

Thermodynamics of the $O(N)$ nonlinear sigma model in 1+1 dimensions

Jens O. Andersen

Nordita, Blegdamsvej 17, 2100 Copenhagen, Denmark

Daniël Boer and Harmen J. Warringa

Department of Physics and Astronomy, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

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The thermodynamics of the $O(N)$ nonlinear sigma model in 1+1 dimensions is studied. We calculate the pressure to next-to-leading order in the $1/N$ expansion and show that at this order, only the minimum of the effective potential can be rendered finite by temperature-independent renormalization. To obtain a finite effective potential away from the minimum requires an arbitrary choice of prescription, which implies that the temperature dependence is ambiguous. We show that the problem is linked to thermal infrared renormalons.

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I. INTRODUCTION

The $O(N)$ nonlinear sigma model (NLSM) in 1+1 dimensions has been studied extensively at zero temperature as a toy model for QCD. It is a remarkably rich theory, which is asymptotically free and has a dynamically generated mass gap. It is renormalizable both perturbatively and in the $1/N$ expansion. Moreover, for $N=3$ it has instanton solutions. Unlike the NLSM in more than two dimensions, where the theory is no longer renormalizable, there is no spontaneous symmetry breaking of the global $O(N)$ symmetry. This reflects the Mermin-Wagner-Coleman theorem [1,2], which forbids spontaneous breakdown of a continuous symmetry in a homogeneous system in one spatial dimension at any temperature. Moreover, the model suffers from infrared (IR) divergences in perturbation theory, since the fields are massless in that case [3]. However, it was conjectured by Elitzur [4] and shown by David [5] that the infrared divergences cancel in $O(N)$ -invariant correlation functions. In addition, a mass gap is generated nonperturbatively. In the large- N limit, which is equivalent to summing all so-called daisy and superdaisy graphs, $m = \mu \exp(-2\pi/g^2)$, where g is the coupling constant and μ is the renormalization scale.

Dine and Fischler [6] investigated the NLSM in 1+1 dimensions at finite temperature. They calculated the free energy in perturbation theory and in the large- N limit. In the weak-coupling expansion, they showed that the two-loop contribution to the ideal gas vanishes and that the three-loop contribution is infrared finite; the latter in fact also vanishes [7]. The leading-order calculation in the $1/N$ expansion shows that a thermal mass of order Ng^2T arises. This is a nonperturbative result that shows that one is effectively dealing with a gas of massive particles.

In this paper, we extend the analysis of Dine and Fischler to next-to-leading-order (NLO) in the $1/N$ expansion. At zero temperature, the effective potential (or equivalently the Gibbs free energy) has been investigated at this order by Biscari *et al.* [8]. The $1/N$ correction to the thermodynamic potential has the interesting feature of containing renormalon singularities. We will show that it cannot in general be renormalized in a temperature-independent way, except at its minimum as a function of m^2 . Away from the minimum, one

will need to introduce a temperature-dependent prescription to deal with the poles in the Borel plane.

Much is known about IR renormalons in the $O(N)$ NLSM in 1+1 dimensions [9–13], but the consequences at finite temperature have not yet been investigated. Thermal renormalons have been studied by Loewe and Valenzuela [14] in ϕ^4 theory in 3+1 dimensions. In this theory, one deals with ultraviolet (UV) renormalons only and thus it resembles QED rather than QCD. They show that the residues of the UV renormalon poles in the Borel plane (which are on the positive real axis, whereas in QCD they would be on the negative real axis due to asymptotic freedom, such that they do not affect the Borel transform [15]) in general are temperature dependent, but the positions of the poles are not. We will show that this is also the case for IR renormalons, except at the minimum of the effective potential, where also the residues are temperature independent.

Blaizot *et al.* [16] have recently studied the Gross-Neveu model in 1+1 dimensions at finite temperature at NLO in the $1/N$ expansion. While there are similarities between this model and the NLSM, such as dynamical mass generation and asymptotic freedom, no problems related to IR renormalons are encountered in Ref. [16] (see also Ref. [17]). One can uniquely define the effective potential at NLO at nonzero temperature.

The paper is organized as follows. In Sec. II, we discuss the NLSM at zero temperature. In Sec. III, we calculate the finite-temperature pressure at NLO. In Sec. IV, we discuss various approximations and compare them with exact numerical results. In Sec. V, we show that one cannot define an off-shell effective potential and this is related to thermal infrared renormalons. In Sec. VI, we summarize and conclude.

II. ZERO TEMPERATURE

The Euclidean Lagrangian for the nonlinear sigma model is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2} \alpha (\Phi^2 - Ng^{-2}), \quad (1)$$

where the scalar field $\Phi = (\phi_1, \phi_2, \dots, \phi_N)$ forms an N -component vector and α is a Lagrange multiplier that en-

forces the constraint $\Phi^2(x) = Ng^{-2}$. The auxiliary field α is now written as the sum of a space-time independent background m^2 and a quantum fluctuating field $\tilde{\alpha}$; $\alpha = m^2 + \tilde{\alpha}$. The Green's functions of Φ require wave function and coupling constant renormalization in the $1/N$ expansion, as discussed by Rim and Weisberger [18] through NLO.

The Lagrangian in Eq. (1) is quadratic in the fields Φ and the integral over Φ can therefore be done exactly. One then obtains (cf. e.g. Ref. [19])

$$\mathcal{Z} = \int \mathcal{D}\tilde{\alpha} \exp \left\{ -\frac{N}{2} \text{tr} \ln [P^2 + m^2 + \tilde{\alpha}] + N \int_0^\beta d\tau \int dx \left[\frac{1}{2} m^2 g^{-2} + \frac{1}{2} \tilde{\alpha} g^{-2} \right] \right\}, \quad (2)$$

where $\beta = 1/T$, such that at zero temperature $\beta = \infty$. The next step is to expand the functional determinant around the classical solution $\tilde{\alpha} = 0$ and integrate over $\tilde{\alpha}$. By scaling $\tilde{\alpha} \rightarrow \tilde{\alpha}/\sqrt{N}$, it is seen that this expansion is equivalent to a $1/N$ expansion.

It is important to realize that m^2 is by definition the vacuum expectation value of α , i.e. it is the quantity with respect to which we will minimize the effective potential in order to obtain the pressure. Beyond leading order in the $1/N$ expansion m^2 receives divergent contributions. In order to show this, we rewrite the expression for m^2 in terms of m_ϕ^2 , where m_ϕ is defined as the pole of the propagator $D_\phi(P, m)$ of Φ . At NLO, $D_\phi(P, m)$ is given by

$$D_\phi(P, m) = \frac{Z_\phi}{P^2 + m^2 - (1/N)\Sigma(P, m)}, \quad (3)$$

where Z_ϕ is the wave function renormalization constant and

$$\Sigma(P, m) = \int \frac{d^2 Q}{(2\pi)^2} \frac{1}{(P+Q)^2 + m^2} \frac{1}{\Pi(Q, m)} \quad (4)$$

is the self-energy function. Here, $\Pi(P, m)$ is the inverse propagator for $\tilde{\alpha}$:

$$\begin{aligned} \Pi(P, m) &= -\frac{1}{2} \int \frac{d^2 Q}{(2\pi)^2} \frac{1}{Q^2 + m^2} \frac{1}{(P+Q)^2 + m^2} \\ &= -\frac{1}{4\pi P^2 \xi} \ln \left(\frac{\xi + 1}{\xi - 1} \right), \end{aligned} \quad (5)$$

where $\xi = \sqrt{1 + 4m^2/P^2}$. Choosing renormalization point $P^2 = -m_\phi^2$, one obtains [20]

$$m_\phi^2 = m^2 + \frac{m^2}{N} \text{li} \left(\frac{\Lambda^2}{m^2} \right), \quad (6)$$

where $\text{li}(x)$ is the logarithmic integral,

$$\text{li}(x) = \mathcal{P} \int_0^x dt \frac{1}{\ln t}. \quad (7)$$

Here, Λ denotes the ultraviolet momentum cutoff and \mathcal{P} indicates a principal-value prescription for the integral. Solving Eq. (6) for m^2 , we obtain

$$m^2 = m_\phi^2 - \frac{m_\phi^2}{N} \text{li} \left(\frac{\Lambda^2}{m_\phi^2} \right). \quad (8)$$

In order to have a pole with a residue equal to unity, one needs $Z_\phi = 1 - (1/N)\Sigma'(P, m)|_{P^2 = -m_\phi^2}$, which yields

$$Z_\phi = 1 + \frac{1}{N} \ln \ln \left(\frac{\Lambda^2}{\mu^2} \right). \quad (9)$$

The wave function renormalization Eq. (9) is in accordance with that obtained by Flyvbjerg [20]. Rim and Weisberger [18] calculated the wave function renormalization constant in dimensional regularization and it also agrees with Eq. (9) as can be seen by identifying $\ln(\Lambda^2/\mu^2) \rightarrow 2/\epsilon$ where $d = 2 - \epsilon$.

The effective potential \mathcal{V} through next-to-leading order in the $1/N$ expansion is given by

$$\begin{aligned} \mathcal{V} &= \frac{m^2 N}{2g_b^2} - \frac{1}{2} N \int \frac{d^2 P}{(2\pi)^2} \ln [P^2 + m^2] \\ &\quad - \frac{1}{2} \int \frac{d^2 P}{(2\pi)^2} \ln [\Pi(P, m)], \end{aligned} \quad (10)$$

where we have added a subscript b to g to indicate explicitly that it is the bare coupling constant. Evaluating the integrals in Eq. (10) using an ultraviolet momentum cutoff Λ , one obtains

$$\begin{aligned} \mathcal{V} &= \frac{m^2 N}{2g_b^2} - \frac{m^2 N}{8\pi} \left(1 + \ln \frac{\Lambda^2}{m^2} \right) - \frac{1}{8\pi} \left[(\Lambda^2 + 2m^2) \ln \ln \frac{\Lambda^2}{m^2} \right. \\ &\quad \left. - m^2 \text{li} \left(\frac{\Lambda^2}{m^2} \right) + 2m^2 \left(\gamma_E - 1 - \ln \frac{\Lambda^2}{4m^2} \right) \right]. \end{aligned} \quad (11)$$

Here and in the subsequent results we have dropped m -independent divergences and terms that vanish in the limit $\Lambda^2 \rightarrow \infty$.

To obtain the pressure, one evaluates the effective potential at its minimum. The condition for the minimum is given by equation

$$\frac{\partial \mathcal{V}}{\partial m^2} = 0. \quad (12)$$

Equation (12) is often referred to as a gap equation. Differentiating Eq. (11), one obtains

$$\frac{4\pi}{g_b^2} = \left(1 - \frac{2}{N}\right) \ln \frac{\Lambda^2}{m^2} + \frac{1}{N} \left[2 \ln \ln \frac{\Lambda^2}{m^2} - \text{li} \left(\frac{\Lambda^2}{m^2} \right) + 2\gamma_E + 4 \ln 2 \right]. \quad (13)$$

To see that Eq. (13) becomes finite after coupling constant renormalization, we first express it in terms of m_ϕ^2 , using Eq. (8):

$$\frac{4\pi}{g_b^2} = \left(1 - \frac{2}{N}\right) \ln \frac{\Lambda^2}{m_\phi^2} + \frac{2}{N} \left(\ln \ln \frac{\Lambda^2}{m_\phi^2} + \gamma_E + \ln 4 \right). \quad (14)$$

The renormalization constant for g is denoted by $Z_{g^2}^{-1}$ and is given by

$$Z_{g^2}^{-1} = 1 + \frac{g^2}{4\pi} \left(1 - \frac{2}{N}\right) \ln \frac{\Lambda^2}{\mu^2} + \frac{1}{N} \frac{g^2}{2\pi} \ln \ln \frac{\Lambda^2}{\mu^2}. \quad (15)$$

Making the substitution $g_b^2 \rightarrow Z_{g^2} g^2(\mu)$, we obtain the renormalized gap equation:

$$\frac{4\pi}{g^2(\mu)} = \left(1 - \frac{2}{N}\right) \ln \frac{\mu^2}{m_\phi^2} + \frac{2}{N} (\gamma_E + \ln 4). \quad (16)$$

The expression Eq. (15) for $Z_{g^2}^{-1}$ is exact in $g^2(\mu)$ up to order $1/N^2$ corrections and results in the known NLO β function [8,18,21]:

$$\beta(g_b^2) = \Lambda \frac{dg_b^2}{d\Lambda} = - \left(1 - \frac{2}{N}\right) \frac{g_b^4}{2\pi} \left(1 + \frac{1}{N} \frac{g_b^2}{2\pi}\right), \quad (17)$$

$$\beta(g^2) = \mu \frac{dg^2}{d\mu} = - \left(1 - \frac{2}{N}\right) \frac{g^4}{2\pi}. \quad (18)$$

Using the gap equation, one can obtain the value of the effective potential \mathcal{V} at the minimum, where it equals the pressure \mathcal{P} . In terms of bare quantities, we obtain

$$\mathcal{P}^{T=0} = - (N-2) \frac{m^2}{8\pi} - \frac{1}{8\pi} \Lambda^2 \ln \frac{4\pi}{g_b^2}. \quad (19)$$

This equation will be used to subtract the pressure at zero temperature from the pressure at finite temperature.

III. FINITE TEMPERATURE

The results at zero temperature are obtained analytically, but at finite temperature this is in general not possible. Therefore, we will investigate the pressure numerically. However, we are able to isolate the ultraviolet divergences analytically.

The effective potential through next-to-leading order in $1/N$ is now given by

$$\mathcal{V} = \frac{m^2 N}{2g_b^2} - \frac{1}{2} N \oint_P \ln[P^2 + m^2] - \frac{1}{2} \oint_P \ln[\Pi^T(P, m)], \quad (20)$$

where the inverse propagator $\Pi^T(P, m)$ is the finite temperature version of Eq. (5) and we have defined the sum-integral

$$\oint_P \equiv T \sum_{p_0=2n\pi T} \int \frac{dp}{2\pi}. \quad (21)$$

Summing over Matsubara frequencies and averaging over angles, $\Pi^T(P, m)$ reduces to

$$\Pi^T(P, m) = \int_{-\infty}^{\infty} \frac{dq}{E_q} R(P, q) [1 + 2n(E_q)], \quad (22)$$

where $E_q = \sqrt{q^2 + m^2}$ and $n(x) = [\exp(\beta x) - 1]^{-1}$ is the Bose-Einstein distribution. The function $R(P, q)$ is given by

$$R(P, q) = - \frac{1}{4\pi} \frac{P^2 + 2pq}{(P^2 + 2pq)^2 + 4p_0^2 E_q^2}. \quad (23)$$

The inverse propagator cannot be evaluated analytically, so we will evaluate it numerically. For this purpose, it is necessary to isolate ultraviolet divergences analytically. As expected on general grounds, i.e. from the absence of temperature-dependent ultraviolet divergences, and as verified numerically, the quantity

$$F_1 = \oint_P \ln \Pi^T(P, m) - \int \frac{d^2 P}{(2\pi)^2} \ln \Pi^T(P, m), \quad (24)$$

is finite. To calculate F_1 we used an Abel-Plana formula [22]. In order to isolate the divergences we consider the limit $p \gg T$, where we can approximate

$$\begin{aligned} \Pi^T(P, m) &\approx \Pi(P, m) - \frac{1}{4\pi} \frac{P^2}{P^4 + 4m^2 p_0^2} J_1(\beta m) \\ &\equiv \Pi_{\text{HE}}^T(P, m), \end{aligned} \quad (25)$$

where $\Pi(P, m)$ is given in Eq. (5) and

$$J_1(\beta m) = 4 \int_0^{\infty} \frac{dq}{E_q} n(E_q). \quad (26)$$

In order to split off the prefactor of the logarithm in $\Pi^{T=0}$, we define $\tilde{\Pi}(P, m) = -4\pi \sqrt{P^2(P^2 + 4m^2)} \Pi(P, m)$. This gives the following contribution to the free energy:

$$\begin{aligned} D_1 &= - \frac{1}{2} \int \frac{d^2 P}{(2\pi)^2} \ln[P^2 + 4m^2] \\ &= - \frac{m^2}{2\pi} \left(1 + \ln \frac{\Lambda^2}{4m^2}\right). \end{aligned} \quad (27)$$

In order to isolate the infinities, we need the large- P behavior of $\tilde{\Pi}_{\text{HE}}^T(P, m)$:

$$\tilde{\Pi}_{\text{HE}}^T(P, m) = \ln \frac{P^2}{\bar{m}^2} + \frac{2m^2}{P^2}(1 + J_1) - \frac{4m^2 p_0^2}{P^4} J_1 + \mathcal{O}\left(\frac{m^4}{P^4}\right), \quad (28)$$

where $\bar{m}^2 = m^2 \exp(-J_1)$. This yields

$$\int \frac{d^2 P}{(2\pi)^2} \ln[\tilde{\Pi}_{\text{HE}}^T(P, m)] = D_2 + \text{finite terms}, \quad (29)$$

where

$$D_2 = \frac{1}{4\pi} \left[\Lambda^2 \ln \ln \frac{\Lambda^2}{\bar{m}^2} - \bar{m}^2 \text{li} \left(\frac{\Lambda^2}{\bar{m}^2} \right) \right] + \frac{m^2}{2\pi} \ln \ln \frac{\Lambda^2}{\bar{m}^2}. \quad (30)$$

Finally, we define

$$F_2 = \int \frac{d^2 P}{(2\pi)^2} \ln[\tilde{\Pi}(P, m)] - D_2. \quad (31)$$

Again we have checked numerically that the quantity F_2 is finite, demonstrating that we have identified all ultraviolet divergences.

By putting everything together, the finite temperature effective potential becomes

$$\begin{aligned} \mathcal{V} = & \frac{Nm^2}{2g_b^2} - \frac{Nm^2}{8\pi} \left(1 + \ln \frac{\Lambda^2}{m^2} \right) + \frac{N}{8\pi} T^2 J_0(\beta m) \\ & - \frac{1}{2} (F_1 + D_1 + F_2 + D_2), \end{aligned} \quad (32)$$

where

$$J_0(\beta m) = \frac{8}{T^2} \int_0^\infty \frac{dq q^2}{E_q} n(E_q). \quad (33)$$

We again note that we have systematically dropped m -independent divergences and terms that vanish in the limit $\Lambda^2 \rightarrow \infty$.

The gap equation (12) at nonzero temperature now becomes

$$\begin{aligned} \frac{4\pi}{g_b^2} = & \ln \frac{\Lambda^2}{\bar{m}^2} + \frac{1}{N} \left[2 \ln \ln \frac{\Lambda^2}{\bar{m}^2} - \frac{d\bar{m}^2}{dm^2} \text{li} \left(\frac{\Lambda^2}{\bar{m}^2} \right) \right. \\ & \left. - 2 \ln \frac{\Lambda^2}{4m^2} + 4\pi \frac{d(F_1 + F_2)}{dm^2} \right]. \end{aligned} \quad (34)$$

From the fact that g_b^2 is temperature independent, one can conclude that \bar{m}^2 is also temperature independent at leading order in the $1/N$ expansion, when it is a solution to the gap equation. We will use this fact later on to conclude that the pressure can be renormalized in a temperature-independent way.

The calculation of the self-energy $\Sigma(P, m)$ at $p_0=0$ and $p^2 = -m_\phi^2$, at finite temperature yields

$$m^2 = m_\phi^2 - \frac{\bar{m}_\phi^2}{N} \left[\text{li} \left(\frac{\Lambda^2}{\bar{m}_\phi^2} \right) + F_3 \right], \quad (35)$$

where F_3 is a finite function that depends on the temperature as well as m_ϕ . Since we use m_ϕ merely as a way to express the renormalized gap equation in terms of finite quantities, any choice of F_3 will do and we choose $F_3=0$. We have checked numerically that other choices indeed do not alter the final result for the pressure. Using Eq. (35), Eq. (34) now becomes

$$\frac{4\pi}{g_b^2} = \ln \frac{\Lambda^2}{\bar{m}_\phi^2} + \frac{1}{N} \left[2 \ln \ln \frac{\Lambda^2}{\bar{m}_\phi^2} - 2 \ln \frac{\Lambda^2}{4m_\phi^2} + 4\pi \frac{d(F_1 + F_2)}{dm_\phi^2} \right]. \quad (36)$$

To render the gap equation finite, we again only need to make the substitution $g_b^2 \rightarrow Z_g g^2(\mu)$, where Z_g^{-1} is given by Eq. (15). The renormalized gap equation then becomes

$$\begin{aligned} \frac{4\pi}{g^2(\mu)} = & \left(1 - \frac{2}{N} \right) \ln \frac{\mu^2}{\bar{m}_\phi^2} \\ & + \frac{2}{N} \left[J_1(\beta m_\phi) + \ln 4 + 2\pi \frac{d(F_1 + F_2)}{dm_\phi^2} \right]. \end{aligned} \quad (37)$$

Using the gap equation, we obtain the value of the effective potential at the minimum which is equal to the pressure. Using Eq. (35) and expanding the J_0 and J_1 functions, one ultimately obtains for the pressure at nonzero temperature minus the pressure Eq. (19) at zero temperature,

$$\begin{aligned} \mathcal{P} \equiv \mathcal{P}^T - \mathcal{P}^{T=0} = & \frac{N-2}{8\pi} [m_\phi^2(0) - m_\phi^2] \\ & + \frac{N}{8\pi} [T^2 J_0(\beta m_\phi) + m_\phi^2 J_1(\beta m_\phi)] \\ & + \frac{1}{2} \left[m_\phi^2 \frac{d(F_1 + F_2)}{dm_\phi^2} - F_1 - F_2 \right], \end{aligned} \quad (38)$$

where F_1 and F_2 are functions of T^2 and $m_\phi^2 = m_\phi^2(T)$, and $m_\phi^2(0) = m_\phi^2(T=0)$. We have numerically evaluated the expression for the pressure, after solving Eq. (37) for $m_\phi(T)$. The result for different values of N is shown in Fig. 1, for the arbitrary choice $g^2(\mu=500) = 10$; hence T is given in the same units as μ . As can be shown and seen in the figure, \mathcal{P}/NT^2 approaches an N -dependent constant (to be evaluated below) at large temperatures, which for $N \rightarrow \infty$ is $\pi/6$. Moreover, it approaches zero in the limit of zero temperature. If we normalize the pressure, for a given value of N , to its value at $T=\infty$, we find that the normalized pressure has a very small dependence on N .

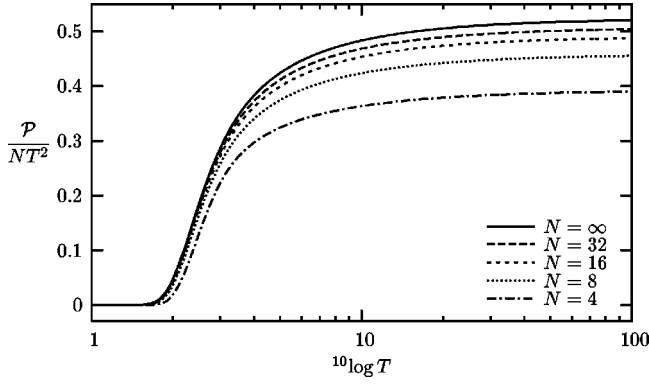


FIG. 1. Pressure as a function of temperature at NLO for different values of N .

IV. HIGH-TEMPERATURE APPROXIMATIONS

In Ref. [23], Bochkarev and Kapusta consider the nonlinear sigma model in 3+1 dimensions, which is nonrenormalizable, at NLO in the $1/N$ expansion. Since the result for the pressure cannot be obtained analytically, they resort to a “high-energy approximation” (HEA). We will make the same approximation and compare it with the exact numerical results obtained in Sec. III.

The idea of the high-energy approximation is that in the part of Π^T proportional to the distribution function $n(E_q)$, the important contribution comes from the region where $p_0, p \gg q$. One can therefore approximate the self-energy $\Pi^T(P, m)$ by its high-energy behavior. In the present case, this amounts to

$$\Pi^T(P, m) \approx \Pi(P, m) - \frac{1}{4\pi} \frac{P^2}{(p_0^2 + \omega_+^2)(p_0^2 + \omega_-^2)} J_1, \quad (39)$$

where $\omega_{\pm} = \sqrt{p^2 + m^2} \pm m$. This expression is identical to Eq. (25). After simply discarding the $T=0$ contribution to $\Pi^T(P, m)$, as done in Ref. [23], the effective potential is approximately given by

$$\begin{aligned} \mathcal{V}_{\text{HEA}} = & \frac{m^2 N}{2g_b^2} - \frac{1}{2} N \int_P \ln[P^2 + m^2] - \frac{1}{2} \int_P \ln P^2 \\ & + \frac{1}{2} \int_P \ln[p_0^2 + \omega_+^2] + \frac{1}{2} \int_P \ln[p_0^2 + \omega_-^2]. \end{aligned} \quad (40)$$

The resulting expression for the gap equation is

$$\begin{aligned} N g_b^{-2} = & N \int_P \frac{1}{P^2 + m^2} - \int_P \frac{\omega_+^2}{m E_p} \frac{1}{p_0^2 + \omega_+^2} \\ & + \int_P \frac{\omega_-^2}{m E_p} \frac{1}{p_0^2 + \omega_-^2}. \end{aligned} \quad (41)$$

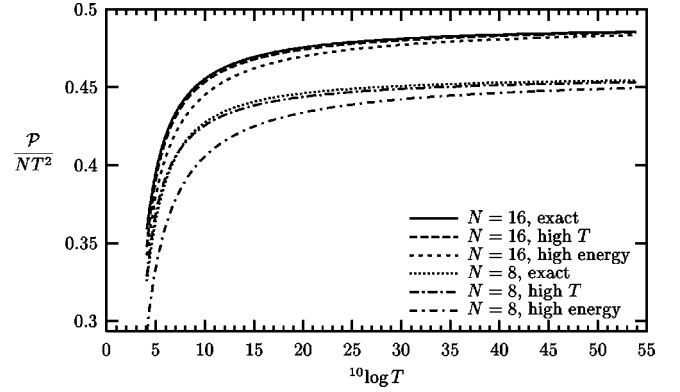


FIG. 2. Pressure as a function of temperature at NLO for different values of N compared with the high-energy and high-temperature approximations.

Again, the gap equation requires coupling constant renormalization. In this approximation, the renormalization constant is

$$Z_{g^2}^{-1} = 1 + \frac{g^2}{4\pi} \left(1 - \frac{2}{N}\right) \ln \frac{\Lambda^2}{\mu^2}, \quad (42)$$

which is consistent with the perturbative renormalization constant to leading order in g^2 . Making the substitution $g_b^2 \rightarrow Z_{g^2} g^2$, we obtain

$$1 = \frac{g^2}{4\pi N} \left[N J_1 - K_1^+ - K_1^- + 2(N-2) \ln \frac{\mu}{m} \right], \quad (43)$$

where the function K_1^{\pm} is

$$K_1^{\pm} = \pm 4 \int_0^{\infty} \frac{dp \omega_{\pm}}{m E_p} n(\omega_{\pm}). \quad (44)$$

Note, however, that the pressure is finite even when we substitute the unrenormalized gap equation (41) into Eq. (40):

$$\begin{aligned} \mathcal{P}_{\text{HEA}} = & \frac{N}{8\pi} [J_0 T^2 + (J_1 - 1)m^2] + \frac{\pi}{6} T^2 \\ & - \frac{1}{8\pi} [(K_0^+ + K_0^-) T^2 + (K_1^+ + K_1^- - 2)m^2], \end{aligned} \quad (45)$$

where the function K_0^{\pm} is

$$K_0^{\pm} = \frac{8}{T^2} \int_0^{\infty} \frac{dp p^2}{E_p} n(\omega_{\pm}). \quad (46)$$

From Fig. 2, one can see that the high-energy approximation underestimates the pressure compared to the exact result. The advantage of an approximation like the high-energy approximation is that the analytic calculations are simpler and that it is easier to implement numerically.

We suggest a different approximation, which is better than the high-energy approximation. We will calculate the inverse $\tilde{\alpha}$ propagator Π^T by first integrating over the momentum. We obtain

$$\begin{aligned} \Pi^T(P, m) = & -\frac{1}{2\beta} \sum_{q_0=2n\pi T} \frac{1}{\sqrt{m^2 + q_0^2}} \\ & \times \frac{P^2 + 2q_0 p_0}{P^4 + 4q_0(q_0 + p_0)P^2 + 4m^2 p^2}. \end{aligned} \quad (47)$$

In the limit $m \ll T$, we can approximate $\Pi^T(P, m)$ by $\Pi_{\text{HT}}^T(P, m)$, where we keep only the $q_0=0$ mode in the sum Eq. (47):

$$\Pi_{\text{HT}}^T(P, m) = -\frac{1}{2} \frac{1}{\beta m} \frac{P^2}{P^4 + 4m^2 p^2}. \quad (48)$$

Since it follows from the leading order gap equation that for high temperature and for all values of the coupling constant, $m \ll T$, we call this approximation the high-temperature (HT) approximation. By using the result that $P^4 + 4m^2 p^2$ can be written as $[p_0^2 + (p + im)^2 + m^2][p_0^2 + (p - im)^2 + m^2]$ and shifting $p \rightarrow p \pm im$ after taking the logarithm, the functions F_1 and F_2 can be approximated by

$$F_1 \approx \frac{1}{2\pi} T^2 J_0(\beta m) - \frac{\pi}{3} T^2, \quad F_2 \approx 0. \quad (49)$$

A numerical calculation of $F_1 + F_2$ shows that for $m/T \leq 0.1$ this approximation has an error smaller than 10 percent. Approximating F_1 and F_2 using the high-energy approximation is less accurate. The result for the pressure in the high-temperature approximation is shown for comparison in Fig. 2 [again for $g^2(\mu=500)=10$].

One can approximate the pressure even further by expanding the functions J_0 and J_1 in the limit $\beta m \rightarrow 0$:

$$\begin{aligned} J_0 = & \frac{4\pi^2}{3} - 4\pi\beta m - 2 \left(\log \frac{\beta m}{4\pi} + \gamma_E - \frac{1}{2} \right) (\beta m)^2 \\ & + \mathcal{O}((\beta m)^4), \end{aligned} \quad (50)$$

$$J_1 = \frac{2\pi}{\beta m} + 2 \left(\log \frac{\beta m}{4\pi} + \gamma_E \right) + \mathcal{O}((\beta m)^2). \quad (51)$$

Inserting the approximations given in Eqs. (49) and (51) into the gap equation (37), one obtains

$$\beta m \approx \pi \left[\left(\frac{2\pi}{g^2(\mu)} - \frac{\ln 4}{N} \right) \left(1 + \frac{2}{N} \right) - \gamma_E - \ln \frac{\mu\beta}{4\pi} \right]^{-1}, \quad (52)$$

which indicates that $\beta m \sim 1/\ln T$ for large T . In the limit $m/T \rightarrow 0$, we obtain for the high-temperature approximation of the pressure

$$\frac{\mathcal{P}}{NT^2} \approx \frac{\pi}{6} \left(1 - \frac{1}{N} \right) - \left(1 - \frac{2}{N} \right) \frac{m}{4T}, \quad (53)$$

where the first term is the pressure of a gas of free massless particles with $N-1$ degrees of freedom.

V. THERMAL RENORMALONS

We have shown that a finite pressure at finite temperature can be obtained after subtraction of the zero-temperature pressure and coupling constant renormalization. This agrees with the general expectation that ultraviolet divergences are connected with short-distance physics and therefore independent of the temperature. While we have shown this explicitly at NLO in the $1/N$ expansion, this is not the case for the effective potential away from its minimum.

In the expression Eq. (11) for the effective potential at zero temperature, the two contributions $\Lambda^2 \ln \ln(\Lambda^2/m^2)$ and $m^2 \text{li}(\Lambda^2/m^2)$ cannot be removed using m -independent counterterms.¹ While this may not be a problem at zero temperature, it would certainly become one at finite temperature when m becomes a function of temperature. This would imply temperature-dependent renormalization, which is not acceptable. In Ref. [8], these two divergences are dealt with by considering the effective potential normalized to its zero-mass value, i.e. $\mathcal{V}_{\text{NLO}}(m) - \mathcal{V}_{\text{NLO}}(0)$. This subtraction is ill-defined due to infrared divergences and therefore one should understand it as subtracting the contributions from $\ln[\Pi(P, m)]$ obtained in the limit $P^2/m^2 \rightarrow \infty$ [24]. This is called the ‘‘perturbative tail.’’ If we denote $\Pi(P, m)$ in this limit by $\Pi_\infty(P, m)$, one finds $\Pi_\infty(P, m) = \ln(P^2/m^2)/(4\pi P^2)$ and

$$\int \frac{d^2 P}{(2\pi)^2} \ln[\Pi_\infty(P, m)] = \frac{1}{4\pi} \left[\Lambda^2 \ln \ln \frac{\Lambda^2}{m^2} - m^2 \text{li} \left(\frac{\Lambda^2}{m^2} \right) \right], \quad (54)$$

where we have implicitly used the principal-value prescription. In Refs. [8,24,25], this subtraction is not motivated, but we point out that the subtracted contribution is associated with IR renormalons. As shown in Ref. [11], the vacuum expectation value of α , i.e. m^2 , is inherently ambiguous, when one tries to separate (in order to subtract) perturbative contributions proportional to Λ^2 from the nonperturbative ones proportional to m^2 in the limit where $\Lambda^2 \gg m^2$. In Ref. [11], it was shown that

$$\langle 0|\alpha|0\rangle = m_{\text{LO}}^2 + \frac{4\pi m_{\text{LO}}^2}{N} \int \frac{d^2 P}{(2\pi)^2} \frac{1}{\Pi} \frac{\partial \Pi}{\partial m_{\text{LO}}^2} + \mathcal{O} \left(\frac{1}{N^2} \right), \quad (55)$$

¹Note that the quantity $\Lambda^2 \ln \ln(\Lambda^2/m^2) - \Lambda^2 \ln \ln(\Lambda^2/\mu^2)$ (with $\mu \neq m$) diverges as $\Lambda^2 \rightarrow \infty$, whereas $\ln \ln(\Lambda^2/m^2) - \ln \ln(\Lambda^2/\mu^2)$ vanishes.

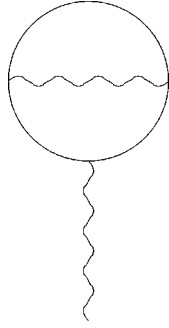


FIG. 3. Tadpole diagram contributing to $\langle\alpha\rangle$ at next-to-leading order in $1/N$. The wavy line represents the $\tilde{\alpha}$ propagator and the solid line the Φ propagator.

where $m_{\text{LO}}^2 \equiv \Lambda^2 \exp(-4\pi/g_b^2)$ and the $1/N$ contribution arises from the tadpole diagram shown in Fig. 3. One can show that this equation is in agreement with the gap equation (13) if we write $m^2 = m_{\text{LO}}^2 + m_{\text{NLO}}^2/N$.

The part of the integral in Eq. (55) that has the IR renormalon pole in the Borel plane is in fact the contribution from the integrand in the limit $P^2/m^2 \rightarrow \infty$:

$$\int^{\Lambda} \frac{d^2P}{(2\pi)^2} \frac{1}{\Pi_{\infty}} \frac{\partial \Pi_{\infty}}{\partial m^2} = -\frac{1}{4\pi} \text{li}\left(\frac{\Lambda^2}{m^2}\right) = -\frac{1}{4\pi} \frac{\Lambda^2}{m^2} e^{-x} \text{Ei}(x), \quad (56)$$

where $x = \ln(\Lambda^2/m^2)$. In the limit $x \rightarrow \infty$, the logarithmic integral has the asymptotic expansion:

$$e^{-x} \text{Ei}(x) = \sum_{n=0}^{\infty} \frac{n!}{x^{n+1}} \mp i\pi e^{-x} = \int_0^{\infty} db \frac{e^{-bx}}{1-b} \mp i\pi e^{-x}, \quad (57)$$

where $\arg(b) = \pm \varepsilon$. From Eq. (57), it is clear that there is a renormalon pole at $b=1$. This shows that when $\Lambda \rightarrow \infty$ the value of $\langle 0|\alpha|0\rangle$ is inherently ambiguous at NLO, due to the freedom in the choice of prescription. David has shown that this ambiguity also arises in dimensional regularization [10].

The same problem appears in the calculation of the effective potential, but *not* in the gap equation. The latter can be seen from the last term of Eq. (10), which contributes to the gap equation as follows:

$$\frac{1}{2} \frac{\partial}{\partial m^2} \int \frac{d^2P}{(2\pi)^2} \ln \Pi = \frac{1}{2} \int \frac{d^2P}{(2\pi)^2} \frac{1}{\Pi} \frac{\partial \Pi}{\partial m^2}. \quad (58)$$

The ambiguity that would arise from this term when removing its perturbative tail [cf. Eq. (56)] cancels in the gap equation (13) against the one arising in m^2 [cf. Eq. (55)].

The perturbative tail of the effective potential, i.e. the first two terms of D_2 defined in Eq. (30), corresponds to poles in the Borel plane at $b=0$ and $b=1$, respectively. Since \bar{m}^2 is only temperature independent at the minimum (at LO only, but that is sufficient since we are working at NLO), the subtraction of the perturbative tail will become temperature dependent, *except* at the minimum. Since subtracting temperature-dependent divergences renders the remaining temperature-dependent terms ambiguous, we refrain from following this strategy and thus from trying to define a finite effective potential at finite temperature. In order to avoid any renormalon ambiguity, we have also not considered obtaining a finite effective potential or even a finite pressure at zero temperature. However, we have calculated the quantity $\mathcal{P}^T - \mathcal{P}^{T=0}$, which is free of renormalon ambiguities and is finite after a temperature-independent coupling constant renormalization.

Finally, we comment on the possible temperature dependence of renormalon contributions to $\langle\alpha\rangle$ and the effective potential. One can show that Eq. (55) at finite temperature has exactly the same renormalon contribution, i.e. neither the pole nor the residue become temperature dependent. Secondly, the perturbative tail of the effective potential which is given by the first two terms of D_2 , corresponds to poles in the Borel plane at $b=0$ and $b=1$. The positions of the renormalon poles are not affected by temperature. Only the residues become temperature dependent, except at the minimum of the potential, as we concluded earlier. The fact that renormalon pole positions are not affected by temperature, but residues are, is also the case for the thermal ultraviolet renormalons in ϕ^4 in 3+1 dimensions studied by the authors of Ref. [14].

VI. SUMMARY AND CONCLUSIONS

To summarize, we have calculated the pressure in the NLSM at finite temperature to NLO in the $1/N$ expansion. Our main result is that we obtain an unambiguous, finite pressure, by subtracting the zero-temperature value of the pressure and renormalization of the coupling constant in a temperature-independent way. This procedure cannot be carried out away from the minimum of the effective potential and we have argued that defining a finite, effective potential by the subtraction of the so-called perturbative tail, leads to ambiguities associated with IR renormalons. In general, these become temperature dependent, and this casts doubt on the usefulness of defining a finite effective potential.

We have calculated the expression for the pressure at finite temperature numerically and observe that the $1/N$ expansion is a meaningful expansion for all temperatures. We have also investigated the high-energy approximation that was originally applied to the NLSM in 3+1 dimensions by Bochkevich and Kapusta. In 1+1 dimension, where one can compare with exact numerical results, we have shown that it underestimates the pressure for all temperatures. We have

suggested an improved approximation, the so-called high-temperature approximation. This approximation has the advantage that it is quite easy to produce numerical results and agrees better with the exact results. At asymptotically high temperatures the pressure approaches that of a gas of $N-1$ free massless particles.

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