Scalar perturbation spectra from warm inflation

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We present a numerical integration of the cosmological scalar perturbation equations in warm inflation. The initial conditions are provided by a discussion of the thermal fluctuations of an inflaton field and thermal radiation using a combination of thermal field theory and thermodynamics. The perturbation equations include the effects of a damping coefficient Γ and a thermodynamic potential V. We give an analytic expression for the spectral index of scalar fluctuations in terms of a new slow-roll parameter constructed from Γ. A series of toy models, inspired by spontaneous symmetry breaking and a known form of the damping coefficient, lead to a spectrum with $n_s > 1$ on large scales and $n_s < 1$ on small scales.

I. INTRODUCTION

Inflation is the most successful idea which we have available for explaining the large scale structure of the Universe. Inflationary dynamics can be realized in two distinct ways. In the original picture, termed supercooled inflation, the Universe rapidly supercools during an inflation phase and subsequently a reheating period is invoked to end inflation and fill the Universe with radiation [1–4]. In the other picture, termed warm inflation, dissipative effects are important during the inflation period, so that radiation production occurs concurrently with inflationary expansion.

The idea of warm inflation was influenced by a calculation of the friction term in the inflaton equation of motion by Hosoya and Sakagami [5]. The magnitude of the damping term suggested the possibility that it could be the dominant effect prolonging inflation [6–9]. The significance of dissipation to inflation was independently realized in [10], but this time with a clearer picture of the associated inflation dynamics, and named warm inflation. This and associated works [11,12] also outlined the nonequilibrium thermodynamical problem underlying this picture, which subsequently has been studied in greater detail [13–19]. These works confirmed that at high temperature the damping was proportional to the relaxation time of the radiation. In addition to these studies of warm inflation dynamics, several phenomenological warm inflation models have been discussed in the literature [20–24].

The density fluctuations in warm inflation arise from thermal, rather than vacuum, fluctuations [6,25,26]. These have their origin in the hot radiation and influence the inflaton through a noise term in the equations of motion [11,26].

The development of cosmological perturbations in warm inflation has been investigated by several authors. Taylor and Berera [27] have analyzed the perturbation spectra by matching the thermally produced fluctuations to gauge invariant parameters when the fluctuations cross the horizon. This technique should be good for order of magnitude estimates. For more accurate treatment of the density perturbations it is necessary to use the cosmological perturbation equations, which can be found in the literature [28–30].

The main focus of the present work is to solve the perturbation equations numerically. We will also set up the equations to take into account some effects which have previously been ignored. We introduce a new slow-roll parameter and include modifications to the equations when the damping term and the potential depend on temperature.

Other improvements have been made in the treatment of thermal fluctuations. The influence of the cosmological expansion on the inflaton fluctuations is considered in detail. This gives a clearer and more accurate picture of the development of the thermal fluctuations than previously. Thermal fluctuations in the radiation, which lead to entropy perturbations, are also analyzed. Entropy perturbations are always present during warm inflation and react back on the curvature fluctuations. In the basic model, the entropy fluctuations disappear before inflation ends.

Section II contains an updated outline of the theory of warm inflation. In Sec. III we obtain expressions for the thermal fluctuations of the inflaton and the radiation. Cosmological perturbations are discussed in Sec. IV. The results of numerical solutions for the cosmological perturbations using the thermal fluctuations as initial conditions are compared with observations in Sec. V. A summary of the main results and some general comments appear in Sec. VI.

II. WARM INFLATION

In this section we shall consider the case of a homogeneous inflaton field interacting with thermalized radiation. If the inflaton is evolving very slowly, then the radiation creates a thermal correction to the inflaton potential [31] and a damping force [5]. The system of equations describing the system includes an equation for the inflaton and an equation describing how the energy lost by the inflaton through the

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damping force is transferred to the radiation.

The equation describing the effects of radiation damping on the evolution of the inflaton field was first obtained by Hosoya [5] using a form of thermodynamic transport theory. This is applicable as long as the inflaton evolves slowly and the radiation remains close to thermal equilibrium. More recent treatments are based on the closed time path formalism, and an account of the effects of radiation damping on an evolving field in flat space can be found, for example, in a paper by Gleiser and Ramos [32]. An interesting variation occurs when the inflaton interacts with the radiation via an intermediate particle decay [19].

When the early Universe is in homogeneous expansion with expansion rate $H(t)$, the evolution equation for the inflaton field is given by

$$\ddot{\phi}+(3H+\Gamma)\dot{\phi}+V_{,\phi}=0, \tag{1}$$

where $\Gamma(\phi,T)$ is the damping term and $V(\phi,T)$ is the thermodynamic potential. The damping term has a generic form given approximately by $\Gamma\sim g^4\phi^2\tau$, where g is a coupling constant. For Hosoya's damping term $\tau\equiv\tau(\phi,T)$ is related to the relaxation time of the radiation and for the models with an intermediate particle decay $\tau\equiv\tau(\phi)$ is related to the lifetime of the intermediate particle. The thermodynamic potential can be expressed in a standardized way by [33]

$$V(\phi,T)=-\frac{\pi^2}{90}g_*T^4+\frac{1}{2}(\delta m_T)^2\phi^2+V_0(\phi), \tag{2}$$

where $g_*(T)$ is the effective particle number and $\delta m_T(\phi,T)$ represents thermal corrections.

The relative strength of the thermal damping compared to the expansion damping can be described by a parameter r,

$$r=\frac{\Gamma}{3H}. \tag{3}$$

Warm inflation is defined to be inflation with large values of r, as opposed to supercooled inflation in which r can be neglected.

The dissipation of the inflaton's motion is associated with the production of entropy. The entropy density of the radiation $s(\phi,T)$ is defined by a thermodynamic relation in terms of the thermodynamic potential,

$$s=-V_{,T}. \tag{4}$$

The rate of entropy production can be deduced from the conservation of energy-momentum. The total density ρ and pressure p are given by

$$\rho=\frac{1}{2}\dot{\phi}^2+V+Ts \tag{5}$$

$$p=\frac{1}{2}\dot{\phi}^2-V. \tag{6}$$

Energy-momentum conservation,

$$\dot{\rho}+3H(p+\rho)=0 \tag{7}$$

now implies entropy production. Making use of Eq. (1) we get

$$T(\dot{s}+3Hs)=\Gamma\dot{\phi}^2. \tag{8}$$

The zero curvature Friedmann equation completes the set of differential equations for ϕ, T and the scale factor a,

$$3H^2=8\pi G\left(\frac{1}{2}\dot{\phi}^2+V+Ts\right). \tag{9}$$

The entropy production has been described in a slightly different way in some previous work on warm inflation [10,26]. We can recover an alternative equation in the case when the temperature corrections to the potential are negligible. If we set $\delta m_T=0$ in Eq. (2), then the radiation density $\rho_r=3sT/4$, and Eq. (8) becomes

$$\dot{\rho}_r+4H\rho_r=\Gamma\dot{\phi}^2. \tag{10}$$

This equation is valid *only* when $\delta m_T=0$.

The slow-roll approximation consists of neglecting terms in the preceding equations with the highest order in time derivatives

$$\dot{\phi}=-\frac{V_{,\phi}}{3H(1+r)}, \tag{11}$$

$$Ts=r\dot{\phi}^2, \tag{12}$$

$$3H^2=8\pi GV. \tag{13}$$

Note that Ts is of the same order as two time derivatives. Slow roll automatically implies inflation, $\ddot{a}>0$.

The consistency of the slow-roll approximation is governed by a set of slow-roll parameters. Warm inflation has extra slow-roll parameters in addition to the usual set for supercooled inflation due to the presence of the damping term Γ. There are four "leading order" slow-roll parameters,

$$\epsilon=\frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2, \quad \eta=\frac{1}{8\pi G}\left(\frac{V_{,\phi\phi}}{V}\right),$$

$$\beta=\frac{1}{8\pi G}\left(\frac{\Gamma_{,\phi}V_{,\phi}}{\Gamma V}\right), \quad \delta=\frac{TV_{,\phi T}}{V_{,\phi}}. \tag{14}$$

The first two parameters are the standard ones introduced for supercooled inflation [34–37]. Two new parameters are required for the ϕ dependence of the damping term and the temperature dependence of the potential. The slow-roll approximation is valid when all of the slow-roll parameters are smaller than $1+r$. In supercooled inflation, the condition is tighter and the slow roll parameters have to be smaller than 1.

As an example, consider the case $\Gamma_{,T} = V_{,\phi T} = 0$. In the warm inflationary regime $r \gg 1$, the rate of change of various slowly varying parameters are given by differentiating Eqs. (11)–(13),

$$\frac{1}{H}\frac{d\ln H}{dt} = -\frac{1}{r}\epsilon, \tag{15}$$

$$\frac{1}{H}\frac{d\ln\dot{\phi}}{dt} = -\frac{1}{r}(\eta - \beta), \tag{16}$$

$$\frac{1}{H}\frac{d\ln Ts}{dt} = -\frac{1}{r}(2\eta - \beta - \epsilon). \tag{17}$$

In each case, the right hand side of these equations gives the relative sizes of terms neglected in the slow-roll approximation. Slow roll therefore requires $\epsilon \ll r$, $\eta \ll r$ and $\beta \ll r$.

Inflation ends at the time when $\ddot{a} = \dot{H} + H^2$ falls to 0. From Eq. (15), the end of inflation corresponds to the condition $\epsilon = r$. Furthermore, the slow-roll equations allow us to rewrite this condition as $\rho_r = V$; in other words inflation ends when the radiation energy density reaches the same value as the vacuum energy.

Similar arguments apply when $\Gamma_{,T}$ and $V_{,\phi T}$ are non-zero, for example,

$$\frac{1}{H}\frac{d\ln\dot{\phi}}{dt} = -\frac{1}{r}\left(\frac{4-c}{4+c}\eta - \frac{4}{4+c}\beta + \frac{c}{4+c}\epsilon - \frac{3c}{4+c}\delta\right). \tag{18}$$

Note that $c = T\Gamma_{,T}/\Gamma$ is not required to be small and therefore does not enter the list of slow-roll parameters. The slow-roll approximation is valid for $\epsilon \ll r$, $\eta \ll r$, $\beta \ll r$ and $\delta \ll r$.

III. THERMAL FLUCTUATIONS

In this section we shall consider the behavior of thermal fluctuations during inflation and their influence on density perturbations on length scales smaller than the size of the horizon. Thermal fluctuations, if they are present, form the predominant source of density perturbations in warm inflation. We will assume throughout that the radiation is close to thermal equilibrium. Surprisingly, this is not a necessary condition for warm inflation but it is an important regime. Conditions for thermalization are model dependent, and are discussed in Refs. [13,14,17,19].

The evolution of a comoving mode of the inflaton fluctuations during a warm inflationary era can be divided into three regimes, depending on the relative effects of different physical processes:

(1) Thermal noise.
(2) Expansion.
(3) Curvature fluctuations.

The transitions between these regimes are called freeze-out and horizon crossing. Where previous calculations have treated each of these regimes seperately, we shall consider the combined effects of the thermal noise and the expansion,

and then the combined effects of expansion and curvature fluctuations. The main results of this section are Eqs. (37), (38) and (51), which provide the initial conditions for the classical evolution equations described in the next section.

A. Inflaton fluctuations

The behavior of a scalar field interacting with radiation can be analyzed using the Schwinger-Keldysh approach to non-equilibrium field theory [38,39]. According to this approach, a simple picture emerges in which the field evolves by a Langevin equation [32,40],

$$-\nabla^2\phi(x,t) + \Gamma\dot{\phi}(x,t) + V_{,\phi} = \xi(x,t), \tag{19}$$

where ξ is a stochastic noise source. Correlation functions can be evaluated by probability averages.

The Schwinger-Keldeysh approach has been studied for curved, as well as flat, spacetime backgrounds [41]. Although the particular form of the Langevin equation has not yet been fully investigated, we can appeal to the equivalence principle to deduce the form of the Langevin equation on scales smaller than the horizon [26].

We shall analyze the inflaton fluctuations on an expanding background over time scales which are longer than the microphysical processes, but short compared to the variation of the expansion rate. We also take modes with physical length scales larger than the microphysical processes. The comoving wave number will be denoted by k and the physical wave number by $q = ka^{-1}$.

The Langevin equation for the Fourier transform of the inflaton fluctuations η_k takes the form [26]

$$\ddot{\eta}_k + 3H\dot{\eta}_k + \Gamma\dot{\eta}_k + k^2a^{-2}\eta_k = \xi_k, \tag{20}$$

where ξ_k is a stochastic noise source. Over the time scales of interest, H and Γ are constant and $a = \exp(Ht)$. As in the previous section, we set $r = \Gamma/3H$.

The correlation function of the noise can be found by converting flat space results [32] to the comoving frame. In the high temperature limit $T \to \infty$, the noise source is Markovian,

$$\langle\xi_k(t)\xi_{-k'}(t')\rangle = 2\Gamma Ta^{-3}(2\pi)^3\delta^3(k-k')\delta(t-t'). \tag{21}$$

The scale factor appears due to the use of comoving coordinates. (The temperature should be large in comparison with the mass of the particles which make up the radiation for this approximation. We shall make the same approximation for the damping term later.)

For the eventual comparison with observational data, we define the power spectra $\mathcal{P}(k)$ by

$$\mathcal{P}_{\phi\phi}(k)(2\pi k^{-1})^3\delta^3(k-k') = \langle\eta_k(t)\eta_{-k'}(t)\rangle, \tag{22}$$

$$\mathcal{P}_{\pi\pi}(k)(2\pi k^{-1})^3\delta^3(k-k') = \langle\dot{\eta}_k(t)\dot{\eta}_{-k'}(t)\rangle, \tag{23}$$

$$\mathcal{P}_{\pi\phi}(k)(2\pi k^{-1})^3\delta^3(k-k') = \langle\dot{\eta}_k(t)\eta_{-k'}(t)\rangle. \tag{24}$$

When comparing to classical perturbation theory it is also useful to define root-mean-square fluctuation amplitudes

$$\delta\phi(k) = |\mathcal{P}_{\phi\phi}(k)|^{1/2}, \qquad (25)$$

$$\delta\dot{\phi}(k) = |\mathcal{P}_{\pi\pi}(k)|^{1/2}. \qquad (26)$$

Defined in this way, $\delta\phi$ has the same dimensions as ϕ.

The power spectra can be found by solving the stochastic equation (20) by Green function methods,

$$\eta_k(t) = -\frac{\pi}{H^2}\int_z^\infty \{J_\nu(z)Y_\nu(z') - J_\nu(z')Y_\nu(z)\}$$

$$\times \left(\frac{z}{z'}\right)^\nu \xi(z')\frac{dz'}{z'} \qquad (27)$$

where $z(t) = k/a(t)H$ and the order of the Bessel functions is

$$\nu = \frac{3}{2}(1+r). \qquad (28)$$

It is now possible to use the relation (21) to obtain the power spectrum,

$$\mathcal{P}_{\phi\phi}(k) = -\left(\frac{\pi}{2}\right)^2 2\Gamma T \int_z^\infty \{J_\nu(z)Y_\nu(z')$$

$$- J_\nu(z')Y_\nu(z)\}^2 z^{2\nu}z'^{2-2\nu}\,dz'. \qquad (29)$$

Similarly, for the time derivatives,

$$\mathcal{P}_{\pi\pi}(k) = -\left(\frac{\pi}{2}\right)^2 H^2 z^2 2\Gamma T \int_z^\infty \{J_{\nu-1}(z)Y_\nu(z')$$

$$- J_\nu(z')Y_{\nu-1}(z)\}^2 z^{2\nu}z'^{2-2\nu}\,dz'. \qquad (30)$$

The formula for the power spectrum can be simplified at late or early times. For early times, when the physical size of the mode a/k is much smaller than the horizon, we can use the $z \gg \nu$ approximation for the Bessel functions to obtain

$$\mathcal{P}_{\phi\phi}(k) \sim ka^{-1}T, \qquad (31)$$

$$\mathcal{P}_{\pi\pi}(k) \sim k^3 a^{-3} T. \qquad (32)$$

Allowing for the transformation from comoving to physical wave numbers, these results are identical to the thermal fluctuations of a field in flat space. The rms fluctuations are proportional to $T^{1/2}$, which is typical of the fluctuations of a classical object in a radiation field [42].

A different approximation holds for later times, when $H < ka^{-1} < (\Gamma H)^{1/2}$, and we can use the $z \ll \nu$ approximation for the Bessel functions. The power spectrum simplifies to

$$\mathcal{P}_{\phi\phi}(k) \sim \frac{\pi^2}{2}\Gamma T z^{2\nu} Y_\nu(z)^2 \int_0^\infty J_\nu(z')^2 z'^{2-2\nu} + O(z^3). \qquad (33)$$

The root-mean-square fluctuation is approaching the dominant solution $z^\nu Y_\nu(z)$ of the homogeneous equation for η_k in this regime as if the noise had no effect. With the warm inflation condition $\Gamma \gg H$, the results further reduce to

$$\mathcal{P}_{\phi\phi}(k) \sim \left(\frac{\pi}{4}\right)^{1/2}(\Gamma H)^{1/2}T\left(1 + \frac{1}{\Gamma H}\frac{k^2}{a^2} + \cdots\right), \qquad (34)$$

$$\mathcal{P}_{\pi\pi}(k) \sim \left(\frac{\pi}{4}\right)^{1/2}\frac{k^4}{\Gamma^2 a^4}(\Gamma H)^{1/2}T. \qquad (35)$$

The cross correlation reduces to

$$\mathcal{P}_{\pi\phi}(k) \sim -[\mathcal{P}_{\phi\phi}(k)]^{1/2}[\mathcal{P}_{\pi\pi}(k)]^{1/2}, \qquad (36)$$

which also indicates that the effects of the noise term are fading away.

The transition between the two regimes has been called freeze-out [26], and occurs at a particular time $t_F(k)$ when $ka^{-1} = (\Gamma H)^{1/2}$. In warm inflation, the freeze-out time will always precede the Hubble crossing time, at which $ka^{-1} = H$. The evolution of the inflaton becomes increasingly deterministic for times $t > t_F$.

The fluctuations approach a constant value,

$$\delta\phi(k) \sim \left(\frac{\pi}{4}\right)^{1/4}(\Gamma H)^{1/4}T^{1/2}, \qquad (37)$$

$$\delta\dot{\phi}(k) \sim -\left(\frac{\pi}{4}\right)^{1/4}\frac{k^2}{\Gamma a^2}(\Gamma H)^{1/4}T^{1/2}. \qquad (38)$$

Apart from the numerical factor, the value of $\delta\phi$ agrees with the results of Berera [26], who analyzed Eq. (20) in the regime $ka^{-1} > (\Gamma H)^{1/2}$. The sign of $\delta\dot{\phi}$ has been chosen for consistency with the cross correlations (36). It is also consistent with the deterministic solution.

B. Energy fluctuations

Now we turn to thermal fluctuations in the energy density in flat space. The classical theory of energy fluctuations is based on fundamental principles of statistical physics. For the energy fluctuation ΔE in a fixed volume V, Lifshitz and Pitaevstii [42] give the formula

$$\langle(\Delta E)^2\rangle_V = C_v T^2, \qquad (39)$$

where C_v is the specific heat. This result can also be expressed in terms of the energy density ρ and the entropy density s,

$$\langle(\Delta\rho)^2\rangle_V = V^{-1}T^3 s_{,T}. \qquad (40)$$

We shall investigate the equivalent result in finite temperature quantum field theory before considering the application to warm inflation.

Consider a scalar field χ in thermal equilibrium at temperature T. Fluctuations ε in the energy density are defined by

$$\varepsilon = \mathcal{H}(x) - \langle\mathcal{H}\rangle_T, \qquad (41)$$

where the energy density operator

$$\mathcal{H} = \frac{1}{2}\dot{\chi}^2 + \frac{1}{2}(\nabla\chi)^2 + \frac{1}{2}m^2\chi^2, \tag{42}$$

and the energy density $\rho_r = \langle\mathcal{H}\rangle_T$ is the thermal ensemble expectation value of $\mathcal{H}$.

The correlation function

$$\langle\varepsilon(x)\varepsilon(x')\rangle_T = \langle\mathcal{H}(x)\mathcal{H}(x')\rangle_T - \langle\mathcal{H}\rangle_T^2. \tag{43}$$

The simplest way to remove any divergences will be to subtract the zero temperature correlations, which in any case are not of interest to us. We therefore define the power spectrum of the radiation $\mathcal{P}_r$ by

$$\mathcal{P}_r(q)(2\pi q^{-1})^3 \delta^3(q-q')$$
$$= \langle\varepsilon_q(t)\varepsilon_{-q'}(t')\rangle_T - \langle\varepsilon_q(t)\varepsilon_{-q'}(t')\rangle_0. \tag{44}$$

As before, we also define a root-mean-square fluctuation

$$\delta\rho_r(k) = |\mathcal{P}_r(q)|^{1/2}. \tag{45}$$

The expectation values for four field operators can be reduced using the cluster decomposition principle [43] to give

$$\mathcal{P}_r(q) = \int \frac{d^3q'}{(2\pi)^3} q^3 [-q(q-q') + \partial_t\partial_{t'} + m^2]^2$$
$$\times [G_T^>(q,t)G_T^>(q-q',t')$$
$$- G_0^>(q,t)G_0^>(q-q',t')], \tag{46}$$

where the expectation values of products of two fields are given by the thermal Green function,

$$\langle\chi_q(t)\chi_{-q'}(t')\rangle = (2\pi)^2 \delta^3(q-q')G_T^>(q,t-t'). \tag{47}$$

For a bosonic field in thermal equilibrium,

$$G_T^>(q,t) = n(\omega_q)e^{i\omega_q t} + (1+n(\omega_q))e^{-i\omega_q t}, \tag{48}$$

where $\omega_q = (m^2+q^2)^{1/2}$ and $n(\omega) = (e^{\omega/T}-1)^{-1}$. The power spectrum reduces to the integral

$$\mathcal{P}_r(q) = \int \frac{d^3q'}{(2\pi)^3} \frac{q^3}{4\omega_{q'}\omega_{q-q'}} (2\omega_{q-q'}^2\omega_{q'}^2 - m^2q^2)$$
$$\times [2n(\omega_{q-q'})n(\omega_{q'}) + n(\omega_{q-q'}) + n(\omega_{q'})]. \tag{49}$$

The classical limit corresponds to $q \ll T$. The integral is then dominated by the region $q' \gg q$. For a massless field, the integral reduces to

$$\mathcal{P}_r(q) = 4q^3 T\rho_r, \tag{50}$$

where ρ_r is the energy density. This is in agreement with the thermodynamic result (40) for pure radiation if we identify q^3 with V^{-1}.

In the expanding Universe, the inflaton and radiation fluctuations are effectively in equilibrium for times $t < t_F$, where

t_F is the freeze-out time mentioned earlier. The total energy fluctuations in this regime are given by Eq. (40). After t_F, the radiation and inflaton fluctuations are uncorrelated and Eq. (50) can be used for the fluctuations in the energy density of the radiation,

$$\delta\rho_r(k) = \left(\frac{2\pi^2}{15}\right)^{1/2} g_*^{1/2}k^{3/2}a^{-3/2}T^{5/2}, \tag{51}$$

where $g*$ is the effective particle number and we have introduced the comoving wave number $k = aq$.

The total energy is defined by Eq. (5). The inflaton energy fluctuations for $t > t_F$ are therefore

$$\delta\rho_\phi = \dot{\phi}\delta\dot{\phi} + V_{,\phi}\delta\phi, \tag{52}$$

where $\delta\phi$ and $\delta\dot{\phi}$ are given above.

IV. COSMOLOGICAL PERTURBATIONS

The thermal fluctuations occurring during warm inflation evolve gradually into cosmological perturbations. The thermal noise has a diminishing effect on the development of the perturbations until the freeze-out time when it becomes insignificant. The perturbations can then be described by a Gaussian random field which grows deterministically.

In warm inflation there are cosmological perturbations in the inflaton field, the radiation and the gravitational field. We shall consider only the scalar gravitational mode, which means that there is one degree of freedom in the metric perturbations. We shall also use a particular gauge, the zero-shear gauge, in which the scalar metric perturbation has the form

$$ds^2 = -(1-2\varphi)dt^2 + a^2(1+2\varphi)\delta_{ij}dx^i dx^j. \tag{53}$$

The inflaton perturbation will be denoted by $\delta\phi$, the temperature perturbation δT and the velocity perturbation v.

For any random perturbation field $g(x)$, the power spectrum is defined by an average

$$\mathcal{P}_g(k)(2\pi k^{-1})^3\delta^3(k-k') = \langle g_k g_{-k'}\rangle, \tag{54}$$

where g_k is the Fourier transform of $g(x)$. The amplitude will be normalized by

$$g(k) = |\mathcal{P}_g(k)|^{1/2}. \tag{55}$$

Note that $g(k)$ satisfies the same linear equations of motion as g_k. We use this normalization for the perturbations and omit explicit reference to k.

The total energy density and pressure perturbations are then

$$\delta\rho = \dot{\phi}\delta\dot{\phi} + V_{,\phi}\delta\phi + \dot{\phi}^2\varphi + T\delta s, \tag{56}$$

$$\delta p = \dot{\phi}\delta\dot{\phi} - V_{,\phi}\delta\phi + \dot{\phi}^2\varphi + s\,\delta T, \tag{57}$$

and the energy momentum flux $(\rho+p)v$.

A complete set of perturbation equations can be obtained from the Einstein field equations and the scalar field equation [29,30]. From the Einstein equation we obtain

$$\dot{\varphi} + H\varphi + 4\pi G k^{-1} a(p+\rho)v = 0, \tag{58}$$

$$3H\dot{\varphi} + (3H^2 + k^2 a^{-2})\varphi - 4\pi G\delta\rho = 0, \tag{59}$$

$$\ddot{\varphi} + 4H\dot{\varphi} + (2\dot{H} + 3H^2)\varphi + 4\pi G\delta p = 0. \tag{60}$$

Perturbations of the scalar field equation give

$$\delta\ddot{\phi} + (3H + \Gamma)\delta\dot{\phi} + \dot{\phi}(\delta\Gamma) + k^2 a^{-2}\delta\phi$$

$$+ \delta V_{,\phi} + 4\dot{\phi}\dot{\varphi} - \Gamma\dot{\phi}\varphi - 2V_{,\phi}\varphi = 0. \tag{61}$$

In general, we can have $\Gamma \equiv \Gamma(\phi, T)$ and $V \equiv V(\phi, T)$,

$$\delta\Gamma = \Gamma_{,\phi}\delta\phi + \Gamma_{,T}\delta T, \tag{62}$$

$$\delta V_{,\phi} = V_{,\phi\phi}\delta\phi + V_{,\phi T}\delta T, \tag{63}$$

$$\delta s = -V_{,\phi T}\delta\phi - V_{,TT}\delta T. \tag{64}$$

The numerical results of the next section are obtained by integrating these equations with the initial conditions set by the thermal fluctuations.

During warm inflation, the background solution rapidly approaches the slow-rolldown approximation (11)–(13). In a similar way, the perturbations also have a slow-roll limit, which begins when their length scales become larger than the horizon, $k < aH$. The $\delta\ddot{\phi}$, $\dot{\varphi}$ and $\ddot{\varphi}$ terms become insignificant, and perturbations approach a slow-roll large-scale limit

$$\delta\phi \sim -CH^{-1}\dot{\phi}, \tag{65}$$

$$\varphi \sim 4\pi GC(1+r)\dot{\phi}^2, \tag{66}$$

$$v \sim -Cka^{-1}H^{-1}, \tag{67}$$

where C is a constant and $r = \Gamma/3H$.

The cosmological perturbations can also be described in terms of gauge invariant quantities, and these are useful for following the development of perturbations after the end of inflation. The curvature perturbation $\mathcal{R}$ [44], is defined by

$$\mathcal{R} = \varphi - k^{-1}aHv. \tag{68}$$

The entropy perturbation e is defined by [45]

$$e = \delta p - c_s^2 \delta\rho, \tag{69}$$

where $c_s^2 = \dot{p}/\dot{\rho}$. In the slow-roll large-scale limit, the curvature perturbation is constant, $\mathcal{R} \sim C$ and the entropy perturbations vanish.

The curvature and entropy perturbations are coupled by a second order equation [30]

$$\ddot{\mathcal{R}} + \Gamma_{\mathcal{R}}\dot{\mathcal{R}} + c_s^2 k^2 a^{-2}\mathcal{R} = -[(He_1)^{\cdot} - \Gamma_{\mathcal{R}}He_1], \tag{70}$$

where $e_1 = e/(p+\rho)$ and

$$\Gamma_{\mathcal{R}} = -3Hc_s^2 - 2H^{-1}\dot{H} - 2c_s^{-1}\dot{c}_s. \tag{71}$$

The perturbation equation (70) implies the classic result [46] that if there is no further entropy generation then the curvature fluctuation $\mathcal{R}$ will retain the constant value it reached during inflation while the wavelength exceeds the horizon size, $k < aH$.

If the entropy perturbation vanishes on small as well as large scales, then we need only solve the cosmological perturbation equation for $\mathcal{R}$. This is a reasonable approximation during supercooled inflation, and the equation is often used in that context [47].

In warm inflation the entropy perturbations are important. Nevertheless, for the case $\Gamma_{,T} = 0$, it is possible to obtain analytic results. It is advantageous to use the inflaton equation for approximate solutions, relying on the fact that the metric perturbations remain small on scales smaller than the horizon, $k > aH$. An analytic approximation to the density perturbation amplitude can be obtained by matching the classical result for $\delta\phi$ Eq. (11) to the thermal fluctuations Eq. (37) at the crossing time t_H when $k = aH$,

$$\mathcal{P}_{\mathcal{R}} \sim \left(\frac{\pi}{4}\right)^{1/2} \frac{H^{5/2}\Gamma^{1/2}T}{\dot{\phi}^2}. \tag{72}$$

This result is analogous to the result $\mathcal{P}_{\mathcal{R}} = H^4/\dot{\phi}^2$ for supercooled inflation.

The spectral index n_s is defined by

$$n_s - 1 = \frac{\partial \ln \mathcal{P}_{\mathcal{R}}}{\partial \ln k}. \tag{73}$$

The slow-roll equations (15)–(17) enable us to express the spectral index for the amplitude (72) in terms of slow-roll parameters,

$$n_s - 1 = \frac{1}{r}\left(-\frac{9}{4}\epsilon + \frac{3}{2}\eta - \frac{9}{4}\beta\right). \tag{74}$$

The first two terms agree with Ref. [26]. The β term shows the dependence of the spectrum on the gradient of the reheating term. For comparison, the spectral index for standard, or supercooled inflation, is $n_s = 1 - 6\epsilon + 2\eta$ [48].

V. NUMERICAL RESULTS

We shall consider the evolution of the cosmological perturbations and the spectrum of density perturbations in a particular toy model which allows us to investigate a number of interesting phenomena. The toy models include some basic features of consistent models of warm inflation, such as [14,19], but they are simplified in order to isolate particular effects.

The same potential for the inflaton field will be used throughout. The thermal corrections to the potential are taken to be insignificant. Three different types of damping term will be considered. We hope to compare the perturbation spectra for different potentials and potentials with large thermal corrections in later work.

The potential we choose is

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - \phi_0^2)^2. \tag{75}$$

The important features of this potential are the maximum at $\phi = 0$ and the minimum at $\phi = \phi_0$. These features can arise in models with spontaneous symmetry breaking and have been used before in models of supercooled inflation [49].

For the damping term Γ, we will take

$$\Gamma = \Gamma_0 \left(\frac{\phi}{\phi_0}\right)^b \left(\frac{T}{V(0)^{1/4}}\right)^c, \tag{76}$$

with different choices of b and c. In the first example, $b = c = 0$, which allows comparison with the analytic results of Taylor and Berera [27]. The second example uses $b = 2$ and $c = 0$ to isolate the effects of the ϕ dependence. The third case, $b = 2$, $c = -1$, is the damping term first calculated by Hosoya [5], and used in some consistent models of warm inflation, for example [14].

A. Homogeneous solution

The inflaton ϕ, entropy s and scale factor a satisfy Eqs. (1)–(9). The numerical solutions shown in Fig. 1 plot the potential and the radiation density ρ_r for the three models listed in Table I. The radiation density drops rapidly at first due to the expansion, but rises as the solutions approach the slow-roll regime.

The end of inflation happens when the slow-roll parameter $\epsilon = 1 + r$, and coincides with the equality between radiation and potential energy, $\rho_r = V$. It is convenient to relate the times of events during inflation to the time of the end of inflation, t_f, and so we plot $N(t)$, where

$$N = \ln(a_f/a), \tag{77}$$

along the horizontal axis.

The radiation temperatures at the end of inflation for the three models are given in Table I. These determine the Hubble length at the end of inflation (when the total energy density is $2\rho_r$),

$$cH_f^{-1} = 1.77 \times 10^{-25} g_*^{-1/2} T_{14}^{-2} \text{ m} \tag{78}$$

where $T_{14} = T_f/(10^{14} \text{ GeV})$. For comparison, a comoving scale which is 500 Mpc at the present epoch would have length

$$3.59 \times 10^{-2} T_{14}^{-1} \text{ m} \tag{79}$$

at the end of inflation.

In the models with $b = 2$, the inflation continues for larger values of ϕ than the model with $b = 0$. A consequence of this is that the reheat temperature is smaller in these models. This can have an important effect on the production of gravitinos [50].
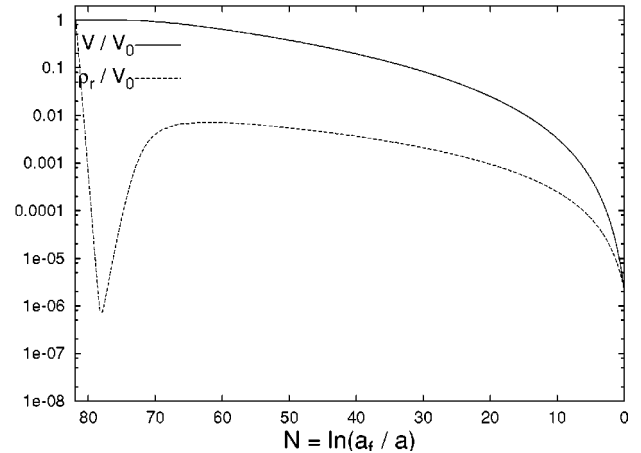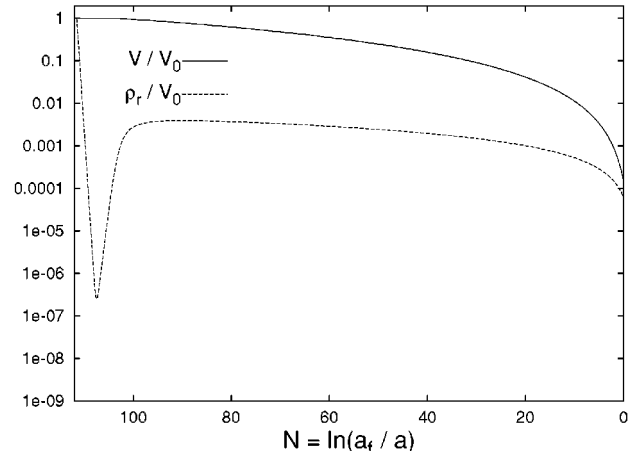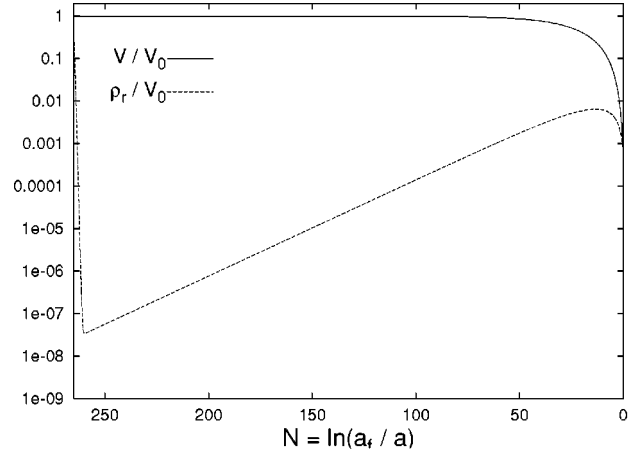

FIG. 1. The potential and the radiation density are plotted against N for the three models described in the text: (I) Constant damping (top), (II) damping depends on ϕ (middle), and (III) damping depends on ϕ and T (bottom). In all cases $r \approx 100$ at $N = 60$.

B. Perturbations on small scales

Our earlier discussion of thermal fluctuations ignored the effects of fluctuations in the metric and the damping term. The cosmological perturbation equations imply that the metric perturbation remains small and has little effect before the horizon crossing time t_H, when $k > aH$. However, for an

TABLE I. Three models with different damping terms. T_f is the temperature at the end of inflation and $g_* = 100$.

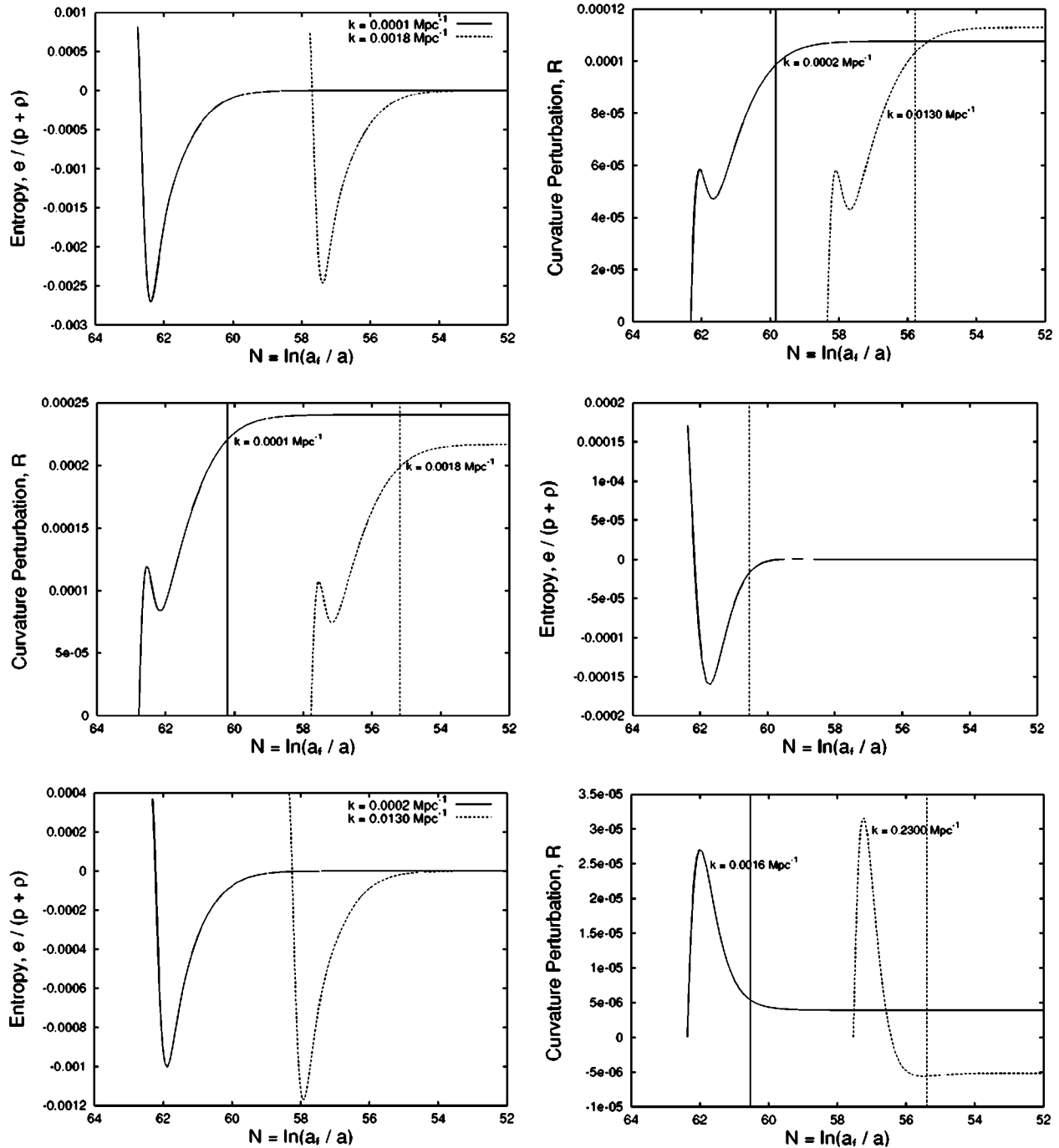| Model	b	c	λ	ϕ_0	$V(0)$	Γ_0 (GeV)	T_f (GeV)
I	0	0	2×10^{-15}	$0.3m_{Pl}$	$(4.48 \times 10^{14}\ \text{GeV})^4$	1.43×10^{13}	3.0×10^{13}
II	2	0	6.4×10^{-16}	$0.6m_{Pl}$	$(6.75 \times 10^{14}\ \text{GeV})^4$	4.3×10^{13}	2.1×10^{13}
III	2	-1	6.4×10^{-16}	$0.6m_{Pl}$	$(6.75 \times 10^{14}\ \text{GeV})^4$	4.3×10^{11}	1.1×10^{13}

FIG. 2. Evolution of the fluctuations for two different values of the comoving wave number k. The three rows represent the three models, constant damping (top), damping depends on ϕ (middle), damping depends on ϕ and T (bottom). The vertical lines indicates horizon crossing, $k = aH$.
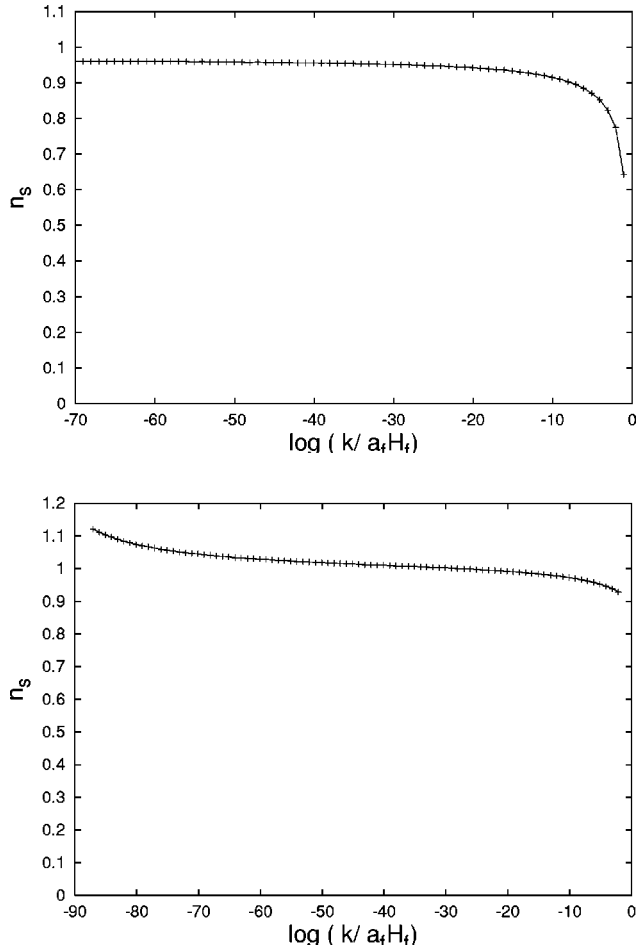
FIG. 3. The spectral index is plotted for the potential $\lambda(\phi^2 - \phi_0^2)^2/4$ with (top) constant damping and (bottom) damping depends on ϕ. The physical wave number k/a has been normalized by the Hubble length cH^{-1} at the end of inflation (see Sec. V A), $\ln[0.002 \text{ Mpc}^{-1}c/(a_fH_f)] = -\ln(g_*^{1/2}T_{14}) - 53.6$, where $T_{14} \times 10^{14}$ GeV is the temperature at the end of inflation.
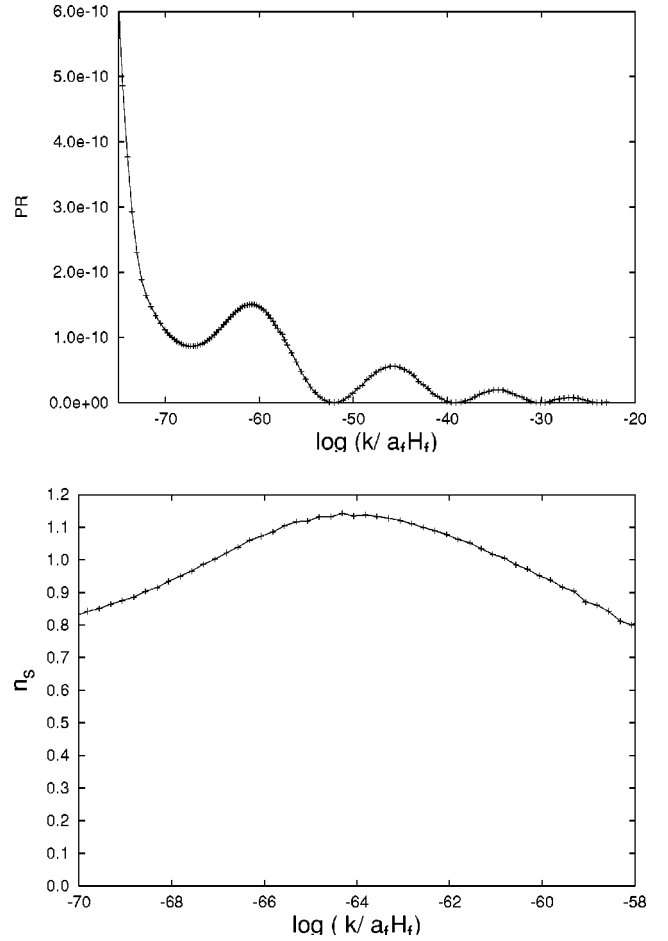

FIG. 4. The scalar fluctuation spectrum for the potential $\lambda(\phi^2 - \phi_0^2)^2/4$ with Hosoya's damping term. The spectral index over a limited range of k is also shown.

accurate treatment, and for dealing with fluctuations in the damping term, we must solve the perturbed Einstein equations and perturbed equation of motion.

Our numerical code simultaneously integrates the equations for the homogeneous background and the perturbation equations for a given wave number k. The integration begins with the homogeneous solution alone. Integration of the perturbation equations only commences at the freeze-out time t_F, defined by the condition $k = a(\Gamma H)^{1/2}$. This is done in order to ensure that the noise term in the Langevin equation which we saw earlier can be neglected. When the integration of the perturbation equations begins, the wavelength of the perturbation is much smaller than the horizon size, and horizon crossing occurs at a later time t_H. The integration is stopped when the radiation energy density reaches the same value as the vacuum energy density.

Initial conditions have to be set for the perturbed quantities at the freeze-out time when the integration of the perturbation equations begins. The initial inflaton fluctuations were given in Eq. (37) and Eq. (38). The initial value of the metric

perturbation is given by the perturbed Einstein equations (58) and (59),

$$\varphi = \frac{a^2}{k^2} 4\pi G \delta\rho, \qquad (80)$$

where $\delta\rho$ is given by Eq. (51). Interestingly, at the freeze-out time, $|\delta\rho_r| \approx |\delta\rho_\phi| \approx |(p+\rho)v|$.

The time evolution of the gauge invariant parameters describing the perturbations is shown in Fig. 2. The entropy perturbations are on the left. In the range $t_F < t < t_f$, the entropy perturbation is dominated by the radiation,

$$e \approx \delta(Ts). \qquad (81)$$

The entropy perturbation has a significant peak but always approaches zero after the perturbation crosses the horizon. In supercooled inflation the entropy perturbation is small compared to the curvature perturbation.

Figure 2 also shows the amplitude of the curvature fluctuations for the three damping models. The amplitudes begin relatively small and grow to approach a constant value very shortly after horizon crossing. The influence of the entropy perturbations can be seen in a small bump on the rising edge

of the plots. The final value of the curvature fluctuation in model I is within 5% of the analytic approximation (72).

The third model, where the damping term depends on the temperature, differs from the other two. In this case, the entropy fluctuation has a direct influence on the inflaton fluctuation. This causes the curvature perturbation to follow the entropy perturbation more closely, and even leads to the final value of the curvature perturbation changing sign. (Note that the root mean square value of the curvature perturbation is given by the modulus in this case.) The analytic approximation for the scalar perturbation amplitude is not applicable to this case and the numerical results confirm this.

C. Perturbation spectra

The spectral index for the first two models is plotted in Fig. 3. The results are in close agreement with the analytical formula (74). When the damping depends on the temperature, the spectrum is rather more complicated and has the oscillatory form shown in Fig. 4. The location of the zeros in this spectrum depends on the parameters Γ_0, λ and ϕ_0. At an extremum in the spectrum, the index changes from blue ($n_s > 1$) to red ($n_s < 1$) with increasing wave number k.

The observational situation is developing rapidly. The first year of cosmic microwave background anisotropy measurements made by the WMAP satellite suggest a spectral index $n_s < 1$ [51]. However, the combination of the microwave data with large scale structure data [52] produce a quite different conclusion, that the spectral index runs from blue ($n_s > 1$) to red ($n_s < 1$) with increasing wave number k. Peiris et al. [52] give their best estimate of the spectral index as $n_s = 1.2$ and $dn_s / d \ln k = -0.077$ at $k = 0.002$ Mpc^{-1}.

The possibility of a spectrum which runs from blue to red is particularly interesting because it is not commonly seen in inflationary models, which typically predict red spectra. Examples of inflation with blue spectra do exist however. The hybrid inflation models, which have two scalar fields, are examples [53,54].

The numerical results show clearly that warm inflation can produce a spectrum of density perturbations with blue runs to red behavior. This behavior is seen quite generally for potentials with spontaneous symmetry breaking when Γ has the form Eq. (76) with $b = 2$ and $c = 0$, and is a possibility when $b = 2$ and $c = -1$.

VI. CONCLUSION

The numerical results presented here show some of the possibilities which we might expect from scalar density fluctuations in the warm inflationary scenario. We have focused on the effects that result from various forms of the damping coefficient Γ. Our main results are as follows:

(1) The nonstochastic evolution begins at the freeze-out time t_F when the fluctuations in the inflaton are given by Eqs. (37) and (38).

(2) The spectral index when $\Gamma \equiv \Gamma(\phi)$ is given by three slow-roll parameters Eq. (74). In this case warm inflation introduces only one extra slow-roll parameter.

(3) The combination of ''spontaneous symmetry breaking'' potentials and $\Gamma \equiv \Gamma(\phi)$ typically leads to $n_s > 1$ on long scales and $n_s < 1$ on short scales.

Emphasis has been placed here on methods for solving the perturbation equations rather than constructing a realistic inflaton potential. However, the equations and the numerical integration have been set up in a way which makes them easily adaptable to different models. The inflaton potentials may include thermal corrections and different forms of the damping terms may also be taken into account, for example using the friction coefficients calculated in Refs. [13,17,19].

A better understanding of non-equilibrium thermodynamics will enable us to calculate the friction term in the inflaton equation for a wider range of conditions than is possible at present. Models of warm inflation are restricted by consistency requirements [13,15,26], many of which are a result of assumptions made due to the difficulties in calculating far from equilibrium effects. Our results suggest that warm inflation can easily produce a spectrum of density fluctuations that fits the observational data and that this is a direction worth pursuing.

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