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Black hole fragmentation and holography

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We discuss the entropy change due to fragmentation for black hole solutions in various dimensions. We find three different types of behavior. The entropy may decrease, increase, or have a mixed behavior, characterized by the presence of a threshold mass. For two-dimensional (2D) black holes we give a complete characterization of the entropy behavior under fragmentation, in the form of sufficient conditions imposed on the function J, which defines the 2D gravitational model. We compare the behavior of the gravitational solutions with that of free field theories in d dimensions. This excludes for a broad class of solutions, including asymptotically flat black holes, the possibility of finding a duality between gravity and a field theory, which realizes the holographic principle. We find that the most natural candidates for holographic duals of the black hole solutions with mixed behavior are field theories with a mass gap. We also discuss the possibility of formulating entropy bounds that make reference only to the energy of a system.

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I. INTRODUCTION

One of the most striking novelties in the research on gravitational physics is the possibility that gravity in d dimensions could be described by a local field theory in $d-1$ dimensions [1–3]. The theoretical evidence for such a holographic description of gravity is mounting. Indications that holography could be a fundamental feature of the gravitational interaction come from different directions: string theory, black hole physics, cosmology [4–6] (for a recent review see [7]). A particularly interesting output of these investigations has been the formulation of stringent holographic bounds for the entropy of a system occupying a given region of space [5,8].

An explicit realization of the holographic principle has been found only in particular cases, essentially for anti–de Sitter (AdS) (and de Sitter) gravity and the so-called anti–de Sitter conformal field theory (AdS-CFT) correspondence [1–3]. A general way to explicitly realize the holographic principle for generic gravitational systems, in particular, for gravity in asymptotically flat spacetimes, is still lacking. In particular, it is not clear if the realizations of the holographic principle always take the form of a correspondence between d-dimensional gravity and a field theory in $d-1$ dimensions, or if there could be some alternative, still unknown, realization of it. If the holographic principle has to be considered a genuine feature of every quantum theory of gravity, one could explain our lack of understanding of the holographic principle as a lack in understanding quantum gravity. However, there are strong indications that holography is a feature of gravity that already appears, and therefore should be explained, at the semiclassical level. The Bekenstein-Hawking area law for the black hole entropy is the most striking example of holographic behavior of a gravitational system that has to be explained already at the semiclassical level.

An alternative strategy one can use in this context is to explore the similarities and the differences between gravity and local field theories in order to check at a fundamental level the possibility of finding correspondences between the two classes of theories. This approach can be very powerful. One nice example, discussed in almost every introductory paper on the holographic principle, is the scaling behavior of the entropy as a function of the volume of the system for a local field theory compared to that of a black hole. For a local field theory the entropy is an extensive quantity; it scales as the volume of the space. On the other hand, the entropy of a black hole scales as the area of the horizon. This simple fact enables one to conclude that the correspondence between gravity and field theory, if it exists, must be *holographic*.

In this paper we will focus on another aspect of the relationship between gravity and local field theory, namely, on the dependence of the entropy for composite systems on the energy. Working in the microcanonical ensemble the entropy-energy relation for a free field theory in d dimensions is given by

$$S \propto E^{(d-1)/d}, \tag{1}$$

which considering two arbitrary excitations with energy E_1, E_2 satisfies the inequality

$$S(E_1+E_2) < S(E_1)+S(E_2). \tag{2}$$

Conversely, asymptotically flat black holes of masses M_1, M_2 in $d \geqslant 4$ dimensions satisfy

$$S(M_1+M_2) > S(M_1)+S(M_2), \tag{3}$$

i.e., fragmentation of a black hole of mass M_1+M_2 into two smaller black holes of masses M_1 and M_2 is entropically not preferred. Assuming the existence of a gravity–field theory correspondence, one has to identify the black hole masses M_1, M_2 as excitations E_1, E_2 of the field theory. It follows that Eq. (2) contradicts Eq. (3) and consequently that the assumed correspondence of gravity and field theory cannot be true. Also, skipping this problem, i.e., assuming the existence of black hole solutions satisfying inequality (2) rather

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than (3), one is faced by another problem. At first sight Eq. (2) seems incompatible with every entropy bound because a black hole could always increase its entropy by fragmenting into smaller black holes.

It is important to notice that the relations (2) and (3) have a different physical meaning when applied to a field theory or to a black hole. For a black hole these inequalities give information about the process of fragmentation of a black hole. Conversely, for a field theory the inequalities do not describe the separation of the system into parts; they have nothing to do with fragmentation. In particular, any interpretation of Eq. (2) as describing the fragmentation of a gas of free particles (for instance, in the form of a process of diffusion) is misleading. For a field theory Eq. (2) tells us simply that the entropy associated with a thermal excitation of energy E_1+E_2 is less than the sum of the entropies associated with two single excitations of energy E_1 and E_2. If we have a duality relating a gravitational theory with a field theory satisfying Eq. (2), we expect that in the field theory excitations will exist that are in correspondence with the single black hole state and with the composite state. Moreover, these field theory excitations must satisfy Eq. (2).

In this paper we will analyze in detail for various classes of black holes in various dimensions the validity of the relation (3). We will show that Eq. (3) holds true only for asymptotically flat black holes. Black holes with different asymptotic behavior (for instance, AdS black holes) may satisfy Eq. (2). For the two-dimensional (2D) case we will be able to give a complete characterization of the entropy for composite black holes in terms of the asymptotic behavior of the solution. We will also show the existence of black hole solutions with mixed behavior, i.e., solutions satisfying either Eq. (2) or Eq. (3) when the masses M_1 and M_2 are above or below some critical value $\tilde{M}_0$. We will argue that this mixed behavior has a natural counterpart in a field theory with a mass gap. Finally, we also discuss the compatibility of Eq. (2) with entropy bounds.

II. ASYMPTOTICALLY FLAT BLACK HOLES

Let us first consider the d-dimensional Schwarzschild black hole ($d\geqslant 4$)

$$ds^2=-\left(1-\frac{k_d GM}{r^{d-3}}\right)dt^2+\left(1-\frac{k_d GM}{r^{d-3}}\right)^{-1}dr^2+r^2 d\Omega_{d-2}^2, \tag{4}$$

where M is the mass and $k_d=16\pi/(d-2)\Omega_{d-2}$, Ω_{d-2} being the volume of the unit S^{d-2} transverse sphere. The black hole entropy is given by the area law

$$S=\frac{A}{4G}=\frac{\Omega_{d-2}}{4}(k_d L_p M)^{(d-2)/(d-3)}, \tag{5}$$

where $L_p=G^{1/(d-2)}$ is the Planck length.

In order to discuss the inequality (3) on physical grounds, we will make the following assumptions: (1) The theory admits, at least in some approximation, multi-black-hole solutions and (2) the gravitational potential energy of the multi-black-hole configuration can be neglected with respect to the black hole masses. Assumption (1) is necessary in order to give to a multi-black-hole configuration a precise meaning, whereas assumption (2) assures us that the multi-black-hole configuration can be treated as a composite system with zero binding energy, whose mass and entropy are simply the sums of those of the elementary constituents.

Assumption (2) is rather restrictive. It can be applied without problems only to gravitational systems that allow for ‘‘asymptotically free’’ states, i.e., multi-black-hole solutions whose binding energy goes to zero when the constituents are pulled apart. This is the case of asymptotically flat geometries whose Newtonian potential, behaving asymptotically as $1/r$, allows for composite states of zero binding energy. When the gravitational potential does not vanish asymptotically, we cannot define asymptotically free states and in general the binding energy cannot be neglected. This is, for instance, the case of asymptotically AdS solutions, which will be discussed in the next section.

If the binding energy of the two-black-hole state cannot be neglected the right hand sides of Eqs. (2) and (3) do not give the correct entropy of the composite state. In principle, one could try to compute the binding energy of the state. For gravitational systems not allowing for asymptotically free states, this is a very hard task unless one knows the exact solution describing the composite state. For this reason, in all the situations in which assumption (2) do not apply, we will make no attempt to evaluate the gravitational potential energy of the multi-black-hole configuration. Although in these situations Eqs. (2), (3) cannot be used to compare the entropy of the single black hole state with the composite one, they can still be very useful to extract information about dualities between the gravitational system and the field theory. Obviously, now the two excitations of energy E_1,E_2 of the field theory are not in correspondence with the bound state of two black holes but with some other state of the gravitational theory.

Using Eq. (5) one easily finds that Eq. (3) is satisfied. As expected, for Schwarzschild black holes the entropy is maximized by the single black hole configuration with mass M_1+M_2. The same holds true for asymptotically flat charged black holes. Considering, for instance, the four-dimensional Reissner-Nordstrom solution, one finds for the entropy

$$S=(ML_p+\sqrt{M^2L_p^2-Q^2})^2, \tag{6}$$

where Q is the electric charge of the black hole. Again, from Eq. (6) it follows that $S(M_1+M_2)>S(M_1)+S(M_2)$. There is one simple argument that can be used to argue that inequality (3) holds in general for asymptotically flat black holes, at least when ML_p is much bigger than the (eventually present) black hole charges. The area law gives $S\propto(r_+/L_p)^{d-2}$, where r_+ is the horizon radius. For asymptotically flat black holes and when $ML_p\gg Q_i$, where Q_i are the charges associated with the black hole, the gravitational potential is dominated by the Newtonian term so that we have $r_+\propto L_p^{(d-2)/(d-3)}M^{1/(d-3)}$. It follows that $S\propto(L_pM)^{(d-2)/(d-3)}$.

III. AdS BLACK HOLES

The argument presented at the end of the previous section does not apply to nonasymptotically flat black holes. For 2D black holes a general discussion will be presented in the next section. Here we will discuss only the most interesting case, namely, asymptotically AdS black holes. The entropy-mass relation for the d-dimensional Schwarzschild–anti–de Sitter black hole ($d \geqslant 4$),

$$ds^2 = -\left(1+\lambda^2 r^2 - \frac{k_d GM}{r^{d-3}}\right)dt^2 + \left(1+\lambda^2 r^2 - \frac{k_d GM}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2, \qquad (7)$$

is rather complicated. However, a simple formula can be found for black holes with $ML_p^{d-2}\lambda^{d-3} \gg 1$. In this latter case we have

$$S \propto \left(\frac{M}{L_p \lambda^2}\right)^{(d-2)/(d-1)}. \qquad (8)$$

Differently from the asymptotically flat case, we see that now inequality (2) is satisfied. This fact is perfectly consistent with the existence of an AdS-CFT correspondence between d-dimensional AdS gravity and $(d-1)$-dimensional conformal field theory. The d-dimensional black hole entropy (8) reproduces correctly, after the $E=M$ identification, the entropy (1) for a free field theory in $d-1$ dimensions.

For the AdS black hole (7) the gravitational potential energy behaves asymptotically as r^2. This implies that the theory does not allow for the asymptotically free states described in the previous section. Our assumption (2) does not apply to this case and the binding energy of the composite state cannot be neglected. This means that, although Eq. (2) is satisfied, fragmentation of the AdS black hole is not necessarily entropically preferred. Hence, the validity of Eq. (2) is not in contradiction with the known classical stability of the AdS black hole (7). The entropy-temperature relation (8) together with Eq. (2) give us an indication of the existence of an AdS–field theory duality but at the same time tell us that the gravitational counterpart of the free field excitations cannot be the bound state of two AdS black holes.

Let us now discuss in some detail the three-dimensional (3D) case, the Banados-Teitelboim-Zanelli (BTZ) black hole [9]. This case is very instructive not only because it is possible to solve exactly the inequality (2) but also because it clarifies the role played by the black hole ground state in our considerations. The BTZ black hole solution with zero angular momentum is

$$ds^2 = -(\lambda^2 r^2 - 8GM)dt^2 + (\lambda^2 r^2 - 8GM)^{-1} dr^2 + r^2 d\phi^2, \qquad (9)$$

where the black hole mass M is defined with reference to the $M=0$ black hole ground state $ds^2 = -\lambda^2 r^2 dt^2 + \lambda^{-2} r^{-2} dr^2 + r^2 d\phi^2$. The entropy is given by $S = 2\pi r_+/4G = (\pi/\lambda)\sqrt{2M/G}$ so that Eq. (2) is satisfied for every value of the mass M. On the other hand, if the black hole mass is defined with reference to the full, geodetically complete AdS spacetime $ds^2 = -(1+\lambda^2 r^2)dt^2 + (1+\lambda^2 r^2)^{-1} dr^2 + r^2 d\phi^2$, the mass spectrum looks rather different. The $M=0$ ground state is separated from the continuous part of the spectrum with $M \geqslant 1/8G$ by a mass gap. For $M \geqslant 1/8G$ the entropy is given by $S = (\pi/2\lambda G)\sqrt{8GM-1}$.

At first sight it may seem rather strange that the thermodynamical features of the BTZ black hole depend so crucially on the choice of the zero mass solution. This behavior can be explained by noticing that the zero mass state of the BTZ solution (9) is separated from the full AdS spacetime by a tower of states describing naked conical singularities. These states are not physically allowed so that a mass gap, separating the full AdS spacetime from the continuous part of the spectrum, is produced. It is the presence of this mass gap that makes the choice of the zero mass solution so crucial for the thermodynamical behavior of the BTZ black hole.

Taking for simplicity $M_1 = M_2 = M$, we see that Eq. (2) is satisfied only for $M \geqslant 3/16G$, whereas for $1/8G \leqslant M \leqslant 3/16G$ we have $S(M_1+M_2) \geqslant S(M_1) + S(M_2)$. We will come back to this point in Sec. V, where we will argue that this behavior is typical for a spectrum with a mass gap.

IV. TWO-DIMENSIONAL BLACK HOLES

In $d \geqslant 3$ spacetime dimensions it is very difficult to formulate general criteria that enable us to decide if a black hole satisfies either Eq. (2) or Eq. (3). These criteria can be found for 2D black holes. The 2D case is interesting for several reasons. Two-dimensional gravity supports a realization of the AdS-CFT correspondence [10]. In two spacetime dimensions we can formulate entropy bounds [11]. Moreover, 2D black holes can be used to describe black holes in higher dimensions. Two-dimensional black holes arise as an effective description of the near-horizon, near-extremal behavior of d-dimensional charged black holes and branes. Every d-dimensional spherically symmetric solution can be described, after dimensional reduction, by a 2D dilaton gravity model.

The generic 2D dilaton gravity model (for a recent review, see [12]) can be completely characterized by a dilaton potential $V(\Phi)$, the action for the model being given by $A = (1/2)\int d^2x \sqrt{-g}[\Phi R + \lambda^2 V(\Phi)]$. The general 2D black hole solution takes the form

$$ds^2 = -\left(J(\Phi) - \frac{2M}{\lambda}\right)dt^2 + \left(J(\Phi) - \frac{2M}{\lambda}\right)^{-1} dr^2, \quad \Phi = \lambda r, \qquad (10)$$

where $J(\Phi) = \int V(\Phi)$ and M is the black hole mass. To be sure that Eq. (10) describes a black hole we will take $M \geqslant 0$, $\Phi > 0$, and $J(\Phi) > 0$ a strictly increasing function of Φ [we will therefore have $J(\infty) = \infty$]. The temperature and entropy associated with the black hole are given by

$$S = 2\pi\Phi_h, \quad T = \frac{\lambda}{4\pi} V(\Phi_h), \qquad (11)$$

where $\Phi_h = J^{-1}(M)$ is the value of the dilaton at the black hole horizon.

Using Eq. (10) one easily finds

$$S = 2\pi J^{-1}(M). \tag{12}$$

One can now find sufficient conditions to be imposed on the function $J(\Phi)$ such that Eq. (2) is satisfied $\forall\ M_1, M_2$. Let us first show that if both conditions (the prime denotes the derivative with respect to M)

$$(J^{-1})''(M) < 0, \quad \forall\ M, \tag{13}$$

$$J^{-1}(M=0) = 0 \tag{14}$$

are satisfied then

$$J^{-1}(M_1+M_2) < J^{-1}(M_1) + J^{-1}(M_2), \tag{15}$$

which, owing to Eq. (12), implies that the inequality (2) for the entropy is also satisfied. The convexity condition (13) implies for every positive $\hat{M}, M_1$ $(J^{-1})'(\hat{M}+M_1) < (J^{-1})'(\hat{M})$, from which it follows that $J^{-1}(\hat{M}+M_1+M_2) + J^{-1}(\hat{M}) < J^{-1}(\hat{M}+M_1) + J^{-1}(\hat{M}+M_2)$. Evaluating the previous inequality at $\hat{M}=0$ and using condition (14) one easily recovers Eq. (15).

Analogously, we can show that if conditions

$$(J^{-1})''(M) > 0, \quad \forall\ M, \tag{16}$$

$$J^{-1}(M=0) = 0 \tag{17}$$

hold then the inequality (3) is satisfied for every M_1, M_2. If $(J^{-1})'' = 0$, identically, then the black hole solution of the model will satisfy $S(M_1+M_2) = S(M_1) + S(M_2)$. If $(J^{-1})''$ changes sign, we cannot make any definite statement about inequalities (2) and (3).

Any function $J^{-1}(M)$ which is everywhere convex (concave) and strictly growing must diverge for $M \to \infty$ more slowly (faster) than $J^{-1} = M$. It follows that $J(\Phi)$ must diverge for $\Phi \to \infty$ faster (more slowly) than $J = \Phi$. We have reached an important result: for models satisfying the required criteria about the derivatives of J^{-1} and $J(0) = 0$, fragmentation of 2D black holes can be entropically preferred or not depending on the asymptotic behavior of the function J. The black hole solutions of the 2D dilaton gravity model characterized by a function J, which asymptotically diverges faster (more slowly) than Φ, will satisfy Eq. (2) [Eq. (3)].

When $J^{-1}(0) \neq 0$ but condition (13) still holds we will have a mixed behavior. The black hole will satisfy Eq. (2) for $M > \tilde{M}_0$ and Eq. (3) for $M < \tilde{M}_0$, where $\tilde{M}_0$ is some threshold mass. Let us first notice that only the case $J^{-1}(0) = \Phi_0 < 0$ has physical relevance. Since J^{-1} is a strictly growing function, $\Phi_0 > 0$ implies $J^{-1}(M_0) = 0$ with M_0 *negative*, i.e., the presence of negative masses in the spectrum. Conversely, $\Phi_0 < 0$ implies

$$J^{-1}(M_0) = 0, \tag{18}$$

with M_0 *positive*, i.e., the black hole spectrum is limited from below by the extremal, nonvanishing value of the mass M_0. In general, the extremal mass M_0 is simply related to the threshold mass $\tilde{M}_0$. Using Eqs. (12) and (18) one easily realizes that the entropy of the extremal state is zero, $S(M_0) = 0$. Depending on the behavior of the dilaton potential at M_0 we can have two cases. (1) $J^{-1}(M_0) = 0$ and $V(M_0) = 0$. The extremal state has zero entropy and from Eq. (11) also zero temperature. (2) $J^{-1}(M_0) = 0$ and $V(M_0) \neq 0$. The extremal state has zero entropy but nonvanishing temperature. In the next section we shall see that in both cases this behavior can be explained by the presence of a mass gap.

Let us now give some examples to illustrate our general results. The simplest model satisfying both conditions (13), (14) is the Jackiw-Teitelboim model, $V = 2\Phi$. We have $J^{-1}(M) = \sqrt{2M/\lambda}$, which satisfies both $(J^{-1})'' < 0$ and $J^{-1}(0) = 0$. For the entropy we get $S \propto \sqrt{M}$, from which Eq. (2) follows for every value of the mass. A more general model is given by [13]

$$V = (h+1)\Phi^h, \quad h > -1. \tag{19}$$

For $h > 0$ the model satisfies the conditions (13) and (14). The black hole entropy is given by

$$S \propto (M/\lambda)^{1/(h+1)}, \tag{20}$$

which as expected satisfies Eq. (2). This class of models contains, as particular cases, 2D gravity models arising as the near-horizon limit of dilatonic zero-branes [14], black three-branes [15], and heterotic string black holes [16]. It is interesting to note that the entropy-energy relation (20) becomes that of a free field theory given by Eq. (1) identifying $h = 1/(d-1)$. This fact gives a hint about the possibility of finding a correspondence between these 2D dilaton gravity models and a free field theory.

For $-1 < h < 0$ the model satisfies the conditions (16) and (17). The black hole entropy now satisfies Eq. (3). An important particular case is given by $h = -1/2$, which describes the spherical dimensional reduction of the Schwarzschild black hole. For $h = 0$ we have the Callan-Giddings-Harvey-Strominger model [17]. Eq. (19) gives $(J^{-1})'' = 0$ identically. The entropy depends linearly on the mass, so that, as expected, $S(M_1+M_2) = S(M_1) + S(M_2)$.

As an example of a model satisfying condition (13) but not (14) let us first consider the exponential potential $V = \beta \exp(\beta\Phi)$, with $\beta > 0$. The black hole horizon is located at $\lambda r = \Phi = J^{-1}(M) = (1/\beta)\ln(2M/\lambda)$. J^{-1} diverges asymptotically more slowly than M but we have $J^{-1}(0) \neq 0$. We therefore expect the presence of a threshold mass $\tilde{M}_0$ separating the two regions of the spectrum where Eq. (2) [Eq. (3)] holds.

The black hole mass has an extremal value $M \geq M_0 = \lambda/2$. However, $J^{-1}(M_0) = 0$ and $V(M_0) \neq 0$. The entropy and temperature of the black hole are

$$S = \frac{2\pi}{\beta}\ln\frac{2M}{\lambda}, \quad T = \frac{\beta}{2\pi}M. \tag{21}$$

The ground state has zero entropy but finite temperature $T(M_0)=(\beta/4\pi)\lambda$. Using Eq. (21) and taking for simplicity $M_1=M_2=M$, one can easily find that the inequality (2) is satisfied only for $M>\tilde{M}_0=\lambda$, whereas for $\lambda/2<M<\lambda$ Eq. (3) holds. Black hole fragmentation becomes entropically preferred for M bigger than the threshold mass $\tilde{M}_0$. Notice that the threshold mass $\tilde{M}_0$, although of the same order, does not coincide with the extremal mass M_0, we have $\tilde{M}_0=2M_0$.

As a second example of models with $J^{-1}(0)\neq 0$ let us consider $J(\Phi)=\Phi^2+1$. This is the 2D analogue of the 3D BTZ black hole discussed at the end of the previous section. In this case also the black hole mass has an extremal value $M_0=\lambda/2$. However, now $S\propto T\propto\sqrt{(2M/\lambda)-1}$, so that the extremal state has zero entropy and temperature, $S(M_0)=T(M_0)=0$. The threshold mass at which black hole fragmentation becomes entropically preferred is given by $\tilde{M}_0=(3/2)M_0=(3/4)\lambda$.

Until now we have considered a system in which the total energy is constant, i.e., fragmentation of a black hole of mass M into two smaller black holes of masses $M_1+M_2=M$. Let us now consider systems at constant volume. We will show that the energy is minimized when the conditions that maximize the entropy are satisfied.

We have to consider the black hole mass M as a function of its radius R and solve the inequality

$$M(R_1+R_2)>M(R_1)+M(R_2). \tag{22}$$

Using Eq. (10) one finds $M(R)=(\lambda/2)J(\Phi_h)=(\lambda/2)J(R)$. Equation (22) becomes $J(R_1+R_2)>J(R_1)+J(R_2)$. Because $(J^{-1})''<0$ and $J^{-1}(0)=0$ imply, respectively, $J''>0$ and $J(0)=0$, it follows that conditions (13), (14) for the function (J^{-1}) are equivalent to conditions (16), (17) for the function J. As a consequence, whenever conditions (13) and (14) are satisfied, we have not only $S(M_1+M_2)<S(M_1)+S(M_2)$ but also $M(R_1+R_2)>M(R_1)+M(R_2)$. The process of fragmentation of a black hole maximizes the entropy if the total mass is constant and minimizes the mass if the total volume is constant. Conversely, if conditions (16) and (17) hold we have $S(M_1+M_2)>S(M_1)+S(M_2)$ but also $M(R_1+R_2)<M(R_1)+M(R_2)$. The configuration with maximal entropy and minimal mass is now given by the single black hole.

V. MASS GAP

It is evident that in what concerns the entropy of composite solutions 2D dilaton gravity models satisfying the conditions (13) and (14) of the previous section are very similar to free field theories. The mass of the gravitational solutions is naturally identified with the energy of the excitation in the field theory. The same is true for generic d-dimensional AdS solutions when the mass of the gravitational solution is much bigger then $L_p^{d-2}\lambda^{d-3}$. We can ask ourselves if the 2D dilaton gravity models satisfying condition (13) but not (14) also have a field theory counterpart. More generally, one would like to find field theoretical counterparts of d-dimensional black hole solutions whose spectrum exhibits extremal $M\neq 0$ solutions. The most natural candidates are field theories with mass gaps. This is rather obvious for the 2D dilaton gravity models discussed at the end of Sec. IV (for instance, the model with an exponential dilaton potential), which are characterized by a ground state of mass M_0 with zero entropy and nonvanishing temperature. The only way to have a nondegenerate ground state at finite temperature is the presence of a mass gap in the spectrum. The energy gap M_0 must be of the order of the temperature. In fact, for the model with exponential potential discussed in Sec. IV we have $E_{gap}=M_0\propto T(M_0)\propto\lambda$.

The relationship between black hole solutions and field theories with mass gaps seems to be more general. Crucial for the existence of this correspondence are both the asymptotic behavior of the metric and the presence of an extremal solution with $M_0\neq 0$. To illustrate this correspondence, let us consider a generic field theory whose spectrum has a mass gap of energy E_0 separating the $E=0$ state from the continuous part of the spectrum $E\geqslant E_0$. We will also assume that for $E\gg E_0$ the spectrum is that of a generic free field theory and that the entropy of the $E=E_0$ state vanishes. For $E\gg E_0$ the entropy-energy relation will be given by Eq. (1) which satisfies $S(E_1+E_2)<S(E_1)+S(E_2)$. On the other hand, for $E\approx E_0$, we have $S=S(E-E_0)$ with $S(E_0)=0$. Thus, the inequality $S(E_1+E_2-E_0)>S(E_1-E_0)+S(E_2-E_0)$ will be satisfied at least for $E_1=E_2=E_0$. Because for $E\gg E_0$ the opposite inequality holds, this implies the existence of a threshold energy $\tilde{E}_0$ separating the two regimes, in complete analogy with what happens for black hole solutions. This behavior is rather intuitive. For small excitations near E_0 the single state of energy $E=E_1+E_2$ has more degeneracy than the two states of energies E_1 and E_2 because the states below the gap do not contribute. For excitations of energy $E\gg E_0$ the contribution of the gap is irrelevant, and the entropy is dominated by the contribution coming from the continuous part of the spectrum.

VI. ENTROPY BOUNDS

The holographic principle in its usual formulation puts an upper bound to the entropy, and hence to the amount of information, which can be stored in a given *region of space* $S\leqslant A/4G$, where A is the area of the surface enclosing the region. At fixed volume the entropy cannot increase by splitting a system into parts, i.e., $S(V_1+V_2)\geqslant S(V_1)+S(V_2)$. This inequality is satisfied not only for extensive systems, when the entropy scales as the volume of the system, but also when the entropy scales as the area of the surface enclosing the system. A holographic bound cannot be violated by fragmentation. Hence the holographic principle is compatible with both Eqs. (2) and (3). This is true for the holographic bound but it is not necessarily true for entropy bounds of different types, which can make reference not only to the volume of the system (or the area enclosing it) but also to its energy. For instance, the Bekenstein bound $S\leqslant 2\pi ER$ [18] refers not only to the linear size R but also to the energy E of the system.

In principle, one could also try to formulate entropy bounds that make reference to nothing but the energy of the system. This could be done following the same line of reasoning one uses to formulate the Bekenstein-Hawking bound. In this latter case one takes a system in a *given region* of space and finds that its entropy is bounded by the area of the surface enclosing the region. In more detail, when one pumps energy into the system keeping its spatial extension constant, the entropy increases until the energy equals the mass pertaining to a black hole fitting in the region. A black hole forms and the bound $S \leqslant A/4G$ is saturated.

Instead, one could take a system with a constant *amount of energy M* and look for a bound on the associated entropy. Spreading the system over regions of space with ever-decreasing volumes, the entropy will increase until the corresponding Schwarzschild radius is reached. A black hole forms and an entropy bound $S \leqslant \pi G M^2$ is found. The problem is that this bound cannot be universal. Although the Schwarzschild black hole cannot increase its entropy by fragmenting, we have seen that there exist many black hole solutions that can increase their entropy through fragmentation. For this simple reason an entropy bound that refers only to the energy of the system cannot be universal.

VII. CONCLUSIONS

In this paper we have discussed the holographic principle and entropy bounds for composite gravitational systems. We have shown that with respect to fragmentation black hole solutions exhibit three different types of behavior. The entropy may decrease, increase, or have a mixed behavior. In the last case there will exist a threshold mass separating the regime where fragmentation is entropically preferred from the regime where it is not. Moreover, for 2D black holes we have been able to find a complete characterization of the entropy behavior, in the form of sufficient conditions imposed on the function J, which defines the 2D gravitational model.

Most of our results rely heavily on the validity of the assumption (2) of Sec. II, which enables us to neglect the binding energy in composite black hole states. This assumption is restrictive and does not apply to a number of cases (for instance, asymptotically AdS black holes). In this situation we cannot use our results to describe fragmentation processes. Nonetheless, the discussion of Eqs. (2) and (3) still gives us information about possible gravity–field theory dualities.

Our results have a strong impact on the way one can realize the holographic principle. In fact, only for gravity theories satisfying relation (2) can one hope to find a realization of the holographic principle in terms of a correspondence of gravity and field theory. For gravity theories satisfying Eq. (3) this correspondence cannot be realized as a gravity–field theory duality and must necessarily have an alternative, still unknown, form. Because asymptotically flat black holes, and in particular the Schwarzschild black hole, belong to this latter case we can exclude the existence of a correspondence between asymptotically flat black holes and a field theory.

On the other hand, we have seen that the class of gravity theories satisfying Eq. (2) is, at least in the 2D case, rather broad. It contains not only AdS gravity, for which the gravity–field theory duality is well established, but also other models, for instance, those with an exponential dilaton potential. The possibility of finding a gravity–field theory duality for these theories is an open question, which deserves further investigation.

Another point of interest is the existence of black holes with mixed behavior, i.e., exhibiting a transition from a regime where Eq. (2) holds to a regime where instead Eq. (3) is satisfied. We have argued that the most natural candidates for holographic duals of these black holes are field theories with a mass gap. Finally, our results seem to exclude the possibility of formulating an entropy bound only in terms of the energy of a system: the existence of black holes satisfying Eq. (2) will always allow the system to violate the bound by black hole fragmentation.

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