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# Superstrings with intrinsic torsion

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We analyze the necessary and sufficient conditions for the preservation of supersymmetry for bosonic geometries of the form $\mathbb{R}^{1,9-d}\times M_d$, in the common Neveu-Schwarz–Neveu-Schwarz (NS-NS) sector of type II string theory and also type I or heterotic string theory. The results are phrased in terms of the intrinsic torsion of $G$ structures and provide a comprehensive classification of static supersymmetric backgrounds in these theories. Generalized calibrations naturally appear since the geometries can always arise as solutions describing NS or type I or heterotic fivebranes wrapping calibrated cycles. Some new solutions are presented. In particular we find $d=6$ examples with a fibered structure which preserve $\mathcal{N}=1,2,3$ supersymmetry in type II and include compact type I or heterotic geometries.

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## I. INTRODUCTION

Supersymmetric backgrounds of string or M theory with nonvanishing fluxes are currently an active area of study for at least two reasons. First, they provide a framework for searching for new models with attractive phenomenology and secondly, they appear in generalizations of the anti–de Sitter (AdS) conformal field theory (CFT) correspondence. For both applications a detailed mathematical understanding of the kinds of geometry that can arise is important for further elucidating physical results. Such an understanding can also lead to new methods for constructing explicit examples.

Here we will analyze supersymmetric geometries of the common Neveu-Schwarz–Neveu-Schwarz (NS-NS) sector of type IIA and IIB supergravity. That is, we consider nonvanishing dilaton $\Phi$ and three-form $H$ but with all Ramond-Ramond (RR) fields and fermions set to zero. The closely related type I and heterotic geometries which allow in addition non-trivial gauge fields will also be considered. Let us introduce the basic conditions. A type II geometry will preserve supersymmetry if and only if there is at least one $\epsilon^+$ or $\epsilon^-$ satisfying

$$\nabla^{\pm}_M\epsilon^{\pm}\equiv\left(\nabla_M\pm\frac{1}{8}H_{MNP}\Gamma^{NP}\right)\epsilon^{\pm}=0,$$

$$\left(\Gamma^M\partial_M\Phi\pm\frac{1}{12}H_{MNP}\Gamma^{MNP}\right)\epsilon^{\pm}=0, \qquad (1.1)$$

where for type IIB (IIA) $\epsilon^{\pm}$ are two Majorana-Weyl spinors of *Spin*(1,9) of the same (opposite) chirality and $\nabla$ is the Levi-Cività connection. Geometrically $\nabla^{\pm}$ are connections with totally anti-symmetric torsion given by $\pm\frac{1}{2}H$. Locally the three-form is given by $H=dB$ and hence satisfies the Bianchi identity

$$dH=0. \qquad (1.2)$$

For heterotic or type I string theory, the bosonic field content also includes a gauge field $A$, with field strength $F$, in the adjoint of $E_8\times E_8$ or $SO(32)/\mathbb{Z}_2$. We choose conventions where a geometry preserves supersymmetry if there is at least one spinor $\epsilon^+$ satisfying Eq. (1.1) and, in addition, the gaugino variation vanishes:

$$\Gamma^{MN}F_{MN}\epsilon^+=0. \qquad (1.3)$$

The Bianchi identity reads

$$dH=2\alpha'(\mathrm{Tr}\,F\wedge F-\mathrm{tr}\,R\wedge R) \qquad (1.4)$$

where the second term on the right hand side is the leading string correction to the supergravity expression. The equations of motion for these conventions can be found in Appendix A.

The geometries we consider here will be of the form $\mathbb{R}^{1,9-d}\times M_d$, and hence with $H,\Phi$ non-vanishing only on $M_d$. When $d=9$ the analysis covers the most general static geometries. Our aim is to determine the necessary and sufficient conditions on the geometry, $H$ and $\Phi$ in order that it admits a certain number of Killing spinors $\epsilon^{\pm}$ and also solves the equations of motion.

As is well known, for the special case when $H=\Phi=0$, the conditions for preservation of supersymmetry are simply that $M_d$ admits at least one covariantly constant spinor and hence has special holonomy. Here, and in the rest of the paper, when we are discussing special holonomy we will assume that $M_d$ is simply connected, otherwise we consider the universal covering space. With this understood, the only non-trivial possibilities for the special holonomy group of $M_d$ are given in Fig. 1. These manifolds are all Ricci flat and hence they automatically also solve the supergravity equations of motion.[1] Note that Fig. 1 presents only the minimal "canonical" dimension $d$ of the manifold in order for it to

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[1]Note that there are also higher order corrections to the equations of motion that give rise to tadpoles for type IIA in $d=8$ and IIB in $d=6$ (via $F$ theory) [1]. The tadpoles can often be cancelled by the addition of spacetime filling strings or D3-branes, respectively. Here we shall not explicitly refer to these corrections further.

 

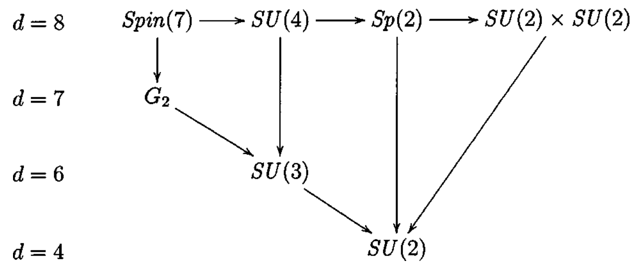


FIG. 1. Special holonomies of manifolds in $d$ dimensions with covariantly constant spinors with respect to either the Levi-Cività connection or a connection with totally anti-symmetric torsion $H$. Only the minimal ''canonical'' dimension $d$ is presented. The arrows represent the different ways the groups can be embedded in each other.

have the corresponding special holonomy. It is also possible to have manifolds of higher dimension with the same special holonomy group: when $H=\Phi=0$, the resulting geometries are simply direct products of special holonomy manifolds in the canonical dimensions given in Fig. 1 with one or more flat directions.

The analysis of a set of necessary and sufficient conditions for the preservation of supersymmetry in certain cases where $H$ and $\Phi$ are non-zero was initiated some time ago in [2] (see also [3,4]). In general, from the first condition in Eq. (1.1), it is necessary that there is at least one spinor which is covariantly constant with respect to one of the connections $\nabla^{\pm}$ with totally anti-symmetric torsion, $\nabla^{+}$ say. This is equivalent to requiring that $\nabla^{+}$ has holonomy given by one of the groups in Fig. 1 (but not necessarily in the canonical dimension, as we shall see). As we discuss in more detail below this implies the existence of various invariant forms on $M_d$ satisfying certain differential constraints. The second equation in (1.1) then imposes additional conditions on the forms. Finally, one shows that the existence of such a set of forms with constraints is in fact sufficient for the existence of one or more solutions to the supersymmetry conditions (1.1).

It is also important to know what extra conditions are required in order that the geometry solves the equations of motion. By analyzing the integrability conditions of Eq. (1.1), it was proved in [5] (see also [6]) for the entire class of geometries under consideration that it is only necessary to impose the Bianchi identity (1.2). Note that it was actually shown that one needs to impose the Bianchi identity for $H$ and the $H$ equation of motion. However, the expression for $H$ implied by supersymmetry, to be discussed below and given in Eq. (1.5), implies that the $H$ equation of motion is automatically satisfied so only the Bianchi identity is required.

Recently it has been appreciated that the necessary and sufficient conditions derived in [2], which just analyzed the $SU(n)$ cases in $d=2n$, can also be phrased in terms of $G$ structures, and this has allowed a number of generalizations [7–10,5]. Similar ideas have been used to analyze other supergravity solutions in [11–15]. The invariant forms on $M_d$ define the $G$ structure, while the differential conditions correspond to restricting the class of the intrinsic torsion of the $G$ structure. We will briefly review some aspects of $G$ structures later, but we refer to, e.g., [16] for further details. The necessary and sufficient conditions for the $G_2$ in $d=7$ [7–9] and $Spin(7)$ in $d=8$ [10] cases have also been analyzed from this point of view. Thus, for the cases when only $\nabla^{+}$, say, has special holonomy $G$, we now have a fairly complete set of results, assuming that $M_d$ has the canonical dimension for $G$ as given in Fig. 1. We shall review all known cases including the results of [2]. Note that the $SU(3)$ case was also recently reviewed in detail from the new perspective of intrinsic torsion in [17]. One new result of this paper will be to analyze the remaining two cases in $d=8$ when $\nabla^{+}$ has holonomy $Sp(2)$ or $SU(2)\times SU(2)$.

One can also ask what happens when $M_d$ does not have the canonical dimension for $G$. For example, we might consider geometries of the form $\mathbb{R}^{1,2}\times M_7$, with $M_7$ admitting two Killing spinors leading to $M_7$ having an $SU(3)$ structure corresponding to $\nabla^{+}$ having $SU(3)$ holonomy. In the case that $H=\Phi=0$, as already noted, this would necessarily imply that $M_7$ is a direct product of a flat direction with a Calabi-Yau three-fold. When $H,\Phi\neq 0$, however, we will show that the geometries can be more general than simply the direct product of a flat direction with a six-manifold $M_6$ with $SU(3)$ structure of the type derived in [2]. In particular, the flat direction can be non-trivially fibered over $M_6$ with the fibration determined by an Abelian $SU(3)$ instanton (i.e. a holomorphic gauge field satisfying the Donaldson-Uhlenbeck-Yau equation).

More generally, we will determine the most general static supersymmetric geometries of the form $\mathbb{R}^{1}\times M_9$ preserving any number of Killing spinors $\epsilon^{+}$. If there is one Killing spinor the geometry will have a $Spin(7)$ structure [and $\nabla^{+}$ will have $Spin(7)$ holonomy] but now in $d=9$. Additional Killing spinors lead to additional $Spin(7)$ structures or equivalently a $G$ structure where $G$ is the maximal common subgroup of them embedded in $SO(9)$. The $G$ structures that arise are still given by the groups as in Fig. 1 but now in $d=9$. We will show that the most general geometries consist of a number of flat directions non-trivially fibered over manifolds $M_d$ that possess $G$ structures in the canonical dimension. The fibration is determined by Abelian generalized $G$ instantons on $M_d$.

Another purpose of this paper is to present the new and the known results in a uniform way. In particular, as emphasized in [7], the expression for the three-form can always be expressed in terms of the $G$ structure in a way related to ''generalized calibrations'' [18,19]. Specifically we always have an expression of the form

$$*H=e^{2\Phi}d(e^{-2\Phi}\Xi) \tag{1.5}$$

where $\Xi$ is an invariant form which specifies, at least partially, the $G$ structure. Generalized calibrations extend the original definition of a calibration form to cases where the background has non-vanishing fluxes. In particular a generalized calibration form, here $\Xi$, is no longer closed and its exterior derivative is related to the flux, here $H$ (and the dilaton $\Phi$) as in Eq. (1.5). The physical significance of generalized calibrated cycles is that they minimize the energy functional of a brane wrapping the cycle in the presence of the fluxes.

TABLE I. $G$ structures for supersymmetric geometries when $\nabla^+$ has special holonomy in canonical dimension.

| dim($M$) | $\mathcal{N}$ | Hol($\nabla^+$) | Hol($\nabla^-$) | $G$ structure | Calibrated cycle |
|---|---|---|---|---|---|
| 4 | (1,0) | $SU(2)$ | $Spin(4)$ | $SU(2)$ | Point in $CY_2$ |
| 6 | 1 | $SU(3)$ | $Spin(6)$ | $SU(3)$ | Kähler-2 in $CY_3$ |
| 7 | 1 | $G_2$ | $Spin(7)$ | $G_2$ | Associative in $G_2$ |
| 8 | (4,0) | $SU(2)^2$ | $Spin(8)$ | $SU(2)^2$ | $CY_2$ and/or $CY_2'$ in $CY_2\times CY_2'$ |
| 8 | (3,0) | $Sp(2)$ | $Spin(8)$ | $Sp(2)$ | Quaternionic in $HK_2$ |
| 8 | (2,0) | $SU(4)$ | $Spin(8)$ | $SU(4)$ | Kähler-4 in $CY_4$ |
| 8 | (1,0) | $Spin(7)$ | $Spin(8)$ | $Spin(7)$ | Cayley in $Spin(7)$ |

That Eq. (1.5) holds might have been anticipated for the following reasons. First one notes that the types of geometries under discussion arise as solutions describing NS fivebranes wrapping supersymmetric cycles in manifolds of special holonomy including the full back reaction of the brane on the geometry. To see this first recall that the geometry of an unwrapped NS fivebrane is a product of $\mathbb{R}^{1,5}$ along the world-volume of the fivebrane with a transverse four-dimensional space with non-vanishing $H$ and $\Phi$. In addition, we know that a *probe* fivebrane with world-volume $\mathbb{R}^{1,5-p}\times\Sigma_p$ will be supersymmetric if $\Sigma_p$ is a calibrated $p$ cycle in some special holonomy background. When we go beyond the probe approximation and consider the back reaction of the fivebrane on the geometry, we thus expect a geometry of the form $\mathbb{R}^{1,5-p}\times M_{p+4}$ with non-vanishing $H$ and $\Phi$. This is precisely the type of geometry we are considering. Now, on physical grounds, we know that we can always add a second probe brane without breaking supersymmetry provided it is wrapping a cycle calibrated by the same calibration form $\Xi$ as the original probe brane. This implies that as we switch on the back reaction, $\Xi$ should still be a calibrating form, though now, since $H$ and $\Phi$ are non-zero, it is a generalized calibration. In other words, if the original probe brane wraps a cycle calibrated by a calibration form $\Xi$, the final geometry $M_{p+4}$ should admit the corresponding generalized calibration form, that is $\Xi$ satisfying Eq. (1.5).

In Table I we have listed the $G$ structures for supersymmetric geometries, preserving $\epsilon^+$ supersymmetries only, when $\nabla^+$ has special holonomy $G$ in canonical dimension. We have also listed the corresponding type of calibrated cycle that a NS fivebrane wraps in order to give the corresponding supersymmetric geometry. The number of minimal $Spin(d)$ spinors preserved in each case is also included. Note that for the $d=4$ and $d=8$ cases we have listed the six- and two-dimensional chirality of the preserved supersymmetry. Also, $CY_n$ corresponds to a Calabi-Yau $n$-fold and $HK_2$ to a hyper-Kähler manifold in $d=8$.

It is interesting to note that the more general geometries in $d=9$ mentioned above, with a number of flat directions fibered over $M_d$, have a fascinating interpretation in this regard. In particular, the flat directions correspond to directions along the world-volume of the fivebrane wrapping a flat direction, and so it is surprising that supersymmetry does not require the fibration to be trivial. Note that this interpretation is mirrored in the refined version of Eq. (1.5) for the flux that one obtains in $d=9$:

$$*H=e^{2\Phi}d(e^{-2\Phi}\Xi\wedge K^1\wedge\cdots\wedge K^{9-d}) \tag{1.6}$$

where $\Xi$, $K^i$ (partly) determine the $G$ structure, with $K^i$ one-forms corresponding to the flat directions of the fivebrane.

The fact that the geometries all satisfy calibration conditions of the form (1.5) connects with a simple vanishing theorem for compact backgrounds [20,6]. Consider the dilaton equation of motion (A4b) as given in Appendix A for the type I case, setting $F=0$ for the type II case. Supposing $M_d$ is compact, integrating the equation of motion gives

$$\int_{M_d}e^{-2\Phi}H\wedge*H+2\alpha'\int_{M_d}e^{-2\Phi}\mathrm{Tr}\,F\wedge*F=0. \tag{1.7}$$

Since the integrand in each term is positive semi-definite, we must have $H=F=0$ and hence $\Phi$ is constant. Thus, we see that there are no compact solutions in type II and type I supergravities with non-zero flux $H$ and dilaton. This vanishing theorem can of course be evaded if one includes leading-order heterotic or type I string corrections which introduce additional $\mathrm{tr}\,R^2$ terms in the dilaton equation of motion.

The theorem is reproduced in the special supersymmetric sub-case as a consequence of the calibration condition (1.5) and the Bianchi identity. This is a reflection of the general result [6,5] that the equations of motion are implied by the preservation of supersymmetry and the Bianchi identity. One has

$$\int_{M_d}e^{-2\Phi}H\wedge*H=\int_{M_d}H\wedge d(e^{-2\Phi}\Xi)$$
$$=-\int_{M_d}e^{-2\Phi}dH\wedge\Xi. \tag{1.8}$$

The simplest case [5] is when $dH=0$ (as is true for any type II background). We then have $H=\Phi=0$ by the same positivity argument as above. [This simplifies and extends[2] an earlier vanishing theorem that was given for the $SU(n)$ cases only in [21].] In the case of type I supergravity, one finds that the Bianchi identity together with the conditions on $F$ for supersymmetry [see Eq. (3.22) below] imply that the last

[2]Note that [21] includes results for the $SU(n)$ case when $dH\neq0$.

TABLE II. $G$ structures in type II theories when both $\nabla^{\pm}$ have special holonomy.

| $\dim(M)$ | $\mathcal{N}_{\mathrm{IIB}}$ | $\mathcal{N}_{\mathrm{IIA}}$ | $\mathrm{Hol}(\nabla^{+})$ | $\mathrm{Hol}(\nabla^{-})$ | $G$ structure | Calibrated cycle |
|---|---|---|---|---|---|---|
| 4 | (1,1) | (2,0) | $SU(2)$ | $SU(2)$ | $\{1\}$ | point in $\mathbb{R}^4$ |
| 6 | 2 | 2 | $SU(3)$ | $SU(3)$ | $SU(2)$ | Kähler-2 in $CY_2$ |
| 7 | 2 | 2 | $G_2$ | $G_2$ | $SU(3)$ | SLAG-3 in $CY_3$ |
| 8 | (2,2) | (4,0) | $SU(4)$ | $SU(4)$ | $SU(3)$ | Kähler-4 in $CY_3$ |
| 8 | (4,0) | (2,2) | $SU(4)$ | $SU(4)$ | $SU(2)^2$ | Kähler-2$\times$Kähler-2 in $CY_2\times CY_2'$ |
| 8 | (3,0) | (2,1) | $SU(4)$ | $Spin(7)$ | $Sp(2)$ | C-LAG-4 in $HK_2$ |
| 8 | (2,0) | (1,1) | $Spin(7)$ | $Spin(7)$ | $SU(4)$ | SLAG-4 in $CY_4$ |
| 8 | (1,1) | (2,0) | $Spin(7)$ | $Spin(7)$ | $G_2$ | co-associative in $G_2$ |

expression in Eq. (1.8) can be rewritten as minus the second term in Eq. (1.7), and again we find $H=\Phi=F=0$.

Up to this point the discussion has focused on geometries admitting one or more Killing spinors of the same type, $\epsilon^{+}$, say. This covers all static cases of the type I and heterotic theories. However, for the type II theories when $H$ and $\Phi$ are non-zero, there are solutions to Eq. (1.1) for both $\epsilon^{+}$ and $\epsilon^{-}$, which requires that both connections $\nabla^{+}$ and $\nabla^{-}$ have special holonomy. This means that the general classification of supersymmetric geometries indicated in Table I, as well as the generalizations to $d=9$, can be refined. In [7] we analyzed the different ways in which probe fivebranes can wrap calibrated cycles in manifolds of special holonomy and determined the holonomies of $\nabla^{\pm}$ that are expected in the corresponding supergravity solutions, after including the back reaction. The results are summarized in Table II. In these cases, $\epsilon^{\pm}$ each define a different structure with groups $G^{\pm}$. Equivalently, together they define a single structure with group $G$ which is the maximal common subgroup of the two embedded in $SO(d)$, and this is also listed in Table II. It is noteworthy that from the wrapped fivebrane perspective, in all cases this minimal $G$ structure is the same as the holonomy of the initial special holonomy manifold that one started with. Since both $\epsilon^{\pm}$ are required to define the $G$ structure, unlike the $G^{\pm}$ structures, it is not covariantly constant with respect to a connection with totally anti-symmetric torsion.

The particular class of geometries with $\nabla^{\pm}$ each having $G_2$ holonomy with a common $SU(3)$ subgroup was analyzed in detail in [5]. The necessary and sufficient conditions on the $SU(3)$ structure in order that the geometry preserves supersymmetry were presented. This case is associated with fivebranes wrapping special Lagrangian (SLAG) three-cycles in manifolds with $SU(3)$ holonomy. It was also shown that the three-form flux can be expressed as a generalized calibration associated with a (3,0) form, as expected for a special Lagrangian cycle. This result again refines that of Eq. (1.5) in a way expected from physical considerations. Here we shall extend the analysis of [5] to cover all cases discussed in [7].

Table II lists the geometries associated with fivebranes wrapping calibrated cycles. Note that explicit solutions corresponding to three more cases were discussed in [22]: $\nabla^{+}$ has $Sp(2)$ holonomy, while $\nabla^{-}$ has $Spin(7)$, $SU(4)$ or $Sp(2)$ holonomy. They correspond to fivebranes wrapping certain quaternionic planes in $\mathbb{R}^8$. Such calibrations are linear and it is plausible that the solutions found in [22] are the most general solutions of this kind. In any case, we will not consider these cases further in this paper.

The geometries listed in Table II are all in their "canonical" dimension. We will argue that they can be generalized to $d=9$, as before, by adding a number of flat directions. In order that both $\epsilon^{+}$ and $\epsilon^{-}$ Killing spinors survive, the fibration must be given by a generalized instanton with respect to the common $G$ structure.

It is natural to wonder if supersymmetric geometries admitting both $\epsilon^{+}$ and $\epsilon^{-}$ Killing spinors are necessarily of the type given in Table II. We shall present an interesting explicit example in $d=6$ which shows that this is not the case. The example is a torus $T^2$ non-trivially fibered over a flat $\mathbb{R}^4$ base with non-vanishing dilaton. For a particular carefully chosen fibration we show that $\nabla^{+}$ has $SU(3)$ holonomy while $\nabla^{-}$ has $SU(2)$ holonomy. This solution thus preserves twelve supercharges which corresponds to $\mathcal{N}=3$ supersymmetry in the remaining four spacetime dimensions. It would be interesting to see how it is related to the type IIB solutions preserving the same amount of supersymmetry with both RR and NS-NS fluxes presented in [23].

In this paper we will not explicitly present many detailed proofs since the arguments follow the same lines as those in [7,5], and also because we do not want to obscure the main results. The plan of the rest of the paper is as follows. In Sec. II we review $G$ structures and their intrinsic torsion. In Sec. III we discuss the necessary and sufficient conditions on the supersymmetric geometries summarized in Table I. We also comment on the additional constraints arising in type I or heterotic string theory. Section IV analyzes the general supersymmetric geometries in $d=9$ when one of the connections $\nabla^{\pm}$ has special holonomy, which generalizes the geometries of Table I. In Sec. V we present some simple explicit solutions of the type discussed in Sec. IV including candidate heterotic or type I compactifications based on fibrations over $K3$ surfaces that preserve eight supersymmetries. Section VI discusses the cases summarized in Table II when both $\nabla^{+}$ and $\nabla^{-}$ have special holonomy. Section VII presents some further explicit solutions in $d=6$ including a type II example preserving 12 supersymmetries corresponding to $\mathcal{N}=3$ supersymmetry and candidate heterotic and type I compactifications based on fibrations over $K3$ surfaces that preserve four supersymmetries. Section VIII concludes with some discussion and a summary of our main results.

## II. *G* STRUCTURES IN CANONICAL DIMENSION

It will be useful first to recall some aspects of the classification of $G$ structures (for further details see e.g. [16]). A manifold $M_d$ admits a $G$ structure if its frame bundle admits a sub-bundle with fiber group $G$. This implies that all tensors and, when appropriate, spinors on $M_d$ can be decomposed globally into representations of $G$. A $G$ structure is typically equivalent to the existence of a set of globally defined $G$-invariant tensors, or alternatively a set of globally defined $G$-invariant spinors. In particular, when $G\subset Spin(d)$ as is the case for $G$-invariant spinors, the structure defines a metric, since the corresponding sub-bundle of the frame bundle can be viewed as a set of orthonormal frames.

The $G$-structure is classified by the intrinsic torsion. When $G\subset Spin(d)$ this is a measure of the failure of the tensors/spinors to be covariantly constant with respect to the Levi-Cività connection of the metric defined by the structure. As a result, all of the components of the intrinsic torsion are encoded in derivatives of the invariant tensors or spinors. Furthermore, the intrinsic torsion, $T$, then takes values in $\Lambda^1\otimes g^\perp$ where $\Lambda^p$ is the space of $p$-forms and $g^\perp\oplus g=spin(d)$ where $g$ is the Lie algebra of $G$. The intrinsic torsion can then be decomposed into irreducible $G$ modules, $T\in\oplus_i\mathcal{W}_i$. We will denote specific components of $T$ in each module $\mathcal{W}_i$ by $W_i$. Only if the intrinsic torsion completely vanishes does the manifold have $G$ holonomy.

For a supersymmetric background $(M_d,g_d,H,\Phi)$, where $g_d$ is the metric on $M_d$, we need some non-trivial globally defined spinors satisfying Eq. (1.1). Note that the spinors are globally defined since $\nabla^\pm\epsilon^\pm=0$ implies they have constant norm, which we take to be unity, $\bar{\epsilon}^\pm\epsilon^\pm=1$, and so are nowhere vanishing. This necessarily defines a $G$ structure with $G\subset Spin(d)$. The possible groups $G$ are precisely the possible special holonomy groups appearing in Fig. 1. The necessary and sufficient conditions for solutions of the particular supersymmetry constraints (1.1) then translate into the $G$ structure being of a particular type with certain components of the intrinsic torsion vanishing. Since $G\subset Spin(d)$ the metric $g_d$ is completely determined by the $G$ structure. Similarly, one finds expressions for $H$ and $\Phi$ in terms of the intrinsic torsion of the $G$ structure.

In this section we will summarize the definition of the structures and how the generic intrinsic torsion is encoded in each case. We will consider only the structures in their canonical dimensions: $Spin(7)$ in $d=8$, $G_2$ in $d=7$, etc. It is straightforward to generalize to the case that the structure is in a higher dimension (for an example, see Appendix E of [13]). In the following sections we then turn to the particular necessary and sufficient conditions on the structure for supersymmetry.

*SU(n)-structure in d=2n.* The structure is completely specified by a real two-form $J$ of maximal rank and a complex $n$-form $\Omega$ satisfying

$$J\wedge\Omega=0,$$
$$\Omega\wedge\bar{\Omega}=i^{n(n+2)}\frac{2^n}{n!}J^n, \tag{2.1}$$

where $J^n$ is defined using the wedge product. Together these define a metric $g_d$ and an orientation chosen as vol$=J^n/n!$. Raising an index on $J$ using this metric defines an almost complex structure satisfying $J^2=-\mathbb{1}$. With respect to this almost complex structure, $\Omega$ is an $(n,0)$-form while the two-form $J$ is of type $(1,1)$. Furthermore the metric $g_d$ is almost Hermitian. Note that the almost complex structure is actually determined solely by the choice of $\Omega$ and is independent of the two-form $J$.

For generic $SU(n)$ structures, the intrinsic torsion decomposes into five modules $\mathcal{W}_i$ [16,24,25]. Consider for instance $SU(4)$. The adjoint representation of $Spin(8)$ decomposes as $\mathbf{28}\to\mathbf{1}+\mathbf{6}+\bar{\mathbf{6}}+\mathbf{15}$ where $\mathbf{15}$ is the adjoint representation of $SU(4)$, and so the remaining representations correspond to $su(4)^\perp$. The one-form $\Lambda^1$ representation decomposes as $\mathbf{8}\to\mathbf{4}+\bar{\mathbf{4}}$. We then have

$$T\in\Lambda^1\otimes su(n)^\perp=\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4\oplus\mathcal{W}_5, \tag{2.2}$$

where the corresponding $SU(4)$ representations of $\mathcal{W}_i$ are given by

$$(\mathbf{4}+\bar{\mathbf{4}})\times(\mathbf{1}+\mathbf{6}+\bar{\mathbf{6}})=(\mathbf{4}+\bar{\mathbf{4}})+(\mathbf{20}+\overline{\mathbf{20}})+(\mathbf{20}+\overline{\mathbf{20}})+(\mathbf{4}+\bar{\mathbf{4}})+(\mathbf{4}+\bar{\mathbf{4}}). \tag{2.3}$$

For $n=2$ and $n=3$ the corresponding representations are

$$(\mathbf{2}+\bar{\mathbf{2}})\times(\mathbf{1}+\mathbf{1}+\mathbf{1})=(\mathbf{2}+\bar{\mathbf{2}})+(\mathbf{2}+\bar{\mathbf{2}})+(\mathbf{2}+\bar{\mathbf{2}}),$$
$$(\mathbf{3}+\bar{\mathbf{3}})\times(\mathbf{1}+\mathbf{3}+\bar{\mathbf{3}})=(\mathbf{1}+\mathbf{1})+(\mathbf{8}+\mathbf{8})+(\mathbf{6}+\bar{\mathbf{6}})+(\mathbf{3}+\bar{\mathbf{3}})+(\mathbf{3}+\bar{\mathbf{3}}), \tag{2.4}$$

respectively. In particular, for $n=2$ the modules $\mathcal{W}_1$ and $\mathcal{W}_3$ are absent. For $n=3$ note that the $\mathcal{W}_1$ and $\mathcal{W}_2$ modules can be further decomposed into real modules $\mathcal{W}_1^\pm$ and $\mathcal{W}_2^\pm$ as discussed in detail in [25].

Each component of the intrinsic torsion $W_i\in\mathcal{W}_i$ can be given in terms of the exterior derivative of $J$ or $\Omega$, or in one case both. Generically, we have the decompositions

$$dJ\in\mathcal{W}_1\oplus\mathcal{W}_3\oplus\mathcal{W}_4,$$
$$d\Omega\in\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_5. \tag{2.5}$$

Explicitly, since $J$ is a (1,1)-form, $dJ$ has a (3,0) piece and a (2,1) piece (plus the complex conjugates). The former defines an irreducible representation (irrep) of $SU(n)$ and gives the $W_1$ component of $T$. The latter splits into a primitive $dJ_0^{(2,1)}$-form, i.e. one satisfying $J\lrcorner dJ_0^{(2,1)}=0$, giving $W_3$, plus a (1,0)-form, giving $W_4$, and which can be written as

$$W_4\equiv J\lrcorner dJ. \tag{2.6}$$

The same expression appears in characterizing any almost Hermitian metric and is known as the Lee form (of $J$). Here we have introduced the notation $\omega\lrcorner\nu$ which contracts a $p$-form $\omega$ into a $(n+p)$-form $\nu$ via

$$(\omega \lrcorner \nu)_{i_1,\ldots,i_n} = \frac{1}{p!}\omega^{j_1,\ldots,j_p}\nu_{j_1,\ldots,j_p i_1,\ldots,i_n}. \quad (2.7)$$

Similarly, since $\Omega$ is an $(n,0)$-form, $d\Omega$ has an $(n,1)$ piece plus an $(n-1,2)$ piece. Let us first consider $n\neq 2$. Again the former defines an irrep, which gives $W_5$ and can be written as a Lee form for either $\mathrm{Re}\,\Omega$ or equivalently $\mathrm{Im}\,\Omega$:

$$\begin{aligned} W_5 &\equiv \frac{1}{4}(\Omega \lrcorner d\bar{\Omega} + \bar{\Omega}\lrcorner d\Omega), \\ &= \mathrm{Re}\,\Omega \lrcorner d(\mathrm{Re}\,\Omega) = \mathrm{Im}\,\Omega \lrcorner d(\mathrm{Im}\,\Omega), \quad n\neq 2. \end{aligned} \quad (2.8)$$

The second line is obtained by noting that $\Omega\lrcorner d\Omega=0$. In general, the $(n-1,2)$ piece of $d\Omega$ splits into a primitive piece $d\Omega_0^{(n-1,2)}$ giving $W_2$ plus another piece that encodes the same $W_1$ component of $T$ as $dJ^{(3,0)}$ due to the second compatibility condition in Eq. (2.1). Note that for $SU(3)$, $W_{1,2}^{\pm}$ can be defined as the real and imaginary parts of $W_{1,2}$, respectively. For $SU(2)$, as noted, the classes $\mathcal{W}_1$ and $\mathcal{W}_3$ are absent. In this case $W_5$ is still given by the first line of Eq. (2.8), while $W_2$ is defined by

$$W_2 = \frac{1}{4}(\Omega \lrcorner d\Omega + \bar{\Omega}\lrcorner d\bar{\Omega}). \quad (2.9)$$

Recall that we have $SU(n)$ holonomy if all the components of the intrinsic torsion vanish. In this case the manifold is Calabi-Yau. Clearly this occurs if and only if $dJ=d\Omega=0$. It will be useful to note some two further cases. First, the almost complex structure is integrable if and only if $W_1=W_2=0$. Second, we note that under a conformal transformation of the $SU(n)$ structure, such that $J\to e^{2f}J$ and $\Omega\to e^{nf}\Omega$, which implies the metric scales as $g\to e^{2f}g$, $W_1, W_2$ and $W_3$ are invariant as is the following combination:

$$(2n-2)W_5 + (-1)^{n+1}2^{n-2}nW_4. \quad (2.10)$$

If this combination together with $W_1$, $W_2$ and $W_3$ all vanish and $W_4$ and $W_5$ are exact, the manifold is conformally Calabi-Yau.

*Spin(7) structures in d=8*. The structure is specified by a $Spin(7)$-invariant Cayley four-form, $\Psi$, which at any given point in $M_8$ can be written as

$$\begin{aligned} \Psi = e^{1234}+e^{1256}+e^{1278}+e^{3456}+e^{3478}+e^{5678}+e^{1357}-e^{1368} \\ -e^{1458}-e^{1467}-e^{2358}-e^{2367}-e^{2457}+e^{2468}, \end{aligned} \quad (2.11)$$

where $e^m$ define a local frame and $e^{mnpq}=e^m\wedge e^n\wedge e^p\wedge e^q$. The structure defines a metric $g_8=(e^1)^2+\cdots+(e^8)^2$ and an orientation which we take to be $\mathrm{vol}=e^1\wedge\cdots\wedge e^8$ implying $*\Psi=\Psi$.

The adjoint representation of $SO(8)$ decomposes under $Spin(7)$ as $\mathbf{28}\to\mathbf{7}+\mathbf{21}$, where $\mathbf{21}$ is the adjoint representation of $Spin(7)$. One then finds that the intrinsic torsion decomposes into two modules [26]

$$\begin{aligned} T\in \Lambda^1\otimes Spin(7)^{\perp} = \mathcal{W}_1\oplus\mathcal{W}_2, \\ \mathbf{8}\times\mathbf{7}=\mathbf{8}+\mathbf{48}. \end{aligned} \quad (2.12)$$

The components $W_i$ of $T$ in $\mathcal{W}_i$ are given in terms of the exterior derivative $d\Psi$ as, again decomposing into $Spin(7)$ representations,

$$\begin{aligned} d\Psi \in \Lambda^5 \cong \mathcal{W}_1\oplus\mathcal{W}_2, \\ \mathbf{56}\to\mathbf{8}+\mathbf{48}. \end{aligned} \quad (2.13)$$

In particular the $W_1$ component in the $\mathbf{8}$ representation is given by

$$W_1 \equiv \Psi \lrcorner d\Psi, \quad (2.14)$$

and is the Lee form for $\Psi$. The $W_2$ component in the $\mathbf{48}$ representation is then given by the remaining pieces of $d\Psi$. Note that the $Spin(7)$ manifold has $Spin(7)$ holonomy only when the intrinsic torsion vanishes, which is equivalent to $d\Psi=0$. In addition, under a conformal transformation we have $\Psi\to e^{4f}\Psi$ for some function $f$, which implies that the metric scales as $g\to e^{2f}g$. Such a transformation leaves the $W_2$ component of $T$ invariant while the Lee form $W_1$ transforms as $W_1\to W_1+28df$.

Given the definition (2.11) one has a number of standard identities, which will be useful in what follows. We have

$$\begin{aligned} \Psi^{m_1m_2m_3p}\Psi_{n_1n_2n_3p} &= 6\,\delta^{m_1m_2m_3}_{n_1n_2n_3} + 9\,\Psi^{[m_1m_2}{}_{[n_1n_2}\delta^{m_3]}_{n_3]}, \\ \Psi^{m_1m_2p_1p_2}\Psi_{n_1n_2p_1p_2} &= 12\,\delta^{m_1m_2}_{n_1n_2} + 4\,\Psi^{m_1m_2}{}_{n_1n_2}, \\ \Psi^{mp_1p_2p_3}\Psi_{np_1p_2p_3} &= 42\,\delta^m_n. \end{aligned} \quad (2.15)$$

*$G_2$ structures in d=7*. The structure is specified by an associative three-form $\phi$. In a local frame this can be given by

$$\phi = e^{246}-e^{235}-e^{145}-e^{136}+e^{127}+e^{347}+e^{567}. \quad (2.16)$$

This defines a metric $g_7=(e^1)^2+\cdots+(e^7)^2$ and an orientation $\mathrm{vol}=e^1\wedge\cdots\wedge e^7$. Explicitly we then have

$$*\phi = e^{1234}+e^{1256}+e^{3456}+e^{1357}-e^{1467}-e^{2367}-e^{2457}. \quad (2.17)$$

The adjoint representation of $SO(7)$ decomposes as $\mathbf{21}\to\mathbf{7}+\mathbf{14}$ where $\mathbf{14}$ is the adjoint representation of $G_2$. The intrinsic torsion then decomposes into four modules [27],

$$\begin{aligned} T\in\Lambda^1\otimes g_2^{\perp} = \mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4, \\ \mathbf{7}\times\mathbf{7}=\mathbf{1}+\mathbf{14}+\mathbf{27}+\mathbf{7}. \end{aligned} \quad (2.18)$$

The components of $T$ in each module $\mathcal{W}_i$ are encoded in terms of $d\phi$ and $d{*}\phi$ which decompose as

$$d\phi \in \Lambda^4 \cong \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4,$$
$$\mathbf{35} \rightarrow \mathbf{1} + \mathbf{27} + \mathbf{7},$$
$$d*\phi \in \Lambda^5 \cong \mathcal{W}_2 \oplus \mathcal{W}_4,$$
$$\mathbf{21} \rightarrow \mathbf{14} + \mathbf{7}. \tag{2.19}$$

Note that the $W_4$ component in the **7** representation appears in both $d\phi$ and $d*\phi$. It is the Lee form, given by

$$W_4 \equiv \phi \lrcorner d\phi = -*\phi \lrcorner d*\phi. \tag{2.20}$$

The $\mathcal{W}_1$ component in the singlet representation can be written as

$$W_1 \equiv *(\phi \wedge d\phi). \tag{2.21}$$

The remaining components of $d\phi$ and $d*\phi$ encode $W_3$ and $W_2$ respectively. The $G_2$ manifold has $G_2$ holonomy if and only if the intrinsic torsion vanishes, which is equivalent to $d\phi = d*\phi = 0$. Note that under a conformal transformation $\phi \rightarrow e^{3f}\phi$ the metric transforms as $g \rightarrow e^{2f}g$ and hence $*\phi \rightarrow e^{4f}*\phi$. Under this transformation $W_1$, $W_2$ and $W_3$ are invariant, while the Lee form transforms as $W_4 \rightarrow W_4 - 12df$. Finally, note that $G_2$ structures of the type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ are called integrable as one can introduce a $G_2$ Dolbeault cohomology [28].

Again there are a number of useful identities given the definition (2.16). We have

$$*\phi^{m_1m_2m_3p}*\phi_{n_1n_2n_3p} = 6\,\delta^{m_1m_2m_3}_{n_1n_2n_3} + 9*\phi^{[m_1m_2}{}_{[n_1n_2}\delta^{m_3]}_{n_3]} - \phi^{m_1m_2m_3}\phi_{n_1n_2n_3},$$
$$*\phi^{m_1m_2p_1p_2}*\phi_{n_1n_2p_1p_2} = 8\,\delta^{m_1m_2}_{n_1n_2} + 2*\phi^{m_1m_2}{}_{n_1n_2},$$
$$*\phi^{mp_1p_2p_3}*\phi_{np_1p_2p_3} = 24\delta^m_n, \tag{2.22}$$

while

$$\phi^{m_1m_2p}\phi_{n_1n_2p} = 2\,\delta^{m_1m_2}_{n_1n_2} + *\phi^{m_1m_2}{}_{n_1n_2},$$
$$\phi^{mp_1p_2}\phi_{np_1p_2} = 6\,\delta^m_n, \tag{2.23}$$

and

$$\phi^{m_1m_2p}*\phi_{n_1n_2n_3p} = \phi^{[m_1}{}_{[n_1n_2}\delta^{m_2]}_{n_3]},$$
$$\phi^{mp_1p_2}*\phi_{n_1n_2p_1p_2} = 4\,\phi^m{}_{n_1n_2}. \tag{2.24}$$

*Sp(n) structures in d=4n.* The structure is specified by three almost complex structures $J^A$ with $A=1,2,3$ satisfying the algebra

$$J^A \cdot J^B = -\delta^{AB}\mathbb{1} + \epsilon^{ABC}J^C. \tag{2.25}$$

Together these define a metric $g_d$. Lowering one index with this metric on each almost complex structure gives a set of maximal rank two-forms $J^A$. Note that the $Sp(n)$ structure could be equally well defined in terms of these forms. We also have a natural orientation given by vol$=(J^A)^{2n}/(2n)!$ for any $J^A$.

For $n=1$ recall that $Sp(1) \cong SU(2)$ and this case has already been considered above. We can make the correspondence by identifying $J \equiv J^3$ and $\Omega \equiv J^2 + iJ^1$. In more detail, first note that one can define nine Lee forms $L^{AB} \equiv J^A \lrcorner dJ^B$, but for $SU(2)$ only the diagonal Lee forms are independent, since $J^A \cdot L^{AB}$ is independent of $A$ for each $B$. The three classes of intrinsic torsion defined above from the $SU(2)$ point of view are given by $W_2 = \frac{1}{2}(L^{22} - L^{11})$, $W_4 = L^{33}$ and $W_5 = \frac{1}{2}(L^{11} + L^{22})$. Note that the almost complex structure $J^3$ is integrable if and only if $L^{11} - L^{22} = 0$ and similarly for $J^1$ and $J^2$ [29].

The only other case of interest in the context of this paper is $Sp(2)$. The adjoint representation of $SO(8)$ decomposes under $Sp(2)$ as $\mathbf{28} \rightarrow 3(\mathbf{1}) + 3(\mathbf{5}) + \mathbf{10}$, where **10** is the adjoint representation of $Sp(2)$. One then finds that the intrinsic torsion decomposes into 9 different $Sp(2)$ modules

$$T \in \Lambda^1 \otimes sp(2)^{\perp} = \bigoplus_{i=1}^{9} \mathcal{W}_i,$$
$$(\mathbf{4}+\mathbf{4}) \times [3(\mathbf{1}) + 3(\mathbf{5})] = 6(\mathbf{4}+\mathbf{4}) + 3(\mathbf{16}+\mathbf{16}), \tag{2.26}$$

where the notation takes into account that while the torsion is real, the representations **4** and **16** are pseudo-real. One can show that all the components of $T$ in $\mathcal{W}_i$ are specified in terms of the exterior derivatives $dJ^A$. Thus the $Sp(2)$ manifold has $Sp(2)$ holonomy if and only if $dJ^A = 0$. In general six of the nine Lee forms $L^{AB} \equiv J^A \lrcorner dJ^B$ are linearly independent [this is actually true for any $Sp(n)$ structure], and these precisely correspond to the six $(\mathbf{4}+\mathbf{4})$ representations appearing in Eq. (2.26). To be more precise, one can show that

$$L^{12} + L^{21} = J^3 \cdot (L^{11} - L^{22}),$$
$$L^{31} + L^{13} = J^2 \cdot (L^{33} - L^{11}),$$
$$L^{23} + L^{32} = J^1 \cdot (L^{22} - L^{33}), \tag{2.27}$$

and hence six independent Lee forms are given by $L^{11}$, $L^{22}$ and $L^{33}$ and $L^{12} - L^{21}$, $L^{31} - L^{13}$ and $L^{23} - L^{32}$. (Note that similar definitions of the independent Lee forms in the case of almost quaternionic manifolds are given in [30].) One also notes the relation

$$*(J^A \wedge J^B \wedge dJ^C) = J^A \cdot L^{BC} + J^B \cdot L^{AC}. \tag{2.28}$$

Finally in later calculations we found it useful to determine the relationships between the ten six-forms $J^A \wedge J^B \wedge J^C$. A general six-form, which is Hodge dual to a two-form, corresponds to the $Sp(2)$ representations in the decomposition $\mathbf{28} \rightarrow \mathbf{10} + 3(\mathbf{5}) + 3(\mathbf{1})$. As the six-forms of interest are constructed from $Sp(2)$ singlets, they must correspond to the three singlets in the decomposition, and hence there must be seven relationships amongst the ten six-forms. They are given by

$$J^1\wedge J^2\wedge J^2=J^1\wedge J^3\wedge J^3=\frac{1}{3}J^1\wedge J^1\wedge J^1,$$
$$J^2\wedge J^3\wedge J^3=J^2\wedge J^1\wedge J^1=\frac{1}{3}J^2\wedge J^2\wedge J^2,$$
$$J^3\wedge J^1\wedge J^1=J^3\wedge J^2\wedge J^2=\frac{1}{3}J^3\wedge J^3\wedge J^3,$$
$$J^1\wedge J^2\wedge J^3=0. \tag{2.29}$$

*SU(2)×SU(2) structures in d=8*. The structure is defined by a pair of orthogonal $SU(2)$ structures which we can write as two triplets of almost complex structures $(J^A,J'^A)$ satisfying

$$J^A\cdot J^B=-\delta^{AB}\mathbb{1}+\epsilon^{ABC}J^C,$$
$$J'^A\cdot J'^B=-\delta^{AB}\mathbb{1}+\epsilon^{ABC}J'^C,$$
$$J^A\cdot J'^B=0. \tag{2.30}$$

Again these define a metric. Lowering one index on the almost complex structures gives six half-maximal rank two-forms. We also have a natural eight-dimensional orientation given by $\mathrm{vol}\wedge\mathrm{vol}'$ where $\mathrm{vol}=(J^A)^2/2$ and $\mathrm{vol}'=(J'^B)^2/2$ for any $A$ and $B$.

Following the usual prescription decomposing the adjoint representation of $SO(8)$ into $SU(2)\times SU(2)$ representations to give $[su(2)\oplus su(2)]^\perp$ one finds 28 different real modules:

$$T\in\Lambda^1\otimes[su(2)\oplus su(2)]^\perp=\bigoplus_{i=1}^{28}\mathcal{W}_i,$$

$$\begin{aligned}&[(\mathbf{2}+\overline{\mathbf{2}},\mathbf{1})+(\mathbf{1},\mathbf{2}+\overline{\mathbf{2}})]\\&\quad\times[6(\mathbf{1},\mathbf{1})+(\mathbf{2}+\overline{\mathbf{2}},\mathbf{2}+\overline{\mathbf{2}})]\\&\quad=10(\mathbf{2}+\overline{\mathbf{2}},\mathbf{1})+10(\mathbf{1},\mathbf{2}+\overline{\mathbf{2}})+4(\mathbf{3},\mathbf{2}+\overline{\mathbf{2}})+4(\mathbf{2}+\overline{\mathbf{2}},\mathbf{3}).\end{aligned} \tag{2.31}$$

Since the $SU(2)$ structures are orthogonal, we necessarily have an almost product structure $\Pi$. This is a tensor $\Pi_m{}^n$ satisfying $\Pi\cdot\Pi=\mathbb{1}$. It can be written in terms of the complex structure as $\Pi=J^A\cdot J^A-J'^B\cdot J'^B$ for any $A$ and $B$. This can be written as the product of two commuting almost complex structures $J^\pm=J^A\pm J'^B$. As discussed in Appendix C, generically the almost product structure is not integrable.

## III. GEOMETRIES WITH $\epsilon^+$ KILLING SPINORS IN CANONICAL DIMENSION

We now consider the generic supersymmetric type II geometries $(M_d,g_d,\Phi,H)$ preserving just one type of spinor, which for definiteness we take to be $\epsilon^+$. This requires that $\nabla^+$ has special holonomy group given by one of the groups given in Fig. 1 (recall that when we discuss holonomy we are referring to the covering space if $M_d$ is not simply connected). In this section we will consider only geometries with $\nabla^+$ having special holonomy in its minimal canonical dimension: the cases are listed in Table I. For each case we present the necessary and sufficient conditions on the $G$ structure in order to preserve supersymmetry and present the generalized calibration expression for the flux. Our aim is to summarize the known cases in a uniform way as well as to present new results on the two remaining cases, $Sp(2)$ and $SU(2)\times SU(2)$. At the end of the section we will also discuss the generalizations needed for the heterotic or type I string theories.

The basic technique to derive the results of this and subsequent sections is to construct tensors from bi-linears in the Killing spinor $\epsilon^+$, which characterize the structure. Differential constraints on the structure are obtained from the vanishing of the dilatino and gravitino variations. The expression for the three-form $H$ as a generalized calibration can easily be obtained using the method of [7]. We will not present any details of these calculations in this section, for reasons of clarity. Note, however, that the next section will contain some representative calculations.

*SU(n)-geometries in d=2n*. We start with the case where $\nabla^+$ has $SU(n)$ holonomy in $d=2n$ first considered in the case of heterotic and type I theories in [2]. In other words we consider supersymmetric geometries in $d=2n$ preserving two complex chiral $d=2n$ spinors related by complex conjugation. For $n=2,4$ both spinors have the same chirality, while for $n=3$ they have opposite chirality. The necessary and sufficient conditions for preservation of supersymmetry are that the manifold $M_{2n}$ has an $SU(n)$ structure $(J,\Omega)$ satisfying the differential conditions

$$d(e^{-2\Phi}\Omega)=0,$$
$$d(e^{-2\Phi}*J)=0, \tag{3.1}$$

with the flux given in terms of the structure, in each case, by [7]

$$*H=-e^{2\Phi}d(e^{-2\Phi}) \quad \text{for } SU(2),$$
$$*H=-e^{2\Phi}d(e^{-2\Phi}J) \quad \text{for } SU(3),$$
$$*H=-e^{2\Phi}d\left(e^{-2\Phi}\frac{1}{2}J\wedge J\right) \quad \text{for } SU(4). \tag{3.2}$$

Note that here and throughout the paper the Hodge star is defined with respect to the canonical orientation fixed by the structure. For $SU(n)$ this is $\mathrm{vol}=J^n/n!$. Our conventions for defining the spinors, and the construction of $J$, $\Omega$ and vol in terms of spinors are given in Appendix B.

These conditions on $J$ and $\Omega$ are equivalent to those in [2] (after setting the gauge field to zero). In particular, as we discuss below, they imply that $J$ is integrable. As a result, the expression for $H$ can be rewritten in the form, as given in [2],

$$H=i(\bar{\partial}-\partial)J, \tag{3.3}$$

where $d=\bar{\partial}+\partial$. (Note that this corrects a sign in the corresponding expression in [2].[3]) However, it is the form (3.2) that naturally generalizes to other cases.

In particular we note that the expression for the three-form flux is that of a generalized Kähler calibration. This is physically reasonable since we expect that geometries with flux should arise as solutions describing fivebranes wrapping supersymmetric cycles, as discussed in detail in [7]. For instance, in the $SU(4)$ case, geometries with non-zero flux with $\nabla^+$ having $SU(4)$ holonomy correspond to a fivebrane wrapped on a Kähler four-cycle in a Calabi-Yau four-fold. Such branes are calibrated by $\frac{1}{2}J\wedge J$ which is precisely the generalized calibration appearing in the expression for $H$. Similarly, the $SU(3)$ geometries correspond to fivebranes wrapping Kähler two-cycles in $CY$ three-folds which are calibrated by $J$. The solutions found in [31] are of this type (see [32] for an explicit discussion). Finally the slightly degenerate $SU(2)$ case corresponds to a fivebrane wrapping a point in a $CY$ two-fold, i.e., the fivebrane is transverse to the $CY_2$. Such configurations are calibrated by the unit function.

The conditions on the $SU(n)$ structure (3.1) can be rephrased in terms of the classification of intrinsic torsion. The first condition in (3.1) implies that $W_1=W_2=0$, and hence the almost complex structure is in fact integrable (as pointed out in [2]). Thus for $SU(3)$ and $SU(4)$ the intrinsic torsion lies in $\mathcal{W}_3\oplus\mathcal{W}_4\oplus\mathcal{W}_5$. For $SU(2)$, since $\mathcal{W}_1$ and $\mathcal{W}_3$ are always absent, we have $T\in\mathcal{W}_4\oplus\mathcal{W}_5$. In all cases the second condition in (3.1) is equivalent to the statement that the Lee form is exact and related to $\Phi$, namely $W_4=2d\Phi$. The first condition also implies that Lee form for $\Omega$ is similarly proportional to $d\Phi$ with $W_5=(-1)^n 2^{n-2}W_4$. For $SU(3)$, this was first noticed in [17].

Note that this relation implies that under a conformal transformation, the invariant combination (2.10) is proportional to $(n-2)W_4$. Thus only when $n=2$ is it possible to have geometries that are conformal to Calabi-Yau $n$-folds, as noticed by [2]. In this case $W_5=W_4=2d\Phi$ with $W_2=0$. The general form of these geometries in ten dimensions is thus given by

$$ds^2=ds^2(\mathbb{R}^{1,5})+e^{2\Phi}d\tilde{s}^2,$$

$$\tilde{\nabla}^2 e^{2\Phi}=0, \tag{3.4}$$

with $H$ given as in Eq. (3.2) and $d\tilde{s}^2$ the metric on $CY_2$. This is just the usual fivebrane solution transverse to $CY_2$. The possibility of conformally $CY_2$ geometries was considered in [2] but here we claim the stronger result that it is in fact necessary.

It is worth emphasizing that if $\Phi=\text{const}$ then the leading order equations of motion imply $H=0$ and in addition $F=0$ for the heterotic or type I case [see for example Eq. (A4b)]. Thus, for instance, the solutions presented in [17] based on the Iwasawa manifold, although supersymmetric, do not solve the leading order equations of motion. In general, solutions with $H\neq 0$, and $\Phi$ non-constant must have $W_4\neq 0$ and $W_5\neq 0$. Similar comments apply to other cases considered below.

*Spin(7)-geometries in d=8*. Now consider the case when $\nabla^+$ has $Spin(7)$ holonomy. This corresponds to supersymmetric geometries in $d=8$ preserving a single chiral spinor of $Spin(8)$. The necessary and sufficient conditions are that $M_8$ admits a $Spin(7)$ structure whose only constraint is that the Lee form is again exact [10]

$$W_1=12d\Phi. \tag{3.5}$$

The flux is then given by [7]

$$*H=-e^{2\Phi}d(e^{-2\Phi}\Psi). \tag{3.6}$$

As in the $SU(n)$ case we can understand these geometries and conditions in terms of wrapped branes. They arise as solutions for fivebranes wrapping Cayley four-cycles in manifolds with $Spin(7)$ holonomy and the expression for $H$ indeed corresponds to a generalized calibration for such a cycle.

It is interesting to note that if we perform a conformal transformation $\tilde{g}\equiv e^{-6/7\Phi}g$, then the corresponding $Spin(7)$ structure defining $\tilde{g}$ has vanishing Lee form, and hence has intrinsic torsion just in the class $\mathcal{W}_2$ [10]. One might entertain the idea of solutions that are conformal to a $Spin(7)$ holonomy manifold, i.e. with $\tilde{g}$ having $Spin(7)$ holonomy. While such a geometry, with non-vanishing flux, certainly admits Killing spinors, we cannot solve the Bianchi identity $dH=0$ with non-zero flux. To see this observe that the geometry has the form

$$g=e^{6/7\Phi}\tilde{g},$$

$$H_{mnp}=-\frac{1}{3}\tilde{\Psi}_{mnp}{}^{q}\tilde{\nabla}_q(e^{6/7\Phi}). \tag{3.7}$$

The expression for $dH$ contains both the **35** and **1** representations of $Spin(7)$. The singlet is proportional to $\tilde{\nabla}^2(e^{6/7\phi})$ while the **35** corresponds to the trace-free part of $\tilde{\nabla}_l\tilde{\nabla}_p(e^{6/7\phi})$. We thus conclude that $dH=0$ implies that $\Phi=\text{const}$ which in turn implies $H=0$.

*$G_2$ geometries in $d=7$*. Next consider the case when $\nabla^+$ has $G_2$ holonomy. These geometries preserve a single $d=7$ spinor. The necessary conditions for supersymmetry were derived in [7–9] and sufficiency was proved in [8,9]. This case was discussed in detail from the point of view of this paper in [5]. The geometry admits a $G_2$ structure satisfying the conditions

$$\phi\wedge d\phi=0,$$

$$d(e^{-2\Phi}*\phi)=0, \tag{3.8}$$

[3]To see this one must take into account that our convention for the definition of $H$ has the opposite sign (and a factor of two) compared to that in [2].

which means that the intrinsic torsion lies in $\mathcal{W}_3 \oplus \mathcal{W}_4$ in the representations $\mathbf{27}+\mathbf{7}$. Moreover it implies that the Lee form is again exact with $W_4 = -6 d\Phi$. The flux is given by [7]

$$*H = e^{2\Phi} d(e^{-2\Phi}\phi). \tag{3.9}$$

It is worth noting that these geometries are special cases of integrable $G_2$ structures in which one can introduce a $G_2$ Dolbeault cohomology [28].

These backgrounds arise as solutions describing fivebranes wrapped on associative three-cycles in manifolds of $G_2$ holonomy. This is reflected in the expression for the flux which is the condition on a generalized calibration for such a cycle. Solutions of this type were presented in [33,34,5] (see [5] for an explicit demonstration of [33]).

If we perform a conformal transformation $\tilde{g} \equiv e^{-\Phi} g$, then the corresponding $G_2$ structure has vanishing Lee form, and hence has intrinsic torsion just in the class $\mathcal{W}_3$ [9]. In particular one can consider an ansatz for solutions that are conformal to a $G_2$ holonomy manifold:

$$g = e^{\Phi}\tilde{g},$$

$$H_{mnp} = -\frac{1}{2}\tilde{*}\tilde{\phi}_{mnp}{}^{q}\nabla_q(e^{\Phi}). \tag{3.10}$$

However, as in the *Spin*(7) case, Eq. (3.7), the Bianchi identity $dH=0$ implies that $\Phi$ is constant and hence $H=0$.

*Sp*(2) *geometries in* $d=8$. Next consider the case when $\nabla^+$ has *Sp*(2) holonomy. The geometries preserve three chiral $d=8$ spinors with the same chirality. The necessary and sufficient conditions for preservation of this supersymmetry are that $M_8$ admits an *Sp*(2) structure satisfying (see Appendix B for the definition of $\Omega^A$)

$$d(e^{-2\Phi}\Omega^A) = 0 \quad \text{for } A = 1,2,3,$$

$$d(e^{-2\Phi}*J^A) = 0 \quad \text{for } A = 1,2,3, \tag{3.11}$$

with the flux being given by

$$*H = -e^{2\Phi} d\left(e^{-2\Phi}\frac{1}{2}J^A \wedge J^A\right) \quad \text{for } A = 1,2,3. \tag{3.12}$$

Note that the conditions on the structure are given by those for the $SU(4)$ case for each complex structure.

The conditions (3.11) imply that the parts of $dJ^A$ transforming in the two $\mathbf{16}$'s are independent of *A*. In addition the 12 $\mathbf{4}$'s are determined by the dilaton. The "diagonal" Lee forms are all equal $L^{11}=L^{22}=L^{33}=2d\Phi$ and hence the off diagonal Lee forms $L^{AB}$, $A\neq B$ are anti-symmetric with $L^{12} = -2J^3 \cdot d\Phi$, $L^{31} = -2J^2 \cdot d\Phi$ and $L^{23} = -2J^1 \cdot d\Phi$.

Since $\nabla^+$ has *Sp*(2) holonomy, these geometries are examples of manifolds known as hyper-Kähler with torsion (HKT). A discussion of these geometries can be found, for example, in [35] and also [30]. Our geometries are special examples since the dilaton places additional constraints as listed above.

It is worth noting that this case arises when fivebranes wrap quaternionic planes in $\mathbb{R}^8$, that is cycles that are complex with respect to all three complex structures. It was shown in [36] that these are linear. In [22] solutions were written down for these configurations and it is plausible that they are the most general, once the Bianchi identity is imposed.

*SU(2)×SU(2) geometries in d=8*. Finally consider the case when $\nabla^+$ has $SU(2)\times SU(2)$ holonomy. This case corresponds to supersymmetric geometries preserving four chiral $d=8$ spinors, all with the same chirality. The necessary and sufficient conditions are that $M_8$ admits an $SU(2) \times SU(2)$ structure satisfying

$$d(e^{-2\Phi}J^A \wedge \mathrm{vol}') = 0,$$

$$d(e^{-2\Phi}J^{A\prime} \wedge \mathrm{vol}) = 0,$$

$$d(e^{-2\Phi}J^A \wedge J^{B\prime}) = 0, \tag{3.13}$$

where, e.g., $\mathrm{vol} = (J^A \wedge J^A)/2$ for each *A*, while the flux is given by

$$*H = -e^{2\Phi} d(e^{-2\Phi}\mathrm{vol} + e^{-2\Phi}\mathrm{vol}'). \tag{3.14}$$

As discussed in Appendix C, the almost product structure defined by $\Pi = (J^A + J'^{\,B})\cdot(J^A - J'^{\,B})$ is not integrable. This is because the mixed components $H_{ija}$ and $H_{abi}$, using the notation of Appendix C, are generically non-zero. A notable subclass of solutions, with integrable products, is given by those corresponding to two orthogonal fivebranes intersecting in a string, one fivebrane wrapping $CY_2$ and the other $CY_2'$ in $CY_2 \times CY_2'$. Such solutions are discussed for instance in [37].

Let us now consider the modifications required for heterotic and type I string theory. In addition to $(g_d, H, \Phi)$ the bosonic field content also includes a gauge field *A*, with field strength *F*, in the adjoint of $E_8\times E_8$ or $SO(32)/\mathbb{Z}_2$. In order to preserve supersymmetry we require the expressions in (1.1) for $\epsilon^+$ only, and thus the cases described in Table I and the above discussion are equally applicable to the heterotic and type I theories. In addition, preservation of supersymmetry requires the vanishing of the gaugino variation (1.3)

$$\Gamma^{MN}F_{MN}\epsilon^+ = 0. \tag{3.15}$$

For each case in Table I, since $\epsilon^+$ is a singlet of the special holonomy group *G* of $\nabla^+$, this is satisfied, breaking no further supersymmetry, if the two-form *F*, considered as the adjoint of $SO(d)$, lies within the adjoint of *G*.

For the *Spin*(7) case we therefore need to consider *F* to be a *Spin*(7) instanton satisfying

$$F_{mn} = -\frac{1}{2}\Psi_{mn}{}^{pq}F_{pq}, \tag{3.16}$$

while for $G_2$ we need

$$F_{mn} = -\frac{1}{2}*\phi_{mn}{}^{pq}F_{pq}. \tag{3.17}$$

For the $SU(n)$ cases, we require

$$F_{mn}=-\frac{1}{2}\left(\frac{1}{2}J\wedge J\right)_{mn}{}^{pq}F_{pq} \tag{3.18}$$

which, in complex coordinates, is equivalent to

$$J^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}}=F_{\alpha\beta}=F_{\overline{\alpha\beta}}=0. \tag{3.19}$$

That is we need a holomorphic gauge field on a holomorphic vector bundle satisfying the Donaldson-Uhlenbeck-Yau equation, as noticed in [2]. For the $Sp(2)$ case we require that the gauge field satisfies Eq. (3.18) for all three complex structures, or equivalently,

$$F_{mn}=J^{A}_{m}{}^{p}J^{A}_{n}{}^{q}F_{pq}, \quad \text{no sum on } A, \tag{3.20}$$

which are the same as the Bogomol'nyi-Prasad-Sommerfield (BPS) equations of [38]. For $SU(2)^2$, with self-dual complex structures, the gauge fields must describe an anti-self-dual instanton for each of the $SU(2)$ structures. This can be written as

$$F_{mn}=-\frac{1}{2}\mathrm{vol}_{mn}{}^{pq}F_{pq}=-\frac{1}{2}\mathrm{vol}'_{mn}{}^{pq}F_{pq}. \tag{3.21}$$

Note that in all cases the instanton condition can be written as

$$*F=\Xi\wedge F \tag{3.22}$$

where $\Xi$ is the invariant form entering the generalized calibration expression for the flux $*H=e^{2\Phi}d(e^{-2\Phi}\Xi)$.

As shown in [5,6] the equations of motion of type I supergravity are automatically satisfied if one imposes the modified Bianchi identity for $H$,

$$dH=2\alpha'\,\mathrm{Tr}\,F\wedge F. \tag{3.23}$$

In type I or heterotic string theory the Bianchi identity is modified by higher order corrections

$$dH=2\alpha'(\mathrm{Tr}\,F\wedge F-\mathrm{tr}\,R\wedge R) \tag{3.24}$$

which allows solutions with $dH=0$ as for the type II theories.

We noted above for the $Spin(7)$ case that the ansatz (3.7) preserves Killing spinors but does not solve the Bianchi identity $dH=0$, and hence the equations of motion, for non-vanishing $H,\Phi$. It is interesting to ask whether there are heterotic solutions solving $dH=2\alpha'\mathrm{Tr}F\wedge F$. Indeed, when $\tilde{g}$ is flat such solutions have already been found [39]. Similarly heterotic solutions for $d=7$ that are conformal to flat space were found in [40]. It would be interesting to construct heterotic solutions when $\tilde{g}$ is conformal to a non-flat $Spin(7)$ or $G_2$-holonomy manifold.

## IV. GENERAL GEOMETRIES WITH $\epsilon^+$ KILLING SPINORS

In the previous section, we gave the necessary and sufficient conditions for preservation of supersymmetry for a geometry of the form $\mathbb{R}^{1,9-d}\times M_d$ when $\nabla^+$ has special holonomy in the corresponding canonical number of dimensions, $Spin(7)$ in $d=8$, $G_2$ in $d=7$ and so on. The analysis for $\nabla^-$ is simply obtained by taking $H\rightarrow -H$. More generally one can ask for the generic static supersymmetric background of the form $\mathbb{R}\times M_9$ preserving some number of supersymmetries. In this section, we give a complete analysis of this question when the spinors are all of the same type and show that in addition to recovering the results of the previous section we find more general classes of geometries. As before, for definiteness we take the Killing spinors to be all of the type $\epsilon^+$ satisfying $\nabla^+\epsilon^+=0$. In the next section we turn to the case where some Killing spinors satisfy $\nabla^+\epsilon^+=0$ and some $\nabla^-\epsilon^-=0$.

Suppose we have $N$ independent spinors $\epsilon^+_{(i)}$ in $d=9$ all satisfying $\nabla^+\epsilon^+_{(i)}=0$. In general, these define a $G$ structure, where $G\subset Spin(9)$ is the stabilizer group of rotations which leave all the spinors invariant. One finds the seven special holonomy groups given in Fig. 1 as possibilities. Furthermore these embed in $Spin(9)$ in the conventional way following the pattern of the dimensional reduction. That is to say $G\subset SO(n)\subset SO(9)$ where $n$ is the canonical dimension for the $G$ structure as given in Fig. 1.

As usual the structures can also be defined in terms of a set of forms which can be constructed out of the spinors. In general, these are of the type $(K^1,\ldots,K^{9-n},\Xi^A)$ with $i_{K^i}\Xi^A=0$. Here $\Xi^A$ are the set of forms used to define the structure in its canonical dimension $n$ as described in Sec. II. The $K^i$ are a set of $9-n$ independent one-forms required to define the additional orthogonal dimensions to give a structure in $d=9$. Thus for instance a $G_2$ structure in $d=9$ is defined by the set $(K^1,K^2,\phi)$ with $i_{K^i}\phi=0$. In a local orthonormal frame $e^m$, we can take the form $\phi$ to have the standard form (2.11) in terms of $e^1,\ldots,e^7$ while $K^1=e^8$ and $K^2=e^9$. Thus, at any given point in $M_9$, the forms $K^1$ and $K^2$ define a reduction of $\mathbb{R}^9$ into $\mathbb{R}^7\oplus\mathbb{R}^2$ and hence define a $SO(7)\subset SO(9)$ structure. The three-form $\phi$ then describes a $G_2\subset SO(7)$ structure on the $\mathbb{R}^7$ subspace in the usual way. Note that the structure always defines a metric. Using this metric we can also view the $K^i$ as vectors. In addition, as we will see, the inner product $K^i\cdot K^j$ is constant for all $i$ and $j$ and so we normalize $K^i$ to be orthonormal.

If the flux $H$ is zero, we have $\nabla K^i=0$ and $M_9$ is then, after going to the covering space, just a product $M_9=\mathbb{R}^{9-n}\times M_n$ where $M_n$ is a $G$ holonomy manifold in the canonical dimension. From this point of view, $G$ holonomy extends trivially to nine dimensions. With flux however, this is no longer the case. We will show that there are new possibilities which are not simply direct products of the geometries given in the previous section with flat space. We discuss the most general case of $G=Spin(7)$, corresponding to one Killing spinor, in detail and then summarize the analogous results for the other structure groups, corresponding to the existence of more than one Killing spinor.

### A. Single Killing spinor: $Spin(7)$ structure in $d=9$

First assume we have a single Killing spinor $\epsilon^+$ on $M_9$, and since $\nabla^+\epsilon^+=0$, we can take $\overline{\epsilon^+}\epsilon^+=1$. It is easy to show that the stability group is $Spin(7)\subset Spin(9)$. Equivalently we have the set of $Spin(7)$-invariant forms $(K,\Psi)$ with $i_K\Psi=0$ and $K^2=1$. In a particular basis $e^m$, we can take $K=e^9$ and $\Psi$ given by the standard form (2.11) in terms of $e^1,\ldots,e^8$. In terms of the spinor $\epsilon^+$, we have

$$K_m=\overline{\epsilon^+}\gamma_m\epsilon^+,\quad \Psi_{mnpq}=-\overline{\epsilon^+}\gamma_{mnpq}\epsilon^+, \tag{4.1}$$

where $\gamma_m$ are nine-dimensional gamma matrices with $\gamma_{1\ldots9}=\mathbb{1}$. From the Killing spinor conditions (1.1), as in the previous section, one derives a set of necessary and sufficient conditions on $(K,\Psi)$. The condition $\nabla^+\epsilon^+=0$ simply translates into $\nabla^+\Psi=\nabla^+K=0$. From the latter constraint we immediately see, since $H$ is totally antisymmetric, that $K$ is a Killing vector, and in addition that the norm of $K$ is constant, as claimed above. In addition one finds

$$dK=G, \tag{4.2}$$

where we have made the generic $SO(8)$ decomposition

$$H\equiv H_0-K\wedge G, \tag{4.3}$$

with $i_KH_0=i_KG=0$. We can now introduce local coordinates such that the metric has the canonical form of a fibration

$$ds^2=ds^2(M_0)+(dy+B)^2, \tag{4.4}$$

with $K=dy+B$, while $dB=G$ is a two-form on $M_0$ and the metric $ds^2(M_0)$ is independent of $y$ and admits a $Spin(7)$ structure defined by $\Psi$, which may, however, at this point, depend on $y$.

Now we turn to the dilatino equation. Following the discussion in [7], given the symmetry properties of the nine-dimensional gamma matrices, one has

$$\partial_m\Phi\overline{\epsilon^+}[A,\gamma^m]_\mp\epsilon^+ +\frac{1}{12}H_{mnp}\overline{\epsilon^+}[A,\gamma^{mnp}]_\pm\epsilon^+=0 \tag{4.5}$$

where $A$ is an operator built out of gamma matrices and $[\cdot,\cdot]_\pm$ refer to the anti-commutator and commutator respectively. By taking $A=\gamma^{m_1}$ with the lower sign and $A=\gamma^{m_1,\ldots,m_6}$ with the upper sign in Eq. (4.5), one finds two constraints on $(K,\Psi)$. First one has the Lee form condition

$$\Psi\lrcorner d\Psi=12d\Phi, \tag{4.6}$$

and then the familiar calibration form for the flux

$$*H=e^{2\Phi}d(e^{-2\Phi}\Psi\wedge K). \tag{4.7}$$

Note that we have fixed our orientation by $\mathrm{vol}_{m_1,\ldots,m_9}=\overline{\epsilon^+}\gamma^{m_1,\ldots,m_9}\epsilon^+$.

If we decompose Eq. (4.7) into $SO(8)$ representations, consistency with Eq. (4.2) requires

$$G_{mn}=-\frac{1}{2}\Psi_{mn}{}^{pq}G_{pq}. \tag{4.8}$$

In other words, $G$ satisfies the $Spin(7)$ instanton equation on $M_0$. As a result, $K$ is not only a Killing vector but actually preserves the $Spin(7)$ structure. That is, the Lie derivative of the spinor $\epsilon^+$ vanishes and hence the Lie derivative of $\Psi$ also vanishes,

$$\mathcal{L}_K\Psi=0, \tag{4.9}$$

which implies similarly that $\mathcal{L}_KH=\mathcal{L}_K\Phi=0$. The Lee form condition in Eq. (4.6) can then be written

$$\Psi\lrcorner d_0\Psi=12d_0\Phi \tag{4.10}$$

where $d_0$ is the exterior derivative on the eight-dimensional space $M_0$. Similarly the condition (4.7) reduces to

$$*_0H_0=-e^{2\Phi}d_0(e^{-2\Phi}\Psi), \tag{4.11}$$

where $*_0$ is the Hodge star on $M_0$. In other words, the $d=8$ $Spin(7)$ structure $\Psi$ on $M_0$ is independent of $y$ and satisfies exactly the same conditions (3.5) and (3.6) as in the last section. In particular, the only constraint on the intrinsic torsion in $d=8$ is that the Lee form is given as in Eq. (4.6). By substituting back into the supersymmetry conditions (1.1) it is easy to see that these conditions are sufficient for supersymmetry. We should point out that it is straightforward to also define and characterize the intrinsic torsion of the $Spin(7)$ structure directly in $d=9$ but as it provides no extra information on how to characterize the geometries we shall not present any details here.

To summarize, the general $d=9$ geometry is simply a flat direction fibered over a $d=8$ $Spin(7)$ geometry, with the fibration determined by an Abelian $Spin(7)$ instanton in $d=8$. The metric is given by Eq. (4.4), the three-form by Eqs. (4.3), (4.11) and the dilaton by Eq. (4.10). In order to obtain a supersymmetric solution to the equations of motion we also need to impose the Bianchi identity for $H$. Explicitly we get

$$d_0H_0-G\wedge G=\begin{cases}0 & \text{for type II,}\\ 2\alpha'(\mathrm{Tr}\,F\wedge F-\mathrm{tr}\,R\wedge R) & \\ \quad\text{for heterotic and type I,} & \end{cases} \tag{4.12}$$

where $F$ is a $Spin(7)$ instanton.

A number of further comments are in order. First, when the flux is zero, we commented above that, after going to the covering space, the geometry is necessarily a direct product of a $d=8$ $Spin(7)$ holonomy manifold with a flat direction. By contrast when the flux is non-zero, it is only in the special case when $dK=G=0$, when the fibration is trivial, that the geometries are simply the product of the $d=8$ $Spin(7)$ geometries considered in the last section with a flat direction.

Second, since $K$ generates a symmetry of the full solution, including the spinors, we can dimensionally reduce a type II solution to get a supersymmetric heterotic solution in $d=8$ with an Abelian instanton $F$ proportional to $G$. Similarly, given a heterotic solution $(g_0,H_0,\Phi,F)$ in $d=8$ with an Abelian $Spin(7)$ instanton $F$, we can oxidize it to obtain a type II solution in $d=9$ with $G$ proportional to $F$, a metric given by Eq. (4.4) and $H=H_0-G\wedge K$.

Third, the solutions are invariant under a $T$ duality in the $y$ direction.

Finally, note that the $d=9$ expression for the flux (4.7) is again that of a generalized calibration. It corresponds to a NS fivebrane wrapping a supersymmetric five-cycle $\Sigma_4\times S^1$ in the product of a $Spin(7)$ manifold $\bar{M}$ with a circle, $\bar{M}\times S^1$, with $\Sigma_4\subset\bar{M}$ being a Cayley four-cycle. (Note that one could equally well replace the circle with a line.) The simplest way of wrapping the fivebrane leads to a $d=9$ geometry consisting of the product of a $d=8$ $Spin(7)$ geometry considered in the last section with an $S^1$. The $S^1$ is a flat direction on the world-volume of the fivebrane. The analysis of this section shows that more complicated geometries can arise leading to the world-volume direction being fibered over the $d=8$ manifold. As wrapped branes have holographic duals, it will be interesting to determine the holographic interpretation of this.

### B. Multiple Killing spinors

The case of multiple $\epsilon^+$ Killing spinors is completely analogous to the $Spin(7)$ case discussed above. As mentioned, the set of spinors $\epsilon^+_{(i)}$ in general define a $G$ structure in $d=9$ with $G$ being one of the standard special holonomy groups $SU(4)$, $Sp(2)$, $SU(2)\times SU(2)$, $G_2$, $SU(3)$ or $SU(2)$. One way to view how these groups appear is to see that the stability group of each $\epsilon^+_{(i)}$ defines a different embedding of $Spin(7)$ in $Spin(9)$. The structure group $G$ is then the common subgroup of this set of embedded $Spin(7)$ groups. From this perspective, each $G$ structure is equivalent to a set of distinct $Spin(7)$ structures.

Recall that the structure can be defined in terms of $(K^i,\Xi^A)$ where $\Xi^A$ are forms used to define the structure in its canonical dimension $n$ and $K^i$ are $9-n$ one-forms. The condition $\nabla^+K^i=0$ implies each $K^i$ is Killing and we can take them to be orthonormal. In addition, as in the $Spin(7)$ case one can always derive a set of necessary and sufficient conditions on $(K^i,\Xi^A)$ using the dilatino constraint. One always finds the familiar calibration condition for $*H$. Explicitly, for the cases where $n=8$ one has

$$*H=\begin{cases} e^{2\Phi}d\left(e^{-2\Phi}\frac{1}{2}J\wedge J\wedge K\right) & \text{for } SU(4),\\ e^{2\Phi}d\left(e^{-2\Phi}\frac{1}{2}J^A\wedge J^A\wedge K\right) & \text{for } Sp(2) \text{ with } A=1,2,3,\\ e^{2\Phi}d(e^{-2\Phi}\mathrm{vol}\wedge K+e^{-2\Phi}\mathrm{vol}'\wedge K) & \text{for } SU(2)\times SU(2),\end{cases} \tag{4.13}$$

where $K$ is the single one-form, while for the $n<8$ cases we have

$$*H=\begin{cases} e^{2\Phi}d(e^{-2\Phi}\phi\wedge K^1\wedge K^2) & \text{for } G_2,\\ e^{2\Phi}d(e^{-2\Phi}J\wedge K^1\wedge K^2\wedge K^3) & \text{for } SU(3),\\ e^{2\Phi}d(e^{-2\Phi}K^1\wedge\cdots\wedge K^5) & \text{for } SU(2).\end{cases} \tag{4.14}$$

The necessary and sufficient conditions also imply that the Killing vectors $K^i$ all commute and furthermore each preserves the underlying $G$ structure $\Xi^A$. This implies that the metric can be put in the canonical fibration form

$$ds^2=ds^2(M_0)+\sum_{i=1}^{9-n}(dy^i+B^i)^2, \tag{4.15}$$

where $M_0$ is an $n$-dimensional manifold and $K^i=dy^i+B^i$. Furthermore, $M_0$ has a $G$ structure defined by $\Xi^A$ independent of $y^i$. The flux $H$ has the related decomposition

$$H\equiv H_0-\sum_{i=1}^{9-n}K^i\wedge G^i, \tag{4.16}$$

where $G^i=dB^i$ are two-forms on $M_0$. In addition one finds a set of constraints on the $G$ structure $\Xi^A$ on $M_0$. As in the $Spin(7)$ case these turn out to be precisely the canonical dimension conditions given in the last section.

The additional freedom in nine-dimensional geometries is given by the two-forms $G^i$ defining the fibration. Again as in the $Spin(7)$ case consistency between the calibration conditions (4.13) and (4.14) and the expansion (4.16) implies that each $G^i$ satisfies the appropriate Abelian $G$ instanton equation on $M_0$.

In summary, general supersymmetric geometries in $d=9$ are closely related to the supersymmetric geometries in the canonical dimensions discussed in the last section. They all have a fibered structure where the base space $M_0$ has a $G$ structure in the canonical dimension satisfying one of the sets of conditions given in Sec. III. The flux is given by a generalized calibration condition (4.13) or (4.14), corresponding to a fivebrane wrapping a five-cycle. The twisting of the fibration is described by two-forms $G^i$ which are all Abelian $G$ instantons on $M_0$. If one makes a dimensional reduction on the $K^i$, the solutions correspond to heterotic solutions in canonical dimension $d=n$ with $9-n$ Abelian instantons. In order to obtain a solution to the equations of motion the flux $H_0$ on $M_0$ must also satisfy a modified Bianchi identity

$$d_0H_0-\sum_{i=1}^{9-n} G^i\wedge G^i=\begin{cases}0 \quad \text{for type II,}\\ 2\alpha'(\mathrm{Tr}\,F\wedge F-\mathrm{tr}\,R\wedge R)\\ \qquad \text{for heterotic and type I.}\end{cases} \tag{4.17}$$

These results provide a comprehensive classification of all the possible supersymmetric heterotic or type I or NS-NS type II bosonic geometries of the form $\mathbb{R}^{1,9-d}\times M_d$ preserving Killing spinors satisfying Eq. (1.1) for $\epsilon^+$. Any solution with $d<9$ can be obtained simply by setting $9-d$ of the $B^i$ twists to zero, so that the fibration becomes, at least partially, a product $M_9=\mathbb{R}^{9-d}\times M_d$.

Note that there is one possible caveat to this analysis which is the existence of geometries with exactly five, six or seven Killing spinors. This necessarily defines an $SU(2)$ structure and would require the existence of a compatible connection $\nabla^+$ without the particular fibration structure described in the text. It is unclear to us whether this is possible or not. Similar comments apply to the existence of solutions with nine or more supersymmetries (so defining an identity structure) which are not simply flat space.

It is interesting to note that particular examples of these general types of solutions have already appeared in the literature. Examples of $SU(2)$ structure in $d=6$ and $SU(2)^2$ in $d=9$ were considered in [41] using conformally Eguchi-Hanson metrics. Similar solutions related to D3-branes were considered in [42]. Further examples will be presented in the next section.

We should also note that $d=6$ geometries of the type discussed here with two flat directions are similar to those studied in [43]. However, the motivation of that work was rather different; namely, the idea was to exploit the fibration structure in order to construct examples of manifolds with $SU(3)$ structures in six dimensions of the type described in the last section.

## V. EXPLICIT EXAMPLES I

We now present explicit solutions of the type described in the last section. For illustration we shall consider here just a single flat direction fibered over a base manifold $M_0$. Additional examples with two flat directions fibered over a four-dimensional base will be considered in Sec. VII. To begin with we consider $M_0$ to be four dimensional, and the three complex structures are taken to be self-dual. As noted in Sec. III, $M_0$ is necessarily conformally hyper-Kähler. The five-dimensional geometry thus takes the form

$$ds^2=e^{2\Phi}(d\tilde{s}^2)+(dy+B)^2, \tag{5.1}$$

$$H_{mnp}=-\tilde{\epsilon}_{mnp}{}^{l}\tilde{\nabla}_l e^{2\Phi}-3B_{[m}G_{np]},$$

$$H_{ymn}=-G_{mn},$$

where $G=dB$ is an Abelian anti-self-dual instanton and $\tilde{\epsilon}$ is the volume form on $d\tilde{s}^2$. Generically, these solutions preserve 1/2 of the $\epsilon^+$ supersymmetries, and none of the $\epsilon^-$ supersymmetries for the type II theories, corresponding to eight supercharges for both the heterotic and the type II theories. For solutions, we must impose the Bianchi identity for $H$. This gives

$$-\tilde{d}\tilde{*}\tilde{d}e^{2\Phi}=\begin{cases}G\wedge G \quad \text{for type II,}\\ G\wedge G+2\alpha'(\mathrm{Tr}\,F\wedge F-\mathrm{tr}\,R\wedge R)\\ \qquad \text{for heterotic and type I.}\end{cases} \tag{5.2}$$

Recall that supersymmetry implies that $F$ is also an anti-self-dual instanton on the base. In the special case that $\mathrm{tr}\,R\wedge R=0$, satisfying the Bianchi identity then implies that the leading equations of motion are automatically satisfied. Otherwise, one must separately check that one has a solution of the equations of motion, including at this order $\alpha'$ corrections.

Particular solutions can be found whenever we have an explicit anti-self-dual Abelian instanton $G$ on a hyper-Kähler manifold. The simplest cases are when the hyper-Kähler metric is flat. Let us present some examples just for the type II case, for simplicity, where the Bianchi identity becomes

$$\tilde{\nabla}^2e^{2\Phi}=-\frac{1}{2}\tilde{G}^2. \tag{5.3}$$

Then a simple anti-self-dual instantons is given for instance by

$$B=\gamma(x^1dx^2-x^3dx^4), \tag{5.4}$$

corresponding to a constant field strength. A radial solution for the dilaton is given by

$$e^{2\Phi}=1+\frac{m}{r^2}-\frac{1}{4}\gamma^2r^2. \tag{5.5}$$

A different radial solution can be obtained by writing the flat metric in terms of left-invariant one-forms on the three-sphere:

$$ds^2=dr^2+\frac{1}{4}r^2[(\sigma_R^1)^2+(\sigma_R^2)^2+(\sigma_R^3)^2] \tag{5.6}$$

with positive orientation given by $dr\wedge\sigma_R^1\wedge\sigma_R^2\wedge\sigma_R^3$ (our conventions are as in [11]). A singular anti-self-dual instanton is then given by

$$B=\frac{\gamma}{4r^2}\sigma_R^3. \tag{5.7}$$

A radial solution for the dilaton is

$$e^{2\Phi}=1+\frac{m}{r^2}-\frac{\gamma^2}{12r^6}. \tag{5.8}$$

When the hyper-Kähler metric is Eguchi-Hanson space or Taub Newman-Unti-Tamborino (NUT) space any of the anti-self-dual harmonic two-forms on these spaces can be used as the Abelian instanton and if they are normalizable they lead to non-singular solutions. These cases have already appeared in the literature [41].

Let us now consider whether we can obtain compact heterotic solutions of the form (5.1). (Recall that there are no compact solutions with flux for the type II cases [6].) The base space $\tilde{M}$ must admit a hyper-Kähler metric so is either $T^4$ or $K3$. In addition, we will compactify the fiber direction on a circle $S^1$ of radius $R$. By construction such a background preserves eight supersymmetries. For a solution we must also satisfy the Bianchi identity. The left-hand side of Eq. (5.2) is exact, thus the sum of the sources on the right-hand side must be trivial in cohomology. Since the manifold is compact, each of the sources is also quantized, being some multiple of the first Pontrjagin $p_1 \in H^2(M_5,\mathbb{Z})$ class (instanton charge) of the corresponding bundle. If $E$ is the bundle describing the $S^1$ fibration and $V$ the bundle of the heterotic and type I gauge fields we have

$$R^2 p_1(E) + 2\alpha' p_1(V) - 2\alpha' p_1(TM_5) = 0 \tag{5.9}$$

in cohomology. Note that given the definition of $G$ the field strength entering $p_1(E)$ is $G/R$, hence the factor of $R^2$ in the first term. Since both $G$ and $F$ are anti-self-dual instantons on the base $p_1(V)$ cannot cancel against $p_1(E)$ and we can satisfy Eq. (5.9) only by including non-trivial $p_1(TM_5)$. The equation for the dilaton on $\tilde{M}$ then becomes

$$\tilde{\nabla}^2 e^{2\Phi} = -\frac{1}{2}\tilde{G}^2 - \alpha'(\mathrm{Tr}\,\tilde{F}^2 - \mathrm{tr}\,\tilde{R}^2). \tag{5.10}$$

One would then have to check whether such a solution for $\Phi$ in fact leads to a background satisfying the full (higher-order) equations for motion. One important point to note is that satisfying Eq. (5.9) with non-vanishing $p_1(E)$ requires $R^2 \sim \alpha'$. In other words the size of the $S^1$ fiber must be of order the string scale. As such the supergravity description of these compactifications is breaking down. (Note, in addition, that $R^2$ is constrained to be a rational multiple of $\alpha'$, so cannot be a modulus.) It would be interesting to find a corresponding conformal field theory description, for instance by taking the orbifold limit of the base $K3$ manifold. Note that it is trivial to extend these solutions to six-dimensional compactifications with $\mathcal{N}=2$ supersymmetry simply by including a second fibered direction.

Now let us consider solutions where the base geometry $M_0$ is in more than four dimensions. Specifically we consider solutions where $M_0$ is conformal to a special holonomy manifold. We noted in Sec. III that this rules out the $SU(n)$ cases for $n \neq 2$. Let us thus consider $M_0$ to be conformal to a $G_2$ holonomy manifold. An eight dimensional geometry preserving two $\epsilon^+$ supersymmetries, one of each $d=8$ chirality, is given by

$$ds^2 = e^{\Phi}(d\tilde{s}^2) + (dy + B)^2,$$

$$H_{mnp} = -\frac{1}{2}\tilde{*}\tilde{\phi}_{mnp}{}^{q}\tilde{\nabla}_q e^{\Phi} - 3B_{[m}G_{np]},$$

$$H_{ymn} = -G_{mn} \tag{5.11}$$

where $G = dB$ is an Abelian $G_2$ instanton on the $G_2$ holonomy manifold $\tilde{M}$. A type II solution is then obtained by solving the Bianchi identity which reads

$$\tilde{*}\tilde{\phi}_{[mnp}{}^{q}\tilde{\nabla}_{l]}\tilde{\nabla}_q e^{\Phi} = 3G_{[mn}G_{pl]}. \tag{5.12}$$

Given that $G$ is a $G_2$ instanton, this is equivalent to

$$\tilde{\nabla}_m\tilde{\nabla}_n e^{\Phi} = -2\tilde{G}_m{}^{k}G_{nk} + \frac{1}{4}\tilde{G}^2\tilde{g}_{mn}. \tag{5.13}$$

To get explicit solutions we need explicit $G_2$ holonomy metrics $d\tilde{s}^2$ and explicit Abelian instantons $G$. One approach is to note that if the $G_2$ holonomy metric admits a Killing vector $v$, then the two-form $dv$ is a $G_2$ instanton if and only if $v$ preserves the $G_2$ structure: $\mathcal{L}_v\phi = di_v\phi = 0$. Since all of the known explicit $G_2$ manifolds have many isometries, this result allows one in principle to find new solutions and would be interesting to investigate further.

If the $G_2$ holonomy manifold is flat, solutions with constant flux can be obtained as follows. We take

$$B = \frac{1}{2}C_{mn}x^m dx^n, \tag{5.14}$$

giving constant field strength $G = C$. This is a $G_2$ Abelian instanton provided $C_{mn} = -\frac{1}{2}\tilde{*}\tilde{\phi}_{mn}{}^{pq}C_{pq}$. In other words, using a suitable projection, we have in general

$$C_{mn} = \frac{2}{3}\left(\delta_{mn}^{pq} - \frac{1}{4}\tilde{*}\tilde{\phi}_{mn}{}^{pq}\right)D_{pq}, \tag{5.15}$$

for an arbitrary constant two-form $D_{mn}$. We then find that

$$e^{\Phi} = -\frac{1}{2}\left(2\tilde{G}_m{}^{k}G_{nk} - \frac{1}{4}\tilde{G}^2\tilde{g}_{mn}\right)x^m x^n + \text{const} \tag{5.16}$$

solves Eq. (5.13).

## VI. GEOMETRIES WITH BOTH $\epsilon^+$ AND $\epsilon^-$ KILLING SPINORS

Let us now turn our attention to the type II cases summarized in Table II. These geometries preserve both $\epsilon^+$ and $\epsilon^-$ Killing spinors and thus define two different structures, $G^{\pm}$, one for each set of Killing spinors, of the type described in Sec. III. Taking both sets together defines a $G$ structure where $G$ is the maximal common subgroup of $G^+$ and $G^-$ given their particular embeddings in $SO(d)$. One can follow the detailed strategy of [5] to derive the necessary and sufficient conditions on this $G$ structure in order that the geometry preserves the corresponding supersymmetry. This is based on direct manipulations of the Killing spinor equations and some details of this approach appear in [7].

Equivalently, we can obtain the conditions on the $G$ structure by writing the $G^{\pm}$ structures in terms of the $G$ structure and then imposing the conditions on the $G^{\pm}$ structures derived in Sec. III. In implementing this strategy it is crucial to recall that the signs presented in Sec. III assumed that the preserved spinors were of the $\epsilon^{+}$ type and also took, in the relevant cases, the preserved spinors to have a definite chirality. In order to get the results of this section, one needs the appropriate generalizations for $\nabla^{-}$ and sometimes the opposite chirality.

*SU(2) geometries in d=6*. This case arises when both $\nabla^{\pm}$ have $SU(3)$ holonomy with a common $SU(2)$ subgroup. These geometries preserve two complex chiral $d=6$ spinors, one $\epsilon^{+}$ and one $\epsilon^{-}$. The necessary and sufficient conditions for preserving this supersymmetry are that $M_6$ admits an $SU(2)$ structure in $d=6$ satisfying the conditions given below. The $SU(2)$ structure in $d=6$ is specified by a two-form $J$, a complex two-form $\Omega$ and two one-forms $K^i$ with $i=1,2$, satisfying Eq. (2.1) for $n=2$ and in addition

$$i_{K^i}\Omega = i_{K^i}J = 0. \tag{6.1}$$

The corresponding $SU(3)$ structures associated with $\nabla^{\pm}$ are given by

$$J^{\pm} = J \pm K^1 \wedge K^2,$$
$$\Omega^{\pm} = \Omega \wedge (K^1 \pm i K^2). \tag{6.2}$$

Demanding that each of the $SU(3)$ structures satisfies the necessary and sufficient conditions for supersymmetry discussed in Eqs. (3.1),(3.2) (with appropriate sign changes for $\nabla^{-}$, as mentioned above) leads to necessary and sufficient conditions on the $SU(2)$ structure. Specifically, we find

$$d(e^{-\Phi}K^i) = 0,$$
$$d(e^{-\Phi}\Omega) = 0,$$
$$dJ \wedge K^1 \wedge K^2 = 0. \tag{6.3}$$

with the flux given by

$$*H = -e^{2\Phi} d(e^{-2\Phi}J). \tag{6.4}$$

These geometries also possess an almost product structure

$$\Pi = 2K^1 \otimes K^{1\#} + 2K^2 \otimes K^{2\#} - \mathbb{1}, \tag{6.5}$$

where $K^{\#}$ is the vector field dual to the one-form $K$, satisfying $\Pi \cdot \Pi = \mathbb{1}$. Since $d(e^{-\Phi}K^i)=0$ this structure is integrable and hence the metric can be cast in the canonical form

$$ds^2 = g^4_{ab}(x,y)dx^a dx^b + e^{2\Phi(x,y)}\delta_{ij}dy^i dy^j. \tag{6.6}$$

The conditions (6.3) then imply that at fixed $y^i$, the $SU(2)$ structure on the four-manifold has $W_2 = W_4 = 0$ and $W_5 = d\Phi$. Such geometries, which in particular are Kähler, are called almost Calabi-Yau.

This case corresponds to fivebranes wrapping Kähler two-cycles in $CY_2$. This is mirrored in the expression for the flux (6.4), and also in the structure of the metric (6.6) with the $y$ directions corresponding to the two directions transverse to the fivebrane and the initial $CY_2$. Explicit examples of such solutions were presented in [44,45] and were further explored from the world-sheet point of view in [46].

*SU(3) geometries in d=7*. This case arises when $\nabla^{\pm}$ each have $G_2$ holonomy and was discussed in [5]. These geometries preserve two $d=7$ spinors, one $\epsilon^{+}$ and one $\epsilon^{-}$. The geometries have an $SU(3)$ structure in $d=7$ is specified by $J$ and $\Omega$ satisfying Eq. (2.1) for $n=3$, and a one-form $K$ such that

$$i_K\Omega = i_K J = 0. \tag{6.7}$$

The two $G_2$ structures are given by

$$\phi^{\pm} = J \wedge K \mp \mathrm{Im}\,\Omega \tag{6.8}$$

and demanding that they satisfy Eqs. (3.8),(3.9) (and their generalization for $\nabla^{-}$) leads to the necessary and sufficient conditions on the $SU(3)$ structure

$$d(e^{-\Phi}K) = 0,$$
$$d(e^{-\Phi}J) = 0,$$
$$d(e^{-\Phi}\mathrm{Re}\,\Omega) \wedge K = 0,$$
$$d(\mathrm{Im}\,\Omega) \wedge \mathrm{Im}\,\Omega = 0 \tag{6.9}$$

with the flux given by

$$*H = -e^{2\Phi} d(e^{-2\Phi}\mathrm{Im}\,\Omega). \tag{6.10}$$

The obvious almost product structure is again integrable and hence the metric can be cast in the canonical form

$$ds^2 = g^6_{ab}(x,y)dx^a dx^b + e^{2\Phi(x,y)}dy^2. \tag{6.11}$$

The six-dimensional slices at fixed $y$ have an $SU(3)$ structure with intrinsic torsion lying in $\mathcal{W}_2 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$, and it is straightforward to see that $W_4 = -W_5 = 2d\Phi$. Recall that for $SU(3)$ the module $\mathcal{W}_2$ splits into two modules $\mathcal{W}_2^{\pm}$. The third condition in Eq. (6.9) implies that while $W_2^{+}$ vanishes $W_2^{-}$ does not. These geometries are not Hermitian, as noted in [5]. This case corresponds to fivebranes wrapping SLAG three-cycles and explicit solutions were given in [7,47].

*SU(3) geometries in d=8*. This is one of the cases when $\nabla^{\pm}$ each have $SU(4)$ holonomy. These geometries preserve two pairs of $d=8$ spinors with opposite chirality, two $\epsilon^{+}$ and two $\epsilon^{-}$. It is in fact very similar to the case of an $SU(2)$ structure in $d=6$ considered above. The $SU(3)$ structure in $d=8$ is specified by $J,\Omega$ satisfying Eq. (2.1) for $n=3$ and two one-forms $K^i$ satisfying Eq. (6.1). The two $SU(4)$ structures are given by

$$J^{\pm} = J \pm K^1 \wedge K^2,$$
$$\Omega^{\pm} = \Omega \wedge (K^1 \pm i K^2). \tag{6.12}$$

Demanding that they satisfy the necessary and sufficient conditions for $SU(4)$ structures given in Eqs. (3.1),(3.2) (and their generalization for $\nabla^-$) leads to the differential conditions (6.3) which are the necessary and sufficient conditions on the $SU(3)$ structure. The flux is given by

$$*H=-e^{2\Phi}d\left(e^{-2\Phi}\frac{1}{2}J\wedge J\right). \tag{6.13}$$

Again there is an integrable product structure and the metric can be written in the form

$$ds^2=g^6_{ab}(x,y)dx^a dx^b+e^{2\Phi(x,y)}\delta_{ij}dy^i dy^j. \tag{6.14}$$

At fixed $y^i$, the $SU(3)$ structure on the six-manifold is almost Calabi-Yau, with the only non-vanishing class being $W_5=d\Phi$. This case corresponds to fivebranes wrapping Kähler four-cycles in $CY_3$ and solutions were found in [46,48].

*SU(2)×SU(2) geometries in d=8*. The second way that $\nabla^\pm$ both have $SU(4)$ holonomy is when they give a common $SU(2)\times SU(2)$ structure. These geometries preserve four $d=8$ spinors with the same chirality, two $\epsilon^+$ and two $\epsilon^-$. The two orthogonal $SU(2)$ structures $J^A$ and $J'^A$ satisfy the conditions (2.30). The two $SU(4)$ structures are given by

$$J^\pm=J^3\pm J'^3,$$

$$\Omega^\pm=\begin{cases}\Omega\wedge\Omega', \\ \Omega\wedge\bar{\Omega}'\end{cases} \tag{6.15}$$

where e.g. $\Omega=J^2+iJ^1$. Demanding that they satisfy the necessary and sufficient conditions for $SU(4)$ structures given in Eqs. (3.1),(3.2) (and their generalization for $\nabla^-$) leads to the necessary and sufficient conditions on the $SU(2)\times SU(2)$ structure given by

$$\mathrm{vol}'\wedge dJ^3=0,$$

$$d(e^{-\Phi}J^A)=0 \quad \text{for} \quad A=1,2,$$

$$\mathrm{vol}\wedge dJ'^3=0,$$

$$d(e^{-\Phi}J'^A)=0 \quad \text{for} \quad A=1,2, \tag{6.16}$$

where e.g. vol$=\frac{1}{2}J^3\wedge J^3$. The flux is given by

$$*H=-e^{2\Phi}d(e^{-2\Phi}J^3\wedge J'^3). \tag{6.17}$$

The almost product structure

$$\Pi=J^+\cdot J^-=J^3\cdot J^3-J'^3\cdot J'^3 \tag{6.18}$$

is integrable[4] since $\nabla^\pm J^\pm=0$, $J^\pm$ commute and $J^\pm$ are integrable (see Appendix C) and implies the canonical form of the metric

$$ds^2=g^4_{ij}(x,y)dx^i dx^j+g'^4_{ab}(x,y)dy^a dy^b, \tag{6.19}$$

each block being $4\times4$. The four-dimensional slices each have an $SU(2)$ structure, with $W_2=W_4=0$ and $W_5=d\Phi$ at any point in their transverse directions. These geometries arise when a fivebrane wraps a two-cycle in one Calabi-Yau two-fold and a second two-cycle in a second Calabi-Yau two-fold and it would be interesting to find explicit examples.

*Sp(2) geometries in d=8*. This case arises when $\nabla^+$ has $SU(4)$ holonomy while $\nabla^-$ has *Spin*(7) holonomy. These geometries preserve three $d=8$ spinors of the same chirality, two $\epsilon^+$ and one $\epsilon^-$. We have a *Sp*(2) structure given by a triplet of complex structures satisfying Eqs. (2.25). The $SU(4)$ structure is given by $(J^3,\Omega^3)$, where

$$\Omega^3=\frac{1}{2}J^2\wedge J^2-\frac{1}{2}J^1\wedge J^1+i(J^1\wedge J^2), \tag{6.20}$$

and satisfies Eq. (3.1) while the *Spin*(7) structure is defined by

$$\Psi=\frac{1}{2}J^1\wedge J^1+\frac{1}{2}J^2\wedge J^2-\frac{1}{2}J^3\wedge J^3 \tag{6.21}$$

which satisfies Eq. (3.5) (with appropriate sign changes). This leads to the following necessary and sufficient conditions on the *Sp*(2) structure:

$$d(e^{-\Phi}J^A)=0 \quad \text{for} \quad A=1,2,$$

$$d^\dagger(e^{-2\Phi}J^3)=0, \tag{6.22}$$

with the flux given by

$$*H=-e^{2\Phi}d(e^{-2\Phi}\mathrm{Re}\,\Omega^1)=e^{2\Phi}d(e^{-2\Phi}\mathrm{Re}\,\Omega^2)$$
$$=-e^{2\Phi}d\left(e^{-2\Phi}\frac{1}{2}J^3\wedge J^3\right). \tag{6.23}$$

Note that the conditions imply that the two **16**'s in each of $dJ^1$ and $dJ^2$ vanish. Moreover, the six independent Lee forms are given by

[4]Note that the existence of a generic pair $J^\pm$ of integrable complex structures satisfying only $[J^+,J^-]=0$ does not guarantee that the almost product structure $\Pi=J^+\cdot J^-$ is integrable. A concrete counterexample is discussed in Appendix C.

$$L^{11}=3d\Phi,\quad L^{22}=3d\Phi,\quad L^{33}=2d\Phi,$$
$$L^{12}-L^{21}=-2J^3\cdot d\Phi,\quad L^{31}-L^{13}=-J^2\cdot d\Phi,$$
$$L^{23}-L^{32}=-J^1\cdot d\Phi. \tag{6.24}$$

It is worth emphasizing that the intrinsic torsion of this $Sp(2)$ structure is not totally anti-symmetric, and hence the geometry is not HKT. These geometries arise when NS five-branes wrap complex-Lagrangian (C-LAG) four-cycles in hyper-Kähler eight-manifolds. Recall that these cycles are complex with respect to one complex structure and special Lagrangian with respect to the remaining two. It would be interesting to find explicit examples (note that a promising ansatz and the corresponding BPS equations for this case were given in [48]).

*SU(4) geometries in d=8.* This is the first case when $\nabla^{\pm}$ each have $Spin(7)$ holonomy. These geometries preserve two $d=8$ spinors with the same chirality, one $\epsilon^+$ and one $\epsilon^-$. In this case we have an $SU(4)$ structure $J,\Omega$ satisfying Eq. (2.1) for $n=4$. The two $Spin(7)$ structures are given by

$$\Psi^{\pm}=\frac{1}{2}J\wedge J\pm \mathrm{Re}\,\Omega \tag{6.25}$$

and satisfy Eq. (3.5) (with sign changes for $\nabla^-$) leading to the following necessary and sufficient conditions on the $SU(4)$ structure:

$$d(e^{-\Phi}J)=0,$$
$$*(*d\,\mathrm{Re}\,\Omega\wedge\mathrm{Re}\,\Omega)=-6d\Phi, \tag{6.26}$$

with the flux given by

$$*H=-e^{2\Phi}d(e^{-2\Phi}\mathrm{Re}\,\Omega). \tag{6.27}$$

The intrinsic torsion of the $SU(4)$ structure lies in $\mathcal{W}_2\oplus\mathcal{W}_4\oplus\mathcal{W}_5$, with $2W_4=W_5=6d\Phi$, and so in particular the geometries are not Hermitian. These geometries arise when fivebranes wrap SLAG four-cycles in $CY_4$ and it would be interesting to find explicit examples. (Again, a promising ansatz and the corresponding BPS equations for this case were given in [48].)

*$G_2$ geometries in d=8.* This is the second case when $\nabla^{\pm}$ each have $Spin(7)$ holonomy. These geometries preserve one $\epsilon^+$ and one $\epsilon^-$ $d=8$ spinor of opposite chirality. In this case the two $Spin(7)$ structures give rise to a $G_2$ structure in $d=8$ with $\phi$ as in Eq. (2.16) and a one-form $K$ satisfying

$$i_K\phi=0. \tag{6.28}$$

The two $Spin(7)$ structures are given by

$$\Psi^{\pm}=-i_K*\phi\pm\phi\wedge K \tag{6.29}$$

and satisfy Eq. (3.5) (and sign changes for $\nabla^-$) leading to the necessary and sufficient conditions on the $G_2$ structure:

$$d(e^{-\Phi}K)=0, \tag{6.30}$$
$$d(e^{-\Phi}\phi)\wedge K=0, \tag{6.31}$$
$$*d(i_K*\phi)\wedge\phi\wedge K=0, \tag{6.32}$$
$$*d(i_K*\phi)\wedge i_K*\phi=4*d\Phi, \tag{6.33}$$

with flux given by

$$*H=e^{2\Phi}d(e^{-2\Phi}i_K*\phi). \tag{6.34}$$

The intrinsic torsion of the $G_2$ structure lies in $\mathcal{W}_2\oplus\mathcal{W}_4$ with $W_4=-4d\Phi$. This means one cannot introduce a $G_2$ Dolbeault cohomology [28]. These geometries arise when fivebranes wrap co-associative four-cycles in $G_2$ manifolds and it would be interesting to find explicit examples. (Again a promising ansatz and the corresponding BPS equations for this case were given in [48].)

*{1} geometries.* For completeness let us briefly mention the case corresponding to the first entry in Table II. This case has two different $SU(2)$ structures each satisfying Eq. (2.1) giving a trivial structure defined by four real one-forms $K^i$. A little calculation reveals that this case can always be put in the canonical form

$$ds^2=e^{2\Phi}ds^2(\mathbb{R}^4),$$
$$*H=-e^{2\Phi}d(e^{-2\Phi}), \tag{6.35}$$

which is just the transverse space to the simple fivebrane solution.

We conclude this section with two comments. First, considering either set of $\epsilon^+$ or $\epsilon^-$ Killing spinors we see that the geometries of this section are special cases of those appearing in Sec. III. It is then clear, from the results of Sec. IV, that supersymmetric geometries in $d=9$ can be obtained by fibering an appropriate number of flat directions over the geometries in this section. In order that the same amount of supersymmetry is preserved, the fibrations are determined by Abelian instantons that satisfy the generalized self-duality conditions for both of the $G^{\pm}$ structures. In other words they must satisfy the generalized self-duality conditions for the maximal common subgroup $G$. Note that in general the Bianchi identity for $H$ may further restrict which fibrations are possible. For instance in the cases where both $\nabla^+$ and $\nabla^-$ have $SU(n+1)$ holonomy, one can show that $dH$ has no components transforming as a four-form under $SO(2n)\supset SU(n)$ for the common $SU(n)$ structure. As such, there are in fact no solutions with non-trivial twisting.

The second comment is to note that we have considered only structures $G^{\pm}$ that are orthogonal in the sense that preserved spinors $\epsilon^+$ and $\epsilon^-$ are orthogonal, that is $\overline{\epsilon^+}\epsilon^-=0$. In fact, as we now show, this is a necessary condition for a non-trivial solution to be supersymmetric. Take any two Killing spinors $\epsilon^+$ and $\epsilon^-$. The vanishing of the gravitini variations implies that

$$\nabla_m(\overline{\epsilon^+}\epsilon^-)=\frac{1}{4}H_{mab}\overline{\epsilon^+}\gamma^{ab}\epsilon^-. \tag{6.36}$$

The dilatino equation implies that for any gamma-matrix operator $A$ we have

$$\partial_m \Phi \overline{\epsilon^+}[A,\gamma^m]_\pm \epsilon^- = \frac{1}{12} H_{mnp} \overline{\epsilon^+}[A,\gamma^{mnp}]_\pm \epsilon^- . \quad (6.37)$$

Taking $A=\gamma^m$ and using the upper sign, we conclude that

$$\nabla_m(\overline{\epsilon^+}\epsilon^-) = \partial_m \Phi(\overline{\epsilon^+}\epsilon^-). \quad (6.38)$$

This is trivially satisfied if the $G^\pm$ structures are orthogonal since then $\overline{\epsilon^+}\epsilon^- = 0$. If the structures are not orthogonal, we have some point where $\overline{\epsilon^+}\epsilon^-$ is non-zero and then by continuity there will be a neighborhood in which it is non-zero. In this neighborhood we have $\overline{\epsilon^+}\epsilon^- = e^{\Phi+\Phi_0}$, for some constant $\Phi_0$.

The two spinors $\epsilon^\pm$ define a pair of $G^\pm$ structures both of which are sub-bundles of the same $SO(d)$ bundle of orthonormal frames defined by the metric $g_d$.[5] Together $\epsilon^\pm$ define a common $G$ structure sub-bundle of the two $G^\pm$ structures. Furthermore, there always exists *some* metric-compatible connection $\tilde{\nabla}$ that preserves this $G$ structure. (Note that this connection generically does not have totally antisymmetric torsion.) Necessarily it preserves the $G^\pm$ structures, so that $\tilde{\nabla}\epsilon^\pm = 0$. Thus in fact we have $\nabla(\overline{\epsilon^+}\epsilon^-) = \tilde{\nabla}(\overline{\epsilon^+}\epsilon^-) = 0$ implying $\Phi$ is a constant. However, the equations of motion then imply that $H$ is constant. We thus conclude that there are no supersymmetric solutions with non-vanishing flux when the structures $G^\pm$ are not orthogonal.

## VII. EXPLICIT EXAMPLES II

In this section, we present some further explicit solutions in $d=6$, some preserving both $\epsilon^+$ and $\epsilon^-$ supersymmetries, for the type II theories, including a solution that preserves the unusual fraction of 12/32 supersymmetry. The basic solutions have two flat directions fibered over a four dimensional base space, with the fibration being specified by two Abelian instantons on the base, and thus generalize those discussed in Sec. V. We shall also discuss compact heterotic geometries in $d=6$ preserving both eight and four supercharges.

It will be convenient in this section to distinguish different six-dimensional solutions by the number of preserved supersymmetries. Let us start with the most supersymmetric case corresponding to a flat NS five-brane as discussed at the end of the last section. Recall that the $d=4$ solution transverse to a simple fivebrane (6.35) preserves eight $\epsilon^+$ spinors and eight $\epsilon^-$ spinors satisfying the projections

$$\gamma^{1234}\epsilon^+ = -\epsilon^+, \quad \gamma^{1234}\epsilon^- = +\epsilon^- . \quad (7.1)$$

As previously noted $\nabla^\pm$ have $SU(2)^\pm$ holonomy in $SO(4) = SU(2)^+ \times SU(2)^-$ with the maximal common subgroup being the identity. We can trivially lift this to a six-dimensional solution by adding two extra flat directions. This still preserves 16 supercharges corresponding to $\mathcal{N}=4$ supersymmetry in the remaining four spacetime dimensions.

We now twist the two flat directions, as in Sec. V, with two Abelian instantons,

$$ds^2 = e^{2\Phi} d\tilde{s}^2 + (dy+B^1)^2 + (dz+B^2)^2,$$

$$H_{mnp} = -\tilde{\epsilon}_{mnp}{}^q \tilde{\nabla}_q e^{2\Phi} + 3B^1_{[m}G^1_{np]} + 3B^2_{[m}G^2_{np]},$$

$$H_{mny} = G^1_{mn}, \quad H_{mnz} = G^2_{mn}, \quad (7.2)$$

giving the dilaton equation

$$\tilde{\nabla}^2 e^{2\Phi} = -\frac{1}{2}[(\tilde{G}^1)^2 + (\tilde{G}^2)^2], \quad (7.3)$$

where $m,n=1,\ldots,4$ and now $G^i = dB^i$ are taken to be *self-dual* instantons on the $\mathbb{R}^4$ base space $d\tilde{s}^2$. This twisting still preserves eight $\epsilon^-$ spinors so that $\nabla^-$ still has $SU(2)^-$ holonomy. For non-zero $G^i$, generically the solution breaks all of the $\epsilon^+$ supersymmetry however. [Note that, simply for convenience of later discussion, we have exchanged the roles of $\nabla^+$ and $\nabla^-$, by taking $H \to -H$ and changing the orientation on the base, as compared to the discussion in Sec. V. There we took anti-self-dual instantons so that $\epsilon^+$ spinors were preserved. This accounts for the difference in signs of the terms involving $B$ and $G$ in Eq. (7.2) compared to those in Eq. (5.1).] Hence, generically these solutions preserve $\mathcal{N}=2$ supersymmetry in the remaining four spacetime dimensions.

Interestingly, it is nonetheless possible to preserve four $\epsilon^+$ Killing spinors corresponding to $\nabla^+$ having $SU(3)$ holonomy, for suitably chosen non-generic instantons. To see this we define an $SU(3)$ structure by

$$J = e^{2\Phi}\tilde{J} + (dy+B^1)\wedge(dz+B^2),$$

$$\Omega = e^{2\Phi}\tilde{\Omega}\wedge[(dy+B^1)+i(dz+B^2)], \quad (7.4)$$

where $\tilde{J} = dx^1\wedge dx^2 + dx^3\wedge dx^4$ and $\tilde{\Omega} = (dx^1 + i dx^2)\wedge(dx^3 + i dx^4)$ define the $SU(2)^+$ structure on $\mathbb{R}^4$. Demanding that the $SU(3)$ structure satisfies the conditions for supersymmetry (3.1), we find that

$$\tilde{J} \lrcorner G^i = 0,$$

$$\tilde{\Omega} \lrcorner (G^1 + iG^2) = 0. \quad (7.5)$$

The generic constant flux solution to these equations is given by

$$G^1 + iG^2 = k\tilde{\Omega} \quad (7.6)$$

for some complex constant $k$. (Note that, as we discuss below, this is the same twisting that appears in the Iwasawa manifold analyzed in [17].) The Bianchi identity then implies the equation for the dilaton

$$\tilde{\nabla}^2 e^{2\Phi} = -8|k|^2, \quad (7.7)$$

[5] Note that this is true only for $D^\pm\epsilon^\pm=0$ with $D^\pm$ a pair of spin connections, compatible with the metric $g_d$, and not, for instance, if $D^\pm$ are general Clifford connections.

which can easily be solved. To summarize, the solution (7.2) with flat base space will preserve eight $\epsilon^-$ and four $\epsilon^+$ spinors for the specific choice of self-dual instantons (7.6) and dilaton satisfying (7.7).

A number of comments are now in order. First, this special solution corresponds to $\mathcal{N}=3$ supersymmetry in the remaining four spacetime dimensions. It would be interesting to relate this solution to those discussed in [23].

Second, the holonomy of the connections $\nabla^\pm$ for the special solution are $SU(3)$ and $SU(2)$, respectively. This is not a combination appearing in Table II. The form of the solution indicates that this solution is related to fivebranes wrapping two flat directions, but a world-volume interpretation of the twisting and preservation of supersymmetry are obscure to us at present.

Third, this special background is also a heterotic or type I solution. In this case, one loses the $\epsilon^-$ supersymmetries and the solution preserves only four $\epsilon^+$ spinors, and so has $\mathcal{N}=1$ supersymmetry in four dimensions. Including additional heterotic instantons simply adds to the source $|k|^2$ in the dilaton equation (7.7). Note that by taking $H\rightarrow -H$ and switching the orientation of the base, we switch $\epsilon^+$ and $\epsilon^-$ and hence we can also obtain a heterotic solution from the generic solution (7.2) with an $SU(2)$ structure and $\mathcal{N}=2$ supersymmetry.

Finally, the metric and three-form obtained by setting the dilaton constant in Eq. (7.2) with $G^1+iG^2=k\tilde{\Omega}$, were first considered in the heterotic case [including an additional Abelian instanton embedded in $E_8\times E_8$ or $SO(32)$] in [17]. There it was demonstrated that the conditions for the preservation of $\epsilon^+$ supersymmetry with $\nabla^+$ having $SU(3)$ holonomy were satisfied. However, given the analysis here, the background in [17] is problematic for the following somewhat subtle reason. As we have already noted when the dilaton is constant and $H\neq 0$, the leading-order type II (or heterotic or type I) equations of motion are not satisfied. As shown in [5], these equations of motion are a direct consequence of the preservation of supersymmetry once the Bianchi identity (3.23) is imposed [or equivalently (3.24) if $\mathrm{tr}\,R\wedge R=0$ as for the geometry considered in [17].] This contradiction is resolved by the fact that the background in [17] actually satisfies a Bianchi identity with the opposite sign to the one arising in type I supergravity. This discrepancy is probably related to the sign discrepancy between the expression (3.3) and the corresponding expression in [2].[6]

The type II solutions we have been discussing can also be generalized by replacing the flat space in Eq. (7.2) with a generic Calabi-Yau two-fold $CY_2$. As usual for type II, the Calabi-Yau two-fold cannot be compact in order to satisfy the Bianchi identity $dH=0$. If we take the orientation of the $CY_2$ to be such that the complex structures are self-dual, we impose the projections $\gamma^{1234}\epsilon^\pm=-\epsilon^\pm$. In this case, the solution preserves no $\epsilon^-$ supersymmetry, and generically no $\epsilon^+$ supersymmetry. However, choosing $G^1+iG^2=k\tilde{\Omega}$, where $\tilde{\Omega}$ is the holomorphic (2,0) form on $CY_2$, we find that $\nabla^+$ has $SU(3)$ holonomy and the solution still preserves four $\epsilon^+$ supersymmetries, corresponding to $\mathcal{N}=1$ supersymmetry in four dimensions. Alternatively, if the orientation of the $CY_2$ is chosen so that the complex structures are anti-self-dual, we impose the projections $\gamma^{1234}\epsilon^\pm=+\epsilon^\pm$. These solutions break all of the $\epsilon^+$ supersymmetry, but preserve eight $\epsilon^-$ spinors. The latter choice of orientation corresponds (after exchanging $\epsilon^+$ with $\epsilon^-$ by taking $H\rightarrow -H$ and switching the orientation on the base) to a simple generalization from $d=5$ to $d=6$ of the solutions discussed in Sec. V and explicitly obtained in [41] for the cases of Taub-NUT and Eguchi-Hanson space. The former choice of orientation on the other hand, gives a new kind of supersymmetric solution that exploits the fact that one is twisting two flat directions and not just one as considered in [41].

Similarly, one can obtain heterotic and type I geometries preserving $\mathcal{N}=1,2$ supersymmetry. By taking the flat directions to be a two-torus, and $M_0$ to to be either conformally $T^4$ or conformally $K3$, we get compact and supersymmetric heterotic geometries. It will be interesting to see whether it is possible to solve the heterotic Bianchi identity for these geometries; if it is, as in Sec. V, the $\mathrm{tr}\,R\wedge R$ contribution will be essential. In addition, one should again find that the radius of the two-torus is required to be of order the string scale and that several of the moduli are fixed.

## VIII. DISCUSSION

In this paper we have studied the necessary and sufficient conditions for static geometries of type I or heterotic string theory, or type II theories with only non-vanishing NS-NS fields, to preserve supersymmetry and solve the equations of motion. The Killing spinors define $G$ structures on the geometries and we determined the intrinsic torsion of the $G$ structure. We emphasized the universal expression for the three-form flux in terms of generalized calibrations and the connection with wrapped branes, following [7,5].

The geometries always have a connection with totally anti-symmetric torsion, $\nabla^+$ (or $\nabla^-$ for the type II theories), which has special holonomy. We first discussed the geometries in the canonical dimension for the special holonomy group, $d=8$ for $Spin(7)$, $d=7$ for $G_2$, etc. We then showed that the most general geometries in $d=9$ have a number of flat directions fibered over these geometries in the canonical dimensions, with the fibration being determined by Abelian generalized instantons. We also discussed the physical interpretation of these geometries in terms of wrapped fivebranes. For example, the eight-dimensional geometries with a single flat dimension fibered over a seven-dimensional geometry with $G_2$ structure correspond to fivebranes wrapping supersymmetric cycles of the form $S^1\times\Sigma_3\subset S^1\times M_{G_2}$ where $\Sigma_3\subset M_{G_2}$ is an associative three-cycle in a $G_2$ holonomy manifold. The fact that the resulting eight-dimensional geometry is not necessarily a direct product of $S^1$ with a seven-dimensional geometry is worth further investigation. We presented some explicit examples, that would be worth studying further and generalizing.

[6]Following recent correspondence the authors of [17] have independently confirmed this discrepancy in [2].

These results provide a comprehensive classification of all of the supersymmetric static geometries of the heterotic or type I theory. For the type II theories, we also analyzed the geometries that arise when both connections $\nabla^{\pm}$ have special holonomy. Our analysis covers all cases of NS fivebranes wrapping calibrated cycles, as listed in Tables I and II.

We also presented an explicit solution with a torus $T^2$ fibered over an $\mathbb{R}^4$ base with $\nabla^+$ having $SU(3)$ holonomy and $\nabla^-$ having $SU(2)$ holonomy. This solution has four $\epsilon^+$ Killing spinors and eight $\epsilon^-$ spinors. The form of the flux suggests that the solution should be interpreted as a flat fivebrane with two of the world-volume directions further wrapped on the two-torus. Naively, one would therefore expect 8 plus 8 Killing spinors and so it would also be interesting to find a physical interpretation of the twisting which leads to this reduction of supersymmetry. In [23] type II solutions on $T^6$ orientifolds with non-vanishing RR and NS-NS fluxes were presented that also preserve 12 Killing spinors and it would be interesting to see if they are related.

Candidate heterotic compactifications in $d=6$ were also presented, preserving both four and eight supersymmetries. They are based on manifolds which are fibrations of $T^2$ over a $K3$ base. The models with four supersymmetries arise for non-generic complex structure on the $K3$ and there are additional constraints on the radii of the circles of the torus. This indicates that many moduli are fixed. We showed that the size of the torus is necessarily of order the string scale, indicating that the supergravity approximation is breaking down. One would also have to check the equations for motion are satisfied. To pursue these models further one might aim to construct a conformal field theory description. It would also be interesting to relate our compactifications to those of [49–52].

We have emphasized that the expression for the three-form flux is easy to understand as a generalized calibration since the geometry should still admit fivebranes wrapping the corresponding cycles. It is very interesting to note that many, and in some cases all, of the other conditions constraining the intrinsic torsion can be interpreted in the same way. For example, consider the case of the $SU(3)$ structure with only $\epsilon^+$ Killing spinors. The expression for the flux (3.2) is the general calibration condition for a fivebrane wrapping a Kähler two-cycle in a Calabi-Yau three-fold. In addition the intrinsic torsion is constrained to satisfy Eq. (3.1). Suppose we consider the trivial product of our $SU(3)$ manifold $M_6$ with a torus $T^2$. Let $K^1=dy^1$ and $K^2=dy^2$ represent the extra directions. The full set of conditions on the structure can then be written on the eight-dimensional space $M_6\times T^2$ as

$$\begin{aligned} d[e^{-2\Phi}J\wedge J]&=0,\\ d[e^{-2\Phi}\Omega\wedge(K^1+iK^2)]&=0,\\ d[e^{-2\Phi}J\wedge K^1\wedge K^2]&=-e^{-2\Phi}{*H}. \end{aligned} \tag{8.1}$$

Given that $H$ lies solely in $M_6$, we see that all three expressions are calibration conditions of the form $*H=e^{2\Phi}d(e^{-2\Phi}\Xi)$ just for wrapping different cycles. The first is for a fivebrane wrapping a Kähler four-cycle in the Calabi-Yau, the second for wrapping a special Lagrangian cycle (and one of the $K^i$ directions), while the last is the familiar expression for the wrapping of a Kähler two-cycle in the Calabi-Yau together with the torus $T^2$. This is physically reasonable, since the geometry $M_6\times T^2$, corresponding to the full back reaction solution around a brane wrapping a Kähler two-cycle, should still admit probe branes wrapping the special Lagrangian three- and Kähler four-cycles. Similar arguments extend to the fibration cases in Sec. IV and the geometries with $\epsilon^+$ and $\epsilon^-$ in Sec. VI.

An important motivation for this work is that a good understanding of the geometry underlying supergravity configurations might allow us to find new explicit solutions. Indeed for the cases listed in Table I a co-homogeneity one ansatz is useful for finding solutions [5]. This is a practical alternative to finding solutions describing wrapped fivebranes using the gauge supergravity approach initiated in [53]. For the cases in Table II, on the other hand, a simple generalization of this technique can lead to co-homogeneity one but also to a co-homogeneity two or more ansatz, and progress in the latter case is much more difficult [5]. At present the gauge supergravity approach is the best available tool to produce solutions for these latter cases. It should be noted, however, that since the configurations in Table II preserve more supersymmetry than those in Table I, one expects that with new techniques, ultimately, they could be easier to analyze.

Finally, it is natural to generalize this work to also include RR fields in the type II theories, as well as to consider Lorentzian geometries. Such geometries will allow one to describe both wrapped NS and D-branes, as well as *pp* waves and general non-static backgrounds. Based on this work and on [13] we expect generalized calibrations to play an important role.

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## APPENDIX A: EQUATIONS OF MOTION

The low-energy effective action for heterotic or type I string theory is given by the type I supergravity action

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}H^2-\alpha'\operatorname{Tr}F^2\right] \tag{A1}$$

where $F$ is in the adjoint of $SO(32)$ or $E_8\times E_8$. In type I supergravity the three-form $H$ satisfies a modified Bianchi identity

$$dH=2\alpha'\operatorname{Tr}F\wedge F. \tag{A2}$$

Including the leading order string correction from anomaly cancellation we get

$$dH = 2\alpha' (\mathrm{Tr}\, F\wedge F - \mathrm{tr}\, R\wedge R) \tag{A3}$$

but to fully consistently implement this one should also include modifications to the action. The equations of motion coming from (A1) are given by

$$R_{MN} - \frac{1}{4} H_{MRS} H_N{}^{RS} + 2\nabla_M \nabla_N \Phi - 2\alpha' \mathrm{Tr}\, F_M{}^R F_{NR} = 0, \tag{A4a}$$

$$\nabla^2 (e^{-2\Phi}) - \frac{1}{6} e^{-2\Phi} H_{MNR} H^{MNR} - \alpha' e^{-2\Phi} \mathrm{Tr}\, F_{MN} F^{MN} = 0, \tag{A4b}$$

$$\nabla_M (e^{-2\Phi} H^{MNR}) = 0, \tag{A4c}$$

$$D^M (e^{-2\Phi} F_{MN}) - \frac{1}{2} e^{-2\Phi} F^{RS} H_{RSN} = 0. \tag{A4d}$$

The action and equations of motion for the type II theories with all RR fields set to zero are obtained by simply setting the gauge field $F$ to zero and using the Bianchi identity $dH = 0$.

## APPENDIX B: SPINOR AND $G$ STRUCTURE CONVENTIONS

In doing calculations it is often useful to have an explicit set of projections defining the Killing spinors and the corresponding $G$ structures. Here we define one possible set of conventions consistent with the expressions given in the paper. In particular, we will use the same set of projectors (or subset of them) to define the invariant spinors in all cases. Specifically, the Killing spinors will be defined by their $\pm 1$ eigenvalues for the set of commuting gamma matrices

$$\gamma^{1234}, \gamma^{5678}, \gamma^{1256}, \gamma^{1357}. \tag{B1}$$

We concentrate on the cases of $G$ structure in canonical dimension. However, in each case we also give how the structure embeds in the next simplest structure group following Fig. 1. Using these embeddings one can obtain conventions for any of the $G$ structures in arbitrary dimensions $d \leqslant 9$.

Note that in all dimensions the gamma matrix algebra is taken to be $\{\gamma_m, \gamma_n\} = 2\delta_{mn}$ and the adjoint spinor is written as $\bar{\epsilon}$ and the conjugate spinor as $\epsilon^{\mathrm{c}}$. We always normalize the Killing spinors to satisfy $\bar{\epsilon}\epsilon = 1$.

*Spin(7)*. In eight dimensions, a $Spin(7)$ structure defines a single real chiral invariant spinor $\epsilon$. For definiteness, we choose $\gamma_{1,\ldots,8}\epsilon = \epsilon$. A possible set of independent, commuting projections defining an $\epsilon$ are

$$\gamma^{1234}\epsilon = \gamma^{5678}\epsilon = \gamma^{1256}\epsilon = \gamma^{1357}\epsilon = -\epsilon. \tag{B2}$$

Writing the Cayley four-form $\Psi$ as

$$\Psi_{mnpq} = -\bar{\epsilon}\gamma_{mnpq}\epsilon \tag{B3}$$

then matches the expression (2.11). The corresponding volume form is given by

$$\mathrm{vol}_{m_1,\ldots,m_8} = \bar{\epsilon}\gamma_{m_1,\ldots,m_8}\epsilon, \tag{B4}$$

Note that one can always choose a real basis for the gamma matrices so that $\bar{\epsilon} = \epsilon^{\mathrm{T}}$. The conventions for lifting a $Spin(7)$ structure to $d=9$ are given in Sec. IV A.

*SU*(4). An $SU(4)$ structure leaves invariant two real orthogonal spinors $\epsilon_{(a)}$ with $a=1,2$ of the same chirality in $d=8$. These can be defined by

$$\gamma^{1234}\epsilon_{(a)} = \gamma^{5678}\epsilon_{(a)} = \gamma^{1256}\epsilon_{(a)} = -\epsilon_{(a)} \tag{B5}$$

with

$$\gamma^{1357}\epsilon_{(1)} = +\epsilon_{(1)}, \quad \gamma^{1357}\epsilon_{(2)} = -\epsilon_{(2)}. \tag{B6}$$

Defining a complex spinor $\eta = (1/\sqrt{2})(\epsilon_{(1)} + i\epsilon_{(2)})$, the forms $J$ and $\Omega$ can then be written as

$$J_{mn} = -i\bar{\eta}\gamma_{mn}\eta,$$

$$\Omega_{mnpq} = \bar{\eta}^{\mathrm{c}}\gamma_{mnpq}\eta. \tag{B7}$$

Note that in the basis where $\bar{\epsilon} = \epsilon^{\mathrm{T}}$, we have the more familiar expressions $J_{mn} = i\eta^{\dagger}\gamma_{mn}\eta$ and $\Omega_{mnpq} = \eta^{\mathrm{T}}\gamma_{mnpq}\eta$. Given $\gamma^{12}\epsilon_{(1)} = -\epsilon_{(2)}$ we get the standard expressions

$$J = e^{12} + e^{34} + e^{56} + e^{78},$$

$$\Omega = (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6)(e^7 + ie^8). \tag{B8}$$

The corresponding volume form is given by Eq. (B4) as above. Note that each real spinor $\epsilon_{(a)}$ also defines a corresponding $Spin(7)$ structure as in Eq. (B3) given by

$$\Psi^{(1)} = \frac{1}{2} J\wedge J - \mathrm{Re}\,\Omega,$$

$$\Psi^{(2)} = \frac{1}{2} J\wedge J + \mathrm{Re}\,\Omega. \tag{B9}$$

*Sp(2).* We now have three real orthogonal invariant spinors $\epsilon_{(a)}$ with $a=1,2,3$ of the same chirality in $d=8$. These can be defined by

$$\gamma^{1234}\epsilon_{(a)} = \gamma^{5678}\epsilon_{(a)} = (\gamma^{1256} + \gamma^{1357} + \gamma^{1458})\epsilon_{(a)} = -\epsilon_{(a)} \tag{B10}$$

with

$$\gamma^{1256}\epsilon_{(1)} = +\epsilon_{(1)}, \quad \gamma^{1458}\epsilon_{(2)} = +\epsilon_{(2)}, \quad \gamma^{1357}\epsilon_{(3)} = +\epsilon_{(3)}. \tag{B11}$$

Note that the eigenvalues under $(\gamma^{1256}, \gamma^{1357}, \gamma^{1458})$ of $\epsilon_{(a)}$ are $(+1,-1,-1)$, $(-1,-1,+1)$ and $(-1,+1,-1)$ for $a=1,2,3$ respectively. The three two-forms $J^A$ are then given by

$$J^1_{mn} = -\overline{\epsilon_{(2)}}\gamma_{mn}\epsilon_{(3)},$$

$$J^2_{mn} = -\overline{\epsilon_{(3)}}\gamma_{mn}\epsilon_{(1)},$$

$$J^3_{mn} = -\overline{\epsilon_{(1)}}\gamma_{mn}\epsilon_{(2)}. \tag{B12}$$

Given $\gamma^{12}\epsilon_{(2)} = \gamma^{56}\epsilon_{(2)} = \epsilon_{(3)}$, $\gamma^{14}\epsilon_{(3)} = \gamma^{58}\epsilon_{(3)} = \epsilon_{(1)}$, and $\gamma^{13}\epsilon_{(1)} = \gamma^{57}\epsilon_{(1)} = \epsilon_{(2)}$, we have the explicit expressions

$$J^1 = e^{12} + e^{34} + e^{56} + e^{78},$$

$$J^2 = e^{14} + e^{23} + e^{58} + e^{67},$$

$$J^3 = e^{13} + e^{42} + e^{57} + e^{86}. \tag{B13}$$

The corresponding volume form is given by Eq. (B4) as above. Note that each almost complex structure $J^A$ as an $SU(4)$ structure has a corresponding (4,0)-form $\Omega^A$ given by

$$\Omega^1 = \frac{1}{2}J^3\wedge J^3 - \frac{1}{2}J^2\wedge J^2 + iJ^2\wedge J^3,$$

$$\Omega^2 = \frac{1}{2}J^1\wedge J^1 - \frac{1}{2}J^3\wedge J^3 + iJ^3\wedge J^1,$$

$$\Omega^3 = \frac{1}{2}J^2\wedge J^2 - \frac{1}{2}J^1\wedge J^1 + iJ^1\wedge J^2. \tag{B14}$$

Each spinor $\epsilon_{(a)}$ also defines a corresponding $Spin(7)$ structure given by

$$\Psi^{(1)} = \frac{1}{2}J^2\wedge J^2 + \frac{1}{2}J^3\wedge J^3 - \frac{1}{2}J^1\wedge J^1,$$

$$\Psi^{(2)} = \frac{1}{2}J^3\wedge J^3 + \frac{1}{2}J^1\wedge J^1 - \frac{1}{2}J^2\wedge J^2,$$

$$\Psi^{(3)} = \frac{1}{2}J^1\wedge J^1 + \frac{1}{2}J^2\wedge J^2 - \frac{1}{2}J^3\wedge J^3. \tag{B15}$$

*SU(2)×SU(2)*. We now have four orthogonal, real invariant spinors all of the same chirality in $d=8$. They can be defined by

$$\gamma^{1234}\epsilon_{(a)} = \gamma^{5678}\epsilon_{(a)} = -\epsilon_{(a)} \tag{B16}$$

with

$$\gamma^{1256}\epsilon_{(a)} = \begin{cases} -\epsilon_{(a)} & \text{for } a=2,3, \\ +\epsilon_{(a)} & \text{for } a=1,4, \end{cases}$$

$$\gamma^{1357}\epsilon_{(a)} = \begin{cases} -\epsilon_{(a)} & \text{for } a=1,2, \\ +\epsilon_{(a)} & \text{for } a=3,4. \end{cases} \tag{B17}$$

The three two-forms $J^A$ are given by combinations, self-dual on the $(a)$ index,

$$J^1_{mn} = -(\overline{\epsilon_{(2)}}\gamma_{mn}\epsilon_{(3)} + \overline{\epsilon_{(1)}}\gamma_{mn}\epsilon_{(4)}),$$

$$J^2_{mn} = -(\overline{\epsilon_{(3)}}\gamma_{mn}\epsilon_{(1)} + \overline{\epsilon_{(2)}}\gamma_{mn}\epsilon_{(4)}),$$

$$J^3_{mn} = -(\overline{\epsilon_{(1)}}\gamma_{mn}\epsilon_{(2)} + \overline{\epsilon_{(3)}}\gamma_{mn}\epsilon_{(4)}). \tag{B18}$$

The second set of $J'^A$ two-forms is given by the corresponding anti-self-dual combinations with minus signs between the first and second terms in parentheses. Given $\gamma^{12}\epsilon_{(2)} = \gamma^{56}\epsilon_{(2)} = \epsilon_{(3)}$, $\gamma^{14}\epsilon_{(3)} = \gamma^{58}\epsilon_{(3)} = \epsilon_{(1)}$, and $\gamma^{13}\epsilon_{(1)} = \gamma^{57}\epsilon_{(1)} = \epsilon_{(2)}$, together with $\gamma^{12}\epsilon_{(1)} = -\gamma^{56}\epsilon_{(1)} = \epsilon_{(4)}$, $\gamma^{14}\epsilon_{(2)} = -\gamma^{58}\epsilon_{(2)} = \epsilon_{(4)}$, and $\gamma^{13}\epsilon_{(3)} = -\gamma^{57}\epsilon_{(3)} = \epsilon_{(4)}$, we have the explicit expressions

$$J^1 = e^{12} + e^{34}, \quad J'^1 = e^{56} + e^{78},$$

$$J^2 = e^{14} + e^{23}, \quad J'^2 = e^{58} + e^{67},$$

$$J^3 = e^{13} + e^{42}, \quad J'^3 = e^{57} + e^{86}. \tag{B19}$$

Again, the corresponding volume form is given by Eq. (B4) as above. Note that there are six $SU(4)$ structures given by $J^A_{\pm} = J^A \pm J'^A$ and similarly each spinor $\epsilon_{(a)}$ defines a corresponding $Spin(7)$ structure given by

$$\Psi^{(1)} = \text{vol} + \text{vol}' - J^1\wedge J'^1 + J^2\wedge J'^2 + J^3\wedge J'^3,$$

$$\Psi^{(2)} = \text{vol} + \text{vol}' + J^1\wedge J'^1 - J^2\wedge J'^2 + J^3\wedge J'^3,$$

$$\Psi^{(3)} = \text{vol} + \text{vol}' + J^1\wedge J'^1 + J^2\wedge J'^2 - J^3\wedge J'^3,$$

$$\Psi^{(4)} = \text{vol} + \text{vol}' - J^1\wedge J'^1 - J^2\wedge J'^2 - J^3\wedge J'^3. \tag{B20}$$

*$G_2$*. A $G_2$ structure defines a single invariant spinor in $d=7$. This can be defined by the projections

$$\gamma^{1234}\epsilon = \gamma^{1256}\epsilon = \gamma^{1357}\epsilon = -\epsilon, \tag{B21}$$

where we have taken $i\gamma_{1,\ldots,7} = \mathbb{1}$. The associative three-form (2.16) is then given by

$$\phi_{mnp} = -i\bar{\epsilon}\gamma_{mnp}\epsilon. \tag{B22}$$

The corresponding volume form is given by

$$\text{vol}_{m_1,\ldots,m_7} = i\bar{\epsilon}\gamma_{m_1,\ldots,m_7}\epsilon. \tag{B23}$$

Note that the relation between $\phi$ and vol is slightly nonstandard. It is the opposite of the conventions given, for instance in [54]. To match the expressions in [54], one replaces $e_7$ with $-e_7$ and permutes the new basis vol$=-e_{1234567}$ to $e_{3254761}$. Note that one can choose an imaginary basis for the $\gamma$ matrices where $\bar{\epsilon} = \epsilon^{\mathrm{T}}$.

Lifting to $d=8$, the $G_2$ structure defines a pair of real spinors $\epsilon_{(a)}$ with $a=1,2$ satisfying Eq. (B21) of opposite chirality. They can be distinguished by

$$\gamma^{5678}\epsilon_{(1)} = -\epsilon_{(1)}, \quad \gamma^{5678}\epsilon_{(2)} = +\epsilon_{(2)}. \tag{B24}$$

The $G_2$ structure is defined by $\phi$ and $K$ given by

$$\phi_{mnp} = -\overline{\epsilon_{(1)}}\gamma_{mnp}\epsilon_{(2)},$$

$$K_m = \overline{\epsilon_{(1)}}\gamma_m\epsilon_{(2)}. \tag{B25}$$

With $\gamma^8\epsilon_{(1)} = \epsilon_{(2)}$, we have $K = e^8$ and $\phi$ takes the standard form (2.16). The corresponding volume form $\mathrm{vol} = e^1 \wedge \cdots \wedge e^8$ is given by

$$\mathrm{vol}_{m_1,\ldots,m_8} = \overline{\epsilon_{(1)}}\gamma_{m_1,\ldots,m_8}\epsilon_{(1)} = -\overline{\epsilon_{(2)}}\gamma_{m_1,\ldots,m_8}\epsilon_{(2)}. \tag{B26}$$

The two $Spin(7)$ structures defined by $\epsilon_{(a)}$ are given by

$$\Psi^{(1)} = -i_K * \phi + \phi \wedge K,$$

$$\Psi^{(2)} = -i_K * \phi - \phi \wedge K. \tag{B27}$$

Note that with these conventions, $i_K * \phi = -*_7\phi$ where $*_7\phi$ is the usual coassociative four-form, which is the Hodge dual of $\phi$ on the seven-dimensional subspace orthogonal to $K$.

*SU(3)*. The $SU(3)$ structure defines a single chiral complex spinor $\epsilon$. This can be defined by the conditions

$$\gamma^{1234}\epsilon = \gamma^{1256}\epsilon = -\epsilon. \tag{B28}$$

We choose the chirality $i\gamma^{1,\ldots,6}\epsilon = \epsilon$ so that $\gamma^{12}\epsilon = i\epsilon$. The forms $J$ and $\Omega$ are then given by

$$J_{mn} = -i\bar{\epsilon}\gamma_{mn}\epsilon,$$

$$\Omega_{mnp} = \overline{\epsilon^c}\gamma_{mnp}\epsilon. \tag{B29}$$

Given $\gamma^{135}\epsilon = \epsilon^c$, we get the standard expressions

$$J = e^{12} + e^{34} + e^{56},$$

$$\Omega = (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6). \tag{B30}$$

The corresponding volume form is

$$\mathrm{vol}_{m_1,\ldots,m_6} = i\bar{\epsilon}\gamma_{m_1,\ldots,m_6}\epsilon. \tag{B31}$$

Again one can always choose a basis where $\bar{\epsilon} = \epsilon^\dagger$ and $\epsilon^c = \epsilon^*$.

Lifting to $d=7$, the $SU(3)$ structure defines a pair of invariant spinors $\epsilon_{(a)}$ with $a = 1,2$ satisfying Eq. (B28). Fixing $i\gamma_{1,\ldots,7} = \mathbb{1}$, they can be distinguished by

$$\gamma^{1357}\epsilon_{(1)} = -\epsilon_{(1)}, \quad \gamma^{1357}\epsilon_{(2)} = +\epsilon_{(2)}. \tag{B32}$$

The $SU(3)$ structure is given by

$$J_{mn} = -\overline{\epsilon_{(1)}}\gamma_{mn}\epsilon_{(2)},$$

$$\Omega_{mnp} = i\overline{\epsilon_{(1)}}\gamma_{mnp}\epsilon_{(2)} - \frac{1}{2}(\overline{\epsilon_{(1)}}\gamma_{mnp}\epsilon_{(1)} - \overline{\epsilon_{(2)}}\gamma_{mnp}\epsilon_{(2)}),$$

$$K_m = -i\overline{\epsilon_{(1)}}\gamma_m\epsilon_{(2)}. \tag{B33}$$

Given $\gamma^{12}\epsilon_{(1)} = \epsilon_{(2)}$, this gives $K = e^7$ and $J$ and $\Omega$ take the standard form (B30). The corresponding volume form $\mathrm{vol} = e^1 \wedge \cdots \wedge e^7$ is given by

$$\mathrm{vol}_{m_1,\ldots,m_7} = i\overline{\epsilon_{(1)}}\gamma_{m_1,\ldots,m_7}\epsilon_{(1)} = i\overline{\epsilon_{(2)}}\gamma_{m_1,\ldots,m_7}\epsilon_{(2)}. \tag{B34}$$

The two $G_2$ structures defined by $\epsilon_{(a)}$ are given by

$$\phi^{(1)} = J \wedge K - \mathrm{Im}\,\Omega,$$

$$\phi^{(2)} = J \wedge K + \mathrm{Im}\,\Omega, \tag{B35}$$

*SU*(2). Finally for $SU(2)$ the structure again defines a single complex spinor of definite chirality. We take the negative chirality

$$\gamma^{1234}\epsilon = -\epsilon. \tag{B36}$$

The forms $J$ and $\Omega$ are then given by

$$J_{mn} \equiv J^3_{mn} = -i\bar{\epsilon}\gamma_{mn}\epsilon,$$

$$\Omega_{mn} \equiv J^2_{mn} + iJ^1_{mn} = \overline{\epsilon^c}\gamma_{mn}\epsilon. \tag{B37}$$

Given $\gamma^{12}\epsilon = i\epsilon$ and $\gamma^{13}\epsilon = \epsilon^c$ we get the self-dual combinations

$$J^1 = e^{14} + e^{23},$$

$$J^2 = e^{13} + e^{42},$$

$$J^3 = e^{12} + e^{34}. \tag{B38}$$

The corresponding volume form is

$$\mathrm{vol}_{m_1,\ldots,m_4} = i\bar{\epsilon}\gamma_{m_1,\ldots,m_4}\epsilon. \tag{B39}$$

Again one can always choose a basis where $\bar{\epsilon} = \epsilon^\dagger$ and $\epsilon^c = \epsilon^*$.

Lifting to $d=6$, the $SU(2)$ structure defines a pair of complex invariant spinors $\epsilon_{(a)}$ with $a = 1,2$ satisfying Eq. (B36). These have opposite chirality and can be distinguished by

$$\gamma^{3456}\epsilon_{(1)} = -\epsilon_{(1)}, \quad \gamma^{3456}\epsilon_{(2)} = +\epsilon_{(2)}. \tag{B40}$$

The $SU(2)$ structure is given by

$$J_{mn} = -\frac{1}{2}i(\overline{\epsilon_{(1)}}\gamma_{mn}\epsilon_{(1)} + \overline{\epsilon_{(2)}}\gamma_{mn}\epsilon_{(2)}),$$

$$\Omega_{mn} = \overline{\epsilon^c_{(1)}}\gamma_{mn}\epsilon_{(2)},$$

$$K^1_m + iK^2_m = \overline{\epsilon_{(2)}}\gamma_m\epsilon_{(1)}. \tag{B41}$$

Given $\gamma^{12}\epsilon_{(i)} = \epsilon_{(i)}$ and $\gamma^{135}\epsilon_{(i)} = \epsilon^c_{(i)}$ while $\gamma^5\epsilon_{(1)} = \epsilon_{(2)}$ and $\gamma^6\epsilon_{(1)} = i\epsilon_{(2)}$, we have $K_1 = e^5$, $K_2 = e^6$ and $J$ and $\Omega$ take the standard form (B38). The corresponding volume form $\mathrm{vol} = e^1 \wedge \cdots \wedge e^6$ is given by

$$\mathrm{vol}_{m_1,\ldots,m_6} = i\bar{\epsilon}_{(1)}\gamma_{m_1,\ldots,m_6}\epsilon_{(1)} = -i\bar{\epsilon}_{(2)}\gamma_{m_1,\ldots,m_6}\epsilon_{(2)}. \tag{B42}$$

The two $SU(3)$ structures defined by $\epsilon_{(a)}$ are given by

$$J^{(1)} = J + K^1\wedge K^2, \quad \Omega^{(1)} = \Omega\wedge(K^1 + iK^2),$$
$$J^{(2)} = J - K^1\wedge K^2, \quad \Omega^{(2)} = \Omega\wedge(K^1 - iK^2). \tag{B43}$$

## APPENDIX C: ALMOST PRODUCT STRUCTURES

An almost complex structure is a $GL(n,\mathbb{C})$ structure on a $2n$-dimensional manifold, which is characterized by a tensor $J_m{}^n$ satisfying $J\cdot J = -\mathbb{1}$. Using this one can split the tangent space $T_pM^{\mathbb{C}}$ at any point in the two subspaces $T_pM^+ \oplus T_pM^-$ corresponding to the $+i$ and $-i$ eigenvalues of $J$ respectively. The Nijenhuis tensor for the almost complex structure is defined by

$$N_{mn}{}^r = J_m{}^s\partial_{[s}J_{n]}{}^r - J_n{}^s\partial_{[s}J_{m]}{}^r. \tag{C1}$$

The almost complex structure is integrable if and only if the Nijenhuis tensor vanishes and in this case one can introduce holomorphic co-ordinates on the manifold. If $J$ is compatible with a metric, namely $J_{mq} \equiv J_m{}^n g_{nq}$ is a two-form, then the metric is called almost Hermitian and Hermitian if $J$ is integrable.

Similarly, an almost product structure is a $GL(P,\mathbb{R})\times GL(Q,\mathbb{R})$ structure on a $(P+Q)$-dimensional manifold, which is characterized by a tensor $\Pi_m{}^n$ satisfying $\Pi\cdot\Pi = +\mathbb{1}$. At any point the tangent space splits accordingly as $T_pM = T_pM^P \oplus T_pM^Q$, where $P$ $(Q)$ is the number of $+1$ $(-1)$ eigenvalues of $\Pi$. The Nijenhuis tensor for the almost product structure is defined again by

$$N_{mn}{}^r = \Pi_m{}^s\partial_{[s}\Pi_{n]}{}^r - \Pi_n{}^s\partial_{[s}\Pi_{m]}{}^r \tag{C2}$$

and the almost product structure is integrable if and only if the Nijenhuis tensor vanishes (see e.g. [55]). If furthermore the almost product structure is metric compatible, i.e. $\Pi_{mq} \equiv \Pi_m{}^n g_{nq}$ is a symmetric tensor, one can introduce "separating co-ordinates" on the manifold such that the metric takes the $(P\times P, Q\times Q)$ block-diagonal form

$$ds^2 = g_{ij}{}^P(x,y)dx^i dx^j + g_{ab}{}^Q(x,y)dy^a dy^b \tag{C3}$$

where $i,j = 1,\ldots,P$ and $a,b = 1,\ldots,Q$.

Two commuting almost complex structures $J, J'$, satisfying $J\cdot J' = J'\cdot J$ give rise to an almost product structure

$$\Pi = J\cdot J'. \tag{C4}$$

Suppose $J$ and $J'$ are metric compatible and satisfy $\nabla^+ J = \nabla^+ J' = 0$, or $\nabla^- J = \nabla^- J' = 0$, where $\nabla^\pm$ is a metric connection with totally anti-symmetric torsion $\pm\frac{1}{2}H$. The Nijenhuis tensor then reads in general

$$N_{mn}{}^r = \pm\frac{1}{2}(H_{mn}{}^r + \Pi_m{}^p\Pi_n{}^q H_{pq}{}^r - \Pi^{rp}\Pi_m{}^q H_{pqn} - \Pi_n{}^p\Pi^{rq}H_{pqm}). \tag{C5}$$

Using the tangent space decomposition, one finds that the only non-zero components are given by

$$N_{ij}{}^c = \pm 2H_{ij}{}^c$$
$$N_{ab}{}^k = \pm 2H_{ab}{}^k. \tag{C6}$$

If instead we assume that $J^+, J^-$ are commuting and are both integrable, and also $\nabla^+ J^+ = \nabla^- J^- = 0$, then all the components of $N_{mn}{}^r$ vanish, hence $\Pi$ is integrable [56]. To see this we first note that given the assumptions, $H$ is a (2,1) + (1,2) form with respect to either complex structure $J^\pm$:

$$H_{mnr} = J_m{}^p J_n{}^q H_{pqr} + J_r{}^p J_m{}^q H_{pqn} + J_n{}^p J_r{}^q H_{pqm}. \tag{C7}$$

To proceed, write $2\Pi = J^+\cdot J^- + J^-\cdot J^+$ to get

$$2\nabla_m\Pi_n{}^p = J_n^{+\,r}J^{-sp}H_{mrs} - J_n^{-\,r}J^{+sp}H_{mrs}. \tag{C8}$$

Then using Eq. (C7) we find

$$4\nabla_{[m}\Pi_{n]}{}^p = -J_m^{+\,r}J_n^{+\,s}H_{rst}\Pi^{tp} + J_m^{-\,r}J_n^{-\,s}H_{rst}\Pi^{tp} \tag{C9}$$

from which it easily follows that $N(\Pi) = 0$.

It is sometimes incorrectly stated in the literature (see for instance [56–58]) that $\Pi$, defined by Eq. (C4), is integrable if and only if the two commuting almost complex structures are integrable. A concrete class of counter-example is provided by the geometry (7.2) for generic instantons $G$. This geometry has an $SU(2)$ structure, built from the $\epsilon^-$ Killing spinors, which can be specified by two $SU(3)$ structures. The corresponding two almost complex structures, written as two-forms, are given by

$$J = e^{2\Phi}(dx^1\wedge dx^2 - dx^3\wedge dx^4) + (dy + B^1)\wedge(dz + B^2),$$
$$J' = e^{2\Phi}(dx^1\wedge dx^2 - dx^3\wedge dx^4) - (dy + B^1)\wedge(dz + B^2). \tag{C10}$$

Both almost complex structures are integrable. A quick way to see this is to note that the geometry is a special example of the canonical $SU(3)$ geometry in $d = 6$ (preserving twice as much supersymmetry) that was discussed in Sec. III (with expressions for $\epsilon^+$ spinors rather than $\epsilon^-$ spinors that we have here) for either $SU(3)$ structure. In particular, as pointed out in Sec. III, the almost complex structures are integrable. Moreover, the two complex structures clearly commute and thus define an almost product structure given by $\Pi = J\cdot J'$. On the other hand, because $\nabla^- J = \nabla^- J' = 0$ and hence $\nabla^-\Pi = 0$, from Eq. (C6) we see that there are non-zero components of the associated Nijenhuis tensor, namely

$$N_{mn}{}^y = -2G^1_{mn}$$
$$N_{mn}{}^z = -2G^2_{mn}. \tag{C11}$$

For definiteness, let us briefly present a simple example very explicitly. In particular, set the dilaton field to zero and $B^1 = B^2 = x^1 dx^2 + x^3 dx^4$. Then the almost complex structures corresponding to Eq. (C10) read

$$J_a{}^b = \begin{bmatrix} 0 & 1 & 0 & 0 & -x^1 & -x^1 \\ -1 & 0 & 0 & 0 & -x^1 & x^1 \\ 0 & 0 & 0 & -1 & x^3 & x^3 \\ 0 & 0 & 1 & 0 & -x^3 & x^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$J'_a{}^b = \begin{bmatrix} 0 & 1 & 0 & 0 & -x^1 & -x^1 \\ -1 & 0 & 0 & 0 & x^1 & -x^1 \\ 0 & 0 & 0 & -1 & x^3 & x^3 \\ 0 & 0 & 1 & 0 & x^3 & -x^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{C12}$$

It is not difficult to check directly that these are both integrable and indeed commute. The corresponding almost product structure is

$$\Pi = J\cdot J' = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2x^1 & 2x^1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2x^3 & 2x^3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C13}$$

Computing the corresponding Nijenhuis tensor, we find that it has the non-zero components given by Eq. (C11) with $G^1 = G^2 = dx^1\wedge dx^2 + dx^3\wedge dx^4$. It would be interesting to investigate the consequences of this counter example, especially in the context of the sigma model literature.

[1] S. Sethi, C. Vafa, and E. Witten, Nucl. Phys. **B480**, 213 (1996).
[2] A. Strominger, Nucl. Phys. **B274**, 253 (1986).
[3] C. M. Hull, in Turin 1985, Proceedings, Superunification and Extra Dimensions, pp. 347–375; Phys. Lett. B **178**, 357 (1986).
[4] B. de Wit, D.J. Smit, and N.D. Hari Dass, Nucl. Phys. **B283**, 165 (1987).
[5] J.P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, ''*G*-structures and Wrapped NS5-Branes,'' Commun. Math. Phys. (to be published), hep-th/0205050.
[6] B. de Wit, D.J. Smit, and N.D. Hari Dass, Nucl. Phys. **B283**, 165 (1987).
[7] J.P. Gauntlett, N. Kim, D. Martelli, and D. Waldram, J. High Energy Phys. **11**, 018 (2001).
[8] T. Friedrich and S. Ivanov, Asian J. Math. **6**, 303 (2002).
[9] T. Friedrich and S. Ivanov, J. Geom. Phys. **48**, 1 (2003).
[10] S. Ivanov, ''Connection with Torsion, Parallel Spinors and Geometry of Spin(7) Manifolds,'' math.dg/0111216.
[11] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis, and H.S. Reall, Class. Quantum Grav. **20**, 4587 (2003).
[12] S. Gurrieri, J. Louis, A. Micu, and D. Waldram, Nucl. Phys. **B654**, 61 (2003).
[13] J.P. Gauntlett and S. Pakis, J. High Energy Phys. **04**, 039 (2003).
[14] P. Kaste, R. Minasian, M. Petrini, and A. Tomasiello, Fortschr. Phys. **51**, 764 (2003); J. High Energy Phys. **09**, 033 (2002).
[15] K. Behrndt and C. Jeschek, J. High Energy Phys. **04**, 002 (2003).
[16] S. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Mathematics, Vol. 201 (Longman, Harlow, England, 1989).
[17] G.L. Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis, and G. Zoupanos, Nucl. Phys. **B652**, 5 (2003).
[18] J. Gutowski and G. Papadopoulos, Phys. Lett. B **462**, 81 (1999).
[19] J. Gutowski, G. Papadopoulos, and P.K. Townsend, Phys. Rev. D **60**, 106006 (1999).
[20] D.Z. Freedman, G.W. Gibbons, and P.C. West, Phys. Lett. **124B**, 491 (1983).
[21] S. Ivanov and G. Papadopoulos, Phys. Lett. B **497**, 309 (2001).
[22] G. Papadopoulos, ''Brane Solitons and Hypercomplex Structures,'' math.dg/0003024.
[23] A.R. Frey and J. Polchinski, Phys. Rev. D **65**, 126009 (2002).
[24] A. Gray and L.M. Hervella, Ann. Mat. Pura Appl. **282**, 1 (1980).
[25] S. Chiossi and S. Salamon, in *Differential Geometry, Valencia, 2001* (World Scientific Publishing, River Edge, N.J., 2002), pp. 115–133, math.dg/0202282.
[26] M. Fernandez, Ann. Mat. Pura Appl. **143**, 101 (1982).
[27] M. Fernandez and A. Gray, Ann. Mat. Pura Appl. **32**, 19 (1982).
[28] M. Fernandez and L. Ugarte, Geom. Dedic. **70**, 57 (1998).
[29] P. Gauduchon and P. Tod, J. Geom. Phys. **25**, 291 (1998).
[30] S. Ivanov, J. Geom. Phys. **41**, 235 (2002).
[31] J.M. Maldacena and C. Nunez, Phys. Rev. Lett. **86**, 588 (2001).

[32] G. Papadopoulos and A.A. Tseytlin, Class. Quantum Grav. **18**, 1333 (2001).
[33] B.S. Acharya, J.P. Gauntlett, and N. Kim, Phys. Rev. D **63**, 106003 (2001).
[34] J.M. Maldacena and H. Nastase, J. High Energy Phys. **09**, 024 (2001).
[35] G. Grantcharov and Y.S. Poon, Commun. Math. Phys. **213**, 19 (2000).
[36] J. Dadok, F.R. Harvey, and F. Morgan, Trans. Am. Math. Soc. **307**, 1 (1988).
[37] J.P. Gauntlett, G.W. Gibbons, G. Papadopoulos, and P.K. Townsend, Nucl. Phys. **B500**, 133 (1997).
[38] R.S. Ward, Nucl. Phys. **B236**, 381 (1984).
[39] J.A. Harvey and A. Strominger, Phys. Rev. Lett. **66**, 549 (1991).
[40] M. Gunaydin and H. Nicolai, Phys. Lett. B **351**, 169 (1995); **376**, 329 (1996).
[41] H. Lu and J.F. Vazquez-Poritz, Phys. Lett. B **534**, 155 (2002).
[42] H. Lu and J.F. Vazquez-Poritz, Nucl. Phys. **B633**, 114 (2002).
[43] E. Goldstein and S. Prokushkin, ''Geometric Model for Complex Non-Kaehler Manifolds with SU(3) Structure,'' hep-th/0212307.
[44] J.P. Gauntlett, N. Kim, D. Martelli, and D. Waldram, Phys. Rev. D **64**, 106008 (2001).
[45] F. Bigazzi, A.L. Cotrone, and A. Zaffaroni, Phys. Lett. B **519**, 269 (2001).
[46] K. Hori and A. Kapustin, J. High Energy Phys. **11**, 038 (2002).
[47] J. Gomis and J.G. Russo, J. High Energy Phys. **10**, 028 (2001).
[48] M. Naka, ''Various Wrapped Branes from Gauged Supergravities,'' hep-th/0206141.
[49] K. Dasgupta, G. Rajesh, and S. Sethi, J. High Energy Phys. **08**, 023 (1999).
[50] K. Becker and K. Dasgupta, J. High Energy Phys. **11**, 006 (2002).
[51] P.K. Tripathy and S.P. Trivedi, J. High Energy Phys. **03**, 028 (2003).
[52] K. Becker, M. Becker, K. Dasgupta, and P.S. Green, J. High Energy Phys. **04**, 007 (2003).
[53] J.M. Maldacena and C. Nunez, Int. J. Mod. Phys. A **16**, 822 (2001).
[54] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 2000).
[55] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces* (Macmillan, New York, 1965).
[56] S.J. Gates, C.M. Hull, and M. Rocek, Nucl. Phys. **B248**, 157 (1984).
[57] A. Sevrin and J. Troost, Nucl. Phys. **B492**, 623 (1997).
[58] U. Lindstrom and M. Zabzine, J. High Energy Phys. **02**, 006 (2003).