Spherically symmetric dissipative anisotropic fluids: A general study

L. Herrera,[1,*] A. Di Prisco,[1] J. Martin,[2,†] J. Ospino,[2] N. O. Santos,[3,4,5,‡] and O. Troconis[1]
[1]*Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela*
[2]*Area de Física Teórica, Facultad de Ciencias, Universidad de Salamanca, 37008 Salamanca, Spain*
[3]*LERMA-UMR 8112-CNRS, ERGA Université PARIS VI, 4 place Jussieu, 75005 Paris Cedex 05, France*
[4]*Laboratório Nacional de Computação Científica, 25651-070 Petrópolis-RJ, Brazil*
[5]*Centro Brasileiro de Pesquisas Físicas, 22290-180 Rio de Janeiro-RJ, Brazil*
(Received 8 December 2003; published 29 April 2004)

The full set of equations governing the evolution of self-gravitating spherically symmetric dissipative fluids with anisotropic stresses is deployed and used to carry out a general study on the behavior of such systems, in the context of general relativity. Emphasis is given to the link between the Weyl tensor, the shear tensor, the anisotropy of the pressure, and the density inhomogeneity. In particular we provide the general, necessary, and sufficient condition for the vanishing of the spatial gradients of energy density, which in turn suggests a possible definition of a gravitational arrow of time. Some solutions are also exhibited to illustrate the discussion.

DOI: 10.1103/PhysRevD.69.084026 PACS number(s): 04.40.Nr

I. INTRODUCTION

This work is devoted to the study of dissipative, locally anisotropic, spherically symmetric self-gravitating fluids, with particular emphasis on a set of physical and geometrical variables which appear to play a fundamental role in the evolution of such systems. These variables are the Weyl tensor, the shear tensor, the local anisotropy of the pressure, and the density inhomogeneity.

The Weyl tensor [1] or some functions of it [2] have been proposed to provide a gravitational arrow of time, the rationale behind this idea being that tidal forces tend to make the gravitating fluid more inhomogeneous as the evolution proceeds, thereby indicating the sense of time. However, some works have thrown doubt on this proposal [3]. Further evidence about the relevance of the Weyl tensor in the evolution of self-gravitating systems may be found in [4].

The role of density inhomogeneities in the collapse of dust [5] and in particular in the formation of naked singularities [6] has been extensively discussed in the literature.

On the other hand, the assumption of local anisotropy of pressure, which seems to be very sensible to describe the matter distribution under a variety of circumstances, has been proven to be very useful in the study of relativistic compact objects (see [7] and references therein).

A clue pointing to the relevance of the above-mentioned three factors in the fate of spherical collapse is also provided by the expression of the active gravitational mass in terms of those factors [8].

Finally, the relevance of the shear tensor in the evolution of self-gravitating systems has been brought out by many authors (see [9] and references therein).

Now, in the study of self-gravitating compact objects it is usually assumed that deviations from spherical symmetry are likely to be incidental rather than basic features of the process involved (see, however, the discussion in [10]). Thus, since the seminal paper by Oppenheimer and Snyder [11], most of the work dedicated to the problem of general relativistic gravitational collapse deal with spherically symmetric fluid distribution. Accordingly we shall consider spherically symmetric fluid distributions.

Also, the fluid distribution under consideration will be assumed to be dissipative. Indeed, dissipation due to the emission of massless particles (photons and/or neutrinos) is a characteristic process in the evolution of massive stars. In fact, it seems that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole, is neutrino emission [12]. Consequently, in this paper, the matter distribution forming the self-gravitating object will be described as a dissipative fluid.

In the diffusion approximation, it is assumed that the energy flux of radiation (as that of thermal conduction) is proportional to the gradient of temperature. This assumption is in general very sensible, since the mean free path of particles responsible for the propagation of energy in stellar interiors is in general very small as compared with the typical length of the object. Thus, for a main sequence star such as the Sun, the mean free path of photons at the center, is of the order of 2 cm. Also, the mean free path of trapped neutrinos in compact cores of densities about $10^{12}\ \mathrm{g\,cm^{-3}}$ becomes smaller than the size of the stellar core [13,14].

Furthermore, the observational data collected from supernova 1987A indicate that the regime of radiation transport prevailing during the emission process is closer to the diffusion approximation than to the streaming out limit [15].

However, in many other circumstances, the mean free path of particles transporting energy may be large enough so as to justify the free streaming approximation. Therefore we shall include simultaneously both limiting cases of radiative transport (diffusion and streaming out), allowing for describing a wide range of situations.

*Postal address: Apartado 80793, Caracas 1080A, Venezuela. Email address: laherrera@telcel.net.ve
†Email address: chmm@usal.es
‡Email address: nos@cbpf.br; santos@ccr.jussieu.fr

It is also worth mentioning that although the most common method of solving Einstein's equations is to use comoving coordinates (e.g., [16]), we shall use noncomoving coordinates, which implies that the velocity of any fluid element (defined with respect to a conveniently chosen set of observers) has to be considered as a relevant physical variable [17].

The paper is organized as follows: In the next section we introduce the notation and write all relevant equations. Section III is devoted to the analysis of different special cases. Finally the results are discussed in the last section.

II. BASIC EQUATIONS

In this section we shall deploy the relevant equations for describing a dissipative self-gravitating locally anisotropic fluid. In spite of the fact that not all these equations are independent [for example, the field equations and the conservation equations (Bianchi identities)] we shall present them all, since depending on the problem under consideration, it may be more advantageous using one set instead of the other.

A. Einstein equations

We consider spherically symmetric distributions of collapsing fluid, which for sake of completeness we assume to be locally anisotropic, undergoing dissipation in the form of heat flow and/or free streaming radiation, bounded by a spherical surface Σ.

The line element is given in Schwarzschild-like coordinates by

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{1}$$

where $\nu(t,r)$ and $\lambda(t,r)$ are functions of their arguments. We number the coordinates: $x^0=t,\ x^1=r,\ x^2=\theta,\ x^3=\phi$.

The metric (1) has to satisfy the Einstein field equations

$$G^{\nu}_{\mu}=8\pi T^{\nu}_{\mu}, \tag{2}$$

which in our case read [18]

$$-8\pi T^0_0=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right), \tag{3}$$

$$-8\pi T^1_1=-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right), \tag{4}$$

$$-8\pi T^2_2=-8\pi T^3_3=-\frac{e^{-\nu}}{4}[2\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})]+\frac{e^{-\lambda}}{4}\left(2\nu''+\nu'^2-\lambda'\nu'+2\frac{\nu'-\lambda'}{r}\right), \tag{5}$$

$$-8\pi T_{10}=-\frac{\dot{\lambda}}{r}, \tag{6}$$

where overdots and primes stand for partial differentiation with respect to t and r, respectively.

In order to give physical significance to the T^{μ}_{ν} components we apply the Bondi approach [18].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates (τ,x,y,z)

$$d\tau=e^{\nu/2}dt,\quad dx=e^{\lambda/2}dr,\quad dy=r\,d\theta,\quad dz=r\sin\theta\, d\phi.$$

Then, denoting the Minkowski components of the energy tensor by an overbar, we have

$$\overline{T^0_0}=T^0_0,\quad \overline{T^1_1}=T^1_1,\quad \overline{T^2_2}=T^2_2,\quad \overline{T^3_3}=T^3_3,$$

$$\overline{T_{01}}=e^{-(\nu+\lambda)/2}T_{01}.$$

Next, we suppose that when viewed by an observer moving relative to these coordinates with proper velocity ω in the radial direction, the physical content of space consists of an anisotropic fluid of energy density ρ, radial pressure P_r, tangential pressure $P_\perp$, radial heat flux q, and unpolarized radiation of energy density ϵ traveling in the radial direction. Thus, when viewed by this moving observer the covariant tensor in Minkowski coordinates is

$$\begin{pmatrix} \rho+\epsilon & -q-\epsilon & 0 & 0\\ -q-\epsilon & P_r+\epsilon & 0 & 0\\ 0 & 0 & P_\perp & 0\\ 0 & 0 & 0 & P_\perp \end{pmatrix}.$$

Then a Lorentz transformation readily shows that

$$T^0_0=\overline{T^0_0}=\frac{\rho+P_r\omega^2}{1-\omega^2}+\frac{2\omega q}{1-\omega^2}+\frac{\epsilon(1+\omega)}{1-\omega}, \tag{7}$$

$$T^1_1=\overline{T^1_1}=-\frac{P_r+\rho\omega^2}{1-\omega^2}-\frac{2\omega q}{1-\omega^2}-\frac{\epsilon(1+\omega)}{1-\omega}, \tag{8}$$

$$T^2_2=T^3_3=\overline{T^2_2}=\overline{T^3_3}=-P_\perp, \tag{9}$$

$$T_{01}=e^{(\nu+\lambda)/2}\overline{T_{01}}=-\frac{(\rho+P_r)\omega e^{(\nu+\lambda)/2}}{1-\omega^2}-\frac{qe^{(\lambda+\nu)/2}}{1-\omega^2}(1+\omega^2)-\frac{e^{(\lambda+\nu)/2}\epsilon(1+\omega)}{1-\omega}. \tag{10}$$

Note that the coordinate velocity in the (t,r,θ,ϕ) system, dr/dt, is related to ω by

$$\omega=\frac{dr}{dt}e^{(\lambda-\nu)/2}. \tag{11}$$

Feeding back Eqs. (7)–(10) into Eqs. (3)–(6), we get field equations in the form

$$\frac{\rho+P_r\omega^2}{1-\omega^2}+\frac{2\omega q}{1-\omega^2}+\frac{\epsilon(1+\omega)}{1-\omega}=-\frac{1}{8\pi}\left\{-\frac{1}{r^2}+e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)\right\}, \tag{12}$$

$$\frac{P_r+\rho\omega^2}{1-\omega^2}+\frac{2\omega q}{1-\omega^2}+\frac{\epsilon(1+\omega)}{1-\omega}$$
$$=-\frac{1}{8\pi}\left\{\frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)\right\}, \qquad (13)$$

$$P_\perp=-\frac{1}{8\pi}\left\{\frac{e^{-\nu}}{4}[2\ddot{\lambda}+\dot{\lambda}(\dot{\lambda}-\dot{\nu})]\right.$$
$$\left.-\frac{e^{-\lambda}}{4}\left(2\nu''+\nu'^2-\lambda'\nu'+2\frac{\nu'-\lambda'}{r}\right)\right\}, \qquad (14)$$

$$\frac{(\rho+P_r)\omega e^{(\lambda+\nu)/2}}{1-\omega^2}+\frac{q e^{(\lambda+\nu)/2}}{1-\omega^2}(1+\omega^2)+\frac{e^{(\lambda+\nu)/2}\epsilon(1+\omega)}{1-\omega}$$
$$=-\frac{\dot{\lambda}}{8\pi r}. \qquad (15)$$

The four-velocity vector is defined as

$$u^\alpha=\left(\frac{e^{-\nu/2}}{(1-\omega^2)^{1/2}},\frac{\omega e^{-\lambda/2}}{(1-\omega^2)^{1/2}},0,0\right), \qquad (16)$$

from which we can calculate the four acceleration $a^\alpha = u^\alpha_{;\beta}u^\beta$ to obtain

$$\omega a_1=-a_0 e^{(\lambda-\nu)/2}=-\frac{\omega}{1-\omega^2}\left[\left(\frac{\omega\omega'}{1-\omega^2}+\frac{\nu'}{2}\right)\right.$$
$$\left.+e^{(\lambda-\nu)/2}\left(\frac{\omega\dot{\lambda}}{2}+\frac{\dot{\omega}}{1-\omega^2}\right)\right]. \qquad (17)$$

For the exterior of the fluid distribution, the spacetime is that of Vaidya, given by

$$ds^2=\left(1-\frac{2M(u)}{R}\right)du^2+2du\,dR-R^2(d\theta^2+\sin^2\theta\, d\phi^2), \qquad (18)$$

where u is a coordinate related to the retarded time, such that $u=$const is (asymptotically) a null cone open to the future and R is a null coordinate ($g_{RR}=0$).

The two coordinate systems (t,r,θ,ϕ) and (u,R,θ,ϕ) are related at the boundary surface and outside it by

$$u=t-r-2M\ln\left(\frac{r}{2M}-1\right), \qquad (19)$$

$$R=r. \qquad (20)$$

In order to match smoothly the two metrics above on the boundary surface $r=r_\Sigma(t)$, we require the continuity of the first and second fundamental forms across that surface, yielding (see [19] for details)

$$e^{\nu_\Sigma}=1-\frac{2M}{R_\Sigma}, \qquad (21)$$

$$e^{-\lambda_\Sigma}=1-\frac{2M}{R_\Sigma}, \qquad (22)$$

$$[P_r]_\Sigma=[q]_\Sigma, \qquad (23)$$

where, from now on, subscript Σ indicates that the quantity is evaluated on the boundary surface Σ, and Eq. (23) expresses the discontinuity of the radial pressure in the presence of heat flow, which is a well-known result [20].

Equations (21), (22), and (23) are the necessary and sufficient conditions for a smooth matching of the two metrics (1) and (18) on Σ.

B. Conservation laws ($T^\mu_{\nu;\mu}=0$)

The energy-momentum tensor (7)–(10) may be written as

$$T^\mu_\nu=\tilde{\rho}u^\mu u_\nu-\hat{P}h^\mu_\nu+\Pi^\mu_\nu+\tilde{q}(s^\mu u_\nu+s_\nu u^\mu), \qquad (24)$$

with

$$h^\mu_\nu=\delta^\mu_\nu-u^\mu u_\nu,$$

$$\Pi^\mu_\nu=\Pi\left(s^\mu s_\nu+\frac{1}{3}h^\mu_\nu\right),$$

$$\hat{P}=\frac{\tilde{P}_r+2P_\perp}{3},$$

$$\tilde{\rho}=\rho+\epsilon,$$

$$\tilde{P}_r=P_r+\epsilon,$$

$$\tilde{q}=q+\epsilon,$$

$$\Pi=\tilde{P}_r-P_\perp$$

and s^μ is defined as

$$s^\mu=\left(\frac{\omega e^{-\nu/2}}{(1-\omega^2)^{1/2}},\frac{e^{-\lambda/2}}{(1-\omega^2)^{1/2}},0,0\right), \qquad (25)$$

with the properties $s^\mu u_\mu=0$, $s^\mu s_\mu=-1$, and $\tilde{q}^\mu=\tilde{q}s^\mu$.

We may write, for the shear tensor

$$\sigma_{\alpha\beta}=\frac{1}{2}\sigma\left(s_\alpha s_\beta+\frac{1}{3}h_{\alpha\beta}\right), \qquad (26)$$

with

$$\sigma=-\frac{1}{(1-\omega^2)^{1/2}}\left[e^{-\nu/2}\left(\dot{\lambda}+\frac{2\omega\dot{\omega}}{1-\omega^2}\right)\right.$$
$$\left.+e^{-\lambda/2}\left(\omega\nu'+\frac{2\omega'}{1-\omega^2}-\frac{2\omega}{r}\right)\right]. \qquad (27)$$

Then from $T^\mu_{\nu;\mu}=0$, using Eq. (24), we find

$$\tilde{\rho}_{;\alpha}u^{\alpha}+(\tilde{\rho}+\hat{P})\theta+\tilde{q}^{\alpha}_{;\alpha}=\Pi_{\alpha\beta}\sigma^{\alpha\beta}+\tilde{q}a^{\nu}s_{\nu} \tag{28}$$

and

$$(\tilde{\rho}+\hat{P})a_{\alpha}+h^{\beta}_{\alpha}(\tilde{q}_{;\nu}u^{\nu}s_{\beta}+\tilde{q}s_{\beta;\nu}u^{\nu}-\hat{P}_{,\beta}+\Pi^{\mu}_{\beta;\mu})+\sigma_{\alpha\beta}\tilde{q}s^{\beta}+\frac{4}{3}\theta\tilde{q}s_{\alpha}=0 \tag{29}$$

or, contracting Eq. (29) with s^{α},

$$\tilde{P}_{r;\mu}s^{\mu}+(\tilde{P}_r-P_{\perp})s^{\mu}_{;\mu}-(\tilde{\rho}+P_{\perp})a_{\mu}s^{\mu}+\frac{4}{3}\theta\tilde{q}+\tilde{q}_{;\nu}u^{\nu}-\tilde{q}s^{\mu}s^{\nu}\sigma_{\mu\nu}=0.$$

C. Ricci identities

Ricci identities for the vector u_{α} read

$$u_{\alpha;\beta;\nu}-u_{\alpha;\nu;\beta}=R^{\mu}_{\alpha\beta\nu}u_{\mu}, \tag{30}$$

or using

$$u_{\alpha;\beta}=a_{\alpha}u_{\beta}+\sigma_{\alpha\beta}+\frac{1}{3}\theta h_{\alpha\beta}, \tag{31}$$

we have

$$\frac{1}{2}R^{\rho}_{\alpha\beta\mu}u_{\rho}=a_{\alpha;[\mu}u_{\beta]}+a_{\alpha}u_{[\beta;\mu]}+\sigma_{\alpha[\beta;\mu]}+\frac{1}{3}\theta_{,[\mu}h_{\beta]\alpha}+\frac{1}{3}\theta h_{\alpha[\beta;\mu]}. \tag{32}$$

1. Raychaudhuri equation

Contracting Eq. (32) with u^{β} and then the indices α and μ, we find the Raychaudhuri equation for the evolution of the expansion:

$$\theta_{;\alpha}u^{\alpha}+\frac{\theta^2}{3}+\sigma_{\alpha\beta}\sigma^{\alpha\beta}-a^{\alpha}_{;\alpha}=-u_{\rho}u^{\beta}R^{\rho}_{\beta}=-4\pi(\tilde{\rho}+3\hat{P}), \tag{33}$$

where

$$\sigma_{\alpha\beta}\sigma^{\alpha\beta}=\frac{1}{6}\sigma^2.$$

2. Constraint equation

If in Eq. (32) we contract first α and μ and then contract with $h^{\alpha\beta}$, we obtain the constraint equation expressing a direct relation between expansion θ, shear $\sigma^{\alpha\beta}$ and the heat flux q:

$$R^{\rho}_{\beta}u_{\rho}h^{\alpha\beta}=h^{\alpha}_{\beta}\left(\sigma^{\beta\mu}_{;\mu}-\frac{2}{3}\theta^{;\beta}\right)+\sigma^{\alpha\beta}a_{\beta}=8\pi\tilde{q}s^{\alpha}. \tag{34}$$

3. Propagation equation of the shear

Contracting Eq. (32) with $u^{\beta}h^{\alpha}_{\gamma}h^{\mu}_{\nu}$ we have

$$u_{\rho}u^{\beta}R^{\rho}_{\alpha\beta\mu}h^{\alpha}_{\gamma}h^{\mu}_{\nu}=h^{\alpha}_{\gamma}h^{\mu}_{\nu}(a_{\alpha;\mu}-\sigma_{\alpha\mu;\beta}u^{\beta})-a_{\gamma}a_{\nu}-u^{\beta}_{;\mu}h^{\mu}_{\nu}\left(\sigma_{\gamma\beta}+\frac{\theta}{3}h_{\gamma\beta}\right)-\frac{\theta_{;\alpha}u^{\alpha}}{3}h_{\gamma\nu}. \tag{35}$$

On the other hand, we know that the Riemann tensor may be expressed through the Weyl tensor $C^{\rho}_{\alpha\beta\mu}$, the Ricci tensor $R_{\alpha\beta}$, and the scalar curvature R, as

$$R^{\rho}_{\alpha\beta\mu}=C^{\rho}_{\alpha\beta\mu}+\frac{1}{2}R^{\rho}_{\beta}g_{\alpha\mu}-\frac{1}{2}R_{\alpha\beta}\delta^{\rho}_{\mu}+\frac{1}{2}R_{\alpha\mu}\delta^{\rho}_{\beta}-\frac{1}{2}R^{\rho}_{\mu}g_{\alpha\beta}-\frac{1}{6}R(\delta^{\rho}_{\beta}g_{\alpha\mu}-g_{\alpha\beta}\delta^{\rho}_{\mu}). \tag{36}$$

Contracting Eq. (36) with $u_{\rho}u^{\beta}h^{\alpha}_{\gamma}h^{\mu}_{\nu}$ and using Einstein equation (2), we find

$$R^{\rho}_{\alpha\beta\mu}u_{\rho}u^{\beta}h^{\alpha}_{\gamma}h^{\mu}_{\nu}=E_{\gamma\nu}+4\pi\Pi_{\gamma\nu}+\frac{4\pi}{3}h_{\gamma\nu}(\tilde{\rho}+3\hat{P}), \tag{37}$$

where $E_{\gamma\nu}$ denotes the "electric" part of the Weyl tensor defined by Eq. (42) below.

From Eqs. (35) and (37), taking into account Eq. (33), it follows that

$$E_{\gamma\nu}+4\pi\Pi_{\gamma\nu}=h^{\alpha}_{\gamma}h^{\mu}_{\nu}(a_{\alpha;\mu}-\sigma_{\alpha\mu;\beta}u^{\beta})-a_{\gamma}a_{\nu}-\sigma^{\beta}_{\nu}\sigma_{\gamma\beta}-\frac{2}{3}\theta\sigma_{\gamma\nu}-\frac{1}{3}\left(a^{\alpha}_{;\alpha}-\frac{1}{6}\sigma^2\right)h_{\gamma\nu}. \tag{38}$$

D. Evolution equations for the Weyl tensor

According to Kundt and Trümper [21], the Bianchi identities

$$R_{\mu\nu\kappa\delta;\lambda}+R_{\mu\nu\lambda\kappa;\delta}+R_{\mu\nu\delta\lambda;\kappa}=0 \tag{39}$$

may be written as

$$C_{\mu\nu\kappa;\lambda}^{\ \ \ \ \ \lambda}=R_{\kappa[\mu;\nu]}-\frac{1}{6}g_{\kappa[\mu}R_{,\nu]}. \tag{40}$$

Then taking into account Einstein equations (2), Eq. (40) reads

$$C_{\mu\nu\kappa;\lambda}^{\ \ \ \ \ \lambda}=8\pi T_{\kappa[\mu;\nu]}-\frac{8\pi}{3}g_{\kappa[\mu}T_{,\nu]}. \tag{41}$$

In the spherically symmetric case the "magnetic" part of the Weyl tensor vanishes ($H_{\alpha\beta}=0$); then, we have

$$C_{\mu\nu\kappa\lambda}=(g_{\mu\nu\alpha\beta}g_{\kappa\lambda\gamma\delta}-\epsilon_{\mu\nu\alpha\beta}\epsilon_{\kappa\lambda\gamma\delta})u^{\alpha}u^{\gamma}E^{\beta\delta}, \tag{42}$$

with $g_{\mu\nu\alpha\beta}=g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha}$, $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Cività symbol multiplied by $\sqrt{-g}$ and $E^{\beta\gamma}$, the "electric" part of Weyl tensor may be written as

$$E_{\alpha\beta}=E\left(s_\alpha s_\beta+\frac{1}{3}h_{\alpha\beta}\right), \tag{43}$$

with

$$E=\frac{e^{-\nu}}{4}\left(\ddot{\lambda}+\frac{\dot{\lambda}(\dot{\lambda}-\dot{\nu})}{2}\right)-\frac{e^{-\lambda}}{4}\left(\nu''+\frac{\nu'^2-\lambda'\nu'}{2}-\frac{\nu'-\lambda'}{r}+\frac{2(1-e^{\lambda})}{r^2}\right). \tag{44}$$

Contracting Eq. (42) with u^ν we obtain

$$u^\nu C_{\mu\nu k\lambda}=E_{\mu k}u_\lambda-E_{\mu\lambda}u_k, \tag{45}$$

from where it follows that

$$u^\nu C_{\mu\nu\kappa;\lambda}^{\ \ \ \ \lambda}+u^\nu_{;\lambda}C^{\lambda}_{\mu\nu\kappa}=\theta E_{\mu\kappa}+u^\alpha E_{\mu\kappa;\alpha}-u_{\kappa;\lambda}E^\lambda_\mu-u_\kappa E^\lambda_{\mu;\lambda}. \tag{46}$$

Also, from Eqs. (31) and (42), we obtain

$$u^\nu_{;\lambda}C_{\mu\nu\kappa}^{\ \ \ \ \lambda}=u_\mu u_\kappa\sigma_{\delta\beta}E^{\delta\beta}-a_\beta u_\mu E^\beta_\kappa-h_{\mu\kappa}\sigma^{\alpha\beta}E_{\alpha\beta}+\sigma_{\kappa\alpha}E^\alpha_\mu+\sigma_{\mu\alpha}E^\alpha_\kappa-\frac{\theta}{3}E_{\mu\kappa}. \tag{47}$$

Replacing Eq. (47) into Eq. (46), it results that

$$u^\nu C_{\mu\nu\kappa;\lambda}^{\ \ \ \ \lambda}=\frac{4\theta}{3}E_{\mu\kappa}+u^\alpha E_{\mu\kappa;\alpha}-u_{\kappa;\lambda}E^\lambda_\mu-u_\kappa E^\lambda_{\mu;\lambda}-u_\mu u_\kappa\sigma_{\delta\beta}E^{\delta\beta}+a_\beta u_\mu E^\beta_\kappa+h_{\mu\kappa}\sigma^{\alpha\beta}E_{\alpha\beta}-\sigma_{\kappa\alpha}E^\alpha_\mu-\sigma_{\mu\alpha}E^\alpha_\kappa. \tag{48}$$

Contracting Eq. (48) with $h^\mu_\alpha h^\kappa_\beta$ we have

$$h^\mu_\alpha h^\kappa_\beta u^\nu C_{\mu\nu\kappa;\lambda}^{\ \ \ \ \lambda}=\frac{4\theta}{3}E_{\alpha\beta}-u_{\beta;\lambda}E^\lambda_\alpha+u^\nu E_{\mu\kappa;\nu}h^\mu_\alpha h^\kappa_\beta+h_{\alpha\beta}\sigma^{\kappa\nu}E_{\kappa\nu}-\sigma_{\kappa\alpha}E^\kappa_\beta-\sigma_{\kappa\beta}E^\kappa_\alpha. \tag{49}$$

On the other hand,

$$h^\mu_\alpha h^\kappa_\beta u^\nu T_{\kappa\mu;\nu}=-u^\nu\hat{P}_{;\nu}h_{\alpha\beta}+u^\nu\Pi_{\kappa\mu;\nu}h^\mu_\alpha h^\kappa_\beta+\tilde{q}_\alpha a_\beta+\tilde{q}_\beta a_\alpha,$$

$$h^\mu_\alpha h^\kappa_\beta u^\nu T_{\kappa\nu;\mu}=(\tilde{\rho}+\hat{P})\left(\sigma_{\alpha\beta}+\frac{\theta}{3}h_{\alpha\beta}\right)-\Pi_{\beta\nu}\left(\sigma^\nu_\alpha+\frac{\theta}{3}h^\nu_\alpha\right)+\tilde{q}_{\kappa;\mu}h^\mu_\alpha h^\kappa_\beta,$$

$$h^\mu_\alpha h^\kappa_\beta u^\nu g_{\kappa[\mu}T_{,\nu]}=\frac{1}{2}u^\nu(\tilde{\rho}_{;\nu}-3\hat{P}_{;\nu})h_{\alpha\beta}. \tag{50}$$

Feeding back Eqs. (49) and (50) into Eq. (41) we find

$$\theta E_{\alpha\beta}+(u^\nu E_{\mu\kappa;\nu}-4\pi u^\nu\Pi_{\mu\kappa;\nu}+4\pi\tilde{q}_{\kappa;\mu})h^\mu_\alpha h^\kappa_\beta+\frac{4\pi}{3}u^\nu\tilde{\rho}_{;\nu}h_{\alpha\beta}+E\sigma_{\alpha\beta}=-4\pi(\tilde{\rho}+\hat{P})\left(\sigma_{\alpha\beta}+\frac{\theta}{3}h_{\alpha\beta}\right)+4\pi(\tilde{q}_\alpha a_\beta+\tilde{q}_\beta a_\alpha)+4\pi\Pi_{\nu\beta}\left(\sigma^\nu_\alpha+\frac{\theta}{3}h^\nu_\alpha\right). \tag{51}$$

Next, contracting Eq. (48) with u^k we have

$$u^\kappa u^\nu C_{\mu\nu k;\lambda}^{\ \ \ \ \lambda}=-E^\lambda_{\mu;\lambda}-a^\lambda E_{\mu\lambda}-\sigma^\nu_\lambda E^\lambda_\nu u_\mu. \tag{52}$$

The following expressions can also be easily calculated:

$$u^k u^\nu T_{k\mu;\nu}=u^\nu\tilde{\rho}_{;\nu}u_\mu+(\tilde{\rho}+\hat{P})a_\mu-a^k(\tilde{q}_k u_\mu+\Pi_{\mu k})+u^\nu\tilde{q}_{\mu;\nu},$$

$$u^k u^\nu T_{k\nu;\mu}=\tilde{\rho}_{;\mu}-2\tilde{q}_k\left(a^k u_\mu+\sigma^k_\mu+\frac{\theta}{3}h^k_\mu\right),$$

$$g_{k[\mu}T_{;\nu]}u^\nu u^k=-\frac{1}{2}T_{,\nu}h^\nu_\mu=-\frac{1}{2}(\tilde{\rho}-3\tilde{P})_{,\nu}h^\nu_\mu. \tag{53}$$

Finally, feeding back Eqs. (52) and (53) into Eq. (41) and contracting with h^μ_α we have

$$E^\lambda_{\mu;\lambda}h^\mu_\alpha+a^\lambda E_{\alpha\lambda}=-8\pi\tilde{q}_\kappa\left(\sigma^\kappa_\alpha+\frac{\theta}{3}h^\kappa_\alpha\right)+\frac{4\pi}{3}(2\tilde{\rho}+3\hat{P})_{;\nu}h^\nu_\alpha-4\pi(\tilde{\rho}+\hat{P})a_\alpha+4\pi a^\kappa\Pi_{\alpha\kappa}-4\pi u^\nu\tilde{q}_{\mu;\nu}h^\mu_\alpha. \tag{54}$$

E. Weyl tensor, mass function, and anisotropy

For the line element (1) we have

$$R^3_{232}=1-e^{-\lambda}=\frac{2m}{r}, \tag{55}$$

where the mass function $m(r,t)$ is defined as

$$m=4\pi\int_0^r r^2T^0_0dr. \tag{56}$$

Then from Eqs. (36), (43) and Einstein equations (2) it follows that

$$\frac{3m}{r^3}=4\pi\tilde{\rho}+4\pi(P_\perp-\tilde{P}_r)+E, \tag{57}$$

which in tensorial form reads

$$E_{\alpha\beta}-4\pi\Pi_{\alpha\beta}=\left(\frac{3m}{r^3}-4\pi\tilde{\rho}\right)\left(s_\alpha s_\beta+\frac{1}{3}h_{\alpha\beta}\right). \tag{58}$$

F. Summary

Equations (28), (29), (33), (34), (38), (52), (55), and (58) are

$$\tilde{\rho}_{;\alpha}u^{\alpha}+(\tilde{\rho}+\hat{P})\theta+\tilde{q}^{\alpha}_{;\alpha}=\Pi_{\alpha\beta}\sigma^{\alpha\beta}+\tilde{q}a^{\nu}s_{\nu}, \quad (59)$$

$$(\tilde{\rho}+\hat{P})a_{\alpha}+h^{\beta}_{\alpha}(\tilde{q}_{;\nu}u^{\nu}s_{\beta}+\tilde{q}s_{\beta;\nu}u^{\nu}-\hat{P}_{,\beta}+\Pi^{\mu}_{\beta;\mu})+\sigma_{\alpha\beta}\tilde{q}s^{\beta}+\frac{4}{3}\theta\tilde{q}s_{\alpha}=0, \quad (60)$$

$$\theta_{;\alpha}u^{\alpha}+\frac{\theta^2}{3}+\sigma_{\alpha\beta}\sigma^{\alpha\beta}-a^{\alpha}_{;\alpha}=-u_{\rho}u^{\beta}R^{\rho}_{\beta}=-4\pi(\tilde{\rho}+3\hat{P}), \quad (61)$$

$$R^{\rho}_{\beta}u_{\rho}h^{\alpha\beta}=h^{\alpha}_{\beta}\left(\sigma^{\beta\mu}_{;\mu}-\frac{2}{3}\theta^{;\beta}\right)+\sigma^{\alpha\beta}a_{\beta}=8\pi\tilde{q}s^{\alpha}, \quad (62)$$

$$E_{\gamma\nu}+4\pi\Pi_{\gamma\nu}=h^{\alpha}_{\gamma}h^{\mu}_{\nu}(a_{\alpha;\mu}-\sigma_{\alpha\mu;\beta}u^{\beta})-a_{\gamma}a_{\nu}-\sigma^{\beta}_{\nu}\sigma_{\gamma\beta}-\frac{2}{3}\theta\sigma_{\gamma\nu}-\frac{1}{3}\left(a^{\alpha}_{;\alpha}-\frac{1}{6}\sigma^2\right)h_{\gamma\nu}, \quad (63)$$

$$\theta E_{\alpha\beta}+(u^{\nu}E_{\mu\kappa;\nu}-4\pi u^{\nu}\Pi_{\mu\kappa;\nu}+4\pi\tilde{q}_{\kappa;\mu})h^{\mu}_{\alpha}h^{\kappa}_{\beta}+\frac{4\pi}{3}u^{\nu}\tilde{\rho}_{;\nu}h_{\alpha\beta}+E\sigma_{\alpha\beta}=-4\pi(\tilde{\rho}+\hat{P})\left(\sigma_{\alpha\beta}+\frac{\theta}{3}h_{\alpha\beta}\right)+4\pi(\tilde{q}_{\alpha}a_{\beta}+\tilde{q}_{\beta}a_{\alpha})+4\pi\Pi_{\nu\beta}\left(\sigma^{\nu}_{\alpha}+\frac{\theta}{3}h^{\nu}_{\alpha}\right), \quad (64)$$

$$E^{\lambda}_{\mu;\lambda}h^{\mu}_{\alpha}+a^{\lambda}E_{\alpha\lambda}=-8\pi\tilde{q}_{\kappa}\left(\sigma^{\kappa}_{\alpha}+\frac{\theta}{3}h^{\kappa}_{\alpha}\right)+\frac{4\pi}{3}(2\tilde{\rho}+3\hat{P})_{;\nu}h^{\nu}_{\alpha}-4\pi(\tilde{\rho}+\hat{P})a_{\alpha}+4\pi a^{\kappa}\Pi_{\alpha\kappa}-4\pi u^{\nu}\tilde{q}_{\mu;\nu}h^{\mu}_{\alpha}, \quad (65)$$

$$E_{\alpha\beta}-4\pi\Pi_{\alpha\beta}=\left(\frac{3m}{r^3}-4\pi\tilde{\rho}\right)\left(s_{\alpha}s_{\beta}+\frac{1}{3}h_{\alpha\beta}\right). \quad (66)$$

In each of Eqs. (59)–(66) there is only one scalar-independent component; thus, contracting with s^{α} we may write the equivalent set

$$\tilde{\rho}^{*}+(\tilde{\rho}+\tilde{P}_r)\theta=\frac{2}{3}\left(\theta+\frac{\sigma}{2}\right)\Pi-\tilde{q}^{\dagger}-2\tilde{q}a-\frac{2s^1}{r}\tilde{q}, \quad (67)$$

$$\tilde{P}^{\dagger}_r+(\tilde{\rho}+\tilde{P}_r)a+\frac{2s^1}{r}\Pi=\frac{\sigma}{3}\tilde{q}-\tilde{q}^{*}-\frac{4\theta}{3}\tilde{q}, \quad (68)$$

$$\theta^{*}+\frac{\theta^2}{3}+\frac{\sigma^2}{6}-a^{\dagger}-a^2-\frac{2as^1}{r}=-4\pi(\tilde{\rho}+3\tilde{P}_r)+8\pi\Pi, \quad (69)$$

$$\left(\frac{\sigma}{2}+\theta\right)^{\dagger}=-\frac{3\sigma s^1}{2r}+12\pi\tilde{q}, \quad (70)$$

$$E+4\pi\Pi=-a^{\dagger}-a^2-\frac{\sigma^{*}}{2}-\frac{\theta\sigma}{3}+\frac{as^1}{r}+\frac{\sigma^2}{12}, \quad (71)$$

$$\left(4\pi\tilde{P}_r+\frac{3m}{r^3}\right)\left(\theta+\frac{\sigma}{2}\right)+(E-4\pi\Pi+4\pi\tilde{\rho})^{*}=-\frac{12\pi s^1}{r}\tilde{q}, \quad (72)$$

$$(E+4\pi\tilde{\rho}-4\pi\Pi)^{\dagger}=\frac{3s^1}{r}(4\pi\Pi-E)+4\pi\tilde{q}\left(\frac{\sigma}{2}+\theta\right), \quad (73)$$

$$\frac{3m}{r^3}=4\pi\tilde{\rho}+4\pi(P_{\perp}-\tilde{P}_r)+E, \quad (74)$$

with $f^{\dagger}=f_{,\alpha}s^{\alpha}$, $f^{*}=f_{,\alpha}u^{\alpha}$, and $a^{\alpha}=as^{\alpha}$, and where the expansion is given by

$$\theta=u^{\mu}_{;\mu}=\frac{1}{2(1-\omega^2)^{1/2}}\left[e^{-\nu/2}\left(\dot{\lambda}+\frac{2\omega\dot{\omega}}{1-\omega^2}\right)+e^{-\lambda/2}\left(\omega\nu'+\frac{2\omega'}{1-\omega^2}+\frac{4\omega}{r}\right)\right]. \quad (75)$$

Then from Eqs. (27) and (75) it follows at once that

$$\frac{\sigma}{2}+\theta=\frac{3\omega s^1}{r}. \quad (76)$$

III. SPECIAL CASES

We shall now apply Eqs. (67)–(74) to analyze different particular cases.

A. Geodesic fluids

If the fluid is geodesic, nondissipative, and locally isotropic, then for bounded configurations, it follows at once from Eq. (68) and the vanishing of the pressure at the boundary that it should be dust. In this case the vanishing of the Weyl tensor implies the shear-free condition as follows from Eq. (72). On the other hand, the shear-free condition implies the conformally flat condition as follows from Eq. (71). Thus in this special case both conditions are equivalent. For nongeodesic fluids this equivalence is not generally true (see below).

B. Locally isotropic perfect fluids

Let us now consider locally isotropic and nondissipative fluids ($\Pi=q=\epsilon=0$) and find the relations linking the Weyl tensor, the shear, and the local density inhomogeneity. Although almost all results in this case are known, we think that it is worthwhile to present them in order to illustrate the general method that will be used later to study more complicated situations.

From Eq. (73), we obtain, after some rearrangements (with $\Pi=\tilde{q}=0$),

$$[r^3E]^\dagger + r^3 4\pi\rho^\dagger = 0. \quad (77)$$

Next, it is convenient to write Eq. (72), with the help of Eqs. (11), (67), (74), and (76), as

$$[r^3E]^{\cdot} + \frac{dr}{dt}[r^3E]' = -2\pi\sigma r^3(\rho+P_r)\sqrt{1-\omega^2}e^{\nu/2}, \quad (78)$$

implying that the vanishing of the Weyl tensor results in the vanishing of spatial gradients of the energy density and shear tensor.

Let us now assume $\rho^\dagger=0$; then, we obtain, from Eq. (77),

$$[r^3E]^\dagger = 0, \quad (79)$$

implying, since the Weyl tensor should be regular inside the fluid distribution, $E=0$. Thus $E=0$ and $\rho^\dagger=0$ are equivalent, and either one of them implies $\sigma=0$. These results were already known (see [22] and references therein).

Next, if $\sigma=0$, it follows from Eq. (70) that

$$\theta^\dagger = 0. \quad (80)$$

Observe that from the above it follows, using Eq. (29), that if the fluid is conformally flat and satisfies a barotropic equation of state of the form $P_r=P_r(\rho)$, then the fluid is geodesic $(a^\alpha=0)$.

Also, assuming the shear-free condition alone $(\sigma=0)$ it follows from Eq. (78) that the convective derivative of Er^3 vanishes, which in turn means that such a quantity remains constant for any fluid element along the fluid lines.

C. Locally anisotropic nondissipative fluids

We shall now relax the condition of local isotropy of the pressure and shall assume $\Pi\neq 0$. Then from Eqs. (76) and (73), it follows that

$$[r^3(E-4\pi\Pi)]^\dagger + r^3 4\pi\rho^\dagger = 0, \quad (81)$$

implying that the vanishing of $E-4\pi\Pi$ results in the vanishing of the spatial gradients of the energy density.

On the other hand, if we assume the vanishing of $\rho^\dagger$, then assuming that all physical variables are regular within the fluid distribution it follows at once that

$$E-4\pi\Pi = 0. \quad (82)$$

Thus $E-4\pi\Pi=0$ and $\rho^\dagger=0$ are equivalent, but neither one of them implies $\sigma=0$.

Therefore if we assume the spacetime to be conformally flat $(E=0)$, then the local anisotropy produces inhomogeneity in the energy density according to the equation

$$(r^3\Pi)^\dagger = r^3\rho^\dagger. \quad (83)$$

Next, it follows from Eq. (72), with the help of Eqs. (11), (67), (74), and (76), that

$$[r^3(E-4\pi\Pi)]^{\cdot} + \frac{dr}{dt}[r^3(E-4\pi\Pi)]' + 8\pi\Pi\omega e^{(\nu-\lambda)/2}r^2 = -2\pi\sigma r^3(\rho+P_r)\sqrt{1-\omega^2}e^{\nu/2}. \quad (84)$$

implying thereby that the convective derivative of $r^3(E-4\pi\Pi)$ is controlled not only by σ, but also by Π.

If $E=4\pi\Pi$, then the following link between the shear and the anisotropy results:

$$4\Pi\omega e^{-\lambda/2} = -\sigma r\sqrt{1-\omega^2}(\rho+P_r). \quad (85)$$

If the fluid is shear free and $E=4\pi\Pi$, then it is either static or locally isotropic. Of course in this last case the fluid is also conformally flat.

D. Locally isotropic dissipative fluids in the quasistatic evolution

We shall now relax the condition of nondissipation by allowing $q\neq 0$ (for simplicity we put $\epsilon=0$), but assuming that the evolution is slow, which means that $\omega^2=\dot{\omega}=\dot{q}=\ddot{\lambda}=\ddot{\nu}=0$ and $q\approx O(\omega)$ (see [19]). Then, from Eqs. (76) and (73), we obtain (in the quasistatic approximation)

$$E'e^{-\lambda/2} + \frac{3Ee^{-\lambda/2}}{r} = -4\pi\rho^\dagger, \quad (86)$$

taking into account that in the quasistatic approximation

$$\rho^\dagger = \rho' e^{-\lambda/2}, \quad (87)$$

we have

$$E' + \frac{3E}{r} = -4\pi\rho'. \quad (88)$$

Next, it follows from Eq. (72), with the help of Eq. (67),

$$\dot{E}e^{-\nu/2} + E'\omega e^{-\lambda/2} + \omega e^{-\lambda/2}\frac{3E}{r} - 4\pi e^{-\lambda/2}q' + 4\pi q e^{-\lambda/2}\left(\frac{1}{r}-\nu'\right) = -2\pi\sigma(\rho+P_r) \quad (89)$$

or, equivalently,

$$\frac{e^{\lambda/2}}{4\pi r^3}(r^3E)^* = q' + q\left(\nu'-\frac{1}{r}\right) - \frac{1}{2}e^{\lambda/2}\sigma(\rho+P_r). \quad (90)$$

From Eq. (88) it follows at once that conformally flat and $\rho'=0$ are equivalent conditions, which is also true in the perfect (nondissipating) fluid, in the quasistatic approximation.

Also, from Eq. (89) or (90) it follows that $E=0$ does not imply the shear-free condition. Indeed, assuming $E=0$ in Eq. (90) we have

$$q' + q\left(\nu'-\frac{1}{r}\right) = \frac{1}{2}e^{\lambda/2}\sigma(\rho+P_r), \quad (91)$$

yielding

$$q=re^{-\nu}\left[\int \frac{e^{\lambda/2+\nu}}{2r}\sigma(\rho+P_r)dr+\beta(t)\right]. \tag{92}$$

If we further impose the shear-free condition, then one obtains, from Eq. (92),

$$q=r\beta(t)e^{-\nu}, \tag{93}$$

leading to a condition on the temperature, which may be obtained using the Landau-Eckart equation

$$q_\mu=\kappa h^\nu_\mu(T_{,\nu}-Ta_\nu) \tag{94}$$

or

$$q=-\kappa e^{-\lambda/2}\left(T'+\frac{T\nu'}{2}\right). \tag{95}$$

Using Eq. (95) in Eq. (93) we obtain

$$T=e^{-\nu/2}\left[C(t)-\frac{\beta(t)}{\kappa}\int_0^r re^{(\lambda-\nu)/2}dr\right]. \tag{96}$$

The two functions $\beta(t)$ and $C(t)$ are simply related to the total luminosity of the sphere and the central temperature, through Eqs. (93) and (96), respectively. A simple model satisfying $E=\sigma=0$ will be next presented.

E. Conformally flat, shear-free sphere, dissipating in the quasistatic regime (with $\epsilon=0$)

From Eq. (5) and $E=0$ we have

$$8\pi P_\perp=\frac{e^{-\lambda}}{r}\left(\nu'-\lambda'-\frac{1}{r}\right)+\frac{1}{r^2}. \tag{97}$$

Then substracting Eq. (4) from Eq. (97) and considering $\Pi=0$, we obtain

$$e^{-\lambda}=r^2c_1+1, \tag{98}$$

where $c_1(t)$ is an arbitrary function of time. Substituting Eq. (98) into Eq. (3) yields

$$8\pi\rho=-3c_1. \tag{99}$$

Considering $E=0$ with Eq. (98) we obtain

$$\nu''+\frac{\nu'^2}{2}-\frac{\nu'}{r(r^2c_1+1)}=0, \tag{100}$$

which has the solution

$$e^{\nu/2}=(r^2c_1+1)^{1/2}c_2+c_3, \tag{101}$$

where $c_2(t)$ and $c_3(t)$ are arbitrary functions of t.

Substituting Eqs. (98) and (101) into Eq. (4) we obtain

$$8\pi P_r=c_1[1+2c_2e^{-(\lambda+\nu)/2}]. \tag{102}$$

From Eq. (27), condition $\sigma=0$ can be rewritten as

$$\left(\frac{\omega e^{\nu/2}}{r}\right)'=-\frac{\dot{\lambda}e^{\lambda/2}}{2r}, \tag{103}$$

which after integration becomes

$$\omega=\left(c_4-\frac{\dot{c}_1}{2c_1}e^{\lambda/2}\right)re^{-\nu/2}, \tag{104}$$

where $c_4(t)$ is an arbitrary function of t.

From Eqs. (98), (99), (101), (102) we have

$$8\pi(\rho+P_r)\omega=-(2c_1c_4-\dot{c}_1e^{\lambda/2})rc_3e^{-\nu}. \tag{105}$$

Now substituting Eqs. (98), (101), (105) into Eq. (6) we obtain

$$8\pi q=(2c_1c_3c_4+\dot{c}_1c_2)re^{-\nu}. \tag{106}$$

Next, using Eqs. (21) and (22) in Eqs. (98) and (101) we obtain

$$c_1=-\frac{2M}{r_\Sigma^3} \tag{107}$$

and

$$c_3=\sqrt{1-\frac{2M}{r_\Sigma}}(1-c_2). \tag{108}$$

Also, from the junction condition (23) and from Eq. (104) evaluated at the boundary surface, it follows that

$$c_1(1+2c_2)=(2c_1c_3c_4+\dot{c}_1c_2)\frac{r_\Sigma}{(1-2M/r_\Sigma)} \tag{109}$$

and

$$\dot{c}_1=2c_1\sqrt{1-\frac{2M}{r_\Sigma}}\left(c_4-\frac{\omega_\Sigma}{r_\Sigma}\sqrt{1-\frac{2M}{r_\Sigma}}\right). \tag{110}$$

Solving algebraically the system (108)–(110) for c_2, c_3, and c_4, we can express these functions in terms of M, r_Σ, ω_Σ, and $\dot{M}$.

We shall further specify our model by assuming

$$c_2=1, \tag{111}$$

$$c_3=0, \tag{112}$$

implying

$$e^{-\lambda}=e^{\nu}=1-\frac{2Mr^2}{r_\Sigma^3}. \tag{113}$$

Using Eqs. (107), (111), (112), and (113) in Eqs. (99) and (102) we obtain

$$8\pi\rho=-8\pi P_r=\frac{6M}{r_\Sigma^3}. \tag{114}$$

Then using Eqs. (111) and (112) in Eq. (106) we obtain

$$8\pi q=r\dot{c}_1 e^{-\nu}, \tag{115}$$

which is our equation (93) with $\dot{c}_1/8\pi=\beta(t)$. By virtue of Eqs. (107) and (109), the expression for q becomes

$$8\pi q=-\frac{6Mr}{r_\Sigma^4}\frac{(1-2M/r_\Sigma)}{(1-2Mr^2/r_\Sigma^3)}. \tag{116}$$

Next, using Eqs. (107), (109), (113) in Eq. (96), we obtain

$$T=\frac{1}{\sqrt{1-2Mr^2/r_\Sigma^3}}\left[T_c-\frac{3}{16\pi\kappa r_\Sigma}\left(1-\frac{2M}{r_\Sigma}\right)\times\log\left(1-\frac{2Mr^2}{r_\Sigma^3}\right)\right], \tag{117}$$

with T_c denoting the temperature at $r=0$, and, from Eq. (104),

$$\omega=\frac{r}{r_\Sigma}\sqrt{\frac{1-2M/r_\Sigma}{1-2Mr^2/r_\Sigma^3}}\left[\frac{3}{2}\left(1-\sqrt{\frac{1-2M/r_\Sigma}{1-2Mr^2/r_\Sigma^3}}\right)+\omega_\Sigma\right]. \tag{118}$$

Thus our model represents a conformally flat sphere of fluid evolving slowly and shear free, with homogeneous energy density and pressure, satisfying the ''inflationary'' equation of state $\rho+P_r=0$, with an inward $(q<0)$ heat flux.

F. General case

First of all, it should be noticed that in the presence of both anisotropic pressure and dissipation for B_1 warped product spacetimes (which include the spherically symmetric case) there exist some restrictions on physical variables, provided by the syzygy [23]. However, in our case, as follows from the definitions in Sec. II B we have $|\tilde{q}^\nu\tilde{q}_\nu|=(s_\nu\tilde{q}^\nu)^2$ (notice that our signature is -2), and therefore the syzygy [see Eq. (37) in [23]] reduces to the identity $0=0$.

Then, we obtain from Eqs. (76) and (73), after some rearrangements,

$$[r^3(E-4\pi\Pi-4\pi\tilde{q}\omega)]^\dagger+r^3 4\pi\tilde{\rho}^\dagger=-4\pi(\tilde{q}\omega)^\dagger r^3 \tag{119}$$

or, equivalently

$$[r^3(E-4\pi\Pi)]^\dagger+4\pi r^3\tilde{\rho}^\dagger=4\pi r^3\tilde{q}\left(\frac{\sigma}{2}+\theta\right), \tag{120}$$

and if $\tilde{q}=0$, we recover Eq. (81).

Also, from Eq. (72), with the help of Eq. (67) we get

$$[r^3(E-4\pi\Pi)]^{\cdot}+\frac{dr}{dt}[r^3(E-4\pi\Pi)]'+8\pi\Pi\omega e^{(\nu-\lambda)/2}r^2+4\pi r^3\sqrt{1-\omega^2}e^{\nu/2}\left[\tilde{q}\left(\frac{s^1}{r}-2a\right)-\tilde{q}^\dagger\right]=-2\pi\sigma r^3(\tilde{\rho}+\tilde{P}_r)\sqrt{1-\omega^2}e^{\nu/2}, \tag{121}$$

which yields Eq. (84) in the nondissipative case.

From Eq. (120) it is clear that the appearance of inhomogeneities in the energy density is controlled by the Weyl tensor, the anisotropy of the pressure, and the dissipation. If we express the heat flux through the temperature, the relaxation time, and the heat conduction coefficient, by means of some transport equation (e.g., [24]), then the aforementioned parameters may be related to the creation of such density inhomogeneities.

Indeed, assuming, for example, the transport equation [25]

$$\tau h^\mu_\nu(q^\nu)^*+q^\mu=\kappa h^{\mu\nu}(T_{,\nu}-Ta_\nu)-\frac{1}{2}\kappa T^2\left(\frac{\tau u^\alpha}{\kappa T^2}\right)_{;\alpha}q^\mu, \tag{122}$$

where τ, κ, and T denote the relaxation time, the thermal conductivity, and the temperature, respectively, we obtain, putting for simplicity $\epsilon=0$ and using Eq. (68),

$$q=\frac{\tau\left[P_r^\dagger+(\rho+P_r)a+\frac{2}{3\omega}\left(\frac{\sigma}{2}+\Theta\right)\Pi\right]-\kappa(T^\dagger+Ta)}{1+\frac{\tau}{2}\left[\frac{1}{3}(2\sigma-5\Theta)+\frac{\tau^*}{\tau}-\frac{\kappa^*}{\kappa}-\frac{2T^*}{T}\right]}. \tag{123}$$

Then replacing Eq. (123) into Eq. (120) (with $\epsilon=0$) we obtain an expression which brings out the role of thermodynamic variables in the appearance of density inhomogeneities: namely,

$$[r^3(E-4\pi\Pi)]^\dagger+4\pi r^3\tilde{\rho}^\dagger=4\pi r^3\left(\frac{\sigma}{2}+\theta\right)\frac{\tau\left[P_r^\dagger+(\rho+P_r)a+\frac{2}{3\omega}\left(\frac{\sigma}{2}+\Theta\right)\Pi\right]-\kappa(T^\dagger+Ta)}{1+\frac{\tau}{2}\left[\frac{1}{3}(2\sigma-5\Theta)+\frac{\tau^*}{\tau}-\frac{\kappa^*}{\kappa}-\frac{2T^*}{T}\right]}. \tag{124}$$

A similar conclusion applies to the shear of the fluid, as follows from Eq. (121).

With the sole purpose of bringing out the role of dissipation in the formation of density inhomogeneities, let us present a simple and highly idealized model.

G. Dissipative model with $E-4\pi\Pi=0$

Since we want to exhibit the role of dissipation in the formation of inhomogeneities, we shall assume $E-4\pi\Pi=0$; then, from Eqs. (55) and (58), it follows that

$$E-4\pi\Pi=0\Leftrightarrow m=\frac{4\pi}{3}\tilde{\rho}r^3\Leftrightarrow e^{-\lambda}=1-\frac{8\pi}{3}\tilde{\rho}r^2. \tag{125}$$

From Eqs. (72), (120), and (76) we obtain

$$\tilde{\rho}^\dagger=\tilde{q}\frac{3\omega s^1}{r} \tag{126}$$

and

$$3s^1r^2(\tilde{q}+\tilde{P}_r\omega)+(\tilde{\rho}r^3)^*=0; \tag{127}$$

then, combining the Einstein equations (12) and (13) with Eq. (125), it follows that

$$\nu'=\frac{8\pi(\tilde{\rho}+\tilde{\rho}'r+3\tilde{P}_r)r}{3\left(1-\frac{8\pi}{3}\tilde{\rho}r^2\right)}. \tag{128}$$

We shall further assume the equation of state

$$\tilde{P}_r=\frac{1}{3}\tilde{\rho}; \tag{129}$$

then, replacing Eq. (129) into Eq. (128) we obtain

$$e^\nu=\frac{\left(1-\frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2\right)^2}{1-\frac{8\pi}{3}\tilde{\rho}r^2}. \tag{130}$$

Next, using (127) and (126) it follows that:

$$\tilde{\rho}^\dagger=\frac{3\omega\tilde{q}}{r}\sqrt{\frac{1-\frac{8\pi}{3}\tilde{\rho}r^2}{1-\omega^2}} \tag{131}$$

and

$$\omega=\frac{-\dot{\tilde{\rho}}r\pm\sqrt{(\dot{\tilde{\rho}}r)^2-r\tilde{\rho}'(r\tilde{\rho}'+4\tilde{\rho})\left(1-\frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2\right)^2}}{(r\tilde{\rho}'+4\tilde{\rho})\left(1-\frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2\right)}. \tag{132}$$

In the particular case $\tilde{\rho}=\tilde{\rho}(t)$ (but $\tilde{\rho}^\dagger\neq0$), these last equations reduce to

$$\omega=\frac{-\dot{\tilde{\rho}}r}{2\tilde{\rho}\left(1-\frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2\right)} \tag{133}$$

and

$$\tilde{q}=\frac{1}{3}\dot{\tilde{\rho}}re^{(\lambda-\nu)/2}. \tag{134}$$

From junction conditions it is very simple to express $\tilde{\rho}$ and $\dot{\tilde{\rho}}$, and thereof all physical and metric variables in terms of the total mass M, the velocity of the surface ω_Σ, the radius r_Σ, and the luminosity of the sphere. Equation (131) shows how dissipation produces density inhomogeneity. It is worth noticing that both kinds of dissipative processes (ϵ and q) may produce such an inhomogeneity.

IV. CONCLUSIONS

The study of general relativistic gravitational collapse, which has attracted the attention of researchers since the seminal paper by Oppenheimer and Snyder [11], is mainly motivated by the fact that the gravitational collapse of massive stars represents one of the few observable phenomena where general relativity is expected to play a relevant role. Ever since that work, much has been written by researchers trying to provide models of evolving gravitating spheres. However, this endeavor proved to be difficult and uncertain. Different kinds of advantages and obstacles appear, depending on the approach adopted for the modeling.

Here, we have established a set of equations governing the structure and evolution of self-gravitating spherically symmetric dissipative anisotropic fluids. For reasons explained in the Introduction, emphasis has been put on the role played by the Weyl tensor, the anisotropy of the pressure, dissipation, density inhomogeneity, and the shear tensor.

The particular simple relation between the Weyl tensor and density inhomogeneity, for perfect fluids (also valid for locally isotropic, dissipative fluids in the quasistatic regime), is at the origin of the Penrose's proposal to provide a gravitational arrow of time. However, the fact that such a relationship is no longer valid in the presence of the local anisotropy of the pressure and/or dissipative processes explains its failure in scenarios where the above-mentioned factors are present.

From Eq. (120) it is apparent that the production of density inhomogeneities is related to a quantity involving all those factors $([r^3(E-4\pi\Pi)]^{\dagger}-4\pi r^3\tilde{q}(\sigma/2+\theta))$. Alternatively, the appearance of such inhomogeneities may be described by means of thermodynamical variables (if a transport equation is assumed), as indicated in Eq. (124).

This situation is illustrated in the example provided in the last section, where density inhomogeneity is produced by dissipative processes alone. Thus, if following Penrose we adopt the point of view that self-gravitating systems evolve in the sense of increasing of density inhomogeneity, then the absolute value of the quantity above (or some function of it) should increase, providing an alternative definition for an arrow of time.

ACKNOWLEDGMENT

J.M. and J.O. acknowledge financial assistance under grant BFM2003-02121 (M.C.T. Spain).

[1] R. Penrose, in *General Relativity, An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), pp. 581–638.

[2] J. Wainwright, Gen. Relativ. Gravit. **16**, 657 (1984); S.W. Goode and J. Wainwright, Class. Quantum Grav. **2**, 99 (1985); W.B. Bonnor, Phys. Lett. **112A**, 26 (1985).

[3] W.B. Bonnor, Phys. Lett. A **122**, 305 (1987); S.W. Goode, A. Coley, and J. Wainwright, Class. Quantum Grav. **9**, 445 (1992); N. Pelavas and K. Lake, Phys. Rev. D **62**, 044009 (2000).

[4] L. Herrera, Gen. Relativ. Gravit. **35**, 437 (2003).

[5] F. Mena and R. Tavakol, Class. Quantum Grav. **16**, 435 (1999).

[6] D.M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979); D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984); R.P.A.C. Newman, Class. Quantum Grav. **3**, 527 (1986); B. Waugh and K. Lake, Phys. Rev. D **38**, 1315 (1988); I. Dwivedi and P. Joshi, Class. Quantum Grav. **9**, L69 (1992); P.S. Joshi and I.H. Dwivedi, Phys. Rev. D **47**, 5357 (1993); T.P. Singh and P. Joshi, Class. Quantum Grav. **13**, 559 (1996); P. Joshi, N. Dadhich, and R. Maartens, Phys. Rev. D **65**, 101501 (2002).

[7] G. Lemaitre, Ann. Soc. Sci. Bruxelles, Ser. 1 **53**, 51 (1933); R. Bowers and E. Liang, Astrophys. J. **188**, 657 (1974); L. Herrera and N.O. Santos, Phys. Rep. **286**, 53 (1997); H. Bondi, Mon. Not. R. Astron. Soc. **262**, 1088 (1993); L. Herrera, Phys. Lett. A **165**, 206 (1992); W. Barreto, Astrophys. Space Sci. **201**, 191 (1993); A. Coley and B. Tupper, Class. Quantum Grav. **11**, 2553 (1994); J. Martínez, D. Pavón, and L. Núñez, Mon. Not. R. Astron. Soc. **271**, 463 (1994); T. Singh, P. Singh, and A. Helmi, Nuovo Cimento Soc. Ital. Fis., B **110**, 387 (1995); A. Das, N. Tariq, and J. Biech, J. Math. Phys. **36**, 340 (1995); R. Maartens, S. Maharaj, and B. Tupper, Class. Quantum Grav. **12**, 2577 (1995); G. Magli, *ibid.* **14**, 1937 (1997); A. Das, N. Tariq, D. Aruliah, and T. Biech, J. Math. Phys. **38**, 4202 (1997); L. Herrera, A. Di Prisco, J. Hernández-Pastora, and N.O. Santos, Phys. Lett. A **237**, 113 (1998); E. Corchero, Class. Quantum Grav. **15**, 3645 (1998); E. Corchero, Astrophys. Space Sci. **259**, 31 (1998); H. Bondi, Mon. Not. R. Astron. Soc. **302**, 337 (1999); H. Hernández, L. Núñez, and U. Percoco, Class. Quantum Grav. **16**, 897 (1999); T. Harko and M. Mak, J. Math. Phys. **41**, 4752 (2000); A. Das and S. Kloster, Phys. Rev. D **62**, 104002 (2000); S. Jhingan and G. Magli, *ibid.* **61**, 124006 (2000); L. Herrera, A. Di Prisco, J. Ospino, and E. Fuenmayor, J. Math. Phys. **42**, 2129 (2001); L. Herrera, J. Martínez, and J. Ospino, *ibid.* **43**, 4889 (2002); J. Krisch and E. Glass, *ibid.* **43**, 1509 (2002); E. Corchero, Class. Quantum Grav. **19**, 417 (2002); T. Harko and M. Mak, Ann. Phys. (Leipzig) **11**, 3 (2002); M. Mak, P. Dobson, and T. Harko, Int. J. Mod. Phys. D **11**, 207 (2002); R. Goswami and P. Joshi, Class. Quantum Grav. **19**, 5229 (2002); K. Devi and M. Gleiser, Gen. Relativ. Gravit. **34**, 1793 (2002); R. Giambo, Class. Quantum Grav. **19**, 4399 (2002); M. Mak and T. Harko, Chin. J. Astrophys. **2**, 248 (2002); M. Mak and T. Harko, Proc. R. Soc. London **459**, 393 (2003); K. Devi and M. Gleiser, Gen. Relativ. Gravit. **35**, 1793 (2003); A. Pérez Martínez, H.P. Rojas, and H.M. Cuesta, Eur. Phys. J. C **29**, 111 (2003); A. Fuzfa, Class. Quantum Grav. **20**, 4753 (2003).

[8] L. Herrera and N.O. Santos, Gen. Relativ. Gravit. **27**, 1071 (1995); L. Herrera, A. Di Prisco, J.L. Hernández-Pastora, and N.O. Santos, Phys. Lett. A **237**, 113 (1998).

[9] C.B. Collins and J. Wainwright, Phys. Rev. D **27**, 1209 (1983); E.N. Glass, J. Math. Phys. **20**, 1508 (1979); P. Joshi, R. Goswami, and N. Dadhich, gr-qc/0308012; L. Herrera and N.O. Santos, Mon. Not. R. Astron. Soc. **343**, 1207 (2003).

[10] L. Herrera, A. Di Prisco, and J. Martínez, Astrophys. Space Sci. **277**, 447 (2001); L. Herrera, A. Di Prisco, and E. Fuenmayor, Class. Quantum Grav. **20**, 1125 (2003).

[11] J. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

[12] D. Kazanas and D. Schramm, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).

[13] W.D. Arnett, Astrophys. J. **218**, 815 (1977).

[14] D. Kazanas, Astrophys. J. **222**, L109 (1978).

[15] J. Lattimer, Nucl. Phys. **A478**, 199 (1988).

[16] M. May and R. White, Phys. Rev. **141**, 1232 (1966); J. Wilson, Astrophys. J. **163**, 209 (1971); A. Burrows and J. Lattimer, *ibid.* **307**, 178 (1986); R. Adams, B. Cary, and J. Cohen, Astrophys. Space Sci. **155**, 271 (1989); W.B. Bonnor, A. Oliveira, and N.O. Santos, Phys. Rep. **181**, 269 (1989); M. Govender, S. Maharaj, and R. Maartens, Class. Quantum Grav. **15**, 323 (1998); D. Schafer and H. Goenner, Gen. Relativ. Gravit. **32**,

2119 (2000); M. Govender, R. Maartens, and S. Maharaj, Phys. Lett. A **283**, 71 (2001); S. Wagh *et al.*, Class. Quantum Grav. **18**, 2147 (2001).

[17] W.B. Bonnor and H. Knutsen, Int. J. Theor. Phys. **32**, 1061 (1993).

[18] H. Bondi, Proc. R. Soc. London **281**, 39 (1964).

[19] L. Herrera, W. Barreto, A. Di Prisco, and N.O. Santos, Phys. Rev. D **65**, 104004 (2002).

[20] N.O. Santos, Mon. Not. R. Astron. Soc. **216**, 403 (1985).

[21] W. Kundt and M. Trumper, Abh. Math.-Naturwiss. Kl., Akad. Wiss. Lit., Mainz **l12**, 966 (1962).

[22] K. Lake, Gen. Relativ. Gravit. **36**, 193 (2003).

[23] K. Santosuosso, D. Pollney, N. Pelavas, P. Musgrave, and K. Lake, Comput. Phys. Commun. **115**, 381 (1998).

[24] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976); W. Israel and J. Stewart, Phys. Lett. **58A**, 2131 (1976); Ann. Phys. (N.Y.) **118**, 341 (1979); D. Pavón, D. Jou, and J. Casas-Vázquez, Ann. Inst. Henri Poincare, Sect. A **36**, 79 (1982).

[25] R. Maartens, astro-ph/9609119.