

Singularity resolution in quantum gravity

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We examine the singularity resolution issue in quantum gravity by studying a new quantization of standard Friedmann-Robertson-Walker geometrodynamics. The quantization procedure is inspired by the loop quantum gravity program, and is based on an alternative to the Schrödinger representation normally used in metric variable quantum cosmology. We show that in this representation for quantum geometrodynamics there exists a densely defined inverse scale factor operator, and that the Hamiltonian constraint acts as a difference operator on the basis states. We find that the cosmological singularity is avoided in the quantum dynamics. We discuss these results with a view to identifying the criteria that constitute “singularity resolution” in quantum gravity.

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I. INTRODUCTION

It is widely believed that a quantum theory of gravity will give insights into the question of what becomes of classical curvature singularities. This is based largely on intuition from the uncertainty principle and fundamental length scale arguments in regions of large spacetime curvatures. What is required to address the problem quantitatively is quantization of model systems that contain classical metrics with curvature singularities. Such models are usually symmetry reductions of general relativity or other generally covariant metric theories. Within a model an obvious approach is to look at classical observables such as curvature scalars, and see if they can be represented as operators on a suitable Hilbert space. Their spectra and quantum dynamics may give an indication of what becomes of the classical singularity.

This question has been studied using models derived from symmetry reductions of general relativity since the late 1960s [1–4]. All of this work used the Arnowitt-Deser-Misner (ADM) (metric variable) Hamiltonian formulation of general relativity (“geometrodynamics”) as the classical starting point, and the Schrödinger representation as the quantum starting point for developing a quantum gravity model. The results obtained from various mini- and mid-superspace models were largely inconclusive. Some indicated singularity avoidance, others did not, but no insights emerged as general and definitive in the sense of transcending the model studied.

After the development of the Ashtekar (connection variable) Hamiltonian formulation (“connection dynamics”), many of the questions studied in the ADM formulation were reexamined, including the general canonical quantum gravity program (for reviews see [5,6]). The different classical variables led to the development of a non-Schrödinger representation program based on holonomy variables (“loop quantum gravity”). Recently results from this program were applied

by Bojowald [7,8] to the old question of quantizing mini-superspace models, with a view to studying what happens to classical curvature singularities upon quantization. This application has produced some interesting results: for Friedmann-Robertson-Walker (FRW) mini-superspace models the Hamiltonian constraint acts like a difference operator on the space of states, and there is an upper bound on the spectrum of the inverse scale factor operator. Taken together, these results lead to the conclusion that the big bang singularity is resolved in the loop quantum gravity approach [9].

A number of questions may be asked at this stage concerning classical variables, quantization procedures, and singularity resolution: What criteria constitute singularity avoidance? Is the singularity avoidance conclusion from the loop quantum gravity program a result of both the classical starting point *and* the choice of representation? Would a non-Schrödinger representation quantization in the geometrodynamical ADM variables give the same results?

Motivated by these questions, we study a new quantization of flat FRW cosmology (the model in which the loop quantum gravity results were first obtained [7,8]). Our classical starting point is the geometrodynamical Hamiltonian formulation, which we quantize via a non-Schrödinger representation motivated by holonomy-like variables. We obtain results qualitatively similar to those obtained in loop quantum cosmology: the Hamiltonian constraint acts like a difference operator on the space of states, and the spectrum of the inverse scale factor has an upper bound.

This paper is organized as follows: The next section introduces the classical theory and a set of basic variables for FRW geometrodynamics. Section III describes the quantization procedure and discusses the volume, inverse scale factor, and Hamiltonian constraint operators. Section IV contains a discussion of the results with the singularity resolution question in mind.

II. CLASSICAL THEORY

The canonical Hamiltonian variables in geometrodynamics are $(q_{ab}, \tilde{\pi}^{ab})$, where the configuration variable q_{ab} is

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the metric on a spatial 3-surface, and the conjugate momentum variable $\tilde{\pi}^{ab}$ is a function of the surface's extrinsic curvature. In terms of these variables, the vacuum Hamiltonian and spatial diffeomorphism constraints are

$$H = \frac{1}{\sqrt{q}} \left(\tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2 \right) - \sqrt{q} R = 0 \quad (1)$$

$$C_a = D_b \tilde{\pi}_a^b = 0, \quad (2)$$

where $q = \det q_{ab}$, $\tilde{\pi} = \tilde{\pi}^{ab} q_{ab}$, and R is the Ricci scalar of q_{ab} .

For the flat FRW model a reduced Hamiltonian theory may be obtained by the ansatz

$$q_{ab} = B(t) e_{ab}, \quad \tilde{\pi}^{ab} = P(t) e^{ab}, \quad (3)$$

where e_{ab} is the flat Euclidean metric $\text{diag}(1,1,1)$. The density weight on $\tilde{\pi}^{ab}$ is obtained using this fiducial metric. Unlike in general relativity where $B(t)$ is dimensionless since it is a metric variable, we will take it to have dimension length square (which means that the spatial coordinates are dimensionless). The conjugate momentum $P(t)$ is then dimensionless in order that the symplectic form

$$\omega = \frac{1}{8\pi G_N} dB \wedge dP \quad (4)$$

has dimensions of action (in $c=1$ units). The phase space topology is that of a particle on the half line, $\mathbb{R}^+ \times \mathbb{R}$, since $B(t) > 0$ in this parametrization. The standard FRW scale factor is $a(t) = \sqrt{B(t)}$.

The Hamiltonian constraint is

$$H = -\frac{3}{2} P^2 \sqrt{B} \quad (5)$$

and the diffeomorphism constraint C_a vanishes identically. It is interesting to note that this Hamiltonian constraint is identical in form to that obtained in the connection variables [7].

To facilitate quantization, however, it is useful to make a canonical transformation that lifts the half line restriction. We proceed in two steps. The first is to extend the classical configuration space to include the singularity $B(t) = 0$. This step is essential for addressing the question of singularity resolution for without it there is “no classical singularity to avoid” in the quantum theory. (This, for example, is done—more or less unconsciously—in the familiar quantization of the hydrogen atom by the choice of Hilbert space $L^2(\mathbb{R}_0^+)$, even though the classical configuration space is only \mathbb{R}^+ .) The second step is to reparametrize the configuration variable such that its domain is the real line.

This is done by introducing new variables (x, p) by writing $x^2 = B$. The symplectic form becomes

$$\omega = \frac{1}{8\pi G_N} 2x dx \wedge dP = \frac{1}{8\pi G_N} dx \wedge d(2xP). \quad (6)$$

Thus the new momentum is $p = 2xP$, for which $\{x, p\} = 8\pi G_N$. (Both coordinates x and p now have dimension length.) Note that in this parametrization, the point $x=0$ is included to give the full real line as the configuration space. This amounts to an extension of the original parametrization to include the degenerate metric with $B(t) = 0$. (This feature is also present in the connection-triad variables, where invertibility of the triad is not a requirement.)

The Hamiltonian constraint (1) as a function of (x, p) is

$$H = -\frac{3}{8} \frac{p^2}{x^2} |x| = -\frac{3}{8} \frac{p^2}{|x|}. \quad (7)$$

Note that this constraint is now quite different in form than the one in the connection-triad variables, although the x variable may be regarded as a “triad.” The reason for this is that the new momentum p is a product of the metric and extrinsic curvature variables, and this is quite unlike the connection variable A made of the triad connection Γ and extrinsic curvature K as $A = \Gamma + K$. (It is this connection A that is essential for formulating the loop representation through holonomies [5,6].)

We now introduce an algebra of classical observables and write quantities of physical interest as functions of these variables. Their form is motivated by the holonomy observables used for quantization in the loop quantum gravity program. We use as the fundamental variables x and

$$U_\gamma(p) := \exp(i\gamma p/L), \quad (8)$$

where γ is a real parameter and L fixes the (arbitrary) unit of length. The parameter γ is necessary in order to separate momentum points in phase space (e.g. fixing $\gamma/L = 1$ say, gives the same value of U for p and $p + 2n\pi$). This variable may be viewed as a *momentum analog of the holonomy variable* of loop quantum gravity.

The pair $[x, U_\gamma(p)]$ has the Poisson bracket algebra

$$\{x, U_\gamma(p)\} = (8\pi G_N) \frac{i\gamma}{L} U_\gamma(p). \quad (9)$$

This is the basic algebra which will be taken over to the quantum theory.

Other quantities of interest are the volume, which up to a multiplicative constant is

$$V(t) = B(t)^{3/2} = |x|^3, \quad (10)$$

and inverse powers of the scale factor $a(t)$. The latter has proven useful in determining whether there is “singularity avoidance” in a quantum theory [7]. It is possible to write these observables, and the Hamiltonian constraint using our choice of fundamental variables $[x, U_\gamma(p)]$.

The standard FRW scale factor is

$$a(t) = \sqrt{B(t)} = |x|. \quad (11)$$

Using the method introduced in [10], inverse powers of the scale factor may be defined classically either via the bracket

$$\begin{aligned} U_\gamma^{-1}\{V^n, U_\gamma\} &= U_\gamma^{-1}\{|x|^{3n}, U_\gamma\} \\ &= i(8\pi G_N) \frac{\gamma}{L} 3n \operatorname{sgn}(x) |x|^{3n-1}, \end{aligned} \quad (12)$$

or more simply by inverse powers of the volume observable. However, the latter definition cannot be carried over to the quantum theory since, as we see will below, the volume operator turns out to have zero as an eigenvalue, so negative powers of it are not densely defined. Therefore the somewhat indirect definition (12), with $n > 0$ will be useful in studying the approach to the singularity in the quantum theory. The requirement that the power of x on the right hand side be negative means that $n < 1/3$. Thus we need $0 < n < 1/3$ in order to obtain a sequence of inverse powers of the scale factor in terms of the basic variables. The choice $n = 1/6$ gives

$$\frac{\operatorname{sgn}(x)}{\sqrt{|x|}} = -\frac{2Li}{(8\pi G_N)\gamma} U_\gamma^{-1}\{V^{1/6}, U_\gamma\}, \quad (13)$$

and $n = 1/4$ gives

$$\frac{1}{|x|} = \left(\frac{4}{3(8\pi G_N)\gamma L} \right)^4 (U_\gamma^{-1}\{V^{1/4}, U_\gamma\})^4. \quad (14)$$

From this it is clear that powers of Poisson bracket in Eq. (12) may be used as a starting point for defining a large class of operators for inverse powers of the scale factor. It is of interest to see whether these all lead to qualitatively similar behavior concerning the quantum nature of the big bang singularity.

The Hamiltonian constraint can be written as a function of the basic variables by using the relation (13) for the inverse scale factor as follows:

$$H = -\frac{3}{8} \frac{p^2}{|x|} = \frac{3L^2}{2(8\pi G_N)^2 \gamma^2} p^2 (U_\gamma^{-1}\{V^{1/6}, U_\gamma\})^2. \quad (15)$$

Note here that there is the alternative choice of using Eq. (14) to define the Hamiltonian constraint. However, this leads to a more complicated form due to the larger number of U factors.

It is of course also possible to write the Hamiltonian constraint in other classically equivalent ways. One alternative is to substitute $|x|/x^2$ rather than directly using the inverse scale factor $1/|x|$ from Eqs. (13) or (14). These choices will clearly lead to inequivalent operators in the quantum theory since the number of factors of U are different. As for the inverse scale factor, it is generally useful to also study various choices for the Hamiltonian constraint in order to identify the main features that are to be associated with ‘‘singularity resolution’’ in the quantum theory.

In the following, we focus primarily on the simplest ordering of the Hamiltonian constraint, and the square of the definition (13) for the inverse scale factor, since both of these

contain the smallest number of U factors. However, we will briefly comment on other choices.

III. QUANTUM THEORY

To construct the quantum theory for the classical system described above, we will proceed in analogy to the procedure used in loop quantum gravity. The first step is to choose an algebra of classical functions that is represented as quantum configuration operators. We take here the algebra generated by the functions

$$W(\lambda) = e^{i\lambda x/L}, \quad (16)$$

where $\lambda \in \mathbb{R}$. It consists of all functions of the form

$$f(x) = \sum_{j=1}^n c_j e^{i\lambda_j x/L}, \quad (17)$$

with $c_j \in \mathbb{C}$ and their limits with respect to the supremum norm. This algebra is known as the algebra of almost periodic functions over \mathbb{R} and we denote it by $AP(\mathbb{R})$.

It is well known that $AP(\mathbb{R})$ is naturally isomorphic to $C(\bar{\mathbb{R}}_{Bohr})$, the algebra of continuous functions on the so-called Bohr compactification of \mathbb{R} [11]. As the name suggests, $\bar{\mathbb{R}}_{Bohr}$ is a compact group which can be obtained as the dual group of \mathbb{R}_{discr} , the real line endowed with the discrete topology. This suggests that taking $L^2(\bar{\mathbb{R}}_{Bohr}, d\mu_0)$, where μ_0 is Haar measure on $\bar{\mathbb{R}}_{Bohr}$, as the Hilbert space for our theory is a viable option. *This is the decisive point where we depart from the traditional approach in geometrodynamics, where the Hilbert space is the conventional Schrödinger space $L^2(\mathbb{R}, dx)$.* Once we adopt this new choice, basis states in our Hilbert space are given by

$$|\lambda\rangle \equiv |e^{i\lambda x/L}\rangle, \quad \lambda \in \mathbb{R}, \quad (18)$$

with the inner product

$$\langle \mu | \lambda \rangle = \delta_{\mu, \lambda}. \quad (19)$$

This representation has been discussed in some mathematical detail in [12], and also in [13] where it is applied to the quantization of a particle. Notice the difference from the standard quantum mechanics of a particle on the real line, where the right hand side is given instead by a delta function $\delta(\mu - \lambda)$. This feature is traceable to the fact that the configuration space is the real line with the discrete topology, which in turn stems from the choice of the algebra of functions.

The action of the configuration operators $\hat{W}(\lambda)$ is defined by

$$\hat{W}(\lambda)|\mu\rangle = e^{i\lambda \hat{x}/L}|\mu\rangle = e^{i\lambda \mu}|\mu\rangle. \quad (20)$$

It is straightforward to verify that these operators are weakly continuous in λ , which procures the existence of a self-adjoint operator \hat{x} , acting on basis states according to

$$\hat{x}|\mu\rangle = L\mu|\mu\rangle. \quad (21)$$

The next step is to construct the operators corresponding to the classical momentum functions $U_\gamma = e^{i\gamma p/L}$. Their action on the basis states is fixed by the definition of the \hat{x} operator and the requirement that the commutator between \hat{x} and \hat{U}_γ reflects the corresponding Poisson bracket (9) between x and U_γ . With the definition

$$\hat{U}_\gamma|\mu\rangle = |\mu - \gamma\rangle, \quad (22)$$

the commutator is

$$[\hat{x}, \hat{U}_\gamma] = -\gamma L \hat{U}_\gamma. \quad (23)$$

Making now the standard commutator-Poisson bracket correspondence $[\cdot, \cdot] \leftarrow i\hbar\{\cdot, \cdot\}$, gives using Eq. (9) the relation

$$-\gamma L = i\hbar(8\pi G_N) \frac{i\gamma}{L}, \quad (24)$$

which fixes the length L to $L = \sqrt{8\pi} l_p$. This shows explicitly how the eigenvalues of \hat{x} arise in Planck units upon quantization.

Obviously, \hat{U}_γ is unitary; however, it is not weakly continuous with respect to γ . As a consequence, *there is no momentum operator in this representation*, in stark contrast to the Schrödinger quantization. With the basic quantum operators now at our disposal, we are in a position to construct the inverse scale factor operator and investigate its spectrum.

A. Volume and inverse scale factor

The operator for the volume \hat{V} is provided directly by the operator \hat{x} defined in Eq. (21). We have

$$\hat{V}|\mu\rangle = (\sqrt{8\pi} l_p)^3 |\mu|^3 |\mu\rangle. \quad (25)$$

The operators corresponding to U and V can be used to obtain an operator for the inverse scale factor. One way to do this is to use the square of the expression in Eq. (13) with $\gamma=1$. The resulting operator is

$$\frac{\hat{1}}{|x|} := \frac{1}{2\pi l_p^2} (\hat{U}^{-1} [\hat{V}^{1/6}, \hat{U}])^2. \quad (26)$$

The key question is whether this operator is unbounded as in standard quantum cosmology, where its eigenvalues diverge when approaching the quantum state corresponding to $a=0$, or whether it is bounded, indicating a (kinematical) resolution of the classical singularity. To decide this we calculate its eigenvalue on a basis state $|\mu\rangle$:

$$\begin{aligned} \frac{\hat{1}}{|x|}|\mu\rangle &= \frac{1}{2\pi l_p^2} [\hat{U}^{-1} (\hat{V}^{1/6} \hat{U} - \hat{U} \hat{V}^{1/6})^2 |\mu\rangle \\ &= \frac{1}{2\pi l_p^2} (\hat{U}^{-1} \hat{V}^{1/3} \hat{U} - \hat{U}^{-1} \hat{V}^{1/6} \hat{U} \hat{V}^{1/6} \\ &\quad - \hat{V}^{1/6} \hat{U}^{-1} \hat{V}^{1/6} \hat{U} + \hat{V}^{1/3}) |\mu\rangle \\ &= \sqrt{\frac{2}{\pi l_p^2}} (|\mu-1| - 2|\mu|^{1/2} |\mu-1|^{1/2} + |\mu|) |\mu\rangle \\ &= \sqrt{\frac{2}{\pi l_p^2}} (|\mu|^{1/2} - |\mu-1|^{1/2})^2 |\mu\rangle. \end{aligned} \quad (27)$$

This result reveals some important properties of the eigenvalues. First, they are always positive or at most zero, as should be the case. Second and more importantly, the spectrum is clearly bounded from above. For $|\mu| \rightarrow \infty$ the eigenvalues approach 0, as would be expected from the behavior of $1/|x|$ for large $|x|$. Moreover, the eigenvalue of the state $|\mu=0\rangle$ corresponding to the classical singularity ($\hat{a}|0\rangle \equiv |\hat{x}| |0\rangle = 0$) is $\sqrt{2/\pi} l_p^2$, and this is the largest possible eigenvalue. (This is notably different from the results in [9], where the eigenvalue of the inverse scale operator for the state $|\mu=0\rangle$ is 0, and the maximal eigenvalue is obtained instead for the state $|\mu=1\rangle$. Although there are no principal reasons why this could not happen in the quantum regime, it seems somewhat unnatural from the classical point of view. It should be pointed out, however, that this result is obtained in our formalism for a different choice of operator ordering.)

In summary, this new quantization of the inverse scale factor in geometrodynamics mimics the expected classical behavior for large values $a(t)$, and departs significantly from the divergence in the standard quantization near the classical singularity $a(t)=0$. In this sense, the quantization resolves the singularity. This “resolution,” however, is so far only kinematical, since we have not investigated the quantum dynamics. It is conceivable that the quantum dynamics breaks down at the state $|0\rangle$, in which case it would be hard to claim a satisfactory resolution of the singularity. As the dynamics is encoded in the Hamiltonian constraint, we now turn our attention to its operator realization.

B. Hamiltonian constraint

As discussed already in the classical section, the Hamiltonian can be written in many different, classically equivalent forms. The one we will focus on in this section is

$$H = -\frac{3}{8} \frac{p^2}{|x|}, \quad (28)$$

as this is in some sense the simplest one, and the spectrum of the inverse scale operator is already known. As p does not exist as an operator in our quantum representation, we have to choose an alternative way to represent p^2 as an operator. One way to do this is motivated by the classical expression

$$p^2 = L^2 \lim_{\gamma \rightarrow 0} \frac{1}{\gamma^2} (2 - U_\gamma - U_\gamma^{-1}). \quad (29)$$

A physical interpretation of this expression is obtained by setting $\gamma = l_F / L_{phys}$, where L_{phys} is the characteristic size of the system under consideration and l_F is a fundamental length scale. (Note that a Hamiltonian naturally introduces a scale L_{phys} for a physical system.) The limit then suggests that the “point” form of the momentum is recoverable in the case $L_{phys} \gg l_F$.

For quantum cosmology these considerations mean $l_F = l_P$ and $\gamma = l_P / L_{phys}$, and lead to a Hamiltonian constraint operator

$$\begin{aligned} \hat{H}_\gamma &= \frac{3\pi l_P^2}{\gamma^2} (\hat{U}_\gamma + \hat{U}_\gamma^{-1} - 2) \frac{\hat{1}}{|x|} \\ &= \frac{3}{2\gamma^2} (\hat{U}_\gamma + \hat{U}_\gamma^{-1} - 2) (\hat{U}^{-1} [\hat{V}^{1/6}, \hat{U}])^2, \end{aligned} \quad (30)$$

where a specific operator ordering has been chosen. The action of \hat{H}_γ on a basis state is given by

$$\begin{aligned} \hat{H}_\gamma |\mu\rangle &= \frac{\sqrt{18}}{\gamma^2} l_P (|\mu|^{1/2} - |\mu - 1|^{1/2})^2 \\ &\quad \times (|\mu + \gamma\rangle + |\mu - \gamma\rangle - 2|\mu\rangle) \\ &\equiv \frac{\sqrt{18}}{\gamma^2} l_P \mathcal{V}(\mu) (|\mu + \gamma\rangle + |\mu - \gamma\rangle - 2|\mu\rangle). \end{aligned} \quad (31)$$

On the eigenstate $|0\rangle$ of volume with zero eigenvalue, which is the classical singularity, we have

$$\hat{H}_\gamma |0\rangle = \frac{\sqrt{18}}{\gamma^2} l_P (|\gamma\rangle + |-\gamma\rangle - 2|0\rangle), \quad (32)$$

$$\frac{\hat{1}}{|x|} |0\rangle = \sqrt{\frac{2}{\pi l_P^2}} |0\rangle. \quad (33)$$

These equations represent the effects of quantization on the classical singularity. In order to probe the dynamical part further we must solve the quantum constraint equation that encodes time evolution.

As is well known in the theory of constrained systems, normalizable solutions of the quantum constraints do not lie in the kinematical Hilbert space \mathcal{H} but rather in a larger space \mathcal{C}^* . This space can be obtained as the dual space of the dense subspace \mathcal{C} of \mathcal{H} , which is spanned by all elements of the form

$$\sum_{i=1}^n \psi(\mu_i) |\mu_i\rangle. \quad (34)$$

A general element of \mathcal{C}^* can thus be written as

$$\langle \psi | = \sum_{\mu} \psi(\mu) \langle \mu |. \quad (35)$$

Notice that, while the sum is continuous as it runs over every real number, its action on an element of \mathcal{C} is well defined by construction. The constraint equation—symbolically written as

$$\hat{H} |\psi\rangle = 0, \quad (36)$$

is now interpreted as an equation in the dual space,

$$\langle \psi | \hat{H}^\dagger = 0. \quad (37)$$

Using the form of a general element of the dual space (35) and the (dualized) action of the (dual) Hamiltonian on (dual) basis elements, we can derive a relation for the coefficients $\psi(\mu)$:

$$\mathcal{V}(\mu + \gamma) \psi(\mu + \gamma) - 2\mathcal{V}(\mu) \psi(\mu) + \mathcal{V}(\mu - \gamma) \psi(\mu - \gamma) = 0. \quad (38)$$

What is the meaning of this equation and in what sense does it encode the quantum dynamics? First of all, it determines the coefficients for those dual states that are physical. As in the classical theory solutions to the constraint equation represent classical spacetimes, these physical dual states can be interpreted as representing “quantum spacetimes.”

The difference equation (38) gives physical states that are linear combinations of a countable number of components of the form

$$\psi(\mu + n\gamma) |\mu + n\gamma\rangle, \quad (39)$$

where γ is fixed at the Planck scale ($\gamma = l_P / L_{phys} \sim 1$) and $n \in \mathbb{Z}$. As each component corresponds to a different eigenvalue for the volume and scale factor, it can be interpreted as the quantum state representing the Universe at the “time” $\mu + \gamma$. A solution of the Hamiltonian constraint therefore represents a linear combination of FRW universes specified at certain discrete volumes, or equivalently, at discrete times. It is in this sense that time evolution is “discrete with fundamental time step” γ . It is also clear that this “discrete evolution” does not represent the state of a single universe at different discrete times, since the term “single universe” has no meaning here. Rather a “discrete solution” of the Hamiltonian constraint, [i.e., one satisfying Eq. (38)], gives the amplitudes that the physical universe is in one or other of the discretely separated components of the physical state.

The state $\langle 0 |$ corresponding to the classical singularity is contained in only one specific “quantum spacetime,” (i.e., the solution of the Hamiltonian constraint). Furthermore, in

that one case we can see that the system evolves right through the singularity without encountering any problems, since the component $\psi(0)$ can be computed in terms of the components $\psi(\gamma)$ and $\psi(-\gamma)$. In all other physical states, the state $\langle 0|$ does not occur, and so in a sense one can say that the discrete evolution “jumps” over the singularity if the state contains components with both positive and negative values of μ . In such cases there is an instance of smallest but finite volume.

From these observations one can perhaps conclude that dynamically the singularity has been resolved. A dynamical non-resolution of the singularity might have occurred had it turned out that the difference equations have no solutions if they contain the $\psi(0)$ component, or if they contain components with both positive and negative μ in the sum (35).

Finally, it is interesting to note that for our representation of \hat{H} , the state at the classical singularity $\psi(0)$ can be determined in contrast to the results in [9]. However, had we chosen to write the classical Hamiltonian using Eq. (14) instead of Eq. (13), which amounts to using double the number of U operators, we would have ended up with the same result: $\psi(0)$ cannot be determined from the difference equation, but a solution is still possible as it turns out that $\psi(\gamma)$ is then given in terms of $\psi(-\gamma)$. *This shows the significant differences that can arise due to quantization ambiguities.* Ultimately, only physical predictions and comparison with known facts or (as yet hypothetical) experiments can determine the “right” choice.

IV. CONCLUSIONS AND DISCUSSION

Our main result is that there is an alternative to the Schrödinger quantization of the FRW cosmology in the standard ADM geometrodynamical variables. This quantization leads to conclusions qualitatively similar to those obtained in loop quantum cosmology starting from the connection-triad variables: (i) the Hamiltonian constraint acts like a difference operator, and (ii) the inverse scale factor can be represented as a densely defined operator. Thus *it is the representation space and the realizations of the basic observables rather than the nature of the classical variables* that are responsible for the similar conclusions for this model.

To what extent is the quantization we have presented different from the one employed in loop quantum cosmology? The differences at the classical level are clear: the phase space variables (x, p) are not the standard mini-superspace variables that arise via standard reduction from the connection-triad canonical variables, as a comparison with [7,8] shows. The key difference at the quantum level is that \hat{U}_γ is not the holonomy operator associated with the Ashtekar-Sen connection for the FRW model. Rather, the U_γ we use is a standard translation generator whose realization on the Hilbert space $L^2(\bar{\mathbb{R}}_{Bohr}, d\mu_0)$ is applicable to any classical theory, as has been discussed in [12]. Thus interpreting our quantization as a “loop representation” would mean that one is generalizing this term to include *all quantizations* on the Hilbert space $L^2(\bar{\mathbb{R}}_{Bohr}, d\mu_0)$.

It is clear that the alternative representation based on the

Bohr compactification is applicable to other mini-superspace models, since in all such models the phase space variables are functions of only a time coordinate. It is also clear that this applicability is independent of whether the classical phase space variables are metric-extrinsic curvature or connection-triad. The main difference between the variables arises in the form and action of the Hamiltonian constraint.

In the flat FRW case we have discussed, the Ricci scalar term in the Hamiltonian constraint vanishes. Thus the action of the constraint as a difference operator is due only to the kinetic term. In other mini-superspace models the Ricci scalar term, which is a purely configuration variable, will have non-trivial action on the basis states. However, in the Bohr representation, this action is multiplicative. Thus it appears that in other mini-superspace models the “difference operator” feature of the Hamiltonian constraint will survive. Similarly it appears that an inverse scale factor operator is definable using volume and U operators, and that it is likely to have a spectrum bounded above. For models where the phase space is more than two dimensional, the new representation can clearly be used for each pair of phase space variables. Extension beyond mini-superspace (quantum mechanics) to midi-superspace (quantum field theory) models, such as the Gowdy cosmology, would be of much interest [14].

“Singularity resolution” appears to consist of two main features, one kinematical and the other dynamical. The kinematical feature is the spectrum of the operator associated with a curvature scalar (or other relevant classical observable) that diverges at a curvature singularity. If the spectrum is bounded, the singularity may be considered kinematically resolved. It is important to identify the largest eigenvalue and corresponding eigenstate of such an operator, since this is the “closest” the quantum theory can get to the singularity. The dynamical feature of singularity resolution concerns the action of the Hamiltonian constraint on the state of largest curvature: this could lead either to no solution of the constraint for zero or negative values of μ , or to a well defined “evolution” through zero to negative values of μ . The former may be taken as an indication of the breakdown of quantum evolution, and hence a dynamical non-resolution of the singularity, regardless of the boundedness of the curvature operator.

An alternative viewpoint is that the kinematical vs dynamical views are artificial in that the question of singularity resolution is relevant only for the physical state space with a well defined physical inner product. The question then becomes whether there are any physical states for which the curvature operator spectrum is unbounded. This appears more compelling, but it has not been addressed here, or in the context of loop quantum cosmology.

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