Time delay interferometry with moving spacecraft arrays

Massimo Tinto,* F. B. Estabrook,† and J. W. Armstrong‡
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109, USA
(Received 3 October 2003; published 5 April 2004)

Space-borne interferometric gravitational wave detectors, sensitive in the low-frequency (millihertz) band, will fly in the next decade. In these detectors the spacecraft-to-spacecraft light-travel-times will necessarily be unequal, time varying, and (due to aberration) have different time delays on up and down links. The reduction of data from moving interferometric laser arrays in solar orbit will in fact encounter nonsymmetric up- and down-link light time differences that are about 100 times larger than has previously been recognized. The time-delay interferometry (TDI) technique uses knowledge of these delays to cancel the otherwise dominant laser phase noise and yields a variety of data combinations sensitive to gravitational waves. Under the assumption that the (different) up- and down-link time delays are constant, we derive the TDI expressions for those combinations that rely only on four interspacecraft phase measurements. We then turn to the general problem that encompasses time dependence of the light-travel times along the laser links. By introducing a set of noncommuting time-delay operators, we show that there exists a quite general procedure for deriving generalized TDI combinations that account for the effects of time dependence of the arms. By applying our approach we are able to re-derive the ''flex-free'' expression for the unequal-arm Michelson combinations X_1, and obtain the generalized expressions for the TDI combinations called relay, beacon, monitor, and symmetric Sagnac.

DOI: 10.1103/PhysRevD.69.082001 PACS number(s): 04.80.Nn, 07.60.Ly, 95.55.Ym

I. INTRODUCTION

Future space-borne gravitational wave (GW) observatories, such as the Laser Interferometer Space Antenna (LISA) [1,2], will have sensitivity in the low-frequency band and will use time-delay interferometry (TDI) to cancel laser phase noise. All the original papers on TDI considered a configuration of three spacecraft interchanging coherent laser beams, and tacitly or explicitly assumed the array to be at rest in an inertial system. TDI was treated in Euclidean 3-space with a universal time, in which the velocity of light is c and isotropic. Recipes were given for combining data (time series) separately recorded at the various spacecraft, delayed by transit times calculated from the inter-spacecraft separations $L_i (i=1,2,3)$, in order to remove the otherwise overwhelming phase noise of the laser sources [3–9]. The aim is the possible detection of incident gravitational waves of galactic or cosmic origin.

The LISA mission [2] will have three spacecraft orbiting the Sun in a triangular array with the $L_i \simeq 5\times 10^6$ km, and GW detection capability in the band $10^{-4}-1$ Hz. Several TDI Michelson-like and Sagnac-like reduced laser-phase-noise-free data streams will have different responses to secondary phase noise sources and to two polarizations of incoming gravitational waves from different directions. A recent study of a linear array of three spacecraft in a single solar orbit (SyZyGy) [10] uses a TDI combination sensitive to a single polarization of incident gravitational waves, and two others sensitive solely to secondary system noises.

In an important development, Shaddock [11] noticed that the rotational motion of an array results in a difference of the light travel times in the two directions around a Sagnac circuit. Two time delays along each arm must be used, say L_i and L_i', for clockwise or counterclockwise propagation as they enter in any of the TDI combinations. Shaddock emphasized the need for careful distinguishing of primed and unprimed delays in the TDI combinations for Michelson-like combinations and, to eliminate laser noise from the Sagnac-type combinations when the array is moving, he presented new TDI variables related to those originally given by being ''double differenced.''

Cornish and Hellings [12] also considered the effect of rotation of the LISA triangle around its centroid on the TDI combinations, and reported the new data combinations. Summers [13] and Cornish and Hellings [12] further pointed out that the LISA array is not rigid, that L_i and L_i' not only differ from one another but can be time dependent (they ''flex''), and that again the laser phase noise (at least with present laser stability requirements) can enter at a level above the secondary noises. For LISA, and assuming $\dot{L}_i \simeq 10$ m/sec [14], they estimated the magnitude of the remaining frequency fluctuations from the laser to be about 30 times larger than the level set by the secondary noise sources in the center of the frequency band. This may not be as serious a problem with SyZyGy [10].

Finally Shaddock *et al.* [1] addressed the ''flexing'' complication by showing that it becomes of higher order if the sequence of various time delays in the new doubly differenced Sagnac combinations is respected in the TDI recipe, and they introduced a new doubly differenced Michelson-type TDI combination to achieve the same result. They stressed that although all these combinations are consider-

*Electronic address: Massimo.Tinto@jpl.nasa.gov; Also at Space Radiation Laboratory, California Institute of Technology, Pasadena, CA 91125.
†Electronic address: Frank.B.Estabrook@jpl.nasa.gov
‡Electronic address: John.W.Armstrong@jpl.nasa.gov

ably more complicated than those originally given for a non-moving array, and their GW response functions are similarly complex, the final sensitivity—calculated from GW signal strengths and secondary phase noises—is unaffected.

All the analyses above, however, assumed the clocks onboard the three LISA spacecraft to be synchronized to each other in a reference frame attached to the LISA array. It is well known [15], however, that the spacetime geometry—here the Sagnac effect—prevents the self-consistent synchronization of a network of clocks by the transmission of electromagnetic signals in a rotating reference frame. This implies that the time adopted by the LISA onboard clocks and the time used for TDI has to be referenced to an inertial reference frame and that the onboard LISA receivers have to properly convert time information received from Earth to the time in this inertial reference frame. Within this frame, which we can assume to be Solar System Barycentric (SSB), the differences between back-forth delay times that occur are in fact thousands of kilometers, very much larger than has been previously recognized by us or others. The problem is not rotation *per se*, but rather aberration due to motion and changes of orientation in the SSB frame.

In Sec. II we further discuss the need for synchronizing the LISA clocks with respect to a common inertial reference frame (SSB), and the resulting GW response transfer functions. We turn in Sec. III to the derivation of the four-link TDI combinations valid for constant time delays. We first obtain the ''unequal-arm Michelson'' response X as an example of how time-delay operators can be used for deriving TDI data combinations. The operator formalism for TDI was introduced by Dhurandhar *et al.* [9]. We use it in conjunction with the usual subscripted delay notation to achieve a systematic understanding of the ''relay'' (U,V,W), ''beacon'' (P,Q,R), and ''monitor'' (E,F,G) combinations. With laser stabilization at a level somewhat improved from that used in the original LISA study [2], these combinations, now involving different up- and down-link delays, will satisfy sensitivity requirements.

In Sec. IV, however, we go on to use delay time/operator notation to derive ''second generation'' TDI combinations, which account for both the inequality and time dependence of the back/forth optical paths. Following Shaddock *et al.* [1], the resulting doubly differenced combinations, immune to first order shearing (flexing, or constant rate of change of delay times) are denoted X_i, U_i, P_i, E_i, $i=1,2,3$. All these new combinations suppress the nominal LISA laser phase noise to levels lower than those of the secondary (proof-mass and optical-path) noise sources, and their gravitational wave sensitivities are the same as previously computed for the stationary case. For completeness, we calculate the remaining shearing effect on the doubly differenced versions of the system-noise-monitoring combination ζ, denoted ζ_1, ζ_2, ζ_3. Laser noise enters these combinations multiplied by $\sin^2(\pi fL)$, where f is a Fourier frequency in the LISA band 10^{-4}–1 Hz. We plot the laser noise in ζ_i for the nominal LISA system and show it as a result also to be below the level of secondary noises.

II. ABERRATION, TIME DELAYS AND SYNCHRONIZATION OF THE LISA CLOCKS

The kinematics of the LISA and SyZyGy orbits brings in the effects of motion at several orders of magnitude larger than any previous papers on TDI have addressed. The instantaneous rotation axis of LISA, and the SyZyGy array, both swing about the Sun at 30 km/sec, and on any leg the transit times of light signals in opposing directions, say L_i and L_i' $(c=1)$, can differ by as much as 1000 km. Aberration due to LISA's orbit about the Sun dominates its instantaneous rotation. This observation reinforces the requirement that the new TDI combinations of Secs. III and IV *must* be used. Indeed, L_i and L_i' interchange periodically and so are also time dependent; this effect is, however, of order 0.1 m/sec and is dominated by the effect of shearing (''flexing'') already recognized.

This large motional effect has been overlooked because intuitively up/down laser links between two spacecraft moving inertially on parallel geodesics certainly appear symmetric in a co-moving frame. The spacecraft are then seen ''at rest'' and the elapsed light times, or delays, in either direction are the same. Consider, however, an inertial frame in which two spacecraft are moving with speed V along a line, with constant separation. The times of transit of a photon from one to the other, forward or back, clearly must differ by $2VL/c$. This is just an extreme case of aberration. There is no paradox. We have taken the speed of light to be c and isotropic in *both* frames, and special relativity has taught us that is fine so long as we properly re-synchronize the clocks that we use as time coordinates when changing frame (using light beams). The spatial and temporal separations of two successive events along the null world line of a ray of light *depend on choice of frame*. Rays traveling in opposite senses between two moving spacecraft yield different separations in all frames except the co-moving one.

An orbiting array is best described *not* by attempting a sequence of co-moving tangent ''rest'' frames, but rather in the barycentric non-rotating Euclidean frame moving with the Sun (of course we ignore tiny general relativistic distortions). The usual time coordinate of positional solar system astronomy, in principle, uses clocks such that c is isotropic. In LISA and SyZyGy data at the three spacecraft will undoubtedly be taken and time tagged in this solar system barycentric frame, and all the up/down delay times used in the new TDI combinations must be calculated from the coordinates of emission and reception events in the SSB inertial frame. (This is exactly parallel to the time synchronization problem, and its resolution, that has been met by the designers of the GPS satellite array [15] in geocentric orbit.)

Since the motion of the LISA array around the Sun introduces a difference between (and a time dependence in) the co-rotating and counter-rotating light travel times, the correct expressions for the GW contributions to the various first-generation TDI combinations will differ from the expressions valid for a stationary array [4]. The magnitude of the corrections introduced by the inequality of the light-travel times is proportional to the product between the time derivative of the GW amplitude and the differences between the actual

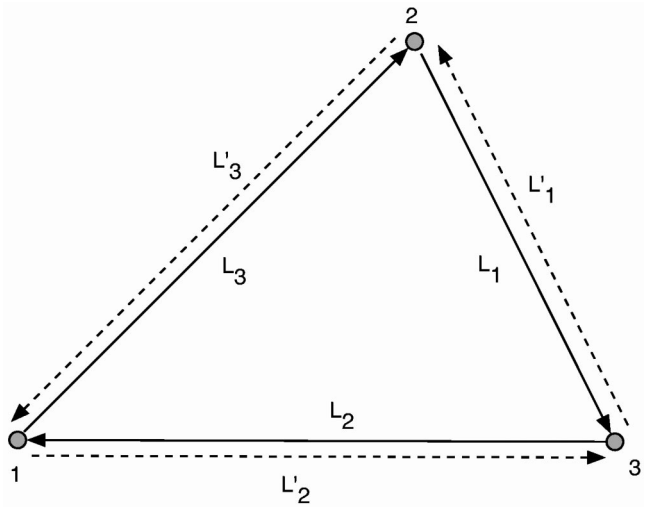

FIG. 1. Schematic diagram of LISA configurations involving six laser beams. Optical path delays taken in the counter-clockwise sense are denoted with a prime, while unprimed delays are in the clockwise sense. See the text for details.

light travel times. At one mHz, for instance, the correction to the expression of the signal valid for a stationary array is five orders of magnitude smaller. Since the amplitude of this correction scales linearly with the Fourier frequency, we can completely disregard this effect (and also the weaker effect due to the time dependence of the light travel times) over the entire LISA band.

It is clear, however, that over many months of continuous observation of a quasi-periodic signal, the TDI responses have to account for the motion of the array around the Sun (and relative to the GW source), which introduces secular modulations in the phase, frequency, and amplitude of the GW responses [16,17].

III. THE FOUR-LINK TDI COMBINATIONS: CONSTANT TIME DELAYS

The notation we will adopt is the same as used in the paper by Shaddock *et al.* [1] (i.e. it is *different* from the original TDI notation, e.g. Ref. [5].) We distinguish time-of-flight delays by denoting with a prime those taken in the counter-clockwise sense and unprimed those taken in the clockwise sense (see Fig. 1).

There are six beams exchanged between the LISA spacecraft, together with the six phase measurements s_{ij} $(i,j=1,2,3)$ recorded when each transmitted beam is mixed with the laser light of the receiving optical bench. The phase fluctuations from the six lasers, which need to be canceled, can be represented by six random processes p_{ij}, where p_{ij} is the phase of the laser in spacecraft j on the optical bench facing spacecraft i. In what follows we assume the center frequencies of the lasers are all equal, and denote it with ν_0. Explicitly, s_{23} is the one-way phase shift measured at spacecraft 3, coming from spacecraft 2, along arm 1. The laser phase noise in s_{23} is $p_{32}(t-L_1)-p_{23}(t)$, where we take $c=1$, so that L_1 is the light time in the direction from spacecraft 2 to spacecraft 3. Similarly, s_{32} is the phase shift measured on arrival at spacecraft 2 along arm $1'$ of a signal transmitted from spacecraft 3. The laser phase noise in s_{32} is $p_{23}(t-L_1')-p_{32}(t)$, where L_1' is the light time in the sense from 3 to 2 along arm $1'$. For the further delays used in the TDI combinations we use the same conventions, being careful to distinguish light travel along arms with primes or not, depending on the sense of the measurement. For example, our notation for delaying the time series $s_{32}(t)$ by the clockwise light time in arm 1 would be $s_{32,1}$ while delaying by the counterclockwise light time in arm $1'$ would be $s_{32,1'}$. As before, we denote six further data streams, τ_{ij} $(i,j=1,2,3)$, as the intra-spacecraft metrology data used to monitor the motion of the two optical benches and the relative phase fluctuations of the two lasers on each of the three spacecraft. The phase fluctuations of the lasers and optical benches enter into the measurements s_{ij} and τ_{ij} in the following form [6] (*henceforth disregarding contributions from other noise sources and the gravitational wave signal*)

$$s_{31}=[p_{13}-\nu_0\hat{n}_2\cdot\vec{\Delta}_{13}]_{,2}-[p_{31}+\nu_0\hat{n}_2\cdot\vec{\Delta}_{31}], \tag{1}$$

$$s_{21}=[p_{12}+\nu_0\hat{n}_3\cdot\vec{\Delta}_{12}]_{,3'}-[p_{21}-\nu_0\hat{n}_3\cdot\vec{\Delta}_{21}], \tag{2}$$

$$\tau_{31}=p_{21}-p_{31}-2\nu_0\hat{n}_3\cdot\vec{\Delta}_{21}+\mu_1, \tag{3}$$

$$\tau_{21}=p_{31}-p_{21}+2\nu_0\hat{n}_2\cdot\vec{\Delta}_{31}+\mu_1. \tag{4}$$

In the above equations we have denoted with μ_i the phase fluctuations introduced by the optical fibers used for exchanging the laser beams between adjacent benches, and with the vector random processes $\vec{\Delta}_{ij}$ the phase fluctuations introduced by the mechanical vibrations of the optical benches.

In order to simplify the derivation of the new TDI combinations, we note that by subtracting Eq. (3) from Eq. (4) we can rewrite the resulting expression (and those obtained from it by permutation of the spacecraft indices) in the following form:

$$\tfrac{1}{2}[\tau_{21}-\tau_{31}]=[p_{31}+\nu_0\hat{n}_2\cdot\vec{\Delta}_{31}]-[p_{21}-\nu_0\hat{n}_3\cdot\vec{\Delta}_{21}], \tag{5}$$

$$\tfrac{1}{2}[\tau_{32}-\tau_{12}]=[p_{12}+\nu_0\hat{n}_3\cdot\vec{\Delta}_{12}]-[p_{32}-\nu_0\hat{n}_1\cdot\vec{\Delta}_{32}], \tag{6}$$

$$\tfrac{1}{2}[\tau_{13}-\tau_{23}]=[p_{23}+\nu_0\hat{n}_1\cdot\vec{\Delta}_{23}]-[p_{13}-\nu_0\hat{n}_2\cdot\vec{\Delta}_{13}]. \tag{7}$$

If we now define the following combinations of laser and optical bench noises appearing in Eqs. (1)–(7) [9]:

$$\phi_1^*\equiv[p_{31}+\nu_0\hat{n}_2\cdot\vec{\Delta}_{31}], \tag{8}$$

$$\phi_1\equiv[p_{21}-\nu_0\hat{n}_3\cdot\vec{\Delta}_{21}], \tag{9}$$

together with those obtained by permuting the spacecraft indices, it is possible to reduce the derivation of the new TDI combinations to the equivalent problem of removing the

three random processes, ϕ_1, ϕ_2, and ϕ_3, from the following six linear combinations of the one-way measurements s_{ij} and τ_{ij}:

$$\eta_{21} \equiv s_{21} - \tfrac{1}{2}[\tau_{32} - \tau_{12}]_{,3'} = \phi_{2,3'} - \phi_1,$$

$$\eta_{31} \equiv s_{31} + \tfrac{1}{2}[\tau_{21} - \tau_{31}] = \phi_{3,2} - \phi_1, \tag{10}$$

$$\eta_{12} \equiv s_{12} + \tfrac{1}{2}[\tau_{32} - \tau_{12}] = \phi_{1,3} - \phi_2,$$

$$\eta_{32} \equiv s_{32} - \tfrac{1}{2}[\tau_{13} - \tau_{23}]_{,1'} = \phi_{3,1'} - \phi_2, \tag{11}$$

$$\eta_{13} \equiv s_{13} - \tfrac{1}{2}[\tau_{21} - \tau_{31}]_{,2'} = \phi_{1,2'} - \phi_3,$$

$$\eta_{23} \equiv s_{23} + \tfrac{1}{2}[\tau_{13} - \tau_{23}] = \phi_{2,1} - \phi_3. \tag{12}$$

A. The unequal-arm Michelson

Here we derive the unequal-arm Michelson combination X, valid for the rigid-rotation case. We use X as an example for deriving TDI data combinations by using an alternative and powerful method based on the use of properly defined time-delay operators.

The X combination relies on the four measurements η_{12}, η_{21}, η_{13}, and η_{31}. Note that the two combinations $\eta_{21} + \eta_{12,3'}$, $\eta_{31} + \eta_{13,2}$, which represent the two synthesized two-way data measured onboard spacecraft 1, can be written in the following form:

$$\eta_{21} + \eta_{12,3'} = (D_{3'}D_3 - I)\phi_1, \tag{13}$$

$$\eta_{31} + \eta_{13,2} = (D_2 D_{2'} - I)\phi_1, \tag{14}$$

where we have denoted with D_j the time-delay operator that shifts by L_j the function it is applied to, and with I the identity operator. Note that in the stationary case any pairs of these operators commute, i.e. $D_i D_{j'} - D_{j'} D_i = 0$ (while they do not when the delays are functions of time [12,1]).

From Eqs. (13), (14) it is easy to derive the following expression for X, by requiring the elimination of ϕ_1:

$$\begin{aligned} X &= [D_2 D_{2'} - I](\eta_{21} + \eta_{12,3'}) - [(D_{3'}D_3 - I)](\eta_{31} + \eta_{13,2}) \\ &= [(\eta_{31} + \eta_{13,2}) + (\eta_{21} + \eta_{12,3'})_{,2'2}] - [(\eta_{21} + \eta_{12,3'}) \\ &\quad + (\eta_{31} + \eta_{13,2})_{,33'}]. \end{aligned} \tag{15}$$

After replacing Eqs. (10),(11),(12) into Eq. (15), we obtain the final expression for X valid in the case of rigid rotation of the LISA array [11]

$$\begin{aligned} X &= [(s_{31} + s_{13,2}) + (s_{21} + s_{12,3'})_{,2'2}] - [(s_{21} + s_{12,3'}) \\ &\quad + (s_{31} + s_{13,2})_{,33'}] + \tfrac{1}{2}[(\tau_{21} - \tau_{31})_{,2'233'} \\ &\quad - (\tau_{21} - \tau_{31})_{,33'} - (\tau_{21} - \tau_{31})_{,2'2} + (\tau_{21} - \tau_{31})]. \end{aligned} \tag{16}$$

As pointed out in [13] and [1], Eq. (15) shows that X is the difference of two sums of phase measurements, each corresponding to a specific light path from a laser onboard spacecraft 1 having phase noise ϕ_1. The first square-bracket term in Eq. (15) represents a synthesized light-beam transmitted from spacecraft 1 and made to bounce once at spacecrafts 3 and 2, respectively. The second square-bracket term instead correspond to another beam also originating from the same laser, experiencing the same overall delay as the first beam, but bouncing off spacecraft 2 first and then spacecraft 3. When they are recombined they will cancel the laser phase fluctuations exactly, having both experienced the same total delays (assuming stationary spacecraft).

B. The relay

The TDI "relay" configurations were called (U,V,W) [Eq. (A4) of [5]]. In what follows, let us consider, as a specific example, the U combination, which has to rely only on the four measurements η_{31}, η_{12}, η_{32} and η_{23}. The idea we will follow for identifying the expression for U is to select combinations of some of these four measurements that contain only one phase noise. By then applying iteratively the time-delay procedure we introduced for the X combination, we will be able to remove all the phase noises $\phi_i, i = 1,2,3$. Note that the obvious combinations that contain only one of the three phase noises ϕ_i are the synthesized two-way Doppler data measured onboard spacecrafts 2 and 3. They in fact contain only the phase noises ϕ_2 and ϕ_3, respectively. Since the remaining two measurements η_{12} and η_{31} can be combined in such a way as to eliminate the phase noise ϕ_1, we can start with the following set of three data combinations:

$$\eta_{12} + \eta_{31,3} = D_3 D_2 \phi_3 - \phi_2, \tag{17}$$

$$\eta_{32,1} + \eta_{23} = [D_1 D_{1'} - I]\phi_3, \tag{18}$$

$$\eta_{23,1'} + \eta_{32} = [D_{1'}D_1 - I]\phi_2. \tag{19}$$

It is then easy to see that the expression for U is given by the following linear combination of the properly delayed Eqs. [(17),(18), and (19)]:

$$\begin{aligned} U &= [D_{1'}D_1 - I](\eta_{12} + \eta_{31,3}) + (\eta_{23,1'} + \eta_{32}) \\ &\quad - D_3 D_2(\eta_{32,1} + \eta_{23}) \\ &= (\eta_{12,11'} + \eta_{31,311'}) - (\eta_{12} + \eta_{31,3}) + (\eta_{23,1'} \\ &\quad + \eta_{32}) - (\eta_{32,123} + \eta_{23,23}), \end{aligned} \tag{20}$$

which, in terms of the one-way measurements s_{ij} and τ_{ij}, becomes

$$\begin{aligned} U &= s_{31,311'} - s_{31,3} + s_{12,11'} - s_{12} + s_{23,1'} + s_{32} - s_{32,123} - s_{23,23} \\ &\quad + \frac{1}{2}[(\tau_{21} - \tau_{31})_{,311'} - (\tau_{21} - \tau_{31})_{,3} - (\tau_{32} - \tau_{12}) \\ &\quad + (\tau_{32} - \tau_{12})_{,11'} + (\tau_{13} - \tau_{23})_{,1'123} - (\tau_{13} - \tau_{23})_{,23}] \end{aligned} \tag{21}$$

with V, W obtained by cycling the spacecraft indices.

C. The beacon

In the ''beacon'' combination, one spacecraft transmits (only) to the other two while those other two exchange one-way beams as usual. These were called the (P,Q,R) combinations, depending on which spacecraft was the transmit-only element of the array [5]. In order to derive the expression for P, which involves only the four data streams η_{12}, η_{13}, η_{32}, and η_{23}, we will proceed according to the above considerations, and use in this case the following data combinations:

$$\eta_{12,2'}-\eta_{13,3}=D_3\phi_3-D_{2'}\phi_2, \tag{22}$$

$$\eta_{32,1}+\eta_{23}=[D_1D_{1'}-I]\phi_3, \tag{23}$$

$$\eta_{23,1'}+\eta_{32}=[D_{1'}D_1-I]\phi_2. \tag{24}$$

By taking advantage of the commutativity of the delay operators in this constant time delay case, it is easy to see that the expression for P is given by the following linear combination of the properly delayed equations [(22),(23), and (24)]:

$$\begin{aligned}P=&D_3(\eta_{32,1}+\eta_{23})-D_{2'}(\eta_{23,1'}+\eta_{32})\\&-[D_{1'}D_1-I](\eta_{12,2'}-\eta_{13,3})\\=&(\eta_{32,13}+\eta_{23,3})-(\eta_{23,1'2'}+\eta_{32,2'})\\&-(\eta_{12,2'11'}-\eta_{13,311'})+(\eta_{12,2'}-\eta_{13,3}).\end{aligned} \tag{25}$$

Equation (25) can be rewritten in terms of the one-way measurements s_{ij}, τ_{ij}

$$\begin{aligned}P=&s_{12,2'}-s_{13,3}-s_{32,2'}+s_{23,3}+s_{32,13}-s_{23,1'2'}+s_{13,311'}\\&-s_{12,2'11'}+\tfrac{1}{2}[(\tau_{21}-\tau_{31})_{,2'3}-(\tau_{21}-\tau_{31})_{,11'2'3}\\&+(\tau_{32}-\tau_{12})_{,2'}-(\tau_{32}-\tau_{12})_{,11'2'}+(\tau_{13}-\tau_{23})_{,3}\\&-(\tau_{13}-\tau_{23})_{,11'3}],\end{aligned} \tag{26}$$

with Q, R obtained by cycling the spacecraft indices in Eq. (26).

D. The monitor

Similarly, there are three combinations where one spacecraft is listen-only [5]. In order to derive these ''monitor'' combinations (E,F,G) [Eq. (A1) of [5]], let us consider the following combinations of the four data streams that enter into E:

$$\eta_{31}-\eta_{21}=D_2\phi_3-D_{3'}\phi_2, \tag{27}$$

$$\eta_{32,1}+\eta_{23}=[D_1D_{1'}-I]\phi_3, \tag{28}$$

$$\eta_{23,1'}+\eta_{32}=[D_{1'}D_1-I]\phi_2. \tag{29}$$

Similarly to the derivations made for the two previous combinations, it is easy to see that the expression for E is given by the following linear combination of the properly delayed equations [(27),(28), and (29)]:

$$\begin{aligned}E=&D_2(\eta_{32,1}+\eta_{23})-D_{3'}(\eta_{23,1'}+\eta_{32})\\&-[D_{1'}D_1-I](\eta_{31}-\eta_{21})\\=&(\eta_{32,12}+\eta_{23,2})-(\eta_{23,1'3'}+\eta_{32,3'})\\&-(\eta_{31,11'}-\eta_{21,1'1})+(\eta_{31}-\eta_{21}),\end{aligned} \tag{30}$$

which in terms of the one-way measurements s_{ij} and τ_{ij} becomes

$$\begin{aligned}E=&s_{32,12}+s_{23,2}-s_{23,1'3'}-s_{32,3'}+s_{21,1'1}-s_{31,11'}-s_{21}+s_{31}\\&+\tfrac{1}{2}[(\tau_{21}-\tau_{31})-(\tau_{21}-\tau_{31})_{,11'}+(\tau_{32}-\tau_{12})_{,3'}\\&-(\tau_{32}-\tau_{12})_{,11'3'}+(\tau_{13}-\tau_{23})_{,2}-(\tau_{13}-\tau_{23})_{,11'2}]\end{aligned} \tag{31}$$

with F, G obtained by cycling the indices.

E. The ζ combinations

In all the above we have used the same symbol (e.g. X for the unequal-arm Michelson combination) for both the rotating (i.e. constant delay times) and stationary cases. This emphasizes that, for these TDI combinations, the forms of the equations do not change going from systems at rest to the moving or rotating case. One need only distinguish between the time-of-flight variations in the clockwise and counterclockwise senses (primed and unprimed delays).

In the case of an array at rest there is one symmetric data combination that cancels exactly all laser noise and optical bench motions and has the property that each of the η_{ij} enters exactly once and is lagged by exactly one of the one-way light times. We called this ζ [[5], Eq. (3.5)] and showed how to take advantage of its relative immunity to GWs in order to assess on-orbit instrumental noise performance and distinguish instrumental noise from a confusion-limited background [7]. Although now the rotation of the array breaks the symmetry and therefore the uniqueness of a ''ζ-like'' combination, it has been shown ([11,12]) that there still exist three generalized TDI laser-noise-free data combinations that have properties very similar to ζ, and which can be used for the same scientific purposes. Here we derive these combinations, which we call $(\zeta_1,\zeta_2,\zeta_3)$, by applying our time-delay operator approach. As we will see in the following section, our derivation will automatically identify the ''correct'' order of the delays that has to be applied to the one-way data. In other words, the expressions lead to an order of time delays such that even with shearing the remaining laser noise is below the level identified by the secondary noise sources. ζ_1 will not have to be further generalized.

Let us consider the following combination of the η_{ij} measurements:

$$\eta_{13,3'}-\eta_{23,3'}+\eta_{21,1}=[D_{3'}D_{2'}-D_1]\phi_1, \tag{32}$$

$$\eta_{31,1'}-\eta_{32,2}+\eta_{12,2}=[D_3D_2-D_{1'}]\phi_1, \tag{33}$$

where we have used the commutativity property of the delay operators in order to cancel the ϕ_2 and ϕ_3 terms. Since both

sides of the two equations above contain only the ϕ_1 noise, ζ_1 is found by the following expression:

$$\zeta_1=[D_{3'}D_{2'}-D_1](\eta_{31,1'}-\eta_{32,2}+\eta_{12,2})-[D_2D_3-D_{1'}] \times(\eta_{13,3'}-\eta_{23,3'}+\eta_{21,1}). \quad (34)$$

In terms of the one-way measurements s_{ij} and τ_{ij}, Eq. (34) becomes

$$\begin{aligned}\zeta_1=&[s_{31,1'}-s_{32,2}+s_{12,2}]_{,2'3'}-[s_{13,3'}-s_{23,3'}+s_{21,1}]_{,32}\\&-[s_{31,1'}-s_{32,2}+s_{12,2}]_{,1}+[s_{13,3'}-s_{23,3'}+s_{21,1}]_{,1'}\\&+\tfrac{1}{2}[(\tau_{32}-\tau_{12})_{,22'3'}-(\tau_{32}-\tau_{12})_{,21}+(\tau_{32}-\tau_{12})_{,13'32}\\&-(\tau_{32}-\tau_{12})_{,13'1'}+(\tau_{13}-\tau_{23})_{,22'3'1'}-(\tau_{13}-\tau_{23})_{,211'}\\&+(\tau_{13}-\tau_{23})_{,233'}-(\tau_{13}-\tau_{23})_{,3'1'}+(\tau_{21}-\tau_{31})_{,22'33'}\\&-(\tau_{21}-\tau_{31})_{,11'}] \end{aligned} \quad (35)$$

together with its cyclic permutations. [This expression for ζ_1 was given (but not derived) in [11] and independently by [12].] If the light times in the arms are equal in the clockwise and counterclockwise senses (e.g. no rotation) there is no distinction between primed and unprimed delay times. In this case, ζ_1 is related to our original symmetric Sagnac ζ by $\zeta_1=\zeta_{,23}-\zeta_{,1}$. Thus for the practical LISA case (arm length difference $<2\%$), the SNR of ζ_1 will be the same as the SNR of ζ.

IV. THE SECOND-GENERATION TDI COMBINATIONS

Generalizations of the original unequal-arm Michelson, (X,Y,Z), and Sagnac, (α,β,γ) TDI combinations to an array with systematic spacecraft velocities, showing that they effectively cancel all laser phase noises, have been derived in [1]. Here we complete that set of TDI combinations by deriving generalized expressions for the ''relay,'' ''beacon,'' and ''monitor'' combinations that are unaffected by the rotation and time dependence of the light-path delays. These TDI combinations rely only on four of the six possible one-way measurements LISA will make, and for this reason they add robustness and trade-off options to the LISA design. Like the unequal-arm Michelson combination X_1 [1], these new combinations involve the four one-way inter-spacecraft measurements at 16 different times.

The *order* of the time-delay operators now becomes important for laser phase terms. The operators can no longer be permuted freely to show cancellation of laser noises in the TDI combinations (they no longer commute). In order to derive the new, ''flex-free'' relay, beacon, and monitor combinations we will start by taking specific combinations of the one-way data entering in each of the expressions derived in the previous section for the rigid-rotation case. These combinations are chosen in such a way to retain only one of the three noises $\phi_i, i=1,2,3$ if possible. In this way we can then implement an iterative procedure based on the use of these basic combinations and of time-delay operators, to cancel the laser noises after dropping terms that are quadratic in $\dot{L}/c$ or linear in the accelerations. This iterative time-delay method, to first order in the velocity, is illustrated abstractly as follows. Given a function of time $\Psi=\Psi(t)$, time delay by L_i is now denoted either with the standard comma notation or by applying the delay operator D_i introduced in the previous section

$$D_i\Psi=\Psi_{,i}\equiv\Psi[t-L_i(t)]. \quad (36)$$

We then impose a second time delay $L_j(t)$:

$$\begin{aligned}D_jD_i\Psi=\Psi_{;ij}&\equiv\Psi\{t-L_j(t)-L_i[t-L_j(t)]\}\\&\simeq\Psi[t-L_j(t)-L_i(t)+\dot{L}_i(t)L_j]\\&\simeq\Psi_{,ij}+\dot{\Psi}_{,ij}\dot{L}_iL_j.\end{aligned} \quad (37)$$

A third time delay $L_k(t)$ gives

$$\begin{aligned}D_kD_jD_i\Psi=\Psi_{;ijk}&=\Psi(t-L_k(t)-L_j[t-L_k(t)]\\&\quad-L_i\{t-L_k(t)-L_j[t-L_k(t)]\})\\&\simeq\Psi_{,ijk}+\dot{\Psi}_{,ijk}[\dot{L}_i(L_j+L_k)+\dot{L}_jL_k]\end{aligned} \quad (38)$$

and so on, recursively; each delay generates a first-order correction proportional to its rate of change times the sum of all delays coming after it in the subscripts. Commas have now been replaced with semicolons [1], to remind us that we consider moving arrays. When the sum of these corrections to the terms of a data combination vanishes, the combination is called flex-free.

Also, note that each delay operator D_i has a unique inverse D_i^{-1}, whose expression can be derived by requiring that $D_i^{-1}D_i=I$, and neglecting quadratic and higher order velocity terms. Its action on a time series $\Psi(t)$ is

$$D_i^{-1}\Psi(t)\equiv\Psi[t+L_i(t+L_i)]. \quad (39)$$

Note that this is not like an advance operator one might expect, since it advances not by $L_i(t)$ but rather $L_i(t+L_i)$.

A. The unequal-arm Michelson

Here we re-derive the generalized unequal-arm Michelson combination [1], X_1, by implementing our method based on the use of time-delay operators. We use again X_1 as an example for showing the effectiveness of this alternative and powerful method for deriving TDI data combinations accounting for rotation and time dependence of the LISA arms.

Let us consider the following two combinations of the one-way measurements entering into the X observable given in the previous section, evaluating them for the noises ϕ_i only [Eq. (15)]:

$$[(\eta_{31}+\eta_{13;2})+(\eta_{21}+\eta_{12;3'})_{;2'2}]=[D_2D_{2'}D_{3'}D_3-I]\phi_1, \quad (40)$$

$$[(\eta_{21}+\eta_{12;3'})+(\eta_{31}+\eta_{13;2})_{;33'}]=[D_{3'}D_3D_2D_{2'}-I]\phi_1. \quad (41)$$

If the time delays were constants, the operators on the right would permute freely; simply differencing of Eqs. (40), (41) eliminates ϕ_1 and indeed is just X. If they do not permute, from Eqs. (40),(41) we can use the delay technique again to write the following expression for X_1:

$$\begin{aligned}X_1=&[D_2D_{2'}D_{3'}D_3-I][(\eta_{21}+\eta_{12;3'})+(\eta_{31}+\eta_{13;2})_{;33'}]\\&-[D_{3'}D_3D_2D_{2'}-I][(\eta_{31}+\eta_{13;2})+(\eta_{21}+\eta_{12;3'})_{;2'2}]\\=&[(\eta_{31}+\eta_{13;2})+(\eta_{21}+\eta_{12;3'})_{;2'2}+(\eta_{21}+\eta_{12;3'})_{;33'2'2}\\&+(\eta_{31}+\eta_{13;2})_{;33'33'2'2}]-[(\eta_{21}+\eta_{12;3'})\\&+(\eta_{31}+\eta_{13;2})_{;33'}+(\eta_{31}+\eta_{13;2})_{;2'233'}\\&+(\eta_{21}+\eta_{12;3'})_{;2'22'233'}].\end{aligned}\tag{42}$$

After substituting Eqs. (10),(11),(12) into Eqs. (42), we obtain the final expression for X_1 [1]

$$\begin{aligned}X_1=&[(s_{31}+s_{13;2})+(s_{21}+s_{12;3'})_{;2'2}+(s_{21}+s_{12;3'})_{;33'2'2}\\&+(s_{31}+s_{13;2})_{;33'33'2'2}]-[(s_{21}+s_{12;3'})\\&+(s_{31}+s_{13;2})_{;33'}+(s_{31}+s_{13;2})_{2'233'}\\&+(s_{21}+s_{12;3'})_{;2'22'233'}]+\frac{1}{2}[(\tau_{21}-\tau_{31})\\&-(\tau_{21}-\tau_{31})_{;33'}-(\tau_{21}-\tau_{31})_{;2'2}\\&+(\tau_{21}-\tau_{31})_{;33'33'2'2}+(\tau_{21}-\tau_{31})_{;2'22'233'}\\&-(\tau_{21}-\tau_{31})_{;2'233'33'2'2}],\end{aligned}\tag{43}$$

As usual, X_2 and X_3 are obtained by cyclic permutation of the spacecraft indices. This expression is readily shown to be laser-noise-free to first order of spacecraft separation velocities $\dot{L}_i$: it is "flex-free."

B. The relay

In order to derive the expressions for the generalized relay combinations (U_1,U_2,U_3) valid for the realistic kinematical configuration of the LISA spacecraft, let us consider the following combinations of the data that enter into the expression for U given in the previous section:

$$[\eta_{12}+\eta_{31;3}+\eta_{23;23}+\eta_{32;123}-\eta_{32}]=[D_3D_2D_1-I]D_{1'}\phi_3,\tag{44}$$

$$[\eta_{12;11'}+\eta_{23;1'}+\eta_{31;311'}]=D_{1'}[D_1D_3D_2-I]\phi_3.\tag{45}$$

In each case we evaluate them for the noises ϕ_i only, as these are what our combinations must remove. The expression for U_1 is then given by the following linear combination of the properly delayed equations (44),(45):

$$\begin{aligned}U_1=&D_{1'}[D_1D_3D_2-I][\eta_{12}+\eta_{31;3}+\eta_{23;23}+\eta_{32;123}-\eta_{32}]\\&-[D_3D_2D_1-I]D_{1'}[\eta_{12;11'}+\eta_{23;1'}+\eta_{31;311'}]\\=&[\eta_{12;2311'}+\eta_{31;32311'}+\eta_{23;232311'}+\eta_{32;1232311'}\\&-\eta_{32;2311'}]-[\eta_{12;1'}+\eta_{31;31'}+\eta_{23;231'}+\eta_{32;1231'}\\&-\eta_{32;1'}]+[\eta_{12;11'1'}+\eta_{23;1'1'}+\eta_{31;311'1'}]\\&-[\eta_{12;11'1'123}+\eta_{23;1'1'123}+\eta_{31;311'1'123}],\end{aligned}\tag{47}$$

which, in terms of the one-way measurements s_{ij} and τ_{ij}, becomes

$$\begin{aligned}U_1=&[s_{12;2311'}+s_{31;32311'}+s_{23;232311'}+s_{32;1232311'}-s_{32;2311'}]\\&-[s_{12;1'}+s_{31;31'}+s_{23;231'}+s_{32;1231'}-s_{32;1'}]\\&+[s_{12;11'1'}+s_{23;1'1'}+s_{31;311'1'}]-[s_{12;11'1'123}\\&+s_{23;1'1'123}+s_{31;311'1'123}]+\frac{1}{2}[(\tau_{32}-\tau_{12})_{;2311'}+(\tau_{21}\\&-\tau_{31})_{;32311'}+(\tau_{13}-\tau_{23})_{;232311'}-(\tau_{13}-\tau_{23})_{;1'1232311'}\\&+(\tau_{13}-\tau_{23})_{;1'2311'}-(\tau_{32}-\tau_{12})_{;1'}-(\tau_{21}-\tau_{31})_{;31'}\\&-(\tau_{13}-\tau_{23})_{;231'}+(\tau_{13}-\tau_{23})_{;1'1231'}+(\tau_{32}-\tau_{12})_{;11'1'}\\&+(\tau_{21}-\tau_{31})_{;311'1'}-(\tau_{32}-\tau_{12})_{;11'1'123}\\&-(\tau_{13}-\tau_{23})_{;1'1'123}-(\tau_{21}-\tau_{31})_{;311'1'123}]\end{aligned}\tag{48}$$

with U_2, U_3 obtained by cycling the spacecraft indices. It can readily be verified using Eqs. (37),(38) that the laser noise remaining in this combination vanishes to first order in the spacecraft relative velocities $\dot{L}_i$.

C. The beacon

In order to derive the expression for P_1 let us consider the following data combinations entering into the expression for P given in Sec. III

$$[\eta_{23}+\eta_{32;1}+\eta_{13;1'1}-\eta_{13}]_{;3}=D_3[D_1D_{1'}-I]D_{2'}\phi_1,\tag{49}$$

$$[\eta_{32}+\eta_{23;1'}+\eta_{12,11'}-\eta_{12}]_{;2'}=D_{2'}[D_{1'}D_1-I]D_3\phi_1,\tag{50}$$

where the expressions on the right-hand-sides follow from the chosen order of the indices appearing on the left-hand-sides of the above equations. By applying our method we obtain the final expression for P_1

$$P_1 = D_{2'}[D_{1'}D_1 - I]D_3[\eta_{23;3} + \eta_{32;13} + \eta_{13;1'13} - \eta_{13;3}] - D_3[D_1D_{1'} - I]D_{2'}[\eta_{32;2'} + \eta_{23;1'2'} + \eta_{12,11'2'} - \eta_{12;2'}] \quad (51)$$

$$= [\eta_{23;3311'2'} + \eta_{32;13311'2'} + \eta_{13;1'13311'2'} - \eta_{13;3311'2'}] - [\eta_{23;332'} + \eta_{32;1332'} + \eta_{13;1'1332'} - \eta_{13;332'}] + [\eta_{32;2'2'3} + \eta_{23;1'2'2'3} + \eta_{12,11'2'2'3} - \eta_{12;2'2'3}] - [\eta_{32;2'2'1'13} + \eta_{23;1'2'2'1'13} + \eta_{12,11'2'2'1'13} - \eta_{12;2'2'1'13}]. \quad (52)$$

Equation (52) can be rewritten in terms of the one-way measurements s_{ij}, τ_{ij}

$$P_1 = [s_{23;3311'2'} + s_{32;13311'2'} + s_{13;1'13311'2'} - s_{13;3311'2'}] - [s_{23;332'} + s_{32;1332'} + s_{13;1'1332'} - s_{13;332'}] + [s_{32;2'2'3} + s_{23;1'2'2'3} + s_{12,11'2'2'3} - s_{12;2'2'3}] - [s_{32;2'2'1'13} + s_{23;1'2'2'1'13} + s_{12,11'2'2'1'13} - s_{12;2'2'1'13}] + \tfrac{1}{2}[(\tau_{13} - \tau_{23})_{;3311'2'} - (\tau_{13} - \tau_{23})_{;1'13311'2'} - (\tau_{21} - \tau_{31})_{;2'1'13311'2'} + (\tau_{21} - \tau_{31})_{;2'3311'2'} - (\tau_{13} - \tau_{23})_{;332'} + (\tau_{13} - \tau_{23})_{;1'1332'} + (\tau_{21} - \tau_{31})_{;2'1'1332'} - (\tau_{21} - \tau_{31})_{;2'332'} + (\tau_{32} - \tau_{12})_{;11'2'2'3} - (\tau_{32} - \tau_{12})_{;2'2'3} - (\tau_{32} - \tau_{12})_{;11'2'2'1'13} + (\tau_{32} - \tau_{12})_{;2'2'1'13}] \quad (53)$$

with P_2, P_3 obtained by cycling the spacecraft indices in Eq. (53). Substituting into Eq. (53) the laser phase noise terms entering the s_{ij} and τ_{ij}, and applying the expansion rules of Eqs. (36)–(38), it can again be shown that, to first order in the systematic relative velocities of the spacecraft, laser phase noise is eliminated.

D. The monitor

The derivation of the generalized ''monitor'' combinations (E_1, E_2, E_3) is more complicated, and rather different from the derivations shown in the previous two sections. One peculiarity of these combinations is that they are not unique. It is indeed possible to derive different expressions for each monitor combination. These combinations cancel the laser noises to the required order in the velocities, and they differ only in the number of terms—delayed data time series—they include. We have derived expressions with 64, 32, and 40 η terms (which we do not provide here). The expression we present in this section shows the same number of η terms (16) as X_1, P_1, and U_1.

Let us consider the following terms entering into the expression for E derived in the previous section:

$$\eta_{21;1'1} - \eta_{21} = [I - D_1D_{1'}]\phi_1 - [I - D_1D_{1'}]D_{3'}\phi_2, \quad (54)$$

$$-\eta_{32,3'} - \eta_{23;1'3'} = D_{3'}[I - D_{1'}D_1]\phi_2, \quad (55)$$

$$\eta_{31} - \eta_{31;11'} = -[I - D_{1'}D_1]\phi_1 + [I - D_{1'}D_1]D_2\phi_3, \quad (56)$$

$$\eta_{23;2} + \eta_{32;12} = -D_2[I - D_1D_{1'}]\phi_3. \quad (57)$$

If the delay operators were constant and commuted, adding these four equations would cancel all laser phase noises and give E. Otherwise the above expressions can be first combined in pairs to remove the ϕ_2, ϕ_3 noises in two shear-free ways

$$D_{3'}[I - D_{1'}D_1][\eta_{21;1'1} - \eta_{21}] - [I - D_1D_{1'}]D_{3'}[\eta_{32,3'} + \eta_{23;1'3'}] = D_{3'}[I - D_{1'}D_1][I - D_1D_{1'}]\phi_1 \quad (58)$$

$$D_2[I - D_1D_{1'}][\eta_{31;11'} - \eta_{31}] - [I - D_{1'}D_1]D_2[\eta_{23;2} + \eta_{32;12}] = D_2[I - D_1D_{1'}][I - D_{1'}D_1]\phi_1. \quad (59)$$

Now we could of course repeat our iterative procedure by properly using the delay operators shown on the right-hand-side of Eqs. (58), (59), and derive the final expression for E_1. However, this expression would include 64 η terms. An alternative, and more elegant way to derive an expression for an E_1 that has only 16 η terms is by noticing that if we first apply inverse operators $D_{3'}^{-1}$ and D_2^{-1} from Eq. (39) to both sides of Eqs. (58) and (59), respectively, and then take the difference of the resulting expressions, we get the following simpler expression for E_1:

$$E_1 \equiv [I - D_{1'}D_1][\eta_{21;1'1} - \eta_{21}] - D_{3'}^{-1}[I - D_1D_{1'}]D_{3'}[\eta_{32,3'} + \eta_{23;1'3'}] + [I - D_1D_{1'}][\eta_{31} - \eta_{31;11'}] + D_2^{-1}[I - D_{1'}D_1]D_2[\eta_{23;2} + \eta_{32;12}]$$

$$= [\eta_{31} - \eta_{31;11'} - \eta_{31;1'1} + \eta_{31;11'1'1}] - [\eta_{21} - \eta_{21;11'} - \eta_{21;1'1} + \eta_{21;1'111'}] + [\eta_{23;2} + \eta_{32;12} - \eta_{32;3'} - \eta_{23;1'3'}] - [\eta_{23;2211'\bar{2}} + \eta_{32;12211'\bar{2}} - \eta_{32;3'3'1'1\bar{3}'} - \eta_{23;1'3'3'1'1\bar{3}'}]. \quad (60)$$

In Eq. (60) we have introduced a bar over some indices for representing the action of the corresponding inverse operator. It is easy to verify, to first order in the spacecraft relative velocities, that the above expression is laser and optical bench noise-free. Equation (60) can be recast in terms of the one-way measurements s_{ij} and τ_{ij}

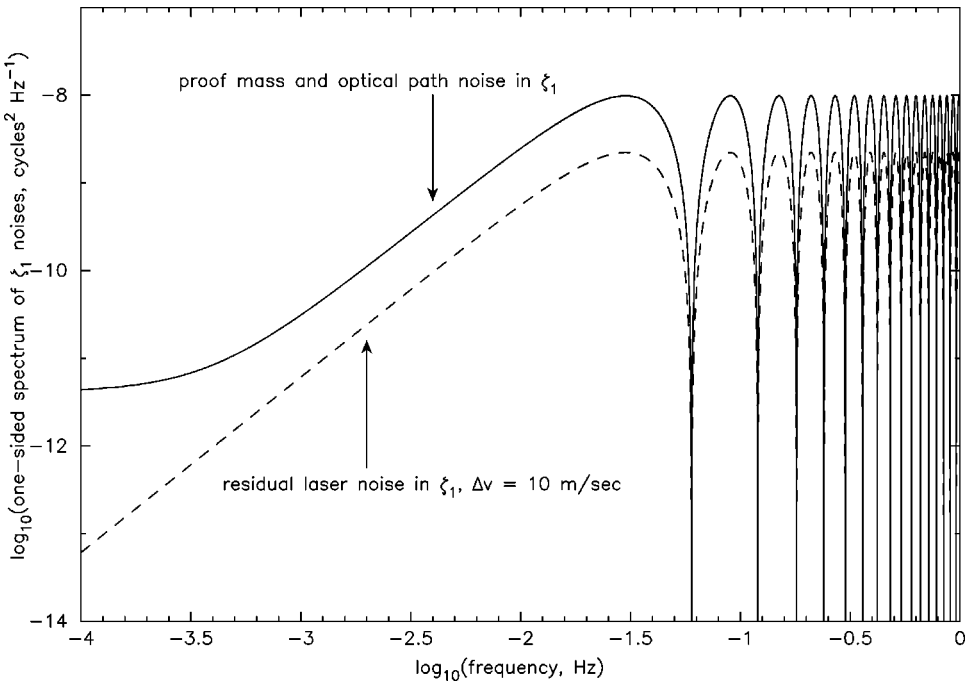

FIG. 2. Spectrum of proof mass and optical path noises in ζ_1 compared with the spectrum of residual laser noise for a shearing array. Spectra are one-sided and expressed as $(\text{cycles})^2/\text{Hz}$. Parameters used are as follows: $30\ \text{Hz}/\sqrt{\text{Hz}}$ for the laser frequency fluctuations, $3\times10^{-15}\ \text{m/sec}^2/\sqrt{\text{Hz}}$ for the proof mass noise, $20\times10^{-12}\ \text{m}/\sqrt{\text{Hz}}$ for aggregate optical path (shot noise, beam-pointing noise, etc.) noise, $L=16.67$ sec, and the velocity differences have been taken to be equal to 10 m/sec. Laser noise does not cancel exactly in ζ_1 for non-zero velocities, but is $\simeq 7$ dB below optical path and proof mass noises.

$$
\begin{aligned}
E_1 = {} & [s_{31} - s_{31;11'} - s_{31;1'1} + s_{31;11'1'1}] - [s_{21} - s_{21;11'} \\
& - s_{21;1'1} + s_{21;1'111'}] + [s_{23;2} + s_{32;12} - s_{32;3'} - s_{23;1'3'}] \\
& - [s_{23;2211'\bar{2}} + s_{32;12211'\bar{2}} - s_{32;3'3'1'1\bar{3}'} - s_{23;1'3'3'1'1\bar{3}'}] \\
& + \tfrac{1}{2}[(\tau_{32} - \tau_{12})_{;3'} - (\tau_{32} - \tau_{12})_{;3'1'1} - (\tau_{32} - \tau_{12})_{;3'11'} \\
& + (\tau_{32} - \tau_{12})_{;3'1'111'} + (\tau_{21} - \tau_{31}) - (\tau_{21} - \tau_{31})_{;11'} \\
& - (\tau_{21} - \tau_{31})_{;1'1} + (\tau_{21} - \tau_{31})_{;11'1'1} + (\tau_{13} - \tau_{23})_{;2} \\
& - (\tau_{13} - \tau_{23})_{;1'12} - (\tau_{13} - \tau_{23})_{;2211'\bar{2}} \\
& + (\tau_{13} - \tau_{23})_{;1'12211'\bar{2}}],
\end{aligned}
\tag{61}
$$

with E_2, E_3 obtained by cycling the indices.

E. The ζ combinations

The expression for ζ_1 derived in the previous section cancels the laser noise exactly under the assumption of constant time delays. Although perfect cancellation is no longer achieved when relative motion between the spacecraft is included, the ordering of the delays determined by our derivation of the expression for ζ_1 given in the previous section implies a minimization of the magnitude of the remaining laser noises at least for the equilateral LISA case.

Consider the expression for ζ_1 given in Eq. (34), now, however, with semicolons rather then simple colons. After some algebra, it is possible to derive the leading order contribution due to the residual laser noises remaining into ζ_1:

$$
\zeta_1 \simeq [\dot{\phi}_{2,4L} - \dot{\phi}_{2,3L}](\dot{L}_3 - \dot{L}_1)L + [\dot{\phi}_{3,4L} - \dot{\phi}_{3,3L}](\dot{L}_2 - \dot{L}_1)L,
\tag{62}
$$

where we have assumed the arm lengths to differ from a nominal LISA arm length L by only a few percents [2].

This residual laser noise can be compared with the optical path and proof mass noises in ζ_1. Using the derivative theorem for Fourier transforms and taking the arm lengths to be the same, the spectrum of the residual laser noise in ζ_1 can be expressed in terms of the spectrum of the raw laser phase noise S_ϕ and the velocities $\dot{L}_i$:

$$
16\pi^2 f^2 \sin^2(\pi f L) S_\phi(f)[(\dot{L}_2 - \dot{L}_1)^2 + (\dot{L}_3 - \dot{L}_1)^2]L^2.
\tag{63}
$$

From Sec. III and [5], the spectrum of ζ_1 due to proof mass and optical path noises is equal to

$$
4\sin^2(\pi f L)[24\sin^2(\pi f L) S^{\text{proof mass}}(f) + 6S^{\text{opt. path}}(f)].
\tag{64}
$$

In Fig. 2 we compare the spectrum of residual laser noise in ζ_1 and the optical path and proof mass noises in ζ_1. The parameters used were $30\ \text{Hz}/\sqrt{\text{Hz}}$ for the raw laser frequency fluctuations, $3\times10^{-15}\ \text{m/sec}^2/\sqrt{\text{Hz}}$ for the proof mass noise, and $20\times10^{-12}\ \text{m}/\sqrt{\text{Hz}}$ for aggregate optical path (shot noise, beam-pointing noise, etc.) noise. All the above spectra are one sided. Figure 2 shows this comparison using nominal $L=16.67$ sec arm lengths and (pessimistically) taking the velocity differences to be 10 m/sec. From Fig. 2 the residual laser noise in ζ_1 for a shearing array [but with the time delays applied as given in Eq. (34)] is $\simeq 7$ dB below the optical path and proof mass noises.

ACKNOWLEDGMENT

This research was performed at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

[1] D.A. Shaddock, M. Tinto, F.B. Estabrook, and J.W. Armstrong, Phys. Rev. D **68**, 061303 (2003).
[2] P.L. Bender, K. Danzmann, and the LISA Study Team, ''Laser Interferometer Space Antenna for the Detection of Gravitational Waves, Pre-Phase A Report,'' Max-Planck-Institüt für Quantenoptik, Report No. MPQ 233, 1998.
[3] M. Tinto and J.W. Armstrong, Phys. Rev. D **59**, 102003 (1999).
[4] J.W. Armstrong, F.B. Estabrook, and M. Tinto, Astrophys. J. **527**, 814 (1999).
[5] F.B. Estabrook, M. Tinto, and J.W. Armstrong, Phys. Rev. D **62**, 042002 (2000).
[6] M. Tinto, F.B. Estabrook, and J.W. Armstrong, Phys. Rev. D **65**, 082003 (2002).
[7] M. Tinto, J.W. Armstrong, and F.B. Estabrook, Phys. Rev. D **63**, 021101 (2001).
[8] M. Tinto, D.A. Shaddock, J. Sylvestre, and J.W. Armstrong, Phys. Rev. D **67**, 122003 (2003).
[9] S.V. Dhurandhar, K.R. Nayak, and J.-Y. Vinet, Phys. Rev. D **65**, 102002 (2002).
[10] F.B. Estabrook, J.W. Armstrong, M. Tinto, and W. Folkner, Phys. Rev. D **68**, 062001 (2003).
[11] D.A. Shaddock, Phys. Rev. D **69**, 1022001 (2004).
[12] N.J. Cornish and R.W. Hellings, Class. Quantum Grav. **20**, 4851 (2003).
[13] D. Summers, ''Algorithm tradeoffs,'' oral presentation, 3rd progress meeting of the ESA funded LISA PMS Project. ESTEC, NL, 2003.
[14] W.M. Folkner, F. Hechler, T.H. Sweetser, M.A. Vincent, and P.L. Bender, Class. Quantum Grav. **14**, 1543 (1997).
[15] N. Ashby, ''The Sagnac Effect in the Global Positioning System,'' available at http://digilander.libero.it/solciclos/
[16] C. Cutler, Phys. Rev. D **57**, 7089 (1998).
[17] N.J. Cornish and L.J. Rubbo, Phys. Rev. D **67**, 022001 (2003).