{Im}\,U^g(s_\pi) \\ &\quad =\delta_1^1(s_\pi)+\frac{\Delta_\pi}{32\pi F_0^2}\left(1-\frac{4M_\pi^2}{s_\pi}\right)^{1/2} \\ &\qquad +\frac{e^2}{32\pi t_\pi}\lambda^{-1/2}(t_\pi,m_\ell^2,M_\pi^2) \\ &\qquad \times[m_\ell^2(5t_\pi+M_\pi^2-m_\ell^2)+4t_\pi(M_\pi^2-t_\pi)] \\ &\qquad -2e^2(t_\pi-M_\pi^2-m_\ell^2)\mathrm{Im}\,C(m_\ell^2,t_\pi,M_\pi^2,m_\gamma^2,m_\ell^2,M_\pi^2), \end{aligned} \tag{63}$$

where

$$\delta_1^1(s_\pi)=\frac{s_\pi}{96\pi F_0^2}\left(1-\frac{4M_{\pi^\pm}^2}{s_\pi}\right)^{3/2}, \tag{64}$$

we define the subtracted phase as

$$\begin{aligned} \delta_P(s_\pi,\xi) &\doteq \mathrm{Im}\,U^g(s_\pi)+\frac{e^2}{8\pi}(t_\pi-M_\pi^2-m_\ell^2) \\ &\quad \times\lambda^{-1/2}(t_\pi,m_\ell^2,M_\pi^2)\xi\ln m_\gamma^2 \\ &\quad +2e^2(t_\pi-M_\pi^2-m_\ell^2)(1-\xi) \\ &\quad \times\mathrm{Im}\,C(m_\ell^2,t_\pi,M_\pi^2,m_\gamma^2,m_\ell^2,M_\pi^2). \end{aligned} \tag{65}$$

VI. CONCLUSION

In this work we proposed a possible splitting between strong and electromagnetic interactions in $K_{\ell 4}$ decay form factors. The technique was applied to the decay of the neutral kaon, $K^0\to\pi^0\pi^-\ell^+\nu_\ell$. It consists of working at the production threshold for the lepton pair, $s_\ell=m_\ell^2$. The latter assumption simplifies the splitting significantly by allowing a partial wave expansion of form factors with exactly the same structure as in pure strong theory. This constitutes a good approximation as long as the dependence of form factors on s_ℓ remains linear and the slope poor.

The interest in the present process is at first theoretical. In fact, the partial wave expansion of form factors involves the P-wave isovector $\pi\pi$ phase shift, $\delta_1^1(s_\pi)$, which can be related to $\pi\pi$ scattering lengths via Roy equations. In turn, scattering lengths are sensitive to the way chiral symmetry is spontaneously broken. Consequently, a theoretical study of the process in question including all possible contributions is imperative. We gave here the first analytic and numerical evaluation of the isospin breaking contribution. This would allow the extraction of $\delta_1^1(s_\pi)$ from the experimental measurement of form factors.

We started with the evaluation of the first term in the partial wave expansion for the f form factor. This term vanishes in the absence of isospin breaking and is free from infrared divergences in its presence. Motivated by these two features, we studied the sensitivity of the isospin breaking

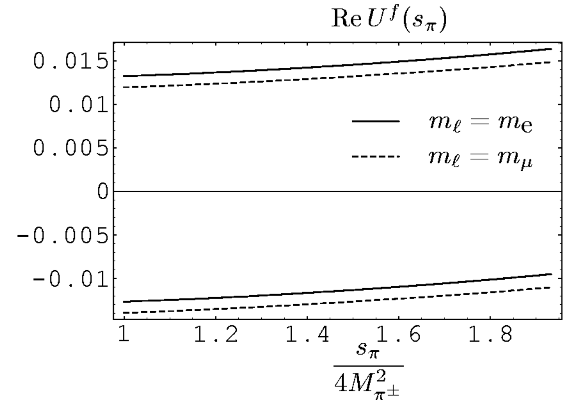

FIG. 2. The real part of the first term in the partial wave expansion for f form factor under the assumptions $s_\ell=m_\ell^2$, $F_0=67.53$ MeV. The error band comes exclusively from the uncertainty in the determination of low-energy constants and has been developed in quadrature.

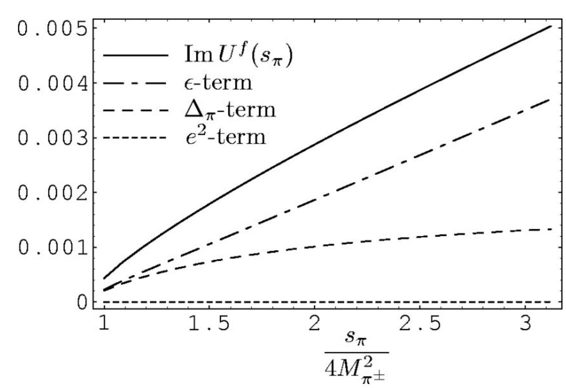

FIG. 3. The imaginary part (in radians) of the first term in the partial wave expansion for the f form factor under the assumptions $s_\ell=m_\ell^2=m_e^2$, $F_0=F_\pi=92.419$ MeV.

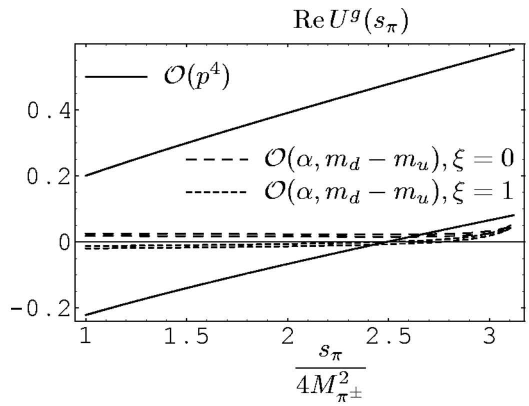

FIG. 4. Radiative correction to the real part of the first term in the partial wave expansion for g form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_0 = 67.53$ MeV. The infrared divergence has been removed applying a minimal, $\xi = 1$, as well as a reasonable, $\xi = 0$, subtraction scheme. Error bands come exclusively from the uncertainty in the determination of low-energy constants and have been developed in quadrature.

correction to variations of F_0 and m_ℓ. This was achieved by plotting the graph of the correction as a function of s_π for two values of F_0 in Fig. 1 and for $m_\ell = m_e$, m_μ in Fig. 2. We then compared in Fig. 3 the relative size for the different contributions to the correction coming from virtual photons, $\mathcal{O}(e^2)$, mass square difference between charged and neutral mesons, $\mathcal{O}(Z_0 e^2)$, and mass difference between up and down quarks, $\mathcal{O}(m_d - m_u)$.

We pursued the evaluation of the first term in the partial wave expansion for g form factor. The comparison between the size of isospin breaking correction to the real part of the term in question and the one-loop level correction to the same quantity and in the absence of isospin breaking was made in Fig. 4. $\mathcal{O}(m_d - m_u)$ and $\mathcal{O}(\alpha)$ contributions to the preceding correction were compared in Fig. 5. Finally, the

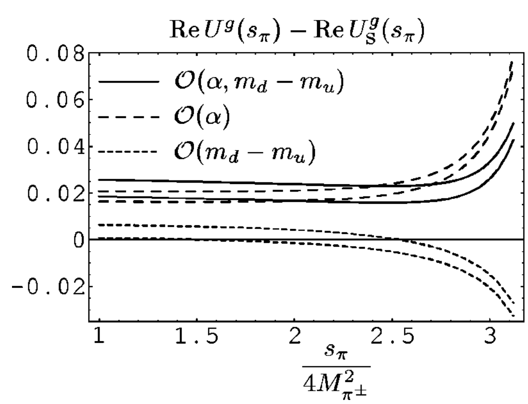

FIG. 5. Isospin breaking correction to the real part of the first term in the partial wave expansion for g form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_0 = 67.53$ MeV. The infrared divergence has been removed applying a reasonable, $\xi = 0$, subtraction scheme. Error bands come exclusively from the uncertainty in the determination of low-energy constants and have been developed in quadrature.

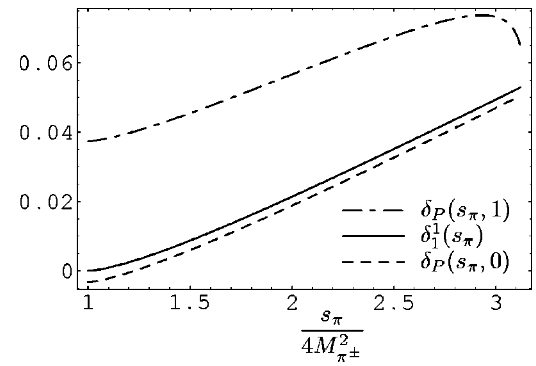

FIG. 6. The imaginary part (in radians) of the first term in the partial wave expansion for g form factor under the assumptions $s_\ell = m_\ell^2 = m_e^2$, $F_0 = F_\pi = 92.419$ MeV. The infrared divergence has been removed applying a minimal, $\xi = 1$, as well as a reasonable, $\xi = 0$, subtraction scheme.

isospin breaking correction to the P-wave isovector $\pi\pi$ phase shift was plotted in Fig. 6. Our results are of great utility for the interpretation of the outgoing data from the KTeV experiment at Fermilab.

APPENDIX: LOOP INTEGRALS

1. *B* integrals

We have

$$B(M_\pi^2, 0, M_\pi^2) = -2\bar{\lambda} + \frac{1}{16\pi^2}\left[1 - \ln\left(\frac{M_\pi^2}{\mu^2}\right)\right], \tag{A1}$$

$$B(m_\ell^2, 0, m_\ell^2) = -2\bar{\lambda} + \frac{1}{16\pi^2}\left[1 - \ln\left(\frac{m_\ell^2}{\mu^2}\right)\right], \tag{A2}$$

$$B(0, m_\ell^2, M_K^2) = -2\bar{\lambda} - \frac{1}{16\pi^2}\left[\ln\left(\frac{m_\ell^2}{\mu^2}\right) - \frac{M_K^2}{M_K^2 - m_\ell^2}\ln\left(\frac{m_\ell^2}{M_K^2}\right)\right], \tag{A3}$$

$$B(m_\ell^2, 0, M_K^2) = -2\bar{\lambda} + \frac{1}{16\pi^2}\left[1 - \ln\left(\frac{M_K^2}{\mu^2}\right)\right] - \frac{1}{16\pi^2}\left(1 - \frac{M_K^2}{m_\ell^2}\right)\ln\left(1 - \frac{m_\ell^2}{M_K^2}\right), \tag{A4}$$

$$\mathrm{Re}\,B(s_\pi, M_\pi^2, M_\pi^2) = \frac{1}{M_\pi^2}A(M_\pi^2) + \frac{1}{16\pi^2}\left[1 - \sigma_\pi \ln\left(\frac{1+\sigma_\pi}{1-\sigma_\pi}\right)\right], \tag{A5}$$

$$\mathrm{Im}\,B(s_\pi, M_\pi^2, M_\pi^2) = \frac{\sigma_\pi}{16\pi}, \tag{A6}$$

where

$$\sigma_\pi = \sqrt{1-\frac{4M_\pi^2}{s_\pi}}, \tag{A7}$$

$$B(s_\pi,M_K^2,M_K^2) = \frac{1}{M_K^2}A(M_K^2) + \frac{1}{16\pi^2} - \frac{1}{8\pi^2}\left(\frac{4M_K^2}{s_\pi}-1\right)^{1/2}\arctan\left(\frac{4M_K^2}{s_\pi}-1\right)^{-1/2}. \tag{A8}$$

For the following integral, we shall distinguish between two cases.

(a) The lepton is an electron:

$$\begin{aligned}B(s_\pi,M_\eta^2,M_\pi^2) ={}& \frac{1}{2M_\eta^2}A(M_\eta^2) + \frac{1}{2M_\pi^2}A(M_\pi^2) + \frac{1}{16\pi^2}\left[1-\frac{1}{2s_\pi}(M_\eta^2-M_\pi^2)\ln\left(\frac{M_\eta^2}{M_\pi^2}\right)\right]\\ &+\mathrm{If}[4M_\pi^2<s_\pi<(M_\eta-M_\pi)^2]\frac{1}{16\pi^2 s_\pi}\sqrt{(M_\eta+M_\pi)^2-s_\pi}\sqrt{(M_\eta-M_\pi)^2-s_\pi}\\ &\times\ln\frac{\sqrt{(M_\eta+M_\pi)^2-s_\pi}+\sqrt{(M_\eta-M_\pi)^2-s_\pi}}{\sqrt{(M_\eta+M_\pi)^2-s_\pi}-\sqrt{(M_\eta-M_\pi)^2-s_\pi}} - \mathrm{If}[(M_\eta-M_\pi)^2<s_\pi<(M_K-m_e)^2]\\ &\times\frac{1}{8\pi^2 s_\pi}\sqrt{(M_\eta+M_\pi)^2-s_\pi}\sqrt{s_\pi-(M_\eta-M_\pi)^2}\arctan\frac{\sqrt{s_\pi-(M_\eta-M_\pi)^2}}{\sqrt{(M_\eta+M_\pi)^2-s_\pi}},\end{aligned} \tag{A9}$$

(b) The lepton is a muon:

$$\begin{aligned}B(s_\pi,M_\eta^2,M_\pi^2) ={}& \frac{1}{2M_\eta^2}A(M_\eta^2) + \frac{1}{2M_\pi^2}A(M_\pi^2) + \frac{1}{16\pi^2}\left[1-\frac{1}{2s_\pi}(M_\eta^2-M_\pi^2)\ln\left(\frac{M_\eta^2}{M_\pi^2}\right)\right]\\ &+\frac{1}{16\pi^2 s_\pi}\sqrt{(M_\eta+M_\pi)^2-s_\pi}\sqrt{(M_\eta-M_\pi)^2-s_\pi}\ \ln\frac{\sqrt{(M_\eta+M_\pi)^2-s_\pi}+\sqrt{(M_\eta-M_\pi)^2-s_\pi}}{\sqrt{(M_\eta+M_\pi)^2-s_\pi}-\sqrt{(M_\eta-M_\pi)^2-s_\pi}}.\end{aligned} \tag{A10}$$

$$\begin{aligned}\mathrm{Re}\,B(t_\pi,m_\ell^2,M_\pi^2) ={}& \frac{1}{2m_\ell^2}A(m_\ell^2) + \frac{1}{2M_\pi^2}A(M_\pi^2) + \frac{1}{16\pi^2}\left[1-\frac{1}{2t_\pi}(m_\ell^2-M_\pi^2)\ln\left(\frac{m_\ell^2}{M_\pi^2}\right)\right]\\ &-\frac{1}{16\pi^2 t_\pi}\sqrt{t_\pi-(m_\ell+M_\pi)^2}\sqrt{t_\pi-(m_\ell-M_\pi)^2}\ \ln\frac{\sqrt{t_\pi-(m_\ell-M_\pi)^2}+\sqrt{t_\pi-(m_\ell+M_\pi)^2}}{\sqrt{t_\pi-(m_\ell-M_\pi)^2}-\sqrt{t_\pi-(m_\ell+M_\pi)^2}},\end{aligned} \tag{A11}$$

$$\mathrm{Im}\,B(t_\pi,m_\ell^2,M_\pi^2) = \frac{1}{16\pi t_\pi}\sqrt{t_\pi-(m_\ell+M_\pi)^2}\sqrt{t_\pi-(m_\ell-M_\pi)^2}. \tag{A12}$$

$$\begin{aligned}B(t_\pi,M_\pi^2,M_K^2) ={}& \frac{1}{2M_\pi^2}A(M_\pi^2) + \frac{1}{2M_K^2}A(M_K^2) + \frac{1}{16\pi^2}\left[1-\frac{1}{2t_\pi}(M_\pi^2-M_K^2)\ln\left(\frac{M_\pi^2}{M_K^2}\right)\right]\\ &+\frac{1}{16\pi^2 t_\pi}\sqrt{(M_\pi+M_K)^2-t_\pi}\sqrt{(M_\pi-M_K)^2-t_\pi}\ \ln\frac{\sqrt{(M_\pi+M_K)^2-t_\pi}+\sqrt{(M_\pi-M_K)^2-t_\pi}}{\sqrt{(M_\pi+M_K)^2-t_\pi}-\sqrt{(M_\pi-M_K)^2-t_\pi}},\end{aligned} \tag{A13}$$

$$\begin{aligned}B(t_\pi,M_\eta^2,M_K^2) ={}& \frac{1}{2M_\eta^2}A(M_\eta^2) + \frac{1}{2M_K^2}A(M_K^2) + \frac{1}{16\pi^2}\left[1-\frac{1}{2t_\pi}(M_\eta^2-M_K^2)\ln\left(\frac{M_\eta^2}{M_K^2}\right)\right]\\ &-\frac{1}{8\pi^2 t_\pi}\sqrt{(M_\eta+M_K)^2-t_\pi}\sqrt{t_\pi-(M_\eta-M_K)^2}\arctan\frac{\sqrt{t_\pi-(M_\eta-M_K)^2}}{\sqrt{(M_\eta+M_K)^2-t_\pi}}.\end{aligned} \tag{A14}$$

2. τ integrals

These integrals appeared while splitting strong and electromagnetic parts in two-point functions. Their definition is given by Eq. (43). We are interested in the following particular τ integrals:

$$\mathrm{Re}\,\tau(s_\pi,M_\pi^2,M_\pi^2)=-\frac{2}{s_\pi\sigma_\pi}\ln\left(\frac{1+\sigma_\pi}{1-\sigma_\pi}\right), \tag{A15}$$

$$\mathrm{Im}\,\tau(s_\pi,M_\pi^2,M_\pi^2)=\frac{2\pi}{s_\pi\sigma_\pi}. \tag{A16}$$

$$\tau(s_\pi,M_K^2,M_K^2)=\frac{4}{s_\pi}\left(\frac{4M_K^2}{s_\pi}-1\right)^{-1/2}\times\arctan\left(\frac{4M_K^2}{s_\pi}-1\right)^{-1/2}. \tag{A17}$$

$$\tau(t_\pi,M_\pi^2,M_K^2)=\frac{2}{\sqrt{(M_\pi-M_K)^2-t_\pi}\sqrt{(M_\pi+M_K)^2-t_\pi}}\times\ln\frac{\sqrt{(M_\pi+M_K)^2-t_\pi}+\sqrt{(M_\pi-M_K)^2-t_\pi}}{\sqrt{(M_\pi+M_K)^2-t_\pi}-\sqrt{(M_\pi-M_K)^2-t_\pi}}. \tag{A18}$$

$$\tau(t_\pi,M_\eta^2,M_K^2)=\frac{4}{\sqrt{t_\pi-(M_\eta-M_K)^2}\sqrt{(M_\eta+M_K)^2-t_\pi}}\times\arctan\frac{\sqrt{t_\pi-(M_\eta-M_K)^2}}{\sqrt{(M_\eta+M_K)^2-t_\pi}}. \tag{A19}$$

3. C integrals

These are scalar three-point functions whose definition and expressions were given in the appendix of Ref. [1]. In what follows, we sketch some of the particular cases that we need for the numerical evaluation of isospin breaking corrections:

$$C(m_\ell^2,0,m_\ell^2,0,m_\ell^2,M_K^2)=\frac{1}{16\pi^2}\left[\frac{1}{m_\ell^2}\ln\left(1-\frac{m_\ell^2}{M_K^2}\right)+\frac{1}{M_K^2-m_\ell^2}\ln\left(\frac{m_\ell^2}{M_K^2}\right)\right], \tag{A20}$$

$$\begin{aligned}
C(t_\pi,t_\pi,0,m_\ell^2,M_\pi^2,M_K^2)=&\frac{1}{32\pi^2t_\pi}\frac{1}{M_K^2-m_\ell^2}\\
&\times\Bigg\{(M_K^2-M_\pi^2+t_\pi)\ln\left(\frac{m_\ell^2}{M_K^2}\right)+x_0\ln\frac{M_K^2-M_\pi^2+t_\pi+x_0}{M_K^2-M_\pi^2+t_\pi-x_0}-x_1\ln\frac{M_K^2-M_\pi^2+t_\pi+x_1}{M_K^2-M_\pi^2+t_\pi-x_1}\\
&-x_0\ln\frac{(x_0+M_K^2-m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}{(x_0-M_K^2+m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}+x_1\ln\frac{(x_1+M_K^2-m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}{(x_1-M_K^2+m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}\\
&-(M_K^2-m_\ell^2)\ln\frac{(x_0+M_K^2-m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}{(x_1+M_K^2-m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}-(M_K^2-m_\ell^2)\ln\frac{(x_0-M_K^2+m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}{(x_1-M_K^2+m_\ell^2)^2-\lambda(t_\pi,m_\ell^2,M_\pi^2)}\\
&-\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)\ln\frac{M_K^2-m_\ell^2+x_0+\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}{M_K^2-m_\ell^2+x_0-\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}\\
&-\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)\ln\frac{M_K^2-m_\ell^2-x_0+\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}{M_K^2-m_\ell^2-x_0-\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}\\
&+\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)\ln\frac{M_K^2-m_\ell^2-x_1+\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}{M_K^2-m_\ell^2-x_1-\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}\\
&+\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)\ln\frac{M_K^2-m_\ell^2+x_1+\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}{M_K^2-m_\ell^2+x_1-\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)}\Bigg\},
\end{aligned} \tag{A21}$$

where

$$\lambda^{1/2}(t_\pi,m_\ell^2,M_\pi^2)=\sqrt{t_\pi-(m_\ell-M_\pi)^2}\sqrt{t_\pi-(m_\ell+M_\pi)^2} \tag{A22}$$

and

$$x_0=\sqrt{\lambda(t_\pi,M_\pi^2,M_K^2)+4t_\pi(M_K^2-m_\ell^2)}, \tag{A23}$$

$$x_1=\lambda^{1/2}(t_\pi,M_\pi^2,M_K^2)=\sqrt{(M_\pi-M_K)^2-t_\pi}\sqrt{(M_\pi+M_K)^2-t_\pi}. \tag{A24}$$

$$\begin{aligned}C(M_\pi^2,t_\pi,m_\ell^2,0,M_\pi^2,M_K^2)&=\frac{1}{16\pi^2}\frac{1}{m_\ell M_\pi}\frac{\sigma_{\ell\pi}}{1-\sigma_{\ell\pi}^2}\\&\times\Bigg\{\ln(-\sigma_{\ell\pi})\left[\ln\left(\frac{m_\ell M_K}{M_K^2-m_\ell^2}\right)+\ln\left(\frac{M_\pi M_K}{M_K^2-m_\ell^2}\right)\right]-\frac{\pi^2}{6}+\frac{1}{2}\ln^2\left(\frac{m_\ell}{M_\pi}\right)-\ln^2\left(\frac{m_\ell}{M_K}\right)-\frac{1}{2}\ln^2(-\sigma_{\ell\pi})\\&-\ln^2(\sigma_{\pi K})-\frac{1}{2}\ln^2\left(1-\frac{m_\ell}{M_\pi}\sigma_{\ell\pi}\right)-\frac{1}{2}\ln^2\left(1-\frac{M_\pi}{m_\ell}\sigma_{\ell\pi}\right)+\frac{1}{2}\ln^2\left(1-\frac{m_\ell}{M_K}\frac{\sigma_{\ell\pi}}{\sigma_{\pi K}}\right)\\&+\frac{1}{2}\ln^2\left(1-\frac{M_K}{m_\ell}\frac{\sigma_{\ell\pi}}{\sigma_{\pi K}}\right)+\frac{1}{2}\ln^2\left(1-\frac{m_\ell}{M_K}\sigma_{\ell\pi}\sigma_{\pi K}\right)+\frac{1}{2}\ln^2\left(1-\frac{M_K}{m_\ell}\sigma_{\ell\pi}\sigma_{\pi K}\right)\\&-\mathrm{Li}_2\left(\frac{m_\ell}{m_\ell-M_\pi\sigma_{\ell\pi}}\right)-\mathrm{Li}_2\left(\frac{M_\pi}{M_\pi-m_\ell\sigma_{\ell\pi}}\right)+\mathrm{Li}_2\left(\frac{m_\ell}{m_\ell-M_K\sigma_{\ell\pi}\sigma_{\pi K}}\right)+\mathrm{Li}_2\left(\frac{M_K}{M_K-m_\ell\sigma_{\ell\pi}\sigma_{\pi K}}\right)\\&+\mathrm{Li}_2\left(\frac{m_\ell\sigma_{\pi K}}{m_\ell\sigma_{\pi K}-M_K\sigma_{\ell\pi}}\right)+\mathrm{Li}_2\left(\frac{M_K\sigma_{\pi K}}{M_K\sigma_{\pi K}-m_\ell\sigma_{\ell\pi}}\right)\Bigg\},\end{aligned} \tag{A25}$$

where

$$\sigma_{\ell\pi}=\frac{\sqrt{t_\pi-(m_\ell+M_\pi)^2}-\sqrt{t_\pi-(m_\ell-M_\pi)^2}}{\sqrt{t_\pi-(m_\ell+M_\pi)^2}+\sqrt{t_\pi-(m_\ell-M_\pi)^2}}, \tag{A26}$$

$$\sigma_{\pi K}=\frac{\sqrt{(M_\pi+M_K)^2-t_\pi}-\sqrt{(M_\pi-M_K)^2-t_\pi}}{\sqrt{(M_\pi+M_K)^2-t_\pi}+\sqrt{(M_\pi-M_K)^2-t_\pi}}. \tag{A27}$$

$$\begin{aligned}\mathrm{Re}\,C(m_\ell^2,t_\pi,M_\pi^2,m_\gamma^2,m_\ell^2,M_\pi^2)&=\frac{1}{16\pi^2}\frac{1}{m_\ell M_\pi}\frac{\sigma_{\ell\pi}}{1-\sigma_{\ell\pi}^2}\\&\times\Bigg\{\ln(-\sigma_{\ell\pi})\left[2\ln(1-\sigma_{\ell\pi}^2)-\ln\left(\frac{m_\gamma^2}{m_\ell M_\pi}\right)\right]+\pi^2+\frac{1}{2}\ln^2\left(\frac{m_\ell}{M_\pi}\right)-\frac{1}{2}\ln^2(-\sigma_{\ell\pi})\\&-\frac{1}{2}\ln^2\left(1-\frac{m_\ell}{M_\pi}\sigma_{\ell\pi}\right)-\frac{1}{2}\ln^2\left(1-\frac{M_\pi}{m_\ell}\sigma_{\ell\pi}\right)+\mathrm{Li}_2(\sigma_{\ell\pi}^2)-\mathrm{Li}_2\left(\frac{M_\pi}{M_\pi-m_\ell\sigma_{\ell\pi}}\right)\\&-\mathrm{Li}_2\left(\frac{m_\ell}{m_\ell-M_\pi\sigma_{\ell\pi}}\right)\Bigg\},\end{aligned} \tag{A28}$$

$$\begin{aligned}\mathrm{Im}\,C(m_\ell^2,t_\pi,M_\pi^2,m_\gamma^2,m_\ell^2,M_\pi^2)&=\frac{1}{16\pi}\frac{1}{\sqrt{t_\pi-(m_\ell-M_\pi)^2}\sqrt{t_\pi-(m_\ell+M_\pi)^2}}\Big[\ln(m_\gamma^2)+\ln(t_\pi)-2\ln\sqrt{t_\pi-(m_\ell-M_\pi)^2}\\&-2\ln\sqrt{t_\pi-(m_\ell+M_\pi)^2}\Big].\end{aligned} \tag{A29}$$

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