Retarded coordinates based at a world line and the motion of a small black hole in an external universe

Eric Poisson
Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1
and Perimeter Institute for Theoretical Physics, 35 King Street North, Waterloo, Ontario, Canada N2J 2W9
(Received 7 November 2003; published 8 April 2004)

In the first part of this article I present a system of retarded coordinates based at an arbitrary world line of an arbitrary curved spacetime. The retarded-time coordinate labels forward light cones that are centered on the world line, the radial coordinate is an affine parameter on the null generators of these light cones, and the angular coordinates are constant on each of these generators. The spacetime metric in the retarded coordinates is displayed as an expansion in powers of the radial coordinate and expressed in terms of the world line's acceleration vector and the spacetime's Riemann tensor evaluated at the world line. The formalism is illustrated in two examples, the first involving a comoving world line of a spatially flat cosmology, the other featuring an observer in circular motion in the Schwarzschild spacetime. The main application of the formalism is presented in the second part of the article, in which I consider the motion of a small black hole in an empty external universe. I use the retarded coordinates to construct the metric of the small black hole perturbed by the tidal field of the external universe, and the metric of the external universe perturbed by the presence of the black hole. Matching these metrics produces the MiSaTaQuWa equations of motion for the small black hole.

DOI: 10.1103/PhysRevD.69.084007 PACS number(s): 04.20.−q, 04.25.−g, 04.40.−b, 04.70.−s

I. INTRODUCTION

In the first part of this article I present a system of retarded coordinates (u,r,θ,ϕ) based at an arbitrary world line γ of an arbitrary spacetime with metric $g_{\alpha\beta}$. The coordinates are adapted to the forward light cone of each point $z(\tau)$ of the world line (τ is the proper-time parameter on γ). The retarded-time coordinate u is constant on each light cone, and it agrees with τ at the cone's apex. The radial coordinate r is an affine parameter on the cone's null generators, and it gives a measure of distance away from the world line. The angular coordinates $\theta^A=(\theta,\phi)$ are constant on each of these generators.

This formalism complements a line of research that was initiated by Synge [1] and pursued by Ellis and his collaborators [2–7] in their work on observational cosmology. While the central ideas exploited here are the same as with Synge and Ellis, my implementation is substantially different: While Synge and Ellis sought definitions for their optical or observational coordinates that apply to large regions of the spacetime, my considerations are limited to a small neighborhood of the world line; and while Ellis favored a construction based on past light cones (because of the cosmological context in which information at γ is gathered from the past), here the preference is given to future light cones.

The introduction of retarded coordinates is motivated by the desire to construct solutions of wave equations for massless fields that are produced by pointlike sources moving on the world line. The retarded coordinates naturally incorporate the causal relation that exists between the source and the field, and for this reason the solution takes a simple explicit form (in the neighborhood in which the coordinates are defined). The point of view developed here, therefore, is opposite to the cosmological view developed by Ellis: here γ is the cause and the effect is propagated along forward light cones.

A quasi-Cartesian version of the retarded coordinates is introduced in Sec. II B, after reviewing some necessary geometrical elements (a tetrad transported on γ, Synge's world function, and the bitensor of parallel transport) in Sec. II A. In Sec. II C, I explore the significance of r as an affine parameter on the generators of the light cones, and describe the vector k^α that is tangent to the congruence of generators. The metric in retarded coordinates is gradually constructed in Secs. II D through II F, and its quasi-Cartesian form is displayed in Eqs. (2.39)–(2.41). In Sec. II G, I carry out a transformation to angular coordinates, and this form of the metric is displayed in Eqs. (2.55)–(2.58). In Sec. II H, I present an important simplification of the formalism that occurs when γ is a geodesic and the Ricci tensor vanishes on the world line. Two examples are worked out in Secs. II I and II J: first I apply the formalism to a comoving world line of a spatially flat cosmology, and then I examine the metric near an observer in circular motion in the Schwarzschild spacetime.

The main application of the formalism is presented in the second part of the paper. Here I consider the motion of a nonrotating black hole of small mass m in a background spacetime with metric $g_{\alpha\beta}$; the metric is assumed to be a solution to the Einstein field equations in vacuum. By employing the powerful method of matched asymptotic expansions (see, for example, Ref. [8] for a discussion), I derive equations of motion for the black hole. If $z^\mu(\tau)$ are the parametric relations describing the black hole's world line, $u^\mu=dz^\mu/d\tau$ is the velocity vector, and $a^\mu=Du^\mu/d\tau$ the acceleration vector, then the equations of motion take the form of

$$a^\mu=-\frac{1}{2}(g^{\mu\nu}+u^\mu u^\nu)(2h^{\rm tail}_{\nu\lambda\rho}-h^{\rm tail}_{\lambda\rho\nu})u^\lambda u^\rho, \qquad (1.1)$$

where

$$h^{\rm tail}_{\mu\nu\lambda} = 4m \int_{-\infty}^{\tau^-} \nabla_\lambda \left(G_{\mu\nu\mu'\nu'} - \frac{1}{2} g_{\mu\nu} G^{\rho}{}_{\rho\mu'\nu'} \right)(\tau,\tau') \times u^{\mu'} u^{\nu'} d\tau' \tag{1.2}$$

is an integration over the past portion of the world line; the integral involves the retarded Green's function $G_{\mu\nu\mu'\nu'}(z(\tau),z(\tau'))$ for the metric perturbation [9] associated with the small black hole, and it is cut off at $\tau' = \tau^- \equiv \tau - 0^+$ to avoid the singular behavior of the Green's function at coincidence.

Equation (1.1) is not new, and the method of derivation presented here also is not new; they both originate in a 1997 paper by Mino, Sasaki, and Tanaka [10]. [The equations were later rederived by Quinn and Wald [11], and they are now known as the MiSaTaQuWa equations of motion.] The method of matched asymptotic expansions was first used to establish that in the limit $m \to 0$, the motion of a black hole is geodesic in the background spacetime [12–14]. Using this approach, Mino, Sasaki, and Tanaka (see also Ref. [15]) were able to compute the first-order correction to this result, and showed that at order m, the motion is accelerated and governed by Eq. (1.1). Their derivation, however, did not rely on a specific choice of coordinates, and it left many details obscured. By adopting retarded coordinates, these details can be revealed and the matching procedure clarified. This was my aim here: to provide a clearer, more explicit implementation of the matching procedure employed by Mino, Sasaki, and Tanaka.

The method of matched asymptotic expansions is explained in Sec. III A. Essentially, the metric of the small black hole perturbed by the tidal gravitational field of the external Universe is matched to the metric of the external Universe perturbed by the black hole; ensuring that this metric is a valid solution to the vacuum Einstein field equations determines the motion of the black hole. The perturbed metric of the small black hole is constructed in an *internal zone* in terms of internal retarded coordinates $(\bar{u},\bar{t},\bar{\theta}^A)$; this is carried out in Sec. III B. The perturbed metric of the external Universe is constructed in an *external zone* in terms of external retarded coordinates (u,r,θ^A); this is carried out in Sec. III C. The matching of the two metrics is performed in a *buffer zone* that overlaps with both the internal and external zones; this involves a transformation from the external to the internal coordinates that is explicitly worked out in Appendix C. The matching is carried out in Sec. III D, where we see that it does indeed produce Eq. (1.1).

This concludes the summary of the results contained in this paper. Various technical details are relegated to Appendixes A and B. Throughout the paper I work in geometrized units ($G=c=1$) and with the conventions of Misner, Thorne, and Wheeler [16]. Other applications of the retarded coordinates (including the fields produced by point particles, and their motion in curved spacetime) are featured in a recent review article [17].

II. RETARDED COORDINATES

A. Geometrical elements

To construct the retarded coordinates we must first introduce some geometrical elements on the world line γ at which the coordinates are based. The world line is described by relations $z^\mu(\tau)$ with τ denoting proper time, its normalized tangent vector is $u^\mu = dz^\mu/d\tau$, and its acceleration vector is $a^\mu = Du^\mu/d\tau$ with $D/d\tau$ denoting covariant differentiation along the world line. Throughout we use Greek indices μ, ν, λ, ρ, etc. to refer to tensor fields defined, or evaluated, on the world line.

We install on γ an orthonormal tetrad that consists of the tangent vector u^μ and three spatial vectors e^μ_a. These are transported on the world line according to

$$\frac{De^\mu_a}{d\tau} = a_a u^\mu + \omega_a{}^b e^\mu_b , \tag{2.1}$$

where $a_a(\tau) = a_\mu e^\mu_a$ are the frame components of the acceleration vector and $\omega_{ab}(\tau) = -\omega_{ba}(\tau)$ is a prescribed rotation tensor. Setting $\omega_{ab}=0$ would make the triad Fermi-Walker transported on the world line, and in many applications this would be a sensible choice. We nevertheless allow the spatial vectors to rotate as they are transported on the world line; this makes the formalism more general, and we will need this flexibility in Sec. III of the paper. It is easy to check that Eq. (2.1) is compatible with the requirement that the tetrad (u^μ, e^μ_a) be orthonormal everywhere on γ.

From the tetrad on γ we define a dual tetrad (e^0_μ, e^a_μ) with the relations

$$e^0_\mu = -u_\mu , \quad e^a_\mu = \delta^{ab} g_{\mu\nu} e^\nu_b . \tag{2.2}$$

The dual vectors e^a_μ satisfy a transport law that is very similar to Eq. (2.1). The tetrad and its dual give rise to the completeness relations

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ab} e^\mu_a e^\nu_b ,$$
$$g_{\mu\nu} = -e^0_\mu e^0_\nu + \delta_{ab} e^a_\mu e^b_\nu \tag{2.3}$$

for the metric and its inverse evaluated on the world line.

The retarded coordinates are constructed with the help of a null geodesic that links a given point x to the world line. This geodesic must be unique, and we thus restrict x to be within the normal convex neighborhood of γ. We denote by β the unique, future-directed null geodesic that goes from the world line to x, and $x' \equiv z(u)$ is the point at which β intersects the world line; u is the value of the proper-time parameter at this point. To tensors at x we shall assign the Greek indices α, β, γ, δ, etc.; to tensors at x' we shall assign the indices α', β', γ', δ', and so on.

From the tetrad $(u^{\alpha'}, e^{\alpha'}_a)$ at x' we construct another tetrad (e^α_0, e^α_a) at x by parallel transport on β. By raising the frame index and lowering the vectorial index, we obtain also a dual tetrad at x: $e^0_\alpha = -g_{\alpha\beta} e^\beta_0$ and $e^a_\alpha = \delta^{ab} g_{\alpha\beta} e^\beta_b$. The metric at x can then be expressed as

$$g_{\alpha\beta} = -e^0_\alpha e^0_\beta + \delta_{ab} e^a_\alpha e^b_\beta , \tag{2.4}$$

and the parallel propagator [1] (also known as the bivector of geodetic parallel displacement [18]) from x' to x is given by

$$g^{\alpha}{}_{\alpha'}(x,x') = -e_0^{\alpha} u_{\alpha'} + e_a^{\alpha} e^a_{\alpha'},$$

$$g^{\alpha'}{}_{\alpha}(x',x) = u^{\alpha'} e_{\alpha}^0 + e_a^{\alpha'} e^a_{\alpha}. \tag{2.5}$$

This is defined such that if A^{α} is a vector that is parallel transported on β, then $A^{\alpha}(x) = g^{\alpha}{}_{\alpha'}(x,x')A^{\alpha'}(x')$ and $A^{\alpha'}(x') = g^{\alpha'}{}_{\alpha}(x',x)A^{\alpha}(x)$. Similarly, if p_{α} is a parallel-transported dual vector, then $p_{\alpha}(x) = g^{\alpha'}{}_{\alpha}(x',x)p_{\alpha'}(x')$ and $p_{\alpha'}(x') = g^{\alpha}{}_{\alpha'}(x,x')p_{\alpha}(x)$.

The last ingredient we shall need is Synge's world function $\sigma(z,x)$ [1] (also known as the biscalar of geodetic interval [18]). This is defined as half the squared geodesic distance between the world-line point $z(\tau)$ and a neighboring point x. The derivative of the world function with respect to z^{μ} is denoted $\sigma_{\mu}(z,x)$; this is a vector at z (and a scalar at x) that is known to be tangent to the geodesic linking z and x. The derivative of $\sigma(z,x)$ with respect to x^{α} is denoted $\sigma_{\alpha}(z,x)$; this vector at x (and scalar at z) is also tangent to the geodesic. We use a similar notation for multiple derivatives; for example, $\sigma_{\mu\alpha} \equiv \nabla_{\alpha}\nabla_{\mu}\sigma$ and $\sigma_{\alpha\beta} \equiv \nabla_{\beta}\nabla_{\alpha}\sigma$, where ∇_{α} denotes a covariant derivative at x while ∇_{μ} indicates covariant differentiation at z.

The vector $-\sigma^{\mu}(z,x)$ can be thought of as a separation vector between x and z, pointing from the world line to x. When x is close to γ, $-\sigma^{\mu}(z,x)$ is small and can be used to express bitensors in terms of ordinary tensors at z [1,18]. For example,

$$\sigma_{\mu\nu} = g_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}\sigma^{\lambda}\sigma^{\rho} + \cdots, \tag{2.6}$$

$$\sigma_{\mu\alpha} = -g^{\nu}{}_{\alpha}\left(g_{\mu\nu} + \frac{1}{6}R_{\mu\lambda\nu\rho}\sigma^{\lambda}\sigma^{\rho} + \cdots\right), \tag{2.7}$$

$$\sigma_{\alpha\beta} = g^{\mu}{}_{\alpha}g^{\nu}{}_{\beta}\left(g_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}\sigma^{\lambda}\sigma^{\rho} + \cdots\right), \tag{2.8}$$

where $g^{\mu}{}_{\alpha} \equiv g^{\mu}{}_{\alpha}(z,x)$ is the parallel propagator and $R_{\mu\lambda\nu\rho}$ is the Riemann tensor evaluated on the world line.

B. Definition of the retarded coordinates

The quasi-Cartesian version of the retarded coordinates are defined by

$$\hat{x}^0 = u, \quad \hat{x}^a = -e^a_{\alpha'}(x')\sigma^{\alpha'}(x,x'), \quad \sigma(x,x') = 0. \tag{2.9}$$

The last statement indicates that $x' \equiv z(u)$ and x are linked by a null geodesic.[1]

From the fact that $\sigma^{\alpha'}$ is a null vector we obtain

$$r \equiv (\delta_{ab}\hat{x}^a\hat{x}^b)^{1/2} = u_{\alpha'}\sigma^{\alpha'}, \tag{2.10}$$

and r is a positive quantity by virtue of the fact that β is a future-directed null geodesic—this makes $\sigma^{\alpha'}$ past-directed. In flat spacetime, $\sigma^{\alpha'} = -(x-x')^{\alpha}$, and in a Lorentz frame that is momentarily comoving with the world line, $r = t - t' > 0$; with the speed of light set equal to unity, r is also the spatial distance between x' and x as measured in this frame. This gives us an interpretation of $r = u_{\alpha'}\sigma^{\alpha'}$ as a *retarded distance* between x and the world line, and we shall keep this interpretation even in curved spacetime. The claim that r gives a measure of distance between x' and x will be substantiated in Sec. II C.

Another consequence of Eq. (2.9) is that

$$\sigma^{\alpha'} = -r(u^{\alpha'} + \Omega^a e_a^{\alpha'}), \tag{2.11}$$

where $\Omega^a \equiv \hat{x}^a/r$ is a frame vector that satisfies $\delta_{ab}\Omega^a\Omega^b = 1$.

A straightforward calculation reveals that under a displacement of the point x, the retarded coordinates change according to

$$du = -k_{\alpha}dx^{\alpha}, \tag{2.12}$$

$$d\hat{x}^a = -(ra^a - \omega^a{}_b\hat{x}^b + e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'})du - e^a_{\alpha'}\sigma^{\alpha'}{}_{\beta}dx^{\beta}, \tag{2.13}$$

where $k_{\alpha} \equiv \sigma_{\alpha}/r$ is a future-directed null vector at x that is tangent to the geodesic β. To obtain these results we must keep in mind that a displacement of x typically induces a simultaneous displacement of x', because the new points $x + \delta x$ and $x' + \delta x'$ must also be linked by a null geodesic. We therefore have $0 = \sigma(x+\delta x, x'+\delta x') = \sigma_{\alpha}\delta x^{\alpha} + \sigma_{\alpha'}\delta x^{\alpha'}$, and Eq. (2.12) follows from the fact that a displacement along the world line is described by $\delta x^{\alpha'} = u^{\alpha'}\delta u$.

C. Retarded distance; null vector field

If we keep x' linked to x by the relation $\sigma(x,x') = 0$, then

$$r(x) = \sigma_{\alpha'}(x,x')u^{\alpha'}(x') \tag{2.14}$$

can be viewed as an ordinary scalar field defined in a neighborhood of γ. We can compute the gradient of r by finding how r changes under a displacement of x (which again induces a displacement of x'). The result is

$$\partial_{\beta}r = -(\sigma_{\alpha'}a^{\alpha'} + \sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'})k_{\beta} + \sigma_{\alpha'\beta}u^{\alpha'}. \tag{2.15}$$

Similarly, we can view

[1] A similar definition can be given for Fermi normal coordinates [1,19]. Here x is linked to γ by a spacelike geodesic that intersects the world line orthogonally. The intersection point is such that $\sigma_{\mu}(x,z)u^{\mu} = 0$, and this replaces the last condition of Eq. (2.9). The other relations are unchanged.

$$k^\alpha(x) = \frac{\sigma^\alpha(x,x')}{r(x)} \tag{2.16}$$

as an ordinary vector field, which is tangent to the congruence of null geodesics that emanate from x'. It is easy to check that this vector satisfies the identities

$$\sigma_{\alpha\beta}k^\beta = k_\alpha, \quad \sigma_{\alpha'\beta}k^\beta = \frac{\sigma_{\alpha'}}{r}, \tag{2.17}$$

from which we also obtain $\sigma_{\alpha'\beta}u^{\alpha'}k^\beta = 1$. From this last result and Eq. (2.15) we deduce the important relation

$$k^\alpha \partial_\alpha r = 1. \tag{2.18}$$

In addition, combining the general statement $\sigma^\alpha = -g^\alpha{}_{\alpha'}\sigma^{\alpha'}$ with Eq. (2.11) gives

$$k^\alpha = g^\alpha{}_{\alpha'}(u^{\alpha'} + \Omega^a e_a^{\alpha'}); \tag{2.19}$$

the vector at x is therefore obtained by parallel transport of $u^{\alpha'} + \Omega^a e_a^{\alpha'}$ on β. From this and Eq. (2.4) we get the alternative expression

$$k^\alpha = e_0^\alpha + \Omega^a e_a^\alpha, \tag{2.20}$$

which confirms that k^α is a future-directed null vector field (recall that $\Omega^a = \hat{x}^a/r$ is a unit frame vector).

The covariant derivative of k_α can be computed by finding how the vector changes under a displacement of x. (It is in fact easier to first calculate how rk_α changes, and then substitute our previous expression for $\partial_\beta r$.) The result is

$$rk_{\alpha;\beta} = \sigma_{\alpha\beta} - k_\alpha\sigma_{\beta\gamma'}u^{\gamma'} - k_\beta\sigma_{\alpha\gamma'}u^{\gamma'} + (\sigma_{\alpha'}a^{\alpha'} + \sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'})k_\alpha k_\beta. \tag{2.21}$$

From this we infer that k^α satisfies the geodesic equation in affine-parameter form, $k^\alpha{}_{;\beta}k^\beta = 0$, and Eq. (2.18) informs us that the affine parameter is in fact r. A displacement along a member of the congruence is therefore described by $dx^\alpha = k^\alpha dr$. Specializing to retarded coordinates, and using Eqs. (2.12), (2.13), and (2.17), we find that this statement becomes $du = 0$ and $d\hat{x}^a = (\hat{x}^a/r)dr$, which integrate to $u = \text{const}$ and $\hat{x}^a = r\Omega^a$, respectively, with Ω^a representing a constant unit vector. We have found that the congruence of null geodesics emanating from x' is described by

$$u = \text{const}, \quad \hat{x}^a = r\Omega^a(\theta^A) \tag{2.22}$$

in the retarded coordinates. Here, the two angles θ^A ($A = 1,2$) serve to parametrize the unit vector Ω^a, which is independent of r.

Equation (2.21) also implies that the expansion of the congruence is given by

$$\Theta = k^\alpha{}_{;\alpha} = \frac{\sigma^\alpha{}_\alpha - 2}{r}. \tag{2.23}$$

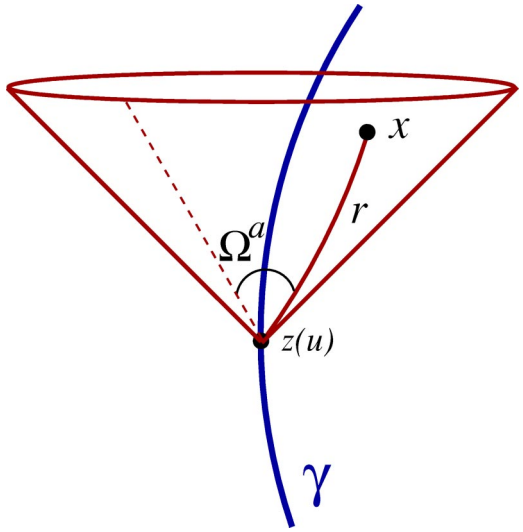

FIG. 1. Retarded coordinates of a point x relative to a world line γ. The retarded time u selects a particular null cone, the unit vector $\Omega^a \equiv \hat{x}^a/r$ selects a particular generator of this null cone, and the retarded distance r selects a particular point on this generator.

Using the expansion for σ^α_α given by Eq. (2.8), we find that this becomes $r\Theta = 2 - \frac{1}{3}R_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} + O(r^3)$, or

$$r\Theta = 2 - \frac{1}{3}r^2(R_{00} + 2R_{0a}\Omega^a + R_{ab}\Omega^a\Omega^b) + O(r^3) \tag{2.24}$$

after using Eq. (2.11). Here, $R_{00} = R_{\alpha'\beta'}u^{\alpha'}u^{\beta'}$, $R_{0a} = R_{\alpha'\beta'}u^{\alpha'}e_a^{\beta'}$, and $R_{ab} = R_{\alpha'\beta'}e_a^{\alpha'}e_b^{\beta'}$ are the frame components of the Ricci tensor evaluated at x'. This result confirms that the congruence is singular at $r = 0$, because Θ diverges as $2/r$ in this limit; the caustic coincides with the point x'.

Finally, we infer from Eq. (2.21) that k^α is hypersurface orthogonal. This, together with the property that k^α satisfies the geodesic equation in affine-parameter form, implies that there exists a scalar field $u(x)$ such that

$$k_\alpha = -\partial_\alpha u. \tag{2.25}$$

This scalar field was already identified in Eq. (2.12): it is numerically equal to the proper-time parameter of the world line at x'. We may thus conclude that the geodesics to which k^α is tangent are the generators of the null cone $u = \text{const}$. As Eq. (2.22) indicates, a specific generator is selected by choosing a direction Ω^a (which can be parametrized by two angles θ^A), and r is an affine parameter on each generator. The geometrical meaning of the retarded coordinates is now completely clear, and r is recognized as a meaningful measure of distance between x and the world line. The construction is illustrated in Fig. 1.

D. Frame components of tensors on the world line

The metric at x in the retarded coordinates will be expressed in terms of frame components of vectors and tensors evaluated on γ. For example, if $a^{\alpha'}$ is the acceleration vector at x', then as we have seen,

$$a_a(u) = a_{\alpha'} e_a^{\alpha'} \tag{2.26}$$

are the frame components of the acceleration at proper time u.

Similarly,

$$\begin{aligned} R_{a0b0}(u) &= R_{\alpha'\gamma'\beta'\delta'} e_a^{\alpha'} u^{\gamma'} e_b^{\beta'} u^{\delta'}, \\ R_{a0bd}(u) &= R_{\alpha'\gamma'\beta'\delta'} e_a^{\alpha'} u^{\gamma'} e_b^{\beta'} e_d^{\delta'}, \\ R_{acbd}(u) &= R_{\alpha'\gamma'\beta'\delta'} e_a^{\alpha'} e_c^{\gamma'} e_b^{\beta'} e_d^{\delta'} \end{aligned} \tag{2.27}$$

are the frame components of the Riemann tensor evaluated on γ. From these we form the useful combinations

$$S_{ab} = R_{a0b0} + R_{a0bc}\Omega^c + R_{b0ac}\Omega^c + R_{acbd}\Omega^c\Omega^d = S_{ba}, \tag{2.28}$$

$$S_a = S_{ab}\Omega^b = R_{a0b0}\Omega^b - R_{ab0c}\Omega^b\Omega^c, \tag{2.29}$$

$$S = S_a\Omega^a = R_{a0b0}\Omega^a\Omega^b, \tag{2.30}$$

in which the quantities $\Omega^a \equiv \hat{x}^a/r$ depend on the angles θ^A only—they are independent of u and r.

We have previously introduced the frame components of the Ricci tensor in Eq. (2.24). The identity

$$R_{00} + 2R_{0a}\Omega^a + R_{ab}\Omega^a\Omega^b = \delta^{ab}S_{ab} - S \tag{2.31}$$

follows easily from Eqs. (2.28)–(2.30) and the definition of the Ricci tensor.

E. Coordinate displacements near γ

The changes in the quasi-Cartesian retarded coordinates under a displacement of x are given by Eqs. (2.12) and (2.13). In these we substitute the expansions for $\sigma_{\alpha'\beta'}$ and $\sigma_{\alpha'\beta}$ that appear in Eqs. (2.6) and (2.7), as well as Eqs. (2.11) and (2.19). After a straightforward calculation, we obtain the following expressions for the coordinate displacements:

$$du = (e_\alpha^0 dx^\alpha) - \Omega_a(e_\alpha^b dx^\alpha), \tag{2.32}$$

$$\begin{aligned} d\hat{x}^a = &-\left[ra^a - r\omega^a{}_b\Omega^b + \frac{1}{2}r^2S^a + O(r^3)\right](e_\alpha^0 dx^\alpha) \\ &+\left[\delta^a{}_b + \left(ra^a - r\omega^a{}_c\Omega^c + \frac{1}{3}r^2S^a\right)\Omega_b \right. \\ &\left. + \frac{1}{6}r^2S^a{}_b + O(r^3)\right](e_\alpha^b dx^\alpha). \end{aligned} \tag{2.33}$$

Notice that the result for du is exact, but that $d\hat{x}^a$ is expressed as an expansion in powers of r.

These results can also be expressed in the form of gradients of the retarded coordinates:

$$\partial_\alpha u = e_\alpha^0 - \Omega_a e_\alpha^a, \tag{2.34}$$

$$\begin{aligned} \partial_\alpha \hat{x}^a = &-\left[ra^a - r\omega^a{}_b\Omega^b + \frac{1}{2}r^2S^a + O(r^3)\right]e_\alpha^0 \\ &+\left[\delta^a{}_b + \left(ra^a - r\omega^a{}_c\Omega^c + \frac{1}{3}r^2S^a\right)\Omega_b \right. \\ &\left. + \frac{1}{6}r^2S^a{}_b + O(r^3)\right]e_\alpha^b. \end{aligned} \tag{2.35}$$

Notice that Eq. (2.34) follows immediately from Eqs. (2.20) and (2.25). From Eq. (2.35) and the identity $\partial_\alpha r = \Omega_a \partial_\alpha \hat{x}^a$ we also infer

$$\begin{aligned} \partial_\alpha r = &-\left[ra_a\Omega^a + \frac{1}{2}r^2S + O(r^3)\right]e_\alpha^0 \\ &+\left[\left(1 + ra_b\Omega^b + \frac{1}{3}r^2S\right)\Omega_a + \frac{1}{6}r^2S_a + O(r^3)\right]e_\alpha^a, \end{aligned} \tag{2.36}$$

where we have used the facts that $S_a = S_{ab}\Omega^b$ and $S = S_a\Omega^a$; see Eqs. (2.29) and (2.30). It may be checked that Eq. (2.36) agrees with Eq. (2.15).

F. Metric near γ

It is straightforward to invert the relations of Eqs. (2.32) and (2.33) and solve for $e_\alpha^0 dx^\alpha$ and $e_\alpha^a dx^\alpha$. The results are

$$\begin{aligned} e_\alpha^0 dx^\alpha = &\left[1 + ra_a\Omega^a + \frac{1}{2}r^2S + O(r^3)\right]du \\ &+\left[\left(1 + \frac{1}{6}r^2S\right)\Omega_a - \frac{1}{6}r^2S_a + O(r^3)\right]d\hat{x}^a, \end{aligned} \tag{2.37}$$

$$\begin{aligned} e_\alpha^a dx^\alpha = &\left[r(a^a - \omega^a{}_b\Omega^b) + \frac{1}{2}r^2S^a + O(r^3)\right]du \\ &+\left[\delta^a{}_b - \frac{1}{6}r^2S^a{}_b + \frac{1}{6}r^2S^a\Omega_b + O(r^3)\right]d\hat{x}^b. \end{aligned} \tag{2.38}$$

These relations, when specialized to the retarded coordinates, give us the components of the dual tetrad (e_α^0, e_α^a) at x. The metric is then computed by using the completeness relations of Eq. (2.3). We find

$$\begin{aligned} g_{uu} = &-(1 + ra_a\Omega^a)^2 + r^2(a_a - \omega_{ab}\Omega^b)(a^a - \omega^a{}_c\Omega^c) \\ &- r^2S + O(r^3), \end{aligned} \tag{2.39}$$

$$g_{ua} = -\left(1 + r a_b \Omega^b + \frac{2}{3} r^2 S\right)\Omega_a + r(a_a - \omega_{ab}\Omega^b) + \frac{2}{3} r^2 S_a + O(r^3), \tag{2.40}$$

$$g_{ab} = \delta_{ab} - \left(1 + \frac{1}{3} r^2 S\right)\Omega_a \Omega_b - \frac{1}{3} r^2 S_{ab} + \frac{1}{3} r^2 (S_a \Omega_b + \Omega_a S_b) + O(r^3). \tag{2.41}$$

We see that the metric possesses a directional ambiguity on the world line: the metric at $r=0$ still depends on the vector $\Omega^a = \hat{x}^a/r$ that specifies the direction to the point x. The retarded coordinates are therefore singular on the world line, and tensor components cannot be defined on γ. Because we are working with *frame components* of tensors, this poses no particular difficulty.

By setting $S_{ab} = S_a = S = 0$ in Eqs. (2.39)–(2.41) we obtain the metric of flat spacetime in the retarded coordinates. This we express as

$$\eta_{uu} = -(1 + r a_a \Omega^a)^2 + r^2 (a_a - \omega_{ab}\Omega^b)(a^a - \omega^a{}_c \Omega^c),$$

$$\eta_{ua} = -(1 + r a_b \Omega^b)\Omega_a + r(a_a - \omega_{ab}\Omega^b), \tag{2.42}$$

$$\eta_{ab} = \delta_{ab} - \Omega_a \Omega_b,$$

and we see that the directional ambiguity persists. This should not come as a surprise: the ambiguity is present even when $a_a = \omega_{ab} = 0$, and is generated simply by performing the coordinate transformation $u = t - \sqrt{x^2 + y^2 + z^2}$. The retarded coordinates are therefore necessarily singular at the world line. But in spite of the directional ambiguity, the metric of flat spacetime has a unit determinant everywhere, and it is easily inverted:

$$\eta^{uu} = 0,$$

$$\eta^{ua} = -\Omega^a, \tag{2.43}$$

$$\eta^{ab} = \delta^{ab} + r(a^a - \omega^a{}_c \Omega^c)\Omega^b + r\Omega^a (a^b - \omega^b{}_c \Omega^c).$$

The inverse metric also is ambiguous on the world line.

To invert the curved-spacetime metric of Eqs. (2.39)–(2.41) we express it as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(r^3)$ and treat $h_{\alpha\beta} = O(r^2)$ as a perturbation. The inverse metric is then $g^{\alpha\beta} = \eta^{\alpha\beta} - \eta^{\alpha\gamma}\eta^{\beta\delta} h_{\gamma\delta} + O(r^3)$, or

$$g^{uu} = 0, \tag{2.44}$$

$$g^{ua} = -\Omega^a, \tag{2.45}$$

$$g^{ab} = \delta^{ab} + r(a^a - \omega^a{}_c \Omega^c)\Omega^b + r\Omega^a (a^b - \omega^b{}_c \Omega^c) + \frac{1}{3} r^2 S^{ab} + \frac{1}{3} r^2 (S^a \Omega^b + \Omega^a S^b) + O(r^3). \tag{2.46}$$

The results for g^{uu} and g^{ua} are exact, and they follow from the general relations $g^{\alpha\beta}(\partial_\alpha u)(\partial_\beta u) = 0$ and $g^{\alpha\beta}(\partial_\alpha u) \times (\partial_\beta r) = -1$ that are derived from Eqs. (2.18) and (2.25).

The metric determinant is computed from $\sqrt{-g} = 1 + \frac{1}{2}\eta^{\alpha\beta} h_{\alpha\beta} + O(r^3)$, which gives

$$\sqrt{-g} = 1 - \frac{1}{6} r^2 (\delta^{ab} S_{ab} - S) + O(r^3) = 1 - \frac{1}{6} r^2 (R_{00} + 2R_{0a}\Omega^a + R_{ab}\Omega^a \Omega^b) + O(r^3), \tag{2.47}$$

where we have substituted the identity of Eq. (2.31). Comparison with Eq. (2.24) gives us the interesting relation $\sqrt{-g} = \frac{1}{2} r\Theta + O(r^3)$, where Θ is the expansion of the generators of the null cones $u = \text{const}$.

G. Transformation to angular coordinates

Because the frame vector $\Omega^a = \hat{x}^a/r$ satisfies $\delta_{ab}\Omega^a \Omega^b = 1$, it can be parametrized by two angles θ^A. A canonical choice for the parametrization is $\Omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. It is then convenient to perform a coordinate transformation from $\hat{x}^a$ to (r, θ^A), using the relations $\hat{x}^a = r\Omega^a(\theta^A)$. (Recall from Sec. II C that the angles θ^A are constant on the generators of the null cones $u = \text{const}$, and that r is an affine parameter on these generators. The relations $\hat{x}^a = r\Omega^a$ therefore describe the behavior of the generators.) The differential form of the coordinate transformation is

$$d\hat{x}^a = \Omega^a dr + r\Omega^a_A d\theta^A, \tag{2.48}$$

where the transformation matrix

$$\Omega^a_A \equiv \frac{\partial \Omega^a}{\partial \theta^A} \tag{2.49}$$

satisfies the identity $\Omega_a \Omega^a_A = 0$.

We introduce the quantities

$$\Omega_{AB} = \delta_{ab}\Omega^a_A \Omega^b_B, \tag{2.50}$$

which act as a (nonphysical) metric in the subspace spanned by the angular coordinates. In the canonical parametrization, $\Omega_{AB} = \text{diag}(1, \sin^2\theta)$. We use the inverse of Ω_{AB}, denoted Ω^{AB}, to raise upper-case Latin indices. We then define the new object

$$\Omega^A_a = \delta_{ab}\Omega^{AB}\Omega^b_B \tag{2.51}$$

which satisfies the identities

$$\Omega^A_a \Omega^a_B = \delta^A_B, \quad \Omega^a_A \Omega^A_b = \delta^a{}_b - \Omega^a \Omega_b. \tag{2.52}$$

The second result follows from the fact that both sides are symmetric in a and b, orthogonal to Ω_a and Ω^b, and have the same trace.

From the preceding results we establish that the transformation from $\hat{x}^a$ to (r,θ^A) is accomplished by

$$\frac{\partial \hat{x}^a}{\partial r}=\Omega^a, \quad \frac{\partial \hat{x}^a}{\partial \theta^A}=r\Omega^a_A, \tag{2.53}$$

while the transformation from (r,θ^A) to $\hat{x}^a$ is accomplished by

$$\frac{\partial r}{\partial \hat{x}^a}=\Omega_a, \quad \frac{\partial \theta^A}{\partial \hat{x}^a}=\frac{1}{r}\Omega^A_a. \tag{2.54}$$

With these rules it is easy to show that in the angular coordinates, the metric takes the form of

$$ds^2=g_{uu}du^2-2dudr+2g_{uA}dud\theta^A+g_{AB}d\theta^Ad\theta^B, \tag{2.55}$$

with

$$g_{uu}=-(1+ra_a\Omega^a)^2+r^2(a_a-\omega_{ab}\Omega^b)(a^a-\omega^a{}_c\Omega^c) -r^2S+O(r^3), \tag{2.56}$$

$$g_{uA}=r\left[r(a_a-\omega_{ab}\Omega^b)+\frac{2}{3}r^2S_a+O(r^3)\right]\Omega^a_A, \tag{2.57}$$

$$g_{AB}=r^2\left[\Omega_{AB}-\frac{1}{3}r^2S_{ab}\Omega^a_A\Omega^b_B+O(r^3)\right]. \tag{2.58}$$

The results $g_{ru}=-1$, $g_{rr}=0$, and $g_{rA}=0$ are exact, and they follow from the fact that in the retarded coordinates, $k_\alpha dx^\alpha=-du$ and $k^\alpha\partial_\alpha=\partial_r$.

The nonvanishing components of the inverse metric are

$$g^{ur}=-1, \tag{2.59}$$

$$g^{rr}=1+2ra_a\Omega^a+r^2S+O(r^3), \tag{2.60}$$

$$g^{rA}=\frac{1}{r}\left[r(a^a-\omega^a{}_b\Omega^b)+\frac{2}{3}r^2S^a+O(r^3)\right]\Omega^A_a, \tag{2.61}$$

$$g^{AB}=\frac{1}{r^2}\left[\Omega^{AB}+\frac{1}{3}r^2S^{ab}\Omega^A_a\Omega^B_b+O(r^3)\right]. \tag{2.62}$$

The results $g^{uu}=0$, $g^{ur}=-1$, and $g^{uA}=0$ are exact, and they follow from the same reasoning as before.

Finally, we note that in the angular coordinates, the metric determinant is given by

$$\sqrt{-g}=r^2\sqrt{\Omega}\left[1-\frac{1}{6}r^2(R_{00}+2R_{0a}\Omega^a+R_{ab}\Omega^a\Omega^b)+O(r^3)\right], \tag{2.63}$$

where Ω is the determinant of Ω_{AB}; in the canonical parametrization, $\sqrt{\Omega}=\sin\theta$.

H. Specialization to $a^\mu=0=R_{\mu\nu}$

In this subsection we specialize our previous results to a situation where γ is a geodesic on which the Ricci tensor vanishes. We therefore set $a^\mu=0=R_{\mu\nu}$ everywhere on γ, and for simplicity we also set ω_{ab} to zero.

It is known that when the Ricci tensor vanishes, the Riemann tensor can be decomposed in terms of a timelike vector u^α and two symmetric-tracefree, spatial tensors $\mathcal{E}_{\alpha\beta}$ and $\mathcal{B}_{\alpha\beta}$ (see, for example, Ref. [8]). In terms of frame components we have

$$\begin{aligned} R_{a0b0}(u)&=\mathcal{E}_{ab}\\ R_{a0bc}(u)&=\varepsilon_{bcd}\mathcal{B}^d{}_a,\\ R_{acbd}(u)&=\delta_{ab}\mathcal{E}_{cd}+\delta_{cd}\mathcal{E}_{ab}-\delta_{ad}\mathcal{E}_{bc}-\delta_{bc}\mathcal{E}_{ad}, \end{aligned} \tag{2.64}$$

where $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$ depend on u, are such that $\mathcal{E}_{ba}=\mathcal{E}_{ab}$, $\delta^{ab}\mathcal{E}_{ab}=0$, $\mathcal{B}_{ba}=\mathcal{B}_{ab}$, $\delta^{ab}\mathcal{B}_{ab}=0$, and ε_{abc} is the three-dimensional permutation symbol. These relations can be substituted into Eqs. (2.28)–(2.30) to give

$$S_{ab}=2\mathcal{E}_{ab}-\Omega_a\mathcal{E}_{bc}\Omega^c-\Omega_b\mathcal{E}_{ac}\Omega^c+\delta_{ab}\mathcal{E}_{bc}\Omega^c\Omega^d +\varepsilon_{acd}\Omega^c\mathcal{B}^d{}_b+\varepsilon_{bcd}\Omega^c\mathcal{B}^d{}_a, \tag{2.65}$$

$$S_a=\mathcal{E}_{ab}\Omega^b+\varepsilon_{abc}\Omega^b\mathcal{B}^c{}_d\Omega^d, \tag{2.66}$$

$$S=\mathcal{E}_{ab}\Omega^a\Omega^b. \tag{2.67}$$

In these expressions the dependence on retarded time u is contained in $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$, while the angular dependence is encoded in the unit vector Ω^a.

It is convenient to introduce the irreducible quantities

$$\mathcal{E}^*=\mathcal{E}_{ab}\Omega^a\Omega^b, \tag{2.68}$$

$$\mathcal{E}^*_a=(\delta_a{}^b-\Omega_a\Omega^b)\mathcal{E}_{bc}\Omega^c, \tag{2.69}$$

$$\mathcal{E}^*_{ab}=2\mathcal{E}_{ab}-2\Omega_a\mathcal{E}_{bc}\Omega^c-2\Omega_b\mathcal{E}_{ac}\Omega^c+(\delta_{ab}+\Omega_a\Omega_b)\mathcal{E}^*, \tag{2.70}$$

$$\mathcal{B}^*_a=\varepsilon_{abc}\Omega^b\mathcal{B}^c{}_d\Omega^d, \tag{2.71}$$

$$\mathcal{B}^*_{ab}=2(\delta^e_{(a}-\Omega_{(a}\Omega^e)\varepsilon_{b)cd}\Omega^c\mathcal{B}^d{}_e. \tag{2.72}$$

These are all orthogonal to Ω^a: $\mathcal{E}^*_a\Omega^a=\mathcal{B}^*_a\Omega^a=0$ and $\mathcal{E}^*_{ab}\Omega^b=\mathcal{B}^*_{ab}\Omega^b=0$. In terms of these Eqs. (2.65)–(2.67) become

$$S_{ab} = \mathcal{E}^*_{ab} + \Omega_a \mathcal{E}^*_b + \mathcal{E}^*_a \Omega_b + \Omega_a \Omega_b \mathcal{E}^* + \mathcal{B}^*_{ab} + \Omega_a \mathcal{B}^*_b + \mathcal{B}^*_a \Omega_b, \tag{2.73}$$

$$S_a = \mathcal{E}^*_a + \Omega_a \mathcal{E}^* + \mathcal{B}^*_a, \tag{2.74}$$

$$S = \mathcal{E}^*. \tag{2.75}$$

When Eqs. (2.73)–(2.75) are substituted into the metric tensor of Eqs. (2.39)–(2.41)—in which a_a and ω_{ab} are both set equal to zero—we obtain the compact expressions

$$g_{uu} = -1 - r^2 \mathcal{E}^* + O(r^3), \tag{2.76}$$

$$g_{ua} = -\Omega_a + \frac{2}{3} r^2 (\mathcal{E}^*_a + \mathcal{B}^*_a) + O(r^3), \tag{2.77}$$

$$g_{ab} = \delta_{ab} - \Omega_a \Omega_b - \frac{1}{3} r^2 (\mathcal{E}^*_{ab} + \mathcal{B}^*_{ab}) + O(r^3). \tag{2.78}$$

The metric becomes

$$g_{uu} = -1 - r^2 \mathcal{E}^* + O(r^3), \tag{2.79}$$

$$g_{ur} = -1, \tag{2.80}$$

$$g_{uA} = \frac{2}{3} r^3 (\mathcal{E}^*_A + \mathcal{B}^*_A) + O(r^4), \tag{2.81}$$

$$g_{AB} = r^2 \Omega_{AB} - \frac{1}{3} r^4 (\mathcal{E}^*_{AB} + \mathcal{B}^*_{AB}) + O(r^5) \tag{2.82}$$

after transforming to angular coordinates using the rules of Eq. (2.53). Here we have introduced the projections

$$\mathcal{E}^*_A \equiv \mathcal{E}^*_a \Omega^a_A = \mathcal{E}_{ab} \Omega^a_A \Omega^b, \tag{2.83}$$

$$\mathcal{E}^*_{AB} \equiv \mathcal{E}^*_{ab} \Omega^a_A \Omega^b_B = 2 \mathcal{E}_{ab} \Omega^a_A \Omega^b_B + \mathcal{E}^* \Omega_{AB}, \tag{2.84}$$

$$\mathcal{B}^*_A \equiv \mathcal{B}^*_a \Omega^a_A = \varepsilon_{abc} \Omega^a_A \Omega^b \mathcal{B}^c{}_d \Omega^d, \tag{2.85}$$

$$\mathcal{B}^*_{AB} \equiv \mathcal{B}^*_{ab} \Omega^a_A \Omega^b_B = 2 \varepsilon_{acd} \Omega^c \mathcal{B}^d{}_b \Omega^a_{(A} \Omega^b_{B)}. \tag{2.86}$$

It may be noted that the inverse relations are $\mathcal{E}^*_a = \mathcal{E}^*_A \Omega^A_a$, $\mathcal{B}^*_a = \mathcal{B}^*_A \Omega^A_a$, $\mathcal{E}^*_{ab} = \mathcal{E}^*_{AB} \Omega^A_a \Omega^B_b$, and $\mathcal{B}^*_{ab} = \mathcal{B}^*_{AB} \Omega^A_a \Omega^B_b$, where Ω^A_a was introduced in Eq. (2.51).

The angular dependence of the quantities listed in Eqs. (2.83)–(2.86) can be made more explicit by expressing them in terms of scalar, vectorial, and tensorial spherical harmonics. Let

$$Y^{\mathsf{m}} = \{Y^0, Y^{1c}, Y^{1s}, Y^{2c}, Y^{2s}\} \tag{2.87}$$

be a set of real, unnormalized, spherical-harmonic functions of degree $l = 2$; explicit expressions are provided in Appendix A. The numerical part of the label m refers to the azimuthal index m and the letter indicates whether the function is proportional to $\cos(m\phi)$ or $\sin(m\phi)$. Vectorial harmonics are defined by

$$Y^{\mathsf{m}}_A = Y^{\mathsf{m}}_{:A}, \quad X^{\mathsf{m}}_A = -\varepsilon_A{}^B Y^{\mathsf{m}}_{:B}, \tag{2.88}$$

where a colon indicates covariant differentiation with respect to a connection compatible with Ω_{AB}, and ε_{AB} is the two-dimensional Levi-Civita tensor. The vectorial harmonics Y^{m}_A have even parity, while X^{m}_A have odd parity. Tensorial harmonics are defined by

$$Y^{\mathsf{m}} \Omega_{AB}, \quad Y^{\mathsf{m}}_{AB} = Y^{\mathsf{m}}_{:AB}, \quad X^{\mathsf{m}}_{AB} = -X^{\mathsf{m}}_{(A:B)}; \tag{2.89}$$

the harmonics $Y^{\mathsf{m}} \Omega_{AB}$ and Y^{m}_{AB} have even parity, while X^{m}_{AB} have odd parity. Apart from notation and normalization, these definitions agree with those of Regge and Wheeler [20], and explicit expressions appear in Appendix A.

We define the harmonic components $\mathcal{E}_{\mathsf{m}}$ of the tensor $\mathcal{E}_{ab}$ with the relations

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}_{33} = -(\mathcal{E}_{11} + \mathcal{E}_{22}), \\ \mathcal{E}_{1c} &= 2\mathcal{E}_{13}, \\ \mathcal{E}_{1s} &= 2\mathcal{E}_{23}, \\ \mathcal{E}_{2c} &= \frac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22}), \\ \mathcal{E}_{2s} &= 2\mathcal{E}_{12}. \end{aligned} \tag{2.90}$$

Similarly, we define the harmonic components $\mathcal{B}_{\mathsf{m}}$ of the tensor $\mathcal{B}_{ab}$ by

$$\begin{aligned} \mathcal{B}_0 &= \mathcal{B}_{33} = -(\mathcal{B}_{11} + \mathcal{B}_{22}), \\ \mathcal{B}_{1c} &= 2\mathcal{B}_{13}, \\ \mathcal{B}_{1s} &= 2\mathcal{B}_{23}, \\ \mathcal{B}_{2c} &= \frac{1}{2}(\mathcal{B}_{11} - \mathcal{B}_{22}), \\ \mathcal{B}_{2s} &= 2\mathcal{B}_{12}. \end{aligned} \tag{2.91}$$

It is then straightforward to prove that Eqs. (2.83)–(2.86) are equivalent to

$$\mathcal{E}^* = \sum_{\mathsf{m}} \mathcal{E}_m Y^{\mathsf{m}},$$

$$\mathcal{E}^*_A = \frac{1}{2} \sum_{\mathsf{m}} \mathcal{E}_m Y^{\mathsf{m}}_A,$$

$$\mathcal{E}^*_{AB} = \sum_{\mathsf{m}} \mathcal{E}_{\mathsf{m}} (Y^{\mathsf{m}}_{AB} + 3 Y^{\mathsf{m}} \Omega_{AB}),$$

$$\mathcal{B}^*_A = \frac{1}{2}\sum_{\mathsf{m}} \mathcal{B}_{\mathsf{m}} X^{\mathsf{m}}_A,$$

$$\mathcal{B}^*_{AB} = -\sum_{\mathsf{m}} \mathcal{B}_{\mathsf{m}} X^{\mathsf{m}}_{AB}. \tag{2.92}$$

This shows that the angular dependence of these quantities is purely quadrupolar ($l=2$).

I. Comoving observer in a spatially flat cosmology

To illustrate how the formalism works we first consider the world line of a comoving observer in a cosmological spacetime with metric

$$ds^2 = -dt^2 + a^2(t)(dx^2+dy^2+dz^2), \tag{2.93}$$

where $a(t)$ is an arbitrary scale factor; for simplicity we take the cosmology to be spatially flat. We take the observer to be at the spatial origin of the coordinate system ($x=y=z=0$), and her velocity vector is given by

$$u^\mu = (1,0,0,0). \tag{2.94}$$

This satisfies the geodesic equation, so $a^\mu = 0$. We wish to transform the metric of Eq. (2.93) to retarded coordinates based at the world line of this observer.

To do so we must first construct a triad of orthonormal spatial vectors e^μ_a. A simple choice is

$$e^\mu_1 = (0,a^{-1},0,0),$$

$$e^\mu_2 = (0,0,a^{-1},0), \tag{2.95}$$

$$e^\mu_3 = (0,0,0,a^{-1});$$

these vectors are all parallel transported on γ, and we have $\omega_{ab}=0$ according to Eq. (2.1).

Using $\Omega^a = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we find that the components of S_{ab}, as defined by Eq. (2.28), are given by

$$S_{11} = -\ddot{a}/a + (\dot{a}/a)^2(\sin^2\theta\sin^2\phi + \cos^2\theta),$$

$$S_{12} = -(\dot{a}/a)^2 \sin^2\theta \sin\phi\cos\phi,$$

$$S_{13} = -(\dot{a}/a)^2 \sin\theta\cos\theta\cos\phi,$$

$$S_{22} = -\ddot{a}/a + (\dot{a}/a)^2(\sin^2\theta\cos^2\phi + \cos^2\theta),$$

$$S_{23} = -(\dot{a}/a)^2 \sin\theta\cos\theta\sin\phi,$$

$$S_{33} = -\ddot{a}/a + (\dot{a}/a)^2 \sin^2\theta,$$

where an overdot indicates differentiation with respect to t; the scale factor and its derivatives are now all evaluated at $t=u$. According to Eq. (2.29), contracting S_{ab} with Ω^b gives S_a, and we obtain

$$S_a = -(\ddot{a}/a)\Omega_a.$$

Another contraction with Ω^a gives

$$S = -\ddot{a}/a,$$

according to Eq. (2.30). From these results it follows that

$$S_{ab}\Omega^a_A\Omega^b_B = [-\ddot{a}/a + (\dot{a}/a)^2]\Omega_{AB},$$

where $\Omega^a_A \equiv \partial\Omega^a/\partial\theta^A$ and $\Omega_{AB} = \mathrm{diag}(1,\sin^2\theta)$ were first introduced in Sec. II G. We also have $S_a\Omega^a_A = 0$.

Substituting these relations into Eqs. (2.56)–(2.58) shows that in the retarded coordinates, the metric components are given by

$$g_{uu} = -1 + r^2(\ddot{a}/a) + O(r^3),$$

$$g_{uA} = O(r^4), \tag{2.96}$$

$$g_{AB} = r^2\Omega_{AB}\left\{1 + \frac{1}{3}r^2[\ddot{a}/a - (\dot{a}/a)^2] + O(r^3)\right\},$$

in addition to $g_{ur} = -1$. Not surprisingly, the metric is spherically symmetric. Recall that the scale factor and its derivatives are all functions of retarded time u. When the scale factor behaves as a power law, $a(t)\propto t^\alpha$ with α a constant, we have $\ddot{a}/a = -\alpha(1-\alpha)/u^2$ and $\ddot{a}/a - (\dot{a}/a)^2 = -\alpha/u^2$. When instead the scale factor behaves as an exponential, $a(t)\propto e^{Ht}$ with H a constant, we have $\ddot{a}/a = H^2$ and $\ddot{a}/a - (\dot{a}/a)^2 = 0$.

To help clarify the meaning of these results, we present next an *ab initio* derivation of Eq. (2.96). We take the metric of Eq. (2.93) and switch to conformal time η, which is defined by the relation $d\eta = dt/a(t)$. The metric becomes

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2).$$

We then introduce spherical coordinates (ρ,θ,ϕ) through the relations $x=\rho\sin\theta\cos\phi$, $y=\rho\sin\theta\sin\phi$, $z=\rho\cos\theta$, and the null coordinate $\bar{u} = \eta - \rho$. The metric now reads

$$ds^2 = a^2(\bar{u}+\rho)(-d\bar{u}^2 - 2d\bar{u}d\rho + \rho^2 d\Omega^2), \tag{2.97}$$

where $d\Omega^2 = \Omega_{AB}d\theta^A d\theta^B = d\theta^2 + \sin^2\theta d\phi^2$. While $\bar{u}$ is a retarded-time coordinate and ρ is a radial coordinate, these are distinct from u and r, and Eq. (2.97) does not match the form of Eq. (2.96). Since u and $\bar{u}$ can both be used to label light cones centered at $\rho=0$, there must exist between them a relation of the form $u = u(\bar{u})$. And since r and ρ can both be used to parametrize the null generators of a light cone $\bar{u} = \mathrm{const}$ (although ρ is not an affine parameter), there must exist between them a relation of the form $r = r(\bar{u},\rho)$. We shall now obtain these relations. We recall that the world line is located at $\rho = r = 0$.

In the coordinates $(\bar{u},\rho,\theta,\phi)$, the observer's velocity vector is given by

$$u^\mu = \left(\frac{1}{a(\bar{u})}, 0, 0, 0 \right), \tag{2.98}$$

where the scale factor is evaluated at $\rho = 0$ and expressed as a function of $\bar{u}$ only. The null vector $\bar{k}_\alpha = -\partial_\alpha \bar{u}$ is tangent to the null cones $\bar{u} = \text{const}$, and its components are given by $\bar{k}^\alpha = (0, a^{-2}, 0, 0)$, where the scale factor is now a function of $\bar{u} + \rho$. We have $\bar{k}_\mu u^\mu = -1/a(\bar{u})$.

We are seeking a null coordinate u and a null vector field $k_\alpha = -\partial_\alpha u$ such that $k_\mu u^\mu = -1$; refer back to Eqs. (2.19) and (2.25). It is easy to see that this should be given by $k_\alpha = a(\bar{u})\bar{k}_\alpha$. We therefore define u with the statement

$$du = a(\bar{u})\, d\bar{u}, \tag{2.99}$$

and

$$k^\alpha = \left(0, \frac{a(\bar{u})}{a^2(\bar{u}+\rho)}, 0, 0 \right) \tag{2.100}$$

is tangent to the light cones $u = \text{const}$. This vector satisfies the geodesic equation $k^\alpha{}_{;\beta} k^\beta = 0$, and the affine parameter on the null generators is r. From Eq. (2.100) we have $d\rho/dr = a(\bar{u})/a^2(\bar{u}+\rho)$, which integrates to

$$r = \int_0^\rho \frac{a^2(\bar{u}+\rho')}{a(\bar{u})}\, d\rho', \tag{2.101}$$

taking into account the boundary values $r(\bar{u}, \rho = 0) = 0$. Equations (2.99) and (2.101) give us the transformation between the old coordinates $(\bar{u}, \rho)$ and the new coordinates (u, r).

After applying the coordinate transformation to the metric we obtain

$$ds^2 = -\left[\frac{a^2(\bar{u}+\rho)}{a^2(\bar{u})} - \frac{2 r_{\bar{u}}}{a(\bar{u})} \right] du^2 - 2\, du\, dr + a^2(\bar{u}+\rho)\rho^2 d\Omega^2, \tag{2.102}$$

where $r_{\bar{u}} \equiv \partial r / \partial \bar{u}$. To show that this matches the results of Eq. (2.96), we must evaluate the integral of Eq. (2.101). It is sufficient to work in a neighborhood of $\rho = 0$, and $a^2(\bar{u}+\rho)$ can be expressed as a Taylor expansion. This yields

$$r = a\rho \left[1 + \frac{a'}{a}\rho + \frac{1}{3}\left(\frac{a''}{a} + \frac{a'^2}{a^2} \right) \rho^2 + O(\rho^3) \right],$$

where a prime indicates differentiation with respect to $\bar{u}$; here and below, the scale factor and its derivatives are evaluated at $\rho = 0$ and expressed as functions of $\bar{u}$ only. From this we gather that $r_{\bar{u}} = a'\rho + a''\rho^2 + O(\rho^3)$, and Eq. (2.102) gives

$$g_{uu} = -1 + \left(\frac{a''}{a} - \frac{a'^2}{a^2} \right) \rho^2 + O(\rho^3)$$

and

$$g_{AB} = a^2 \rho^2 \Omega_{AB} \left[1 + 2\frac{a'}{a}\rho + \left(\frac{a''}{a} + \frac{a'^2}{a^2} \right) \rho^2 + O(\rho)^3 \right].$$

Expressing these results in terms of r instead of ρ, and converting $\bar{u}$ derivatives into u derivatives using Eq. (2.99), returns the results of Eq. (2.96). It should be noted that while Eq. (2.96) gives the metric in a neighborhood of $r = 0$, the expression given in Eq. (2.102) holds globally.

J. Circular motion in Schwarzschild spacetime

As another example we consider the world line of a freely moving observer in circular motion around a Schwarzschild black hole. In the usual Schwarzschild coordinates $(t_s, r_s, \theta_s, \phi_s)$ the metric is given by

$$ds^2 = -(1 - 2M/r_s) dt_s^2 + (1 - 2M/r_s)^{-1} dr_s^2 + r_s^2 (d\theta_s^2 + \sin^2\theta_s d\phi_s^2), \tag{2.103}$$

where M is the mass of the black hole. The observer moves on a circular orbit of radius $r_s = R$ with an angular velocity $d\phi_s/dt_s = \Omega = \sqrt{M/R^3}$, in the equatorial plane $\theta_s = \pi/2$. The velocity vector is

$$u^\mu = \gamma(1, 0, 0, \Omega), \tag{2.104}$$

where $\gamma = (1 - 3M/R)^{-1/2}$ is a normalization factor. The motion is geodesic, and we can once more set $a^\mu = 0$. Because $R_{\alpha\beta} = 0$ for the Schwarzschild spacetime, we will rely on the results presented in Sec. II H.

The vectors

$$e_r^\mu = (0, \beta, 0, 0), \quad e_\theta^\mu = (0, 0, 1/R, 0),$$

and

$$e_\phi^\mu = \gamma(\Omega R/\beta, 0, 0, \beta/R),$$

where $\beta = (1 - 2M/R)^{1/2}$, are normalized, mutually orthogonal, and all orthogonal to u^μ. As such they form a valid set of spatial vectors, but this choice is not optimal because except for e_θ^μ, the vectors are not parallel transported on the world line. By forming linear superpositions, however, and choosing the coefficients appropriately, we can find a set of parallel-transported vectors e_a^μ. We choose $e_1^\mu = \cos(\Omega\tau) e_r^\mu - \sin(\Omega\tau) e_\phi^\mu$, $e_2^\mu = \sin(\Omega\tau) e_r^\mu + \cos(\Omega\tau) e_\phi^\mu$, and $e_3^\mu = -e_\theta^\mu$, or

$$e_1^\mu = \left(-\frac{\gamma\Omega R}{\beta}\sin\Phi, \beta\cos\Phi, 0, -\frac{\beta\gamma}{R}\sin\Phi \right),$$

$$e_2^\mu = \left(\frac{\gamma\Omega R}{\beta}\cos\Phi, \beta\sin\Phi, 0, \frac{\beta\gamma}{R}\cos\Phi \right),$$

$$e_3^\mu = \left(0, 0, -\frac{1}{R}, 0 \right), \tag{2.105}$$

where $\beta = (1-2M/R)^{1/2}$, $\gamma = (1-3M/R)^{-1/2}$, and

$$\Phi = \Omega\tau. \tag{2.106}$$

The vectors e_a^μ are all parallel transported on the world line, and according to Eq. (2.1), we have $\omega_{ab}=0$.

The electric part $\mathcal{E}_{ab}$ of the Riemann tensor is defined by Eq. (2.64), and Eq. (2.90) gives its decomposition into harmonic components $\mathcal{E}_{\mathrm{m}}$. Using the tetrad introduced previously we find that the nonvanishing components are

$$\mathcal{E}_0 = \frac{M}{R^2(R-3M)},$$

$$\mathcal{E}_{2c} = -\frac{3M}{2R^3}\frac{R-2M}{R-3M}\cos 2\Phi,$$

$$\mathcal{E}_{2s} = -\frac{3M}{2R^3}\frac{R-2M}{R-3M}\sin 2\Phi. \tag{2.107}$$

The magnetic part $\mathcal{B}_{ab}$ of the Riemann tensor is defined in Eq. (2.64) and decomposed in harmonic components in Eq. (2.91). Its nonvanishing components are

$$\mathcal{B}_{1c} = -\frac{6M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\cos\Phi,$$

$$\mathcal{B}_{1s} = -\frac{6M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\sin\Phi. \tag{2.108}$$

Equivalent results were obtained by Alvi [21], based on earlier work by Fishbone [22] and Marck [23].

To construct the metric we must form the quantities $\mathcal{E}^*$, $\mathcal{E}_A^*$, $\mathcal{E}_{AB}^*$, $\mathcal{B}_A^*$, and $\mathcal{B}_{AB}^*$, defined by Eqs. (2.68) and (2.83)–(2.86). For this we use Eq. (2.92) and the spherical harmonics listed in Appendix A. We obtain

$$\mathcal{E}^* = \frac{M}{2R^2(R-3M)}(3\cos^2\theta - 1) - \frac{3M}{2R^3}\frac{R-2M}{R-3M}\sin^2\theta\cos 2(\phi-\Phi),$$

$$\mathcal{E}_\theta^* = -\frac{3M}{2R^3(R-3M)}[R+(R-2M)\cos 2(\phi-\Phi)] \times \sin\theta\cos\theta,$$

$$\mathcal{E}_\phi^* = \frac{3M}{2R^3}\frac{R-2M}{R-3M}\sin^2\theta\sin 2(\phi-\Phi),$$

$$\mathcal{E}_{\theta\theta}^* = \frac{3M}{2R^2(R-3M)}\sin^2\theta - \frac{3M}{2R^3}\frac{R-2M}{R-3M}(1+\cos^2\theta)\cos 2(\phi-\Phi),$$

$$\mathcal{E}_{\theta\phi}^* = \frac{3M}{R^3}\frac{R-2M}{R-3M}\sin\theta\cos\theta\sin 2(\phi-\Phi),$$

$$\mathcal{E}_{\phi\phi}^* = -\frac{3M}{2R^2(R-3M)}\sin^4\theta + \frac{3M}{2R^3}\frac{R-2M}{R-3M}\sin^2\theta(1+\cos^2\theta)\cos 2(\phi-\Phi), \tag{2.109}$$

and

$$\mathcal{B}_\theta^* = -\frac{3M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\cos\theta\sin(\phi-\Phi),$$

$$\mathcal{B}_\phi^* = \frac{3M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\sin\theta(1-2\cos^2\theta) \times\cos(\phi-\Phi),$$

$$\mathcal{B}_{\theta\theta}^* = \frac{6M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\sin\theta\sin(\phi-\Phi),$$

$$\mathcal{B}_{\theta\phi}^* = \frac{6M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\sin^2\theta\cos\theta\cos(\phi-\Phi),$$

$$\mathcal{B}_{\phi\phi}^* = -\frac{6M\Omega}{R(R-3M)}\sqrt{1-\frac{2M}{R}}\sin^3\theta\sin(\phi-\Phi). \tag{2.110}$$

In these expressions, the components of the Riemann tensor are all evaluated at $\tau=u$, so that $\Phi=\Omega u$. Substituting them into Eqs. (2.79)–(2.82) gives the Schwarzschild metric in the retarded coordinates (u,r,θ,ϕ); these are based at the world line of an observer moving on a circular orbit of radius R with an angular velocity $\Omega=\sqrt{M/R^3}$.

III. MOTION OF A SMALL BLACK HOLE IN AN EXTERNAL UNIVERSE

A. Matched asymptotic expansions

In this section we consider a nonrotating black hole of small mass m moving in a background spacetime with metric $g_{\alpha\beta}$, and we seek to determine the equations that govern its motion. We will employ the powerful technique of *matched asymptotic expansions* [8,10,12–15,21] and make use of the retarded coordinates developed in Sec. II.

The problem presents itself with a clean separation of length scales, and the method relies heavily on this. On the one hand we have the length scale associated with the small black hole, which is set by its mass m. On the other hand we have the length scale associated with the background spacetime, which is set by the radius of curvature $\mathcal{R}$; this is defined so that a typical component of the background spacetime's Riemann tensor is equal to $1/\mathcal{R}^2$ up to a numerical factor of order unity. We demand that $m/\mathcal{R} \ll 1$. For simplicity we assume that the background spacetime contains no matter, so that its metric is a solution to the Einstein field equations in vacuum.

Let r be a meaningful measure of distance from the small black hole, and let us consider a region of spacetime defined by $r < r_i$, where r_i is a constant that is much smaller than $\mathcal{R}$. This inequality defines a narrow world tube that surrounds the black hole, and we shall call this region the *internal zone*. In the internal zone the gravitational field is dominated by the black hole, and the metric can be expressed as

$$\mathsf{g}(\text{internal zone}) = g(\text{black hole}) + H_1/\mathcal{R} + H_2/\mathcal{R}^2 + \cdots, \tag{3.1}$$

where g(black hole) is the metric of a nonrotating black hole in isolation (as given by the unperturbed Schwarzschild solution), while H_1 and H_2 are corrections associated with the conditions in the external Universe. The metric of Eq. (3.1) represents a black hole that is distorted by the tidal gravitational field of the external Universe, and H_1, H_2 are functions of the spacetime coordinates that can be obtained by solving the Einstein field equations. They must be such that the spacetime possesses a regular event horizon near $r = 2m$, and such that g(internal zone) agrees with the metric of the external Universe—the metric of the background spacetime in the absence of a black hole—when $r \gg m$. As we shall see in Sec. III B, H_1 actually vanishes and the small correction $H_2/\mathcal{R}^2$ can be obtained by employing the well-developed tools of black-hole perturbation theory [15,20,24–26].

Consider now a region of spacetime defined by $r > r_e$, where r_e is a constant that is much larger than m; this region will be called the *external zone*. In the external zone the gravitational field is dominated by the conditions in the external Universe, and the metric can be expressed as

$$\mathsf{g}(\text{external zone}) = g(\text{background spacetime}) + mh_1 + m^2 h_2 + \cdots, \tag{3.2}$$

where g(background spacetime) is the unperturbed metric of the background spacetime in which the black hole is moving, while h_1 and h_2 are corrections associated with the hole's presence; these are functions of the spacetime coordinates that can be obtained by solving the Einstein field equations. We shall truncate Eq. (3.2) to first order in m, and mh_1 will be calculated in Sec. III C by linearizing the field equations about the metric of the background spacetime.

The metric g(external zone) is a functional of a world line γ that represents the motion of the small black hole in the background spacetime. Our goal is to obtain a description of this world line, in the form of equations of motion to be satisfied by the black hole; these equations will be formulated in the background spacetime. It is important to understand that fundamentally, γ exists only as an external-zone construct: It is only in the external zone that the black hole can be thought of as moving on a world line; in the internal zone the black hole is revealed as an extended object and the notion of a world line describing its motion is no longer meaningful.

Equations (3.1) and (3.2) give two different expressions for the metric of the same spacetime; the first is valid in the internal zone $r < r_i \ll \mathcal{R}$, while the second is valid in the external zone $r > r_e \gg m$. The fact that $\mathcal{R} \gg m$ allows us to define a *buffer zone* in which r is restricted to the interval $r_e < r < r_i$. In the buffer zone r is simultaneously much larger than m and much smaller than $\mathcal{R}$—a typical value might be $\sqrt{m\mathcal{R}}$—and Eqs. (3.1), (3.2) are simultaneously valid. Since the two metrics are the same up to a diffeomorphism, these expressions must agree. And since g(external zone) is a functional of a world line γ while g(internal zone) contains no such information, *matching the metrics necessarily determines the motion of the small black hole in the background spacetime*.

Matching the metrics of Eqs. (3.1) and (3.2) in the buffer zone can be carried out in practice only after performing a transformation from the external coordinates used to express g(external zone) to the internal coordinates employed for g(internal zone). The details of this coordinate transformation are presented in Appendix C and the end result of matching is revealed in Sec. III D.

B. Metric in the internal zone

To proceed with the program outlined in the previous subsection we first calculate the internal-zone metric and replace Eq. (3.1) by a more concrete expression. We recall that the internal zone is defined by $r < r_i \ll \mathcal{R}$, where r is a suitable measure of distance from the black hole.

We begin by expressing g(black hole), the Schwarzschild metric of an isolated black hole of mass m, in terms of retarded Eddington-Finkelstein coordinates $(\bar{u}, \bar{r}, \bar{\theta}^A)$, where $\bar{u}$ is retarded time, $\bar{r}$ the usual areal radius, and $\bar{\theta}^A = (\bar{\theta}, \bar{\phi})$ are two angles on the spheres of constant $\bar{u}$ and $\bar{r}$. The metric is given by

$$ds^2 = -f d\bar{u}^2 - 2 d\bar{u} d\bar{r} + \bar{r}^2 d\bar{\Omega}^2, \tag{3.3}$$

where

$$f = 1 - \frac{2m}{\bar{r}}, \tag{3.4}$$

and $d\bar{\Omega}^2 = \bar{\Omega}_{AB} d\bar{\theta}^A d\bar{\theta}^B = d\bar{\theta}^2 + \sin^2\bar{\theta} d\bar{\phi}^2$ is the line element on the unit two-sphere. In the limit $r \gg m$ this metric achieves the asymptotic values $g_{\bar{u}\bar{u}} \to -1$, $g_{\bar{u}\bar{r}} = -1$, $g_{\bar{u}A} = 0$, and $g_{\bar{A}\bar{B}} = \bar{r}^2 \bar{\Omega}_{AB}$; these are appropriate for a black hole immersed in a flat spacetime charted by retarded coordinates.

The corrections H_1 and H_2 in Eq. (3.1) encode the information that our black hole is not isolated but in fact immersed in an external universe whose metric becomes g(background spacetime) asymptotically. In the internal zone the metric of the background spacetime can be expanded in powers of $\bar{r}/\mathcal{R}$ and expressed in a form that can be directly imported from Sec. II. If we assume that the "world line" $\bar{r}=0$ has no acceleration in the background spacetime (a statement that will be justified shortly), then the asymptotic values of g(internal zone) must be given by Eqs. (2.79)–(2.82):

$$\mathsf{g}_{\bar{u}\bar{u}} \to -1 - \bar{r}^2 \bar{\mathcal{E}}^* + O(\bar{r}^3/\mathcal{R}^3),$$

$$\mathsf{g}_{\bar{u}\bar{r}} = -1,$$

$$\mathsf{g}_{\bar{u}A} \to \frac{2}{3}\bar{r}^3(\bar{\mathcal{E}}^*_A + \bar{\mathcal{B}}^*_A) + O(\bar{r}^4/\mathcal{R}^3),$$

$$\mathsf{g}_{\bar{A}\bar{B}} \to \bar{r}^2 \bar{\Omega}_{AB} - \frac{1}{3}\bar{r}^4(\bar{\mathcal{E}}^*_{AB} + \bar{\mathcal{B}}^*_{AB}) + O(\bar{r}^5/\mathcal{R}^3),$$

where

$$\bar{\mathcal{E}}^* = \mathcal{E}_{ab}\bar{\Omega}^a\bar{\Omega}^b,$$

$$\bar{\mathcal{E}}^*_A = \mathcal{E}_{ab}\bar{\Omega}^a_A\bar{\Omega}^b,$$

$$\bar{\mathcal{E}}^*_{AB} = 2\mathcal{E}_{ab}\bar{\Omega}^a_A\bar{\Omega}^b_B + \bar{\mathcal{E}}^*\bar{\Omega}_{AB}$$

and

$$\bar{\mathcal{B}}^*_A = \varepsilon_{abc}\bar{\Omega}^a_A\bar{\Omega}^b\mathcal{B}^c{}_d\bar{\Omega}^d,$$

$$\bar{\mathcal{B}}^*_{AB} = 2\varepsilon_{acd}\bar{\Omega}^c\mathcal{B}^d{}_b\bar{\Omega}^{(a}_A\bar{\Omega}^{b)}_B$$

are the tidal gravitational fields that were first introduced in Sec. II H. Recall that $\bar{\Omega}^a = (\sin\bar{\theta}\cos\bar{\phi}, \sin\bar{\theta}\sin\bar{\phi}, \cos\bar{\theta})$ and $\bar{\Omega}^a_A = \partial\bar{\Omega}^a/\partial\bar{\theta}^A$. Apart from an angular dependence made explicit by these relations, the tidal fields depend on $\bar{u}$ through the frame components $\mathcal{E}_{ab} \equiv R_{a0b0} = O(1/\mathcal{R}^2)$ and $\mathcal{B}^a{}_b \equiv \frac{1}{2}\varepsilon^{acd}R_{0bcd} = O(1/\mathcal{R}^2)$ of the Riemann tensor. (This is the Riemann tensor of the background spacetime evaluated at $\bar{r}=0$.) Notice that we have incorporated the fact that the Ricci tensor vanishes at $\bar{r}=0$: the black hole moves in a vacuum spacetime.

The modified asymptotic values lead us to the following ansatz for the internal-zone metric:

$$\mathsf{g}_{\bar{u}\bar{u}} = -f[1 + \bar{r}^2 e_1(\bar{r})\bar{\mathcal{E}}^*] + O(\bar{r}^3/\mathcal{R}^3), \tag{3.5}$$

$$\mathsf{g}_{\bar{u}\bar{r}} = -1, \tag{3.6}$$

$$\mathsf{g}_{\bar{u}A} = \frac{2}{3}\bar{r}^3[e_2(\bar{r})\bar{\mathcal{E}}^*_A + b_2(\bar{r})\bar{\mathcal{B}}^*_A] + O(\bar{r}^4/\mathcal{R}^3), \tag{3.7}$$

$$\mathsf{g}_{\bar{A}\bar{B}} = \bar{r}^2\bar{\Omega}_{AB} - \frac{1}{3}\bar{r}^4[e_3(\bar{r})\bar{\mathcal{E}}^*_{AB} + b_3(\bar{r})\bar{\mathcal{B}}^*_{AB}] + O(\bar{r}^5/\mathcal{R}^3). \tag{3.8}$$

The five functions e_1, e_2, e_3, b_2, and b_3 can all be determined by solving the Einstein field equations; they must approach unity when $r \gg m$ and be well behaved at $r=2m$ (so that the tidally distorted black hole will have a nonsingular event horizon). In Appendix B, I show that they are given by

$$e_1(\bar{r}) = e_2(\bar{r}) = f, \quad e_3(\bar{r}) = 1 - \frac{2m^2}{\bar{r}^2}, \tag{3.9}$$

and

$$b_2(\bar{r}) = f, \quad b_3(\bar{r}) = 1. \tag{3.10}$$

It is clear from Eqs. (3.5)–(3.8) that the assumed deviation of g(internal zone) with respect to g(black hole) scales as $1/\mathcal{R}^2$. It is therefore of the form of Eq. (3.1) with $H_1=0$. The fact that H_1 vanishes comes as a consequence of our previous assumption that the "world line" $\bar{r}=0$ has a zero acceleration in the background spacetime; a nonzero acceleration of order $1/\mathcal{R}$ would bring terms of order $1/\mathcal{R}$ to the metric, and H_1 would then be nonzero. The perturbed metric of Eqs. (3.5)–(3.10) differs from the one presented by Detweiler [15] only by a transformation from Schwarzschild to Eddington-Finkelstein coordinates, and a transformation from the Zerilli gauge [24] gauge adopted by him to the retarded gauge adopted here.

Why is the assumption of no acceleration justified? As I shall explain more fully in the next paragraph, the reason is simply that it reflects a choice of coordinate system: setting the acceleration to zero amounts to adopting a specific—and convenient—gauge condition.

Inspection of Eqs. (3.5)–(3.8) reveals that the angular dependence of the metric perturbation is generated entirely by scalar, vectorial, and tensorial spherical harmonics of degree $l=2$; this observation was elaborated toward the end of Sec. II H and in Appendix A. In particular, H_2 contains no $l=0$ and $l=1$ modes, and this statement reflects a choice of gauge condition. Zerilli has shown [24] that a perturbation of the Schwarzschild spacetime with $l=0$ corresponds to a shift in the mass parameter. As Thorne and Hartle have shown [8], a black hole interacting with its environment will undergo a change of mass, but this effect is of order $m^3/\mathcal{R}^2$ and thus beyond the level of accuracy of our calculations. There is therefore no need to include $l=0$ terms in H_2. Similarly, it was shown by Zerilli that odd-parity perturbations of degree $l=1$ correspond to a shift in the black hole's angular-momentum parameters. As Thorne and Hartle have shown, a change of angular momentum is quadratic in the hole's angular momentum, and we can ignore this effect when dealing with a nonrotating black hole. There is therefore no need to include odd-parity, $l=1$ terms in H_2. Finally, Zerilli has shown that in a vacuum spacetime, even-parity perturbations of degree $l=1$ correspond to a change of coordinate system—these modes are pure gauge. Since we have the

freedom to adopt any gauge condition, we can exclude even-parity, $l=1$ terms from the perturbed metric. This leads us to Eqs. (3.5)–(3.8), which contain only $l=2$ perturbation modes; the even-parity modes are contained in those terms that involve $\mathcal{E}_{ab}$, while the odd-parity modes are associated with $\mathcal{B}_{ab}$. The perturbed metric contains also higher multipoles, but those come at a higher order in $1/\mathcal{R}$; for example, the terms of order $1/\mathcal{R}^3$ include $l=3$ modes. We conclude that Eqs. (3.5)–(3.8) is a sufficiently general ansatz for the metric in the internal zone.

It shall prove convenient to transform g(internal zone) from the quasispherical coordinates $(\bar{r},\bar{\theta}^A)$ to a set of quasi-Cartesian coordinates $\bar{x}^a=\bar{r}\bar{\Omega}^a(\bar{\theta}^A)$; the transformation rules are worked out in Sec. II G. This gives

$$\mathsf{g}_{\bar{u}\bar{u}}=-f(1+\bar{r}^2 f\bar{\mathcal{E}}^*)+O(\bar{r}^3/\mathcal{R}^3), \tag{3.11}$$

$$\mathsf{g}_{\bar{u}\bar{a}}=-\bar{\Omega}_a+\frac{2}{3}\bar{r}^2 f(\bar{\mathcal{E}}^*_a+\bar{\mathcal{B}}^*_a)+O(\bar{r}^3/\mathcal{R}^3), \tag{3.12}$$

$$\mathsf{g}_{\bar{a}\bar{b}}=\delta_{ab}-\bar{\Omega}_a\bar{\Omega}_b-\frac{1}{3}\bar{r}^2\left(1-2\frac{m^2}{\bar{r}^2}\right)\bar{\mathcal{E}}^*_{ab} -\frac{1}{3}\bar{r}^2\bar{\mathcal{B}}^*_{ab}+O(\bar{r}^3/\mathcal{R}^3), \tag{3.13}$$

where $f=1-2m/\bar{r}$ and where the tidal fields

$$\bar{\mathcal{E}}^*=\mathcal{E}_{ab}\bar{\Omega}^a\bar{\Omega}^b,$$

$$\bar{\mathcal{E}}^*_a=(\delta_a{}^b-\bar{\Omega}_a\bar{\Omega}^b)\mathcal{E}_{bc}\bar{\Omega}^c,$$

$$\bar{\mathcal{E}}^*_{ab}=2\mathcal{E}_{ab}-2\bar{\Omega}_a\mathcal{E}_{bc}\bar{\Omega}^c-2\bar{\Omega}_b\mathcal{E}_{ac}\bar{\Omega}^c+(\delta_{ab}+\bar{\Omega}_a\bar{\Omega}_b)\bar{\mathcal{E}}^*,$$

$$\bar{\mathcal{B}}^*_a=\varepsilon_{abc}\bar{\Omega}^b\mathcal{B}^c{}_d\bar{\Omega}^d,$$

$$\bar{\mathcal{B}}^*_{ab}=\varepsilon_{acd}\bar{\Omega}^c\mathcal{B}^d{}_e(\delta^e{}_b-\bar{\Omega}^e\bar{\Omega}_b)+\varepsilon_{bcd}\bar{\Omega}^c\mathcal{B}^d{}_e(\delta^e{}_a-\bar{\Omega}^e\bar{\Omega}_a),$$

were first introduced in Sec. II H. The metric of Eqs. (3.11)–(3.13) represents the spacetime geometry of a black hole immersed in an external universe and distorted by its tidal gravitational field.

C. Metric in the external zone

We next move on to the external zone and seek to replace Eq. (3.2) by a more concrete expression; recall that the external zone is defined by $m\ll r_e<r$. We take advantage of the fact that in the external zone, the gravitational perturbation associated with the presence of a black hole cannot be distinguished from the perturbation produced by a point particle of the same mass.

The external-zone metric is decomposed as

$$\mathsf{g}_{\alpha\beta}=g_{\alpha\beta}+h_{\alpha\beta}, \tag{3.14}$$

where $g_{\alpha\beta}$ is the metric of the background spacetime and $h_{\alpha\beta}=O(m)$ is the perturbation; we shall work consistently to first order in m and systematically discard all terms of higher order. We relate $h_{\alpha\beta}$ to trace-reversed potentials $\gamma_{\alpha\beta}$,

$$h_{\alpha\beta}=\gamma_{\alpha\beta}-\frac{1}{2}(g^{\gamma\delta}\gamma_{\gamma\delta})g_{\alpha\beta}, \tag{3.15}$$

and we impose the Lorenz gauge condition

$$\gamma^{\alpha\beta}{}_{;\beta}=0; \tag{3.16}$$

indices are raised and lowered with $g^{\alpha\beta}$ and $g_{\alpha\beta}$, respectively. With the understanding that the background spacetime contains no matter, linearizing the Einstein field equations produces the wave equation

$$\Box\gamma^{\alpha\beta}+2R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}\gamma^{\gamma\delta}=-16\pi T^{\alpha\beta} \tag{3.17}$$

for the potentials. Here, $\Box=g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the wave operator and

$$T^{\alpha\beta}(x)=m\int_\gamma g^{\alpha}{}_{\mu}(x,z)g^{\beta}{}_{\nu}(x,z)u^\mu u^\nu\delta_4(x,z)d\tau \tag{3.18}$$

is the stress-energy tensor of a point particle of mass m traveling on a world line γ; $\delta_4(x,z)$ is a scalarized, four-dimensional Dirac functional, the world line is described by relations $z^\mu(\tau)$ in which τ is proper time, and $u^\mu=dz^\mu/d\tau$ is the particle's velocity vector. Solving the linearized field equations produces

$$\gamma_{\alpha\beta}(x)=4m\int_\gamma G_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu d\tau, \tag{3.19}$$

where $G_{\alpha\beta\mu\nu}(x,z)$ is the retarded Green's function[2] [9] associated with Eq. (3.17).

We now place ourselves in the buffer zone (where $m\ll r\ll\mathcal{R}$ and where the matching will take place) and work toward expressing g(external zone) as an expansion in powers of $r/\mathcal{R}$. For this purpose we adopt the retarded coordinates $(u,r\Omega^a)$ of Sec. II and rely on the machinery developed there.

We begin with $g_{\alpha\beta}$, the metric of the background spacetime. We have seen in Sec. II H that if the world line γ is a geodesic, if the vectors e^μ_a are parallel transported on the world line, and if the Ricci tensor vanishes on γ, then the metric takes the form given by Eqs. (2.76)–(2.78). This form, however, is too restrictive for our purposes: We must allow γ to have an acceleration, and allow the basis vectors

[2]The normalization of the gravitational Green's function varies from author to author. Here the normalization is such that the Green's function obeys a wave equation with a right-hand side given by $-4\pi g^{(\alpha}_{\mu}(x,z)g^{\beta)}_{\nu}(x,z)\,\delta_4(x,z)$. The factor of 4π accounts for the factor of 4 on the right-hand side of Eq. (3.19).

to be transported in the most general way compatible with their orthonormality property; this transport law is given by Eq. (2.1),

$$\frac{De_a^\mu}{d\tau} = a_a u^\mu + \omega_a{}^b e_b^\mu ,$$

where $a_a(\tau) = a_\mu e_a^\mu$ are the frame components of the acceleration vector $a^\mu = Du^\mu/d\tau$, and $\omega_{ab}(\tau) = -\omega_{ba}(\tau)$ is a rotation tensor to be determined. Anticipating that a_a and ω_{ab} will both be proportional to m, we express the metric of the background spacetime as

$$g_{uu} = -1 - 2ra_a\Omega^a - r^2\mathcal{E}^* + O(r^3/\mathcal{R}^3), \tag{3.20}$$

$$g_{ua} = -\Omega_a + r(\delta_a{}^b - \Omega_a\Omega^b)a_b - r\omega_{ab}\Omega^b + \frac{2}{3}r^2(\mathcal{E}_a^* + \mathcal{B}_a^*) + O(r^3/\mathcal{R}^3), \tag{3.21}$$

$$g_{ab} = \delta_{ab} - \Omega_a\Omega_b - \frac{1}{3}r^2(\mathcal{E}_{ab}^* + \mathcal{B}_{ab}^*) + O(r^3/\mathcal{R}^3), \tag{3.22}$$

where $\mathcal{E}^*$, $\mathcal{E}_a^*$, $\mathcal{E}_{ab}^*$, $\mathcal{B}_a^*$, and $\mathcal{B}_{ab}^*$ are the tidal gravitational fields first introduced in Sec. II H. The metric of Eqs. (3.20)–(3.22) is obtained from the general form of Eqs. (2.39)–(2.41) by neglecting terms quadratic in a_a and ω_{ab} and specializing to a zero Ricci tensor.

To express the perturbation $h_{\alpha\beta}$ as an expansion in powers of $r/\mathcal{R}$ we assume that x is sufficiently close to γ that a portion of the world line traverses $\mathcal{N}(x)$, the normal convex neighborhood of the point x (see Fig. 2); this assumption is compatible with the condition $r \ll \mathcal{R}$. We then reexpress Eq. (3.19) as

$$\gamma_{\alpha\beta} = 4m\left(\int_{-\infty}^{\tau_<} + \int_{\tau_<}^{\tau_>} + \int_{\tau_>}^{\infty}\right) G_{\alpha\beta\mu\nu}u^\mu u^\nu d\tau,$$

where $\tau_<$ and $\tau_>$ are the values of the proper-time parameter at which γ enters and leaves $\mathcal{N}(x)$, respectively. The third integration contributes nothing because x is then in the past of $z(\tau)$ and the retarded Green's function vanishes. For the second integration, x is the normal convex neighborhood of $z(\tau)$, and the retarded Green's function can be put in the Hadamard form [9,18,27,28]

$$G_{\alpha\beta\mu\nu}(x,z) = U_{\alpha\beta\mu\nu}(x,z)\,\delta_+(\sigma) + V_{\alpha\beta\mu\nu}(x,z)\,\theta_+(-\sigma), \tag{3.23}$$

where $U_{\alpha\beta\mu\nu}(x,z)$ and $V_{\alpha\beta\mu\nu}(x,z)$ are smooth bitensors[3], $\sigma(x,z)$ is Synge's world function [1,18], and $\delta_+(\sigma)$ and $\theta_+(\sigma)$ are Dirac and Heaviside distributions restricted to the future of $z(\tau)$ [18,28]. To integrate over the Dirac term we change variables from τ to σ, noticing that σ increases as

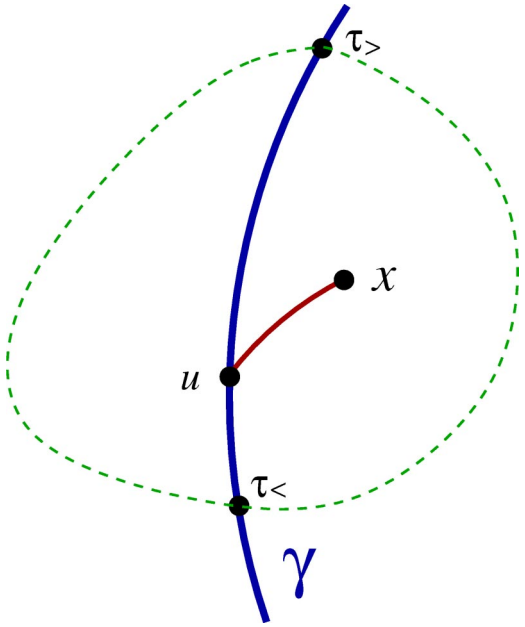

FIG. 2. The region within the dashed boundary represents the normal convex neighborhood of the point x. The world line γ enters the neighborhood at proper time $\tau_<$ and exits at proper time $\tau_>$. Also shown is the retarded point $x' \equiv z(u)$ and the null geodesic that links x to the world line.

$z(\tau)$ passes through the retarded point $z(u) \equiv x'$; recalling the definition of retarded distance given in Eq. (2.10), the integral evaluates to $U_{\alpha\beta\gamma'\delta'}(x,x')u^{\gamma'}u^{\delta'}/r$. The integration over the Heaviside term is cut off at $\tau = u$, and we obtain our final expression for the perturbation:

$$\gamma_{\alpha\beta}(x) = \frac{4m}{r}U_{\alpha\beta\gamma'\delta'}(x,x')u^{\gamma'}u^{\delta'} + \gamma_{\alpha\beta}^{\rm tail}(x). \tag{3.24}$$

Here, primed indices refer to the retarded point $x' \equiv z(u)$ associated with x, and

$$\gamma_{\alpha\beta}^{\rm tail}(x) = 4m\int_{\tau_<}^{u} V_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu d\tau + 4m\int_{-\infty}^{\tau_<} G_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu d\tau \equiv 4m\int_{-\infty}^{u^-} G_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu d\tau \tag{3.25}$$

is the "tail part" of the gravitational potentials. Notice that I have introduced a short-hand notation in the last line of Eq. (3.25); the important point is that the integral of $G_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu$ up to $\tau = u^-$ avoids the singular behavior of the Green's function on the light cone $\sigma(x,z) = 0$.

We must next express Eq. (3.24) in the form of an expansion in powers of $r/\mathcal{R}$. For this we shall need the expansion[4]

[3]The tail part of the Green's function is denoted $V_{\alpha\beta\mu\nu}(x,z)$ in this work, but most authors insert a minus sign and call it $-v_{\alpha\beta\mu\nu}(x,z)$.

[4]This result is implicitly contained in Appendix A of Ref. [10]. It is derived from scratch in Sec. 15 of Ref. [17].

$$U_{\alpha\beta\gamma'\delta'}u^{\gamma'}u^{\delta'} = g^{\alpha'}_{(\alpha}g^{\beta'}_{\beta)}[u_{\alpha'}u_{\beta'} + O(r^3/\mathcal{R}^3)],$$

which contains no terms at order $r/\mathcal{R}$ or $(r/\mathcal{R})^2$. We shall also need the general expansion of a tensor $A_{\alpha\beta}(x)$ in terms of its values at a neighboring point x':[5]

$$A_{\alpha\beta}(x) = g^{\alpha'}_{\alpha}g^{\beta'}_{\beta}(A_{\alpha'\beta'} - A_{\alpha'\beta';\gamma'}\sigma^{\gamma'} + \cdots).$$

Here $A_{\alpha\beta}$ will stand for $\gamma^{\rm tail}_{\alpha\beta}$ and x' will be the retarded point $z(u)$ associated with x; accordingly, Eq. (2.11) gives $\sigma^{\alpha'} = -r(u^{\alpha'} + \Omega^a e^{\alpha'}_a)$. Combining all these results, Eq. (3.24) becomes

$$\gamma_{\alpha\beta}(x) = g^{\alpha'}_{\alpha}g^{\beta'}_{\beta}\left[\frac{4m}{r}u_{\alpha'}u_{\beta'} + \gamma^{\rm tail}_{\alpha'\beta'} + r\gamma^{\rm tail}_{\alpha'\beta'\gamma'}(u^{\gamma'} + \Omega^c e^{\gamma'}_c) + O(mr^2/\mathcal{R}^3)\right], \quad (3.26)$$

where $\gamma^{\rm tail}_{\alpha'\beta'}$ is the tensor of Eq. (3.25) evaluated at x', and

$$\gamma^{\rm tail}_{\alpha'\beta'\gamma'}(x') = 4m\int_{-\infty}^{u^-}\nabla_{\gamma'}G_{\alpha'\beta'\mu\nu}(x',z)u^\mu u^\nu d\tau \quad (3.27)$$

emerges during the computation of

$$\nabla_{\gamma'}\gamma^{\rm tail}_{\alpha'\beta'} = 4m(\partial_{\gamma'}u)V_{\alpha'\beta'\delta'\epsilon'}u^{\delta'}u^{\epsilon'} + \gamma^{\rm tail}_{\alpha'\beta'\gamma'};$$

the term proportional to $\partial_{\gamma'}u$ disappears after contraction with $\sigma^{\gamma'}$.

At this stage we introduce the fields

$$h^{\rm tail}_{\alpha'\beta'} = 4m\int_{-\infty}^{u^-}\left(G_{\alpha'\beta'\mu\nu} - \frac{1}{2}g_{\alpha'\beta'}G^{\delta'}{}_{\delta'\mu\nu}\right)u^\mu u^\nu d\tau, \quad (3.28)$$

$$h^{\rm tail}_{\alpha'\beta'\gamma'} = 4m\int_{-\infty}^{u^-}\nabla_{\gamma'}\left(G_{\alpha'\beta'\mu\nu} - \frac{1}{2}g_{\alpha'\beta'}G^{\delta'}{}_{\delta'\mu\nu}\right)u^\mu u^\nu d\tau \quad (3.29)$$

and recognize that the metric perturbation obtained from Eqs. (3.15) and (3.26) is

$$h_{\alpha\beta}(x) = g^{\alpha'}_{\alpha}g^{\beta'}_{\beta}\left[\frac{2m}{r}(2u_{\alpha'}u_{\beta'} + g_{\alpha'\beta'}) + h^{\rm tail}_{\alpha'\beta'} + rh^{\rm tail}_{\alpha'\beta'\gamma'}(u^{\gamma'} + \Omega^c e^{\gamma'}_c) + O(mr^2/\mathcal{R}^3)\right]. \quad (3.30)$$

This is the desired expansion of the metric perturbation in powers of $r/\mathcal{R}$. Our next task will be to calculate the components of this tensor in the retarded coordinates $(u, r\Omega^a)$.

The first step of this computation is to decompose $h_{\alpha\beta}$ in the tetrad (e^α_0, e^α_a) that is obtained by parallel transport of $(u^{\alpha'}, e^{\alpha'}_a)$ on the null geodesic that links x to its corresponding retarded point $x' \equiv z(u)$ on the world line. (The vectors are parallel transported in the background spacetime.) The projections are

$$h_{00}(u,r,\Omega^a) \equiv h_{\alpha\beta}e^\alpha_0 e^\beta_0 = \frac{2m}{r} + h^{\rm tail}_{00}(u) + r[h^{\rm tail}_{000}(u) + h^{\rm tail}_{00c}(u)\Omega^c] + O(mr^2/\mathcal{R}^3), \quad (3.31)$$

$$h_{0b}(u,r,\Omega^a) \equiv h_{\alpha\beta}e^\alpha_0 e^\beta_b = h^{\rm tail}_{0b}(u) + r[h^{\rm tail}_{0b0}(u) + h^{\rm tail}_{0bc}(u)\Omega^c] + O(mr^2/\mathcal{R}^3), \quad (3.32)$$

$$h_{ab}(u,r,\Omega^a) \equiv h_{\alpha\beta}e^\alpha_a e^\beta_b = \frac{2m}{r}\delta_{ab} + h^{\rm tail}_{ab}(u) + r[h^{\rm tail}_{ab0}(u) + h^{\rm tail}_{abc}(u)\Omega^c] + O(mr^2/\mathcal{R}^3). \quad (3.33)$$

On the right-hand side we have the frame components of $h^{\rm tail}_{\alpha'\beta'}$ and $h^{\rm tail}_{\alpha'\beta'\gamma'}$ taken with respect to the tetrad $(u^{\alpha'}, e^{\alpha'}_a)$; these are functions of retarded time u only.

The perturbation is now expressed as

$$h_{\alpha\beta} = h_{00}e^0_\alpha e^0_\beta + h_{0b}(e^0_\alpha e^b_\beta + e^b_\alpha e^0_\beta) + h_{ab}e^a_\alpha e^b_\beta$$

and its components are obtained by involving Eqs. (2.37) and (2.38), which list the components of the tetrad vectors in the retarded coordinates; this is the second (and longest) step of the computation. Noting that a_a and ω_{ab} can both be set equal to zero in these equations (because they would produce negligible terms of order m^2 in $h_{\alpha\beta}$), and that S_{ab}, S_a, and S can all be expressed in terms of the tidal fields $\mathcal{E}^*$, $\mathcal{E}^*_a$, $\mathcal{E}^*_{ab}$, $\mathcal{B}^*_a$, and $\mathcal{B}^*_{ab}$ using Eqs. (2.73)–(2.75), we arrive at

[5]This is Eq. (A21) of Ref. [10]. This result is derived from scratch in Sec. 5 of Ref. [17].

$$h_{uu}=\frac{2m}{r}+h^{\rm tail}_{00}+r(2m\mathcal{E}^*+h^{\rm tail}_{000}+h^{\rm tail}_{00a}\Omega^a)+O(mr^2/\mathcal{R}^3), \tag{3.34}$$

$$h_{ua}=\frac{2m}{r}\Omega_a+h^{\rm tail}_{0a}+\Omega_a h^{\rm tail}_{00}+r\Big[2m\mathcal{E}^*\Omega_a+\frac{2m}{3}(\mathcal{E}^*_a+\mathcal{B}^*_a)+h^{\rm tail}_{0a0}+\Omega_a h^{\rm tail}_{000}+h^{\rm tail}_{0ab}\Omega^b+\Omega_a h^{\rm tail}_{00b}\Omega^b\Big]+O(mr^2/\mathcal{R}^3), \tag{3.35}$$

$$h_{ab}=\frac{2m}{r}(\delta_{ab}+\Omega_a\Omega_b)+\Omega_a\Omega_b h^{\rm tail}_{00}+\Omega_a h^{\rm tail}_{0b}+\Omega_b h^{\rm tail}_{0a}+h^{\rm tail}_{ab}+r\Big[-\frac{2m}{3}(\mathcal{E}^*_{ab}+\Omega_a\mathcal{E}^*_b+\mathcal{E}^*_a\Omega_b+\mathcal{B}^*_{ab}+\Omega_a\mathcal{B}^*_b+\Omega_b\mathcal{B}^*_a)+\Omega_a\Omega_b(h^{\rm tail}_{000}+h^{\rm tail}_{00c}\Omega^c)+\Omega_a(h^{\rm tail}_{0b0}+h^{\rm tail}_{0bc}\Omega^c)+\Omega_b(h^{\rm tail}_{0a0}+h^{\rm tail}_{0ac}\Omega^c)+(h^{\rm tail}_{ab0}+h^{\rm tail}_{abc}\Omega^c)\Big]+O(mr^2/\mathcal{R}^3). \tag{3.36}$$

These are the *coordinate components* of the metric perturbation $h_{\alpha\beta}$ in the retarded coordinates $(u,r\Omega^a)$, expressed in terms of *frame components* of the tail fields $h^{\rm tails}_{\alpha'\beta'}$ and $h^{\rm tails}_{\alpha'\beta'\gamma'}$. The perturbation is expanded in powers of $r/\mathcal{R}$ and it also involves the tidal gravitational fields of the background spacetime.

The external-zone metric is obtained by adding $g_{\alpha\beta}$ as given by Eqs. (3.20)–(3.22) to $h_{\alpha\beta}$ as given by Eqs. (3.34)–(3.36). The final result is

$$\mathsf{g}_{uu}=-1-r^2\mathcal{E}^*+O(r^3/\mathcal{R}^3)+\frac{2m}{r}+h^{\rm tail}_{00}+r(2m\mathcal{E}^*-2a_a\Omega^a+h^{\rm tail}_{000}+h^{\rm tail}_{00a}\Omega^a)+O(mr^2/\mathcal{R}^3), \tag{3.37}$$

$$\mathsf{g}_{ua}=-\Omega_a+\frac{2}{3}r^2(\mathcal{E}^*_a+\mathcal{B}^*_a)+O(r^3/\mathcal{R}^3)+\frac{2m}{r}\Omega_a+h^{\rm tail}_{0a}+\Omega_a h^{\rm tail}_{00}+r\Big[2m\mathcal{E}^*\Omega_a+\frac{2m}{3}(\mathcal{E}^*_a+\mathcal{B}^*_a)+(\delta_a{}^b-\Omega_a\Omega^b)a_b-\omega_{ab}\Omega^b+h^{\rm tail}_{0a0}+\Omega_a h^{\rm tail}_{000}+h^{\rm tail}_{0ab}\Omega^b+\Omega_a h^{\rm tail}_{00b}\Omega^b\Big]+O(mr^2/\mathcal{R}^3), \tag{3.38}$$

$$\mathsf{g}_{ab}=\delta_{ab}-\Omega_a\Omega_b-\frac{1}{3}r^2(\mathcal{E}^*_{ab}+\mathcal{B}^*_{ab})+O(r^3/\mathcal{R}^3)+\frac{2m}{r}(\delta_{ab}+\Omega_a\Omega_b)+\Omega_a\Omega_b h^{\rm tail}_{00}+\Omega_a h^{\rm tail}_{0b}+\Omega_b h^{\rm tail}_{0a}+h^{\rm tail}_{ab}+r\Big[-\frac{2m}{3}(\mathcal{E}^*_{ab}+\Omega_a\mathcal{E}^*_b+\mathcal{E}^*_a\Omega_b+\mathcal{B}^*_{ab}+\Omega_a\mathcal{B}^*_b+\Omega_b\mathcal{B}^*_a)+\Omega_a\Omega_b(h^{\rm tail}_{000}+h^{\rm tail}_{00c}\Omega^c)+\Omega_a(h^{\rm tail}_{0b0}+h^{\rm tail}_{0bc}\Omega^c)+\Omega_b(h^{\rm tail}_{0a0}+h^{\rm tail}_{0ac}\Omega^c)+(h^{\rm tail}_{ab0}+h^{\rm tail}_{abc}\Omega^c)\Big]+O(mr^2/\mathcal{R}^3). \tag{3.39}$$

D. Matching: motion of the black hole in the background spacetime

Comparison of Eqs. (3.11)–(3.13) and Eqs. (3.37)–(3.39) reveals that the internal-zone and external-zone metrics do no match in the buffer zone. But as the metrics are expressed in different coordinate systems, this mismatch is hardly surprising. A meaningful comparison of the two metrics must therefore come after a transformation from the external coordinates $(u,r\Omega^a)$ to the internal coordinates $(\bar{u},\bar{r}\bar{\Omega}^a)$. This transformation is worked out in Appendix C, and it puts the external-zone metric in its final form

$$\mathsf{g}_{\bar{u}\bar{u}}=-1-\bar{r}^2\bar{\mathcal{E}}^*+O(\bar{r}^3/\mathcal{R}^3)+\frac{2m}{\bar{r}}+\bar{r}\Big[4m\bar{\mathcal{E}}^*-2\Big(a_a-\frac{1}{2}h^{\rm tail}_{00a}+h^{\rm tail}_{0a0}\Big)\bar{\Omega}^a\Big]+O(m\bar{r}^2/\mathcal{R}^3), \tag{3.40}$$

$$\mathsf{g}_{\bar{u}\bar{a}}=-\bar{\Omega}_a+\frac{2}{3}\bar{r}^2(\bar{\mathcal{E}}^*_a+\bar{\mathcal{B}}^*_a)+O(\bar{r}^3/\mathcal{R}^3)+\bar{r}\Big[-\frac{4m}{3}(\bar{\mathcal{E}}^*_a+\bar{\mathcal{B}}^*_a)+(\delta_a{}^b-\bar{\Omega}_a\bar{\Omega}^b)\times\Big(a_b-\frac{1}{2}h^{\rm tail}_{00b}+h^{\rm tail}_{0b0}\Big)-(\omega_{ab}-h^{\rm tail}_{0[ab]})\bar{\Omega}^b\Big]+O(m\bar{r}^2/\mathcal{R}^3), \tag{3.41}$$

$$\mathsf{g}_{\bar{a}\bar{b}}=\delta_{ab}-\bar{\Omega}_a\bar{\Omega}_b-\frac{1}{3}\bar{r}^2(\bar{\mathcal{E}}^*_{ab}+\bar{\mathcal{B}}^*_{ab})+O(\bar{r}^3/\mathcal{R}^3)+O(m\bar{r}^2/\mathcal{R}^3). \tag{3.42}$$

Except for the terms involving a_a and ω_{ab}, this metric is equal to g(internal zone) as given by Eqs. (3.11)–(3.13) linearized with respect to m.

A precise match between g(external zone) and g(internal zone) is produced when we impose the relations

$$a_a = \frac{1}{2} h^{\rm tail}_{00a} - h^{\rm tail}_{0a0} \tag{3.43}$$

and

$$\omega_{ab} = h^{\rm tail}_{0[ab]}. \tag{3.44}$$

While Eq. (3.43) tells us how the black hole moves in the background spacetime, Eq. (3.44) indicates that the vectors e^μ_a are not Fermi-Walker transported on the world line.

The black hole's acceleration vector $a^\mu = a^a e^\mu_a$ can be constructed from the frame components listed in Eq. (3.43). A straightforward computation gives

$$a^\mu = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(2h^{\rm tail}_{\nu\lambda\rho} - h^{\rm tail}_{\lambda\rho\nu}) u^\lambda u^\rho, \tag{3.45}$$

where the tail integral

$$h^{\rm tail}_{\mu\nu\lambda} = 4m \int_{-\infty}^{\tau^-} \nabla_\lambda \left(G_{\mu\nu\mu'\nu'} - \frac{1}{2} g_{\mu\nu} G^{\rho}{}_{\rho\mu'\nu'} \right)(\tau,\tau') \times u^{\mu'} u^{\nu'} d\tau' \tag{3.46}$$

was previously defined by Eq. (3.29). Here, the unprimed indices refer to the current position $z(\tau)$ on the world line, while the primed indices refer to a prior position $z(\tau')$; the integral is cut short at $\tau' = \tau^-$ in the manner defined by Eq. (3.25). These are the MiSaTaQuWa equations of motion, as they were first presented by Mino, Sasaki, and Tanaka [10], and later rederived by Quinn and Wald [11].

Substituting Eqs. (3.43) and (3.44) into Eq. (2.1) gives the following transport equation for the tetrad vectors:

$$\frac{De^\mu_a}{d\tau} = -\frac{1}{2} u^\mu (2h^{\rm tail}_{\nu\lambda\rho} - h^{\rm tail}_{\nu\rho\lambda}) u^\nu e^\lambda_a u^\rho + (g^{\mu\rho} + u^\mu u^\rho) h^{\rm tail}_{\nu[\lambda\rho]} u^\nu e^\lambda_a. \tag{3.47}$$

This can also be written in the alternative form

$$\frac{De^\mu_a}{d\tau} = -\frac{1}{2}(u^\mu e^\lambda_a u^\rho + g^{\mu\lambda} e^\rho_a - g^{\mu\rho} e^\lambda_a) u^\nu h^{\rm tail}_{\nu\lambda\rho} \tag{3.48}$$

that was first proposed by Mino, Sasaki, and Tanaka. Both equations state that in the background spacetime, the tetrad vectors are not Fermi-Walker transported on γ; the rotation tensor is nonzero and given by Eq. (3.44).

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada. Conversations with Claude Barrabès and Werner Israel were instrumental during the early stage of my work on the retarded coordinates. Conversations with Steve Detweiler helped me clarify my thoughts on matched asymptotic expansions.

APPENDIX A: SPHERICAL HARMONICS

In this appendix I provide explicit expressions for the scalar, vectorial, and tensorial spherical harmonics introduced in Sec. II H. All harmonics are of degree $l = 2$.

The scalar harmonics are

$$Y^0 = \frac{1}{2}(3\cos^2\theta - 1),$$

$$Y^{1c} = \sin\theta\cos\theta\cos\phi,$$

$$Y^{1s} = \sin\theta\cos\theta\sin\phi,$$

$$Y^{2c} = \sin^2\theta\cos 2\phi,$$

$$Y^{2s} = \sin^2\theta\sin 2\phi.$$

The vectorial harmonics are defined by $Y^{\rm m}_A = Y^{\rm m}_{:A}$ (even parity) and $X^{\rm m}_A = -\varepsilon_A{}^B Y^{\rm m}_{:B}$ (odd parity), where a colon indicates covariant differentiation with respect to a connection compatible with $\Omega_{AB} = {\rm diag}(1, \sin^2\theta)$, and ε_{AB} is the two-dimensional Levi-Civita tensor. Explicitly,

$$Y^0_\theta = -3\sin\theta\cos\theta,$$

$$Y^0_\phi = 0,$$

$$Y^{1c}_\theta = (2\cos^2\theta - 1)\cos\phi,$$

$$Y^{1c}_\phi = -\sin\theta\cos\theta\sin\phi,$$

$$Y^{1s}_\theta = (2\cos^2\theta - 1)\sin\phi,$$

$$Y^{1s}_\phi = \sin\theta\cos\theta\cos\phi,$$

$$Y^{2c}_\theta = 2\sin\theta\cos\theta\cos 2\phi,$$

$$Y^{2c}_\phi = -2\sin^2\theta\sin 2\phi,$$

$$Y^{2s}_\theta = 2\sin\theta\cos\theta\sin 2\phi,$$

$$Y^{2c}_\phi = 2\sin^2\theta\cos 2\phi,$$

and

$$X^0_\theta = 0,$$

$$X^0_\phi = -3\sin^2\theta\cos\theta,$$

$$X^{1c}_\theta = \cos\theta\sin\phi,$$

$$X^{1c}_\phi = (2\cos^2\theta - 1)\sin\theta\cos\phi,$$

$$X^{1s}_\theta = -\cos\theta\cos\phi,$$

$$X^{1s}_\phi = (2\cos^2\theta - 1)\sin\theta\sin\phi,$$

$$X^{2c}_\theta = 2\sin\theta\sin 2\phi,$$

$$X^{2c}_{\phi}=2\sin^2\theta\cos\theta\cos 2\phi,$$

$$X^{2s}_{\theta}=-2\sin\theta\cos 2\phi,$$

$$X^{2s}_{\phi}=2\sin^2\theta\cos\theta\sin 2\phi.$$

The tensorial harmonics are defined by $Y^{\mathsf{m}}_{AB}=Y^{\mathsf{m}}_{:AB}$ (even parity) and $X^{\mathsf{m}}_{AB}=-X^{\mathsf{m}}_{(A:B)}$ (odd parity). Explicitly,

$$Y^{0}_{\theta\theta}=-3(2\cos^2\theta-1),$$

$$Y^{0}_{\theta\phi}=0,$$

$$Y^{0}_{\phi\phi}=-3\sin^2\theta\cos^2\theta,$$

$$Y^{1c}_{\theta\theta}=-4\sin\theta\cos\theta\cos\phi,$$

$$Y^{1c}_{\theta\phi}=\sin^2\theta\sin\phi,$$

$$Y^{1c}_{\phi\phi}=-2\sin^3\theta\cos\theta\cos\phi,$$

$$Y^{1s}_{\theta\theta}=-4\sin\theta\cos\theta\sin\phi,$$

$$Y^{1s}_{\theta\phi}=-\sin^2\theta\cos\phi,$$

$$Y^{1s}_{\phi\phi}=-2\sin^3\theta\cos\theta\sin\phi,$$

$$Y^{2c}_{\theta\theta}=2(2\cos^2\theta-1)\cos 2\phi,$$

$$Y^{2c}_{\theta\phi}=-2\sin\theta\cos\theta\sin 2\phi,$$

$$Y^{2c}_{\phi\phi}=2\sin^2\theta(\cos^2\theta-2)\cos 2\phi,$$

$$Y^{2s}_{\theta\theta}=2(2\cos^2\theta-1)\sin 2\phi,$$

$$Y^{2s}_{\theta\phi}=2\sin\theta\cos\theta\cos 2\phi,$$

$$Y^{2s}_{\phi\phi}=2\sin^2\theta(\cos^2\theta-2)\sin 2\phi,$$

and

$$X^{0}_{\theta\theta}=0,$$

$$X^{0}_{\theta\phi}=-\frac{3}{2}\sin^3\theta,$$

$$X^{0}_{\phi\phi}=0,$$

$$X^{1c}_{\theta\theta}=\sin\theta\sin\phi,$$

$$X^{1c}_{\theta\phi}=\sin^2\theta\cos\theta\cos\phi,$$

$$X^{1c}_{\phi\phi}=-\sin^3\theta\sin\phi,$$

$$X^{1s}_{\theta\theta}=-\sin\theta\cos\phi,$$

$$X^{1s}_{\theta\phi}=\sin^2\theta\cos\theta\sin\phi,$$

$$X^{1s}_{\phi\phi}=\sin^3\theta\cos\phi,$$

$$X^{2c}_{\theta\theta}=-2\cos\theta\sin 2\phi,$$

$$X^{2c}_{\theta\phi}=-\sin\theta(\cos^2\theta+1)\cos 2\phi,$$

$$X^{2c}_{\phi\phi}=2\sin^2\theta\cos\theta\sin 2\phi,$$

$$X^{2s}_{\theta\theta}=2\cos\theta\cos 2\phi,$$

$$X^{2s}_{\theta\phi}=-\sin\theta(\cos^2\theta+1)\sin 2\phi,$$

$$X^{2c}_{\phi\phi}=-2\sin^2\theta\cos\theta\cos 2\phi.$$

APPENDIX B: CALCULATION OF THE METRIC PERTURBATIONS

In this appendix I derive the form of the functions e_1, e_2, e_3, b_2, and b_3 that appear in Sec. III B. For this it is sufficient to take, say, $\mathcal{E}_{12}=\mathcal{E}_{21}$ and $\mathcal{B}_{12}=\mathcal{B}_{21}$ as the only nonvanishing components of the tidal fields $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$. And since the equations for even-parity and odd-parity perturbations decouple [20,24], each case can be considered separately.

Including only even-parity perturbations, Eqs. (3.5)–(3.8) become

$$\mathsf{g}_{\bar{u}\bar{u}}=-f(1+\bar{r}^2e_1\mathcal{E}_{12}\sin^2\bar{\theta}\sin 2\bar{\phi}),$$

$$\mathsf{g}_{\bar{u}\bar{r}}=-1,$$

$$\mathsf{g}_{\bar{u}\bar{\theta}}=\frac{2}{3}\bar{r}^3e_2\mathcal{E}_{12}\sin\bar{\theta}\cos\bar{\theta}\sin 2\bar{\phi},$$

$$\mathsf{g}_{\bar{u}\bar{\phi}}=\frac{2}{3}\bar{r}^3e_2\mathcal{E}_{12}\sin^2\bar{\theta}\cos 2\bar{\phi},$$

$$\mathsf{g}_{\bar{\theta}\bar{\theta}}=\bar{r}^2-\frac{1}{3}\bar{r}^4e_3\mathcal{E}_{12}(1+\cos^2\bar{\theta})\sin 2\bar{\phi},$$

$$\mathsf{g}_{\bar{\theta}\bar{\phi}}=-\frac{2}{3}\bar{r}^4e_3\mathcal{E}_{12}\sin\bar{\theta}\cos\bar{\theta}\cos 2\bar{\phi},$$

$$\mathsf{g}_{\bar{\phi}\bar{\phi}}=\bar{r}^2\sin^2\bar{\theta}+\frac{1}{3}\bar{r}^4e_3\mathcal{E}_{12}\sin^2\bar{\theta}(1+\cos^2\bar{\theta})\sin 2\bar{\phi}.$$

This metric is then substituted into the vacuum Einstein field equations. Computing the Einstein tensor is simplified by linearizing with respect to $\mathcal{E}_{12}$ and discarding its derivatives with respect to $\bar{u}$: Since the time scale over which $\mathcal{E}_{ab}$ changes is of order $\mathcal{R}$, the ratio between temporal and spatial derivatives is of order $\bar{r}/\mathcal{R}$ and therefore small in the internal zone; the temporal derivatives can be consistently neglected. The field equations produce ordinary differential equations to be satisfied by the functions e_1, e_2, and e_3. Those are easily decoupled, and demanding that the functions all approach unity as $r\to\infty$ and be well behaved at $r=2m$ yields the unique solutions

$$e_1(\overline{r})=e_2(\overline{r})=f, \quad e_3(\overline{r})=1-\frac{2m^2}{\overline{r}^2},$$

as was stated in Eq. (3.9).

Switching now to odd-parity perturbations, Eqs. (3.5)–(3.8) become

$$g_{\overline{uu}}=-f,$$

$$g_{\overline{ur}}=-1,$$

$$g_{\overline{u\theta}}=-\frac{2}{3}\overline{r}^3 b_2\mathcal{B}_{12}\sin\overline{\theta}\cos 2\overline{\phi},$$

$$g_{\overline{u\phi}}=\frac{2}{3}\overline{r}^3 b_2\mathcal{B}_{12}\sin^2\overline{\theta}\cos\overline{\theta}\sin 2\overline{\phi},$$

$$g_{\overline{\theta\theta}}=\overline{r}^2+\frac{2}{3}\overline{r}^4 b_3\mathcal{B}_{12}\cos\overline{\theta}\cos 2\overline{\phi},$$

$$g_{\overline{\theta\phi}}=-\frac{1}{3}\overline{r}^4 b_3\mathcal{B}_{12}\sin\overline{\theta}(1+\cos^2\overline{\theta})\sin 2\overline{\phi},$$

$$g_{\overline{\phi\phi}}=\overline{r}^2\sin^2\overline{\theta}-\frac{2}{3}\overline{r}^4 b_3\mathcal{B}_{12}\sin^2\overline{\theta}\cos\overline{\theta}\cos 2\overline{\phi}.$$

Following the same procedure, we arrive at

$$b_2(\overline{r})=f, \quad b_3(\overline{r})=1,$$

as was stated in Eq. (3.10).

APPENDIX C: TRANSFORMATION FROM EXTERNAL TO INTERNAL COORDINATES

Our task in this appendix is to construct the transformation from the external coordinates $(u,r\Omega^a)$ to the internal coordinates $(\overline{u},\overline{r}\overline{\Omega}^a)$. We shall proceed in three stages. The first stage of the transformation, $(u,r\Omega^a)\to(u',r'\Omega'^a)$, will be seen to remove unwanted terms of order m/r in $g_{\alpha\beta}$, as listed in Eqs. (3.37)–(3.39). The second stage, $(u',r'\Omega'^a)\to(u'',r''\Omega''^a)$, will remove all terms of order $m/\mathcal{R}$ in $g_{\alpha'\beta'}$. Finally, the third stage $(u'',r''\Omega''^a)\to(\overline{u},\overline{r}\overline{\Omega}^a)$ will produce the desired internal coordinates and return the metric in the form of Eqs. (3.40)–(3.42).

The first stage of the coordinate transformation is

$$u'=u-2m\ln r, \quad x'^a=\left(1+\frac{m}{r}\right)x^a,$$

and it affects the metric at orders m/r and $mr/\mathcal{R}^2$. This transformation redefines the radial coordinate—$r\to r'=r+m$—and incorporates in u' the gravitational time delay contributed by the small mass m. After performing the coordinate transformation the metric becomes

$$\begin{aligned}
g_{u'u'}&=-1-r'^2\mathcal{E}'^*+O(r'^3/\mathcal{R}^3)+\frac{2m}{r'}+h^{\rm tail}_{00}\\
&\quad+r'(4m\mathcal{E}'^*-2a_a\Omega'^a+h^{\rm tail}_{000}+h^{\rm tail}_{00a}\Omega'^a)\\
&\quad+O(mr'^2/\mathcal{R}^3),\\
g_{u'a'}&=-\Omega'_a+\frac{2}{3}r'^2(\mathcal{E}'^*_a+\mathcal{B}'^*_a)+O(r'^3/\mathcal{R}^3)+h^{\rm tail}_{0a}\\
&\quad+\Omega'_a h^{\rm tail}_{00}+r'\Big[-\frac{4m}{3}(\mathcal{E}'^*_a+\mathcal{B}'^*_a)\\
&\quad+(\delta_a{}^b-\Omega'_a\Omega'^b)a_b-\omega_{ab}\Omega'^b\\
&\quad+h^{\rm tail}_{0a0}+\Omega'_a h^{\rm tail}_{000}+h^{\rm tail}_{0ab}\Omega'^b+\Omega'_a h^{\rm tail}_{00b}\Omega'^b\Big]\\
&\quad+O(mr'^2/\mathcal{R}^3),\\
g_{a'b'}&=\delta_{ab}-\Omega'_a\Omega'_b-\frac{1}{3}r'^2(\mathcal{E}'^*_{ab}+\mathcal{B}'^*_{ab})+O(r'^3/\mathcal{R}^3)\\
&\quad+\Omega'_a\Omega'_b h^{\rm tail}_{00}+\Omega'_a h^{\rm tail}_{0b}+\Omega'_b h^{\rm tail}_{0a}+h^{\rm tail}_{ab}\\
&\quad+r'\Big[\frac{2m}{3}(\mathcal{E}'^*_{ab}+\Omega'_a\mathcal{E}'^*_b+\mathcal{E}'^*_a\Omega'_b\\
&\quad+\mathcal{B}'^*_{ab}+\Omega'_a\mathcal{B}'^*_b+\Omega'_b\mathcal{B}'^*_a)\\
&\quad+\Omega'_a\Omega'_b(h^{\rm tail}_{000}+h^{\rm tail}_{00c}\Omega'^c)+\Omega'_a(h^{\rm tail}_{0b0}+h^{\rm tail}_{0bc}\Omega'^c)\\
&\quad+\Omega'_b(h^{\rm tail}_{0a0}+h^{\rm tail}_{0ac}\Omega'^c)+(h^{\rm tail}_{ab0}+h^{\rm tail}_{abc}\Omega'^c)\Big]\\
&\quad+O(mr'^2/\mathcal{R}^3).
\end{aligned}$$

This metric matches g(internal zone) at orders 1, $r'^2/\mathcal{R}^2$, and m/r', but there is still a mismatch at orders $m/\mathcal{R}$ and $mr'/\mathcal{R}^2$.

The second stage of the coordinate transformation is

$$\begin{aligned}
u''&=u'-\frac{1}{2}\int^{u'}h^{\rm tail}_{00}(u')du'\\
&\quad-\frac{1}{2}r'[h^{\rm tail}_{00}(u')+2h^{\rm tail}_{0a}(u')\Omega'^a+h^{\rm tail}_{ab}(u')\Omega'^a\Omega'^b],\\
x''_a&=x'_a+\frac{1}{2}h^{\rm tail}_{ab}(u')x'^b,
\end{aligned}$$

and it affects the metric at orders $m/\mathcal{R}$ and $mr/\mathcal{R}^2$. After performing this transformation the metric becomes

$$\begin{aligned}
g_{u''u''}&=-1-r''^2\mathcal{E}''^*+O(r''^3/\mathcal{R}^3)+\frac{2m}{r''}\\
&\quad+r''\Big[4m\mathcal{E}''^*-2\Big(a_a-\frac{1}{2}h^{\rm tail}_{00a}+h^{\rm tail}_{0a0}\Big)\Omega''^a\Big]\\
&\quad+O(mr''^2/\mathcal{R}^3),
\end{aligned}$$

$$\mathsf{g}_{u''a''} = -\Omega''_a + \frac{2}{3} r''^2 (\mathcal{E}''_a{}^* + \mathcal{B}''_a{}^*) + O(r''^3/\mathcal{R}^3)$$
$$+ r''\Bigg[-\frac{4m}{3}(\mathcal{E}''_a{}^* + \mathcal{B}''_a{}^*) - 2m\mathcal{E}_{ab}\Omega''^b$$
$$+ (\delta_a{}^b - \Omega''_a\Omega''^b)\left(a_b - \frac{1}{2}h^{\rm tail}_{00b} + h^{\rm tail}_{0b0}\right)$$
$$- \omega_{ab}\Omega''^b + \frac{1}{2}\Omega''_a h^{\rm tail}_{000} - \frac{1}{2}h^{\rm tail}_{ab0}\Omega''^b + h^{\rm tail}_{0ab}\Omega''^b$$
$$+ \frac{1}{2}(\delta_a{}^b + \Omega''_a\Omega''^b)h^{\rm tail}_{00b}\Bigg] + O(mr''^2/\mathcal{R}^3),$$

$$\mathsf{g}_{a''b''} = \delta_{ab} - \Omega''_a\Omega''_b - \frac{1}{3}r''^2(\mathcal{E}''_{ab}{}^* + \mathcal{B}''_{ab}{}^*) + O(r''^3/\mathcal{R}^3)$$
$$+ r''\Bigg[\frac{2m}{3}(\mathcal{E}''_{ab}{}^* + \Omega''_a\mathcal{E}''_b{}^* + \mathcal{E}''_a{}^*\Omega''_b + \mathcal{B}''_{ab}{}^*$$
$$+ \Omega''_a\mathcal{B}''_b{}^* + \Omega''_b\mathcal{B}''_a{}^*) + \Omega''_a\Omega''_b(h^{\rm tail}_{000} + h^{\rm tail}_{00c}\Omega''^c)$$
$$+ \Omega''_a(h^{\rm tail}_{0b0} + h^{\rm tail}_{0bc}\Omega''^c) + \Omega''_b(h^{\rm tail}_{0a0} + h^{\rm tail}_{0ac}\Omega''^c)$$
$$+ (h^{\rm tail}_{ab0} + h^{\rm tail}_{abc}\Omega''^c)\Bigg] + O(mr''^2/\mathcal{R}^3).$$

To arrive at these expressions we had to involve the relations

$$\frac{d}{du''}h^{\rm tail}_{00} = h^{\rm tail}_{000},$$

$$\frac{d}{du''}h^{\rm tail}_{0a} = h^{\rm tail}_{0a0},$$

$$\frac{d}{du''}h^{\rm tail}_{ab} = 4m\mathcal{E}_{ab} + h^{\rm tail}_{ab0},$$

which are obtained by covariant differentiation of Eq. (3.28) in the direction of $u^{\alpha'}$. The metric now matches g(internal zone) at orders 1, $r''^2/\mathcal{R}^2$, m/r'', and $m/\mathcal{R}$, but there is still a mismatch at order $mr''/\mathcal{R}^2$.

The third and final stage of the coordinate transformation is

$$\bar{u} = u'' - \frac{1}{4}r''^2[h^{\rm tail}_{000} + (h^{\rm tail}_{00a} + 2h^{\rm tail}_{0a0})\Omega''^a$$
$$+ (h^{\rm tail}_{ab0} + 2h^{\rm tail}_{0ab})\Omega''^a\Omega''^b + h^{\rm tail}_{abc}\Omega''^a\Omega''^b\Omega''^c],$$
$$\bar{x}_a = \left(1 + \frac{m}{3}r''\mathcal{E}_{bc}\Omega''^b\Omega''^c\right)x''_a + \frac{1}{2}r''^2\Bigg[-\frac{1}{2}h^{\rm tail}_{00a} + h^{\rm tail}_{0a0}$$
$$+ \left(h^{\rm tail}_{0ab} - h^{\rm tail}_{0ba} + h^{\rm tail}_{ab0} + \frac{4m}{3}\mathcal{E}_{ab}\right)\Omega''^b$$
$$+ (Q_{abc} - Q_{bca} + Q_{cab})\Omega''^b\Omega''^c\Bigg],$$

where

$$Q_{abc} = \frac{1}{2}h^{\rm tail}_{abc} + \frac{m}{3}(\varepsilon_{acd}\mathcal{B}^d{}_b + \varepsilon_{bcd}\mathcal{B}^d{}_a).$$

This produces the metric of Eqs. (3.40)–(3.42).

[1] J. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).
[2] G. Ellis, S. Nel, R. Maartens, W. Stoeger, and A. Whitman, Phys. Rep. **124**, 315 (1985).
[3] W. Stoeger, S. Nel, R. Maartens, and G. Ellis, Class. Quantum Grav. **9**, 493 (1992).
[4] W. Stoeger, G. Ellis, and S. Nel, Class. Quantum Grav. **9**, 509 (1992).
[5] W. Stoeger, S. Stanley, S. Nel, and G. Ellis, Class. Quantum Grav. **9**, 1711 (1992).
[6] W. Stoeger, S. Stanley, S. Nel, and G. Ellis, Class. Quantum Grav. **9**, 1725 (1992).
[7] W. Stoeger, G. Ellis, and C. Xu, Phys. Rev. D **49**, 1845 (1994).
[8] K. Thorne and J. Hartle, Phys. Rev. D **31**, 1815 (1985).
[9] D. Sciama, P. Waylen, and R. Gilman, Phys. Rev. **187**, 1762 (1969).
[10] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[11] T. Quinn and R. Wald, Phys. Rev. D **56**, 3381 (1997).
[12] F. Manasse, J. Math. Phys. **4**, 746 (1963).
[13] R. Kates, Phys. Rev. D **22**, 1853 (1980).
[14] P. D'Eath, *Black Holes: Gravitational Interactions* (Clarendon, Oxford, 1996).
[15] S. Detweiler, Phys. Rev. Lett. **86**, 1931 (2001).
[16] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[17] E. Poisson, The motion of point particles in curved spacetime, 2003, http://xxx.lanl.gov/abs/gr-qc/0306052.
[18] B. DeWitt and R. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[19] F. Manasse and C. Misner, J. Math. Phys. **4**, 735 (1963).
[20] T. Regge and J. Wheeler, Phys. Rev. **108**, 1063 (1957).
[21] K. Alvi, Phys. Rev. D **61**, 124013 (2000).
[22] L. Fishbone, Astrophys. J. **185**, 43 (1973).
[23] J.-A. Marck, Proc. R. Soc. London **A385**, 431 (1983).
[24] F. Zerilli, Phys. Rev. D **2**, 2141 (1970).
[25] C. Vishveshwara, Phys. Rev. D **1**, 2870 (1970).
[26] S. Jhingan and T. Tanaka, Phys. Rev. D **67**, 104018 (2003).
[27] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).
[28] F. Friedlander, *The Wave Equation on a Curved Spacetime* (Cambridge University Press, Cambridge, England, 1975).