PHYSICAL REVIEW D **69**, 074508 (2004)

Different couplings of the chemical potential with an identical partition function in QCD on a lattice

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(Received 25 October 2002; revised manuscript received 11 December 2003; published 29 April 2004)

New couplings of the chemical potential, closer to the continuum, are derived from the transfer matrix formulation of the partition function of QCD on a lattice. In particular the time splitting of the quark fields (and therefore the presence of the temporal gauge fields) can be avoided in some of these couplings.

DOI: 10.1103/PhysRevD.69.074508 PACS number(s): 11.15.Ha, 12.38.Mh

I. INTRODUCTION

The action generally used for QCD on a lattice at finite baryon density [1] was proposed on heuristic grounds by Hasenfratz and Karsch [2] and Kogut *et al.* [3] to avoid the divergence of the free energy which arises when the chemical potential is introduced as in the continuum. In this action the form of the coupling of the chemical potential is based on the analogy between chemical potential and temporal gauge field. Therefore the chemical potential is exponentiated like the temporal gauge field in the temporal link variables, and like these variables it is coupled to the quark fields split in the time direction. Because of gauge invariance, the chemical potential is therefore coupled also to the temporal links.

The possibility of other couplings was investigated [4], but at the end there seemed to be a general consensus about the need of the time splitting with the consequent presence of the temporal links, so that only the form of the function of the chemical potential (not necessarily exponential), can be to some extent arbitrary [5].

Recently some work [6–8] has been done to derive the Hasenfratz-Karsch action from the transfer matrix formulation of the partition function. This is a sound starting point in statistical mechanics of gauge theories because it permits a clear identification of physical degrees of freedom [9]. The chemical potential is then introduced as a Lagrange multiplier in the transfer matrix, but due to the manipulations leading to the path integral it appears in exponential form in the action. These manipulations are also responsible for the time splitting of the fermion fields in the coupling of the chemical potential and therefore for the appearance of the temporal links in this coupling.

The results of Ref. [6], however, do not confirm that the coupling of the chemical potential is essentially unique on the lattice. Indeed Creutz showed in a simple model that the time splitting can be avoided in this coupling. Moreover in the investigation of QCD in a given baryon sector we found that the change in the action which accounts for the projection involves both spatial and temporal links [10].

The settlement of this issue is interesting per se in the framework of the theoretical foundations of the formalism. But it is also of practical relevance. For instance, one would like to know if there is a formulation which might alleviate the difficulties met in numerical simulations [1]. Moreover, one would like to understand why in a perturbative calculation with the Hasenfratz-Karsch Lagrangian one has to collect all the terms which realize a Polyakov loop to get a nonvanishing contribution, in contrast with the coupling of the continuum. Are we facing a mere lattice artifact or a feature of deep physical significance?

We have therefore reexamined the derivation of the Hasenfratz-Karsch action from the transfer matrix. The connection between the action at finite baryon density and the transfer matrix is established in the following way. The path integral at zero baryon density determines the transfer matrix. Starting from this transfer matrix and introducing the chemical potential as the Lagrange multiplier of the baryon charge, one gets a path integral with a new action. But the form of this action depends on how we pass from the trace of the transfer matrix to the path integral. The standard way relies on the normal ordering of fermionic operators (all the creation operators to the left of the annihilation ones) and a specific way of inserting the Grassmannian kernel of the identity in the trace [11]. Following this way the Hasenfratz-Karsch action was derived [6–8]. In the present paper we follow the same strategy but a different path, namely we adopt the antinormal ordering of fermionic operators (all the annihilation operators to the left of the creation ones) and a different insertion of the unity. We then find another coupling without time splitting and temporal gauge fields. In particular for Kogut-Susskind fermions in the flavor basis, the chemical potential is not coupled to the gauge fields at all.

Obviously the quark determinants with the two couplings are identical, since they give the same partition function (also at finite lattice spacing). We have checked this identity by putting the quark matrices in one and the same form. It is remarkable that this form provides still another coupling of the chemical potential which involves both the temporal and the spatial links as in QCD in a given baryon sector [10]. The fact that the gauge fields appear in different ways or disappear altogether confirms that their presence in this coupling is a mere lattice artifact. The identity of the quark determinants can also be exploited to derive some identities for vanishing chemical potential.

The present work can be considered from a different point of view. The path integral is often used only as a means to generate a perturbative expansion. In such a case the action can be that of the continuum. But when one wants to use it in

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a concrete way, to perform analytic or numerical estimates, one needs a concrete definition which requires a discretization of time. There are different ways of discretizing, and different discretizations are equivalent from the point of view of the renormalization group because the corresponding actions differ in general by irrelevant operators. But their renormalizations are different, and the necessary counterterms can be more or less convenient from a practical point of view. It is therefore desirable to have different discretized actions with the same partition function at finite temporal spacing from which to choose the most convenient one for a given purpose. The problem is of general character. We restrict ourselves to the definition of the chemical potential, but our technique can have wider applications, including the many-body theory of fermions and bosons. We will comment on this in our Conclusions.

The paper is organized in the following way. In Sec. II we describe our alternative procedure and derive a new coupling of the chemical potential. In Sec. III we apply our results to QCD at finite baryon density with Wilson and Kogut-Susskind fermions. In Sec. IV we evaluate the energy for free fermions. It is obvious that with a different coupling of the chemical potential but the same quark determinant the free energy must also be the same, but it can be instructive to see how this comes about. In Sec. V we show that the Hasenfratz-Karsch action and the present one give one and the same quark determinant with new couplings of the chemical potential, and we show how some identities for vanishing chemical potential can be obtained. Finally in Sec. VI we present our conclusions.

II. ANTINORMAL ORDERING IN THE CHARGE

Let us introduce some definitions. We denote by a_0 the temporal lattice spacing, by N_0 the number of temporal sites, by x_0 the temporal component of the site position vector x, by T the temperature, by μ the chemical potential, by $\hat{Q}$ the (electric, baryon...) charge operator, and by $\hat{\mathcal{T}}(x_0)$ the fermion transfer matrix. The pure gauge part will be omitted because it does not play any role in the present problem.

We start from the definition of the grand canonical partition function according to the time ordered product

$$Z=\mathrm{Tr}\left\{\exp\left(\frac{\mu}{T}\hat{Q}\right)\prod_{x_0}\hat{\mathcal{T}}(x_0)\right\}, \tag{1}$$

which, using the relation $T^{-1}=a_0N_0$, and assuming the conservation of $\hat{Q}$, is conveniently rewritten as

$$Z=\mathrm{Tr}\left\{\prod_{x_0}[\hat{\mathcal{T}}(x_0)\exp(\mu a_0\hat{Q})]\right\}. \tag{2}$$

$\hat{\mathcal{T}}$ is defined in terms of particle-antiparticle creation-annihilation operators $\hat{c}^\dagger,\hat{d}^\dagger,\hat{c},\hat{d}$ acting in a Fock space. It depends on the time coordinate x_0 only through the dependence on it of the gauge fields. In fact the creation and annihilation operators do not depend on x_0. They depend on the spatial coordinates $\mathbf{x}$ and on the internal quantum numbers (Dirac, color and flavor indices in the case of QCD), comprehensively represented by $I,J\ldots$.

In the transfer matrix formalism often one has to use quantities at a given (Euclidean) time x_0. For this reason we adopt a summation convention over spatial coordinates and intrinsic indices at fixed time. So for instance, for an arbitrary matrix M, we will write

$$\hat{c}^\dagger M(x_0)\hat{c}=\sum_{\mathbf{x},\mathbf{y},I,J}\hat{c}^\dagger_{\mathbf{x},I}M_{\mathbf{x},I;\mathbf{y},J}(x_0)\hat{c}_{\mathbf{y},J}\,. \tag{3}$$

In this notation the charge operator $\hat{Q}$ can be written

$$\hat{Q}=\hat{c}^\dagger\hat{c}-\hat{d}^\dagger\hat{d}. \tag{4}$$

The standard way [11] to obtain the path integral form of Z is to write all the operators in normal order and introduce between the factors in Eq. (2) the identity [12]

$$\mathbb{1}=\int[dc^+dc\,dd^+dd]\exp(-c^+c-d^+d)|cd\rangle\langle cd|, \tag{5}$$

where the basis vectors are coherent states

$$|cd\rangle=|\exp(-c\hat{c}^\dagger-d\hat{d}^\dagger)\rangle. \tag{6}$$

The c^+,c,d^+,d are Grassmann variables. They depend on the time slice where the unit operator is introduced. For the other indices they are subject to the same convention as the creation and annihilation operators. The main property of coherent states is that they are eigenstates of the annihilation operators with Grassmannian eigenvalues

$$\hat{c}|cd\rangle=c|cd\rangle. \tag{7}$$

We will follow a slightly different strategy. We write the exponential of the charge in the following way:

$$\exp(\mu a_0\hat{Q})=\int[dc^+dc\,dd^+dd]\times\exp(\delta S-c^+c-d^+d)|cd\rangle\langle cd| \tag{8}$$

where

$$\delta S=[1-\cosh(\mu a_0)](c^+c+d^+d)+\sinh(\mu a_0)(c^+c-d^+d). \tag{9}$$

The above expression is obtained by expanding the exponential of the charge operator, putting all the terms in antinormal form, inserting in each monomial the unity between the set of annihilation and the set of creation operators and replacing them by their Grassmannian eigenvalues. For the rightmost exponential of the charge before taking the trace one has to move the creation operators to the left of all the operators appearing under trace.

After this, the construction of the path integral proceeds in the standard way, and we get the standard action with the exception of the coupling of the chemical potential where all the fields appear at the same time.

III. QCD AT FINITE DENSITY

The fermion transfer matrix $\hat{\mathcal{T}}$ in the case of QCD can be written [11] in terms of an auxiliary operator $\hat{T}(x_0)$

$$\hat{\mathcal{T}}(x_0)=\mathcal{J}^{-1}\hat{T}^{\dagger}(x_0)\hat{T}(x_0+s), \tag{10}$$

where $\mathcal{J}$ is a function of the gauge fields which will be defined later and $s=\pm 1$ for Wilson/Kogut-Susskind fermions respectively [10]. This difference in sign does not reflect any intrinsic difference, but is only due to the different conventions adopted by Lüscher [11] and Montvay-Münster [5], which we maintain for easy reference.

In our notation the auxiliary operator $\hat{T}(x_0)$ can be written

$$\begin{aligned}\hat{T}(x_0)=&\exp[-\hat{c}^{\dagger}M(x_0)\hat{c}-\hat{d}^{\dagger}M^{T}(x_0)\hat{d}]\\&\times\exp[\hat{d}N(x_0)\hat{c}],\end{aligned} \tag{11}$$

where T means transposed. Its form is the same for Wilson and Kogut-Susskind fermions but the matrices M and N are different in the two cases and will be specified later.

We must notice that the expression of $\hat{T}(x_0)$ was derived [11] in the gauge $U_0=\mathbb{1}$ which is not admissible. But it can be repeated in the gauge $U_0\sim\mathbb{1}$, where $U_0=\mathbb{1}$ with the exception of a single time slice. The temporal links on this time slice generate the Gauss constraint, namely the restriction to gauge invariant states in the trace, which was actually assumed in the quoted work. Its presence does not interfere with the derivation of the new couplings.

In any case in the construction of the path integral formulation of QCD at finite baryon density we do not need to fix the gauge, and only to lighten the formalism we put $U_0=\mathbb{1}$ and reinstate U_0 in the final result. The reader can check that keeping U_0 arbitrary in the intermediate steps one arrives at the same result, provided some care is exercised: for instance when $U_0\neq\mathbb{1}$ the expression $\hat{c}^{\dagger}M(x_0)\hat{c}+\hat{d}^{\dagger}M^{T}(x_0)\hat{d}$ appearing in Eq. (11) changes and does not commute with $\hat{Q}$ any longer. We anticipate that N is Hermitian and also M is Hermitian in the gauge $U_0=\mathbb{1}$.

As already said after use of Eq. (8) the construction of the path integral proceeds in the standard way [11], yielding the standard action for zero chemical potential plus δS. To write the latter in terms of the quark field q we must distinguish the case of Wilson fermions from that of Kogut-Susskind.

A. Wilson fermions

For Wilson fermions we set $s=1$ in Eq. (10) and assume

$$\begin{aligned}M_W(x_0)&=-\ln[(2K)^{1/2}B^{-1/2}(x_0)],\\N_W(x_0)&=2KB(x_0)^{-1/2}C(x_0)B(x_0)^{-1/2},\end{aligned} \tag{12}$$

where K is the hopping parameter and

$$\begin{aligned}B(x_0)&=\mathbb{1}-K\sum_{j=1}^{3}[\hat{U}_j(x_0)T_j^{(+)}+T_j^{(-)}\hat{U}_j^{+}(x_0)]\\C(x_0)&=\frac{1}{2}\sum_{j=1}^{3}i\sigma_j[\hat{U}_j(x_0)T_j^{(+)}-T_j^{(-)}\hat{U}_j^{+}(x_0)].\end{aligned} \tag{13}$$

Let us remark that all the above matrices are understood to have dimension 2 in Dirac space. The link operators $\hat{U}_{\mu}(x_0)$ have the standard Wilson variables $U_{\mu}(x)$ as spatial matrix elements

$$[\hat{U}_{\mu}(x_0)]_{\mathbf{x},\mathbf{y}}=\delta_{\mathbf{x},\mathbf{y}}U_{\mu}(x). \tag{14}$$

We have introduced the translation operators $T_j^{(\pm)}$ with matrix elements

$$(T_{\mu}^{(\pm)})_{x,y}=\delta_{y,x\pm e_{\mu}}, \tag{15}$$

$(e_{\mu})_{\nu}=\delta_{\mu,\nu}$ being the components of the unit vectors. Next we define the quark field q by the transformation

$$c=B^{1/2}P_0^{(+)}qa^{3/2},\quad d^{+}=B^{1/2}P_0^{(-)}qa^{3/2},\quad \bar{q}=q^{\dagger}\gamma_0 \tag{16}$$

where a is the spatial lattice spacing and $P_0^{(\pm)}=\frac{1}{2}(1\pm\gamma_0)$ (but c,d are understood to carry only the indices of the nonvanishing components). The Jacobian of this transformation is the function $\mathcal{J}$ introduced in Eq. (10). The partition function takes the form

$$Z_W=\int[d\bar{q}dq]\exp(S_W+\delta S_W). \tag{17}$$

S_W is the action with zero chemical potential and Wilson parameter $r=1$

$$\begin{aligned}S_W=\sum_x\Bigg\{K\sum_{\mu}&[\bar{q}(x)(1+\gamma_{\mu})U_{\mu}(x)q(x+e_{\mu})+\bar{q}(x+e_{\mu})\\&\times(1-\gamma_{\mu})U_{\mu}^{\dagger}(x)q(x)]-\bar{q}(x)q(x)\Bigg\}\end{aligned} \tag{18}$$

while δS_W is the contribution of the chemical potential which is obtained from Eq. (9)

$$\begin{aligned}\delta S_W=\sum_{x_0}&\bar{q}(x_0)\{[1-\cosh(\mu a_0)]+\sinh(\mu a_0)\gamma_0\}\\&\times B(x_0)q(x_0).\end{aligned} \tag{19}$$

The first term in δS_W breaks the chiral symmetry, but this is only a consequence of the breaking by the Wilson term. Indeed we will see that the corresponding term for Kogut-Susskind fermions does respect the chiral invariance. Notice the “plus” sign in the exponential of the action, to comply with Lüscher’s convention.

B. Kogut-Susskind fermions

Kogut-Susskind fermions will be studied in the flavor basis. The reason is that only in this case is a transfer matrix which defines a Hamiltonian known to us [13]. (In the only other work on the subject of which we are aware [14] the transfer matrix is linear in the creation-annihilation operators of the quarks, so that the resulting Hamiltonian is highly nonlocal, containing the logarithms of these operators.) The case of the spin diagonal basis can be obtained straightforwardly by a change of basis, but in this way nonminimal gauge couplings of the type discussed in [8] are generated. We think that in their absence, namely for the standard spin diagonal action, there exists no transfer matrix. This need not be a serious shortcoming, because the nonminimal couplings are irrelevant from the point of view of the renormalization group. But our method cannot be used with the standard spin diagonal action.

The x_μ are now the block coordinates and the gauge fields are defined on the block links. We set $s=-1$ in Eq. (10) and assume [8]

$$\begin{aligned} M_{KS}(x_0)&=0\\ N_{KS}(x_0)&=\left\{\sum_{j=1}^{3}\{\gamma_5\otimes t_5 t_j+\gamma_j[P_j^{(-)}\hat{U}_j(x_0)T_j^{(+)}\right.\\ &\quad -P_j^{(+)}T_j^{(-)}\hat{U}_j^{+}(x_0)]\}+\frac{m}{K}\mathbb{1}\otimes\mathbb{1}+\gamma_5\\ &\quad\left.\otimes t_5 t_0\right\}, \end{aligned} \tag{20}$$

where m is the quark mass parameter and K the hopping parameter. In the tensor product the γ-matrices act on Dirac indices, while the matrices $t_\mu=\gamma_\mu^T$ act on flavor indices. The projection operators $P_\mu^{(\pm)}$ are given by

$$P_\mu^{(\pm)}=\frac{1}{2}[\mathbb{1}\otimes\mathbb{1}\pm\gamma_\mu\gamma_5\otimes t_5 t_\mu]. \tag{21}$$

The quark field q is obtained by the transformation

$$\begin{aligned} c&=4\sqrt{K}P_0^{(+)}qa^{3/2},\\ d^{+}&=4\sqrt{K}P_0^{(-)}qa^{3/2},\quad \overline{q}=q^{\dagger}\gamma_0 \end{aligned} \tag{22}$$

whose Jacobian is the function $\mathcal{J}$ introduced in Eq. (10). We must remark that here the notation has a slightly different meaning from the Wilson case. c,d carry all their 16 flavor-Dirac indices so that N_{KS} has dimension 16 in this space. The partition function takes the form

$$Z_{KS}=\int[d\overline{q}dq]\exp[-16(S_{KS}+\delta S_{KS})]. \tag{23}$$

S_{KS} is the Kogut-Susskind action with zero chemical potential

$$\begin{aligned} S_{KS}=\frac{1}{2}K\Bigg\{\sum_x &\overline{q}(x)(\gamma_\mu\otimes\mathbb{1}-\gamma_5\otimes t_5 t_\mu)U_\mu(x)q(x+e_\mu)\\ &-\overline{q}(x)(\gamma_\mu\otimes\mathbb{1}+\gamma_5\otimes t_5 t_\mu)U_\mu^{\dagger}(x-e_\mu)q(x-e_\mu)\\ &+2\overline{q}(x)\gamma_5\otimes t_5 t_\mu q(x)+am\overline{q}(x)\mathbb{1}\otimes\mathbb{1}q(x)\Bigg\} \end{aligned} \tag{24}$$

while the chemical potential contribution is obtained from Eq. (9)

$$\begin{aligned} \delta S_{KS}=-K\sum_x &\overline{q}(x)(\{[1-\cosh(\mu a_0)]\gamma_5\otimes t_5 t_0\\ &+\sinh(\mu a_0)\gamma_0\otimes\mathbb{1}\})q(x). \end{aligned} \tag{25}$$

The factor 16 in front of the action accounts for the fact that the volume element with Kogut-Susskind fermions is 16 times larger than in the Wilson case.

IV. ENERGY DENSITY FOR FREE WILSON FERMIONS

There is no need to emphasize that the partition function with the present action is identical to that of Hasenfratz and Karsch. Nevertheless we deem it is instructive to compare to the "naive" definition to see how the divergences of the latter disappear. For simplicity we will consider only the Wilson case neglecting the mass and the spatial Wilson term. But notice that we cannot omit also the temporal one, because otherwise we cannot construct the transfer matrix. Then the quark matrix is

$$Q=\gamma_0(a_0\nabla_0+\sigma)+a_0\gamma\cdot\nabla+\frac{1}{2}a_0^2\Box_0+1-R \tag{26}$$

where

$$\begin{aligned} (\nabla_\mu f)(x)&=\frac{1}{2a_\mu}[f(x+e_\mu)-f(x-e_\mu)]\\ (\Box_\mu f)(x)&=\frac{1}{a_\mu}[f(x+e_\mu)+f(x-e_\mu)-2f(x)] \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\sigma=\sinh(\mu a_0),\quad R=\cosh(\mu a_0),\quad \text{new definition}\\ &\sigma=\mu a_0,\quad R=1,\quad \text{``naive'' definition}. \end{aligned} \tag{28}$$

We now evaluate the energy density $\mathcal{E}$ at zero temperature, setting $a_0=a$. Normalizing at zero baryon density

$$\mathcal{E}=-\frac{1}{2\pi^3a^4}\int_{-\pi}^{\pi}d^3qs^2[\mathcal{I}(\mu)-\mathcal{I}(0)], \tag{29}$$

where $s^2=\Sigma_j(1-\cos q_j)$ and

$$\mathcal{I}(\mu)=\frac{1}{2\pi}\int_{-\pi}^{\pi}dq_0[(\sin q_0-i\sigma)^2+s^2+(\cos q_0-R)^2]^{-1}$$
$$=\frac{1}{2\sqrt{A^2-B^2}}[1-\theta(R-A)]. \tag{30}$$

θ is the step function. With the new coupling of the chemical potential

$$R-A=\frac{1}{2}[s^2+2(1-\cosh(\mu a))]$$
$$\sim\frac{1}{2}[s^2-(\mu a)^2]$$
$$A^2-B^2=s^2\left[1+\frac{1}{4}s^2\right] \tag{31}$$

while with the ‘‘naive’’ definition

$$R-A=\frac{1}{2}[s^2-(\mu a)^2]$$
$$A^2-B^2=s^2\left[1+\frac{1}{4}s^2-\frac{1}{2}(\mu a)^2+\frac{(\mu a)^4}{4s^2}\right]. \tag{32}$$

Therefore in the ‘‘naive’’ case the energy density has a quadratic divergence [2], while with the new coupling

$$\mathcal{E}=\frac{2}{\pi^2}\mu^4\theta(\mu^2), \tag{33}$$

equal to the value obtained by the Hasenfratz-Karsch prescription in the presence of the temporal Wilson term (without the spatial one).

V. IDENTITY OF THE QUARK DETERMINANT WITH THE HASENFRATZ-KARSCH AND THE PRESENT ACTIONS

In this section we will prove the identity of the quark determinant with the Hasenfratz-Karsch and the present actions. We think it is worth reporting because it provides us with a further coupling of the chemical potential. We will then show how identities for vanishing chemical potential can be derived.

We perform the Grassmann integration in a way which leads in both cases to one and the same quark determinant. Again for simplicity we restrict ourselves to Wilson fermions but the results hold also in the case of Kogut-Susskind. We start by writing the quark action in a form common to the two cases

$$S=\sum_{x_0} c^+(x_0)M_+(x_0)c(x_0+1)$$
$$+d^+(x_0)M_-(x_0)d(x_0+1)+c^+(x_0)N(x_0)d^+(x_0)$$
$$+d(x_0)N(x_0)c(x_0)-\alpha_+c^+(x_0)c(x_0)$$
$$-\alpha_-d^+(x_0)d(x_0), \tag{34}$$

where

$$M_\pm=M\exp(\pm\mu),\quad \alpha_\pm=1\quad \text{for the Hasenfratz-Karsch action,}$$
$$M_\pm=1,\quad \alpha_\pm=\cosh\mu\mp\sinh\mu\quad \text{for the present action,} \tag{35}$$

and M is given by Eq. (12). Then we rewrite S as the sum of a term quadratic in the variables c^+ and d^+ properly shifted, plus a term quadratic in the variables c,d

$$S=\sum_{x_0}[c^+(x_0)-d(x_0+1)M_-^T(x_0)N^{-1}(x_0)+\alpha_-d(x_0)N^{-1}(x_0)]N(x_0)[d^+(x_0)+N^{-1}M_+c(x_0+1)-\alpha_-N^{-1}(x_0)c(x_0)]$$
$$+d(x_0)N(x_0)c(x_0)-\alpha_+d(x_0+1)M_-^T(x_0)N^{-1}(x_0)c(x_0)-\alpha_-d(x_0)N^{-1}(x_0)M_+(x_0)c(x_0+1)$$
$$-d(x_0+1)M_-^T(x_0)N^{-1}(x_0)M_+(x_0)c(x_0+1)+\alpha_+\alpha_-d(x_0)N^{-1}(x_0)c(x_0). \tag{36}$$

Now we integrate first over c^+,d^+ and then on c,d, getting the determinant

$$\det N\det\{N+\alpha_+\alpha_-N^{-1}+T_0^{(-)}M_-^TN^{-1}M_+T_0^{(+)}$$
$$-\alpha_+T_0^{(-)}M_-^TN^{-1}-\alpha_-N^{-1}M_+T_0^{(+)}\}, \tag{37}$$

which can be rewritten in the form

$$\det\{\mathbb{1}+N^2+N^{1/2}T_0^{(-)}M_-^TN^{-1}M_+T_0^{(+)}N^{1/2}$$
$$-\alpha_+N^{1/2}T_0^{(-)}M_-^TN^{-1/2}-\alpha_-N^{-1/2}M_+T_0^{(+)}N^{1/2}\}. \tag{38}$$

We note that the presence of N^{-1} is not dangerous, because from the standard form of the determinant we know that there cannot be divergencies due to zeros of N. Introducing the values of $M_\pm$ and $\alpha_\pm$ we get in both cases

$$\begin{aligned}\det\{\mathbb{1}+N^2+N^{1/2}T_0^{(-)}M^+N^{-1}MT_0^{(+)}N^{1/2}\\ -\cosh\mu[N^{1/2}T_0^{(-)}M^+N^{-1/2}+N^{-1/2}MT_0^{(+)}N^{1/2}]\\ +\sinh\mu[N^{1/2}T_0^{(-)}M^+N^{-1/2}-N^{-1/2}MT_0^{(+)}N^{1/2}]\}.\end{aligned}\tag{39}$$

We remind the reader that in the above equation the matrices have dimension 2 in Dirac space and the temporal links must be reinstated. We see a new coupling of the chemical potential which does involve both the temporal and the spatial links.

Finally we observe that a set of functional identities can be obtained by equating the derivatives of the partition function with the present action and that of Hasenfratz and Karsch with respect to the chemical potential at zero chemical potential. For instance with Wilson fermions by taking the second derivative we get

$$\begin{aligned}\langle\overline{q}[B+2K(P_0^{(+)}U_0T_0^{(+)}+P_0^{(-)}T_0^{(-)}U_0^+)]q\rangle\\ +a^3\langle(\overline{q}\gamma_0Bq)^2\rangle+4K^2a^3\langle[\overline{q}(P_0^{(+)}U_0T_0^{(+)}\\ -P_0^{(-)}T_0^{(-)}U_0^+)q]^2\rangle=0.\end{aligned}\tag{40}$$

VI. CONCLUSION

The motivation of the present work was to go deeper in the foundations of the formalism of QCD at finite baryon density. We reconsidered the derivation of the path integral from the transfer matrix formalism, which is a sound starting point when doing statistical mechanics in gauge theories because the physical degrees of freedom are clearly exhibited. We have then found that such a derivation is not unique, and in addition to the Hasenfratz-Karsch action we can get another coupling of the chemical potential which is closer to the continuum, since it does not contain time splitting and temporal gauge fields. This generalizes the results of [6]. We must emphasize that the fermion determinant is exactly equal, at finite lattice spacing, to that of the Hasenfratz-Karsch action. In the course of a check of this identity, we found a still different coupling, which has some features in common with the change in the action due to the projection in a given baryon sector.

Needless to say, our results do not constitute progress towards the solution of the sign problem, even though once a solution will be found, the present formulation might prove helpful. But they can be of practical relevance in other ways. Examples are perturbative calculations and the functional identities at zero chemical potential reported in Sec. V. Moreover the present procedure can be applied to many-body theories of fermions and bosons and to other interaction terms. In the standard formulation of the path integral of these theories in fact the fields and their conjugates appear at split time in all the potential terms. With our form of the action instead the time splitting can be avoided in the potential terms which retain their continuum expression. This makes the action closer to the continuum, even though the discrete form of the kinetic terms cannot be eliminated.

ACKNOWLEDGMENT

This work has been partially supported by EEC under the contract HPRN-CT-2000-00131.

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