Coulomb energy, remnant symmetry, and the phases of non-Abelian gauge theories

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We show that the confining property of the one-gluon propagator, in the Coulomb gauge, is linked to the unbroken realization of a remnant gauge symmetry which exists in this gauge. An order parameter for the remnant gauge symmetry is introduced, and its behavior is investigated in a variety of models via numerical simulations. We find that the color-Coulomb potential, associated with the gluon propagator, grows linearly with distance both in the confined and—surprisingly—in the high-temperature deconfined phase of pure Yang-Mills theory. We also find a remnant symmetry-breaking transition in SU(2) gauge-Higgs theory which completely isolates the Higgs region from the (pseudo)confinement region of the phase diagram. This transition exists despite the absence, pointed out long ago by Fradkin and Shenker, of a genuine thermodynamic phase transition separating the two regions.

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I. INTRODUCTION

It is well known that the phase transition from a confined to an unconfined phase in a non-Abelian gauge theory is associated with the breaking of global center symmetry. If a nontrivial center symmetry exists and is unbroken, then Polyakov line expectation values are zero, and as a consequence the free energy of a static color charge is infinite in an infinite volume. But the confined phase may be associated with other global symmetries as well. In this article we will focus on the remnant gauge symmetry which is found after imposing the Coulomb gauge. What is interesting about this symmetry is that its unbroken realization implies the existence of a confining color-Coulomb potential, and this in turn is a necessary (but not sufficient) condition for confinement.

The color-Coulomb potential arises from the energy of the longitudinal color-electric field in Coulomb gauge, and corresponds diagrammatically to instantaneous dressed one-gluon exchange between static sources. We have previously studied this potential numerically in pure lattice Yang-Mills theory at zero temperature, and found that the potential rises linearly with color charge separation [1,2], albeit with a string tension $\sigma_{\rm Coul}$ which is significantly higher (by about a factor of 3 [2]) than the string tension σ of the static quark potential. In this article we introduce an order parameter for remnant gauge symmetry breaking and study its behavior, as well as the behavior of the color-Coulomb potential, in the following cases.

(1) SU(2) gauge-Higgs theory, with the Higgs field in the adjoint representation.
(2) Pure SU(2) gauge theory, in the confined and the high-temperature deconfined phases.
(3) SU(2) gauge-Higgs theory, with the Higgs field in the fundamental representation.
(4) Compact four-dimensional QED (QED_4).

In the first, second, and fourth theories the action has a nontrivial global center symmetry, which may be broken spontaneously in some range of couplings or temperature. In the third case center symmetry is explicitly broken, and the asymptotic string tension vanishes at all couplings. By studying these different cases, we can explore to what extent remnant symmetry breaking is correlated with center symmetry breaking, and whether the confining Coulomb potential is always associated with a confining static quark potential.

This article is organized as follows. In Sec. II we introduce the remnant symmetry order parameter, and relate it to the confining properties of the color-Coulomb potential. We also examine scaling of $\sigma_{\rm Coul}$ with β. Section III concerns the gauge-adjoint Higgs model, where we find perfect correspondence between remnant symmetry and center symmetry breaking. But this correspondence is lost already in pure Yang-Mills theory at high temperature, studied in Sec. IV, where we find that Coulomb confinement and unbroken remnant symmetry persist in the deconfined phase. A possible explanation of this phenomenon is discussed. In Sec. V we review the Gribov confinement scenario in Coulomb gauge and show that it accords with the vortex dominance scenario, by gauge transforming from the maximal center gauge to the minimal Coulomb gauge. In Sec. VI we present our results for the gauge-fundamental Higgs model, where the gauge-Higgs interaction breaks global center symmetry explicitly. In this case we find very clear numerical evidence of a remnant symmetry-breaking transition which is unaccompanied by a true thermodynamic phase transition, and also argue for the existence of such a transition from a lattice strong-

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coupling analysis. This result is in complete accord with the earlier work of Langfeld [3,4], which found remnant symmetry breaking in Landau gauge, and it implies that a sharp distinction can be made between the Higgs and the (pseudo)confining regions of the gauge-Higgs coupling plane. This distinction exists despite the fact, pointed out by Fradkin and Shenker [5], that these regions are continuously connected in the usual sense of thermodynamics. The remnant symmetry breaking in the adjoint Higgs theory is reexamined in Sec. VII, where we draw some conclusions about the measure of Abelian configurations in the fundamental modular region. Our methods are applied to compact QED_4 in Sec. VIII.

As in the previous work of Ref. [2], we also investigate numerically the relevance of center vortices to the existence of a confining Coulomb potential, particularly in the high-temperature deconfined phase of pure Yang-Mills theory, and in the (pseudo)confined phase of gauge-fundamental Higgs theory. In Sec. V it is shown that thin center vortices lie on the Gribov Horizon, which may be relevant to their dramatic effect on the Coulomb potential. Vortex removal, by the de Forcrand–D'Elia procedure [6], is found in every case to convert a confining Coulomb potential to a nonconfining, asymptotically flat potential.

In Sec. IX, we provide a 2-way translation between the temporal gauge ($A_0=0$) and the minimal Coulomb gauge. This allows our measurements, which are made by gauge-fixing to the minimal Coulomb gauge, to be equivalently described in temporal gauge. We show that the state in which our measurements are made is, in temporal gauge, a quark-pair state of the type introduced by Lavelle and McMullan [7]. It has correct gauge-transformation properties, although it does not make use of Wilson lines running between the sources. Section X contains some concluding remarks.

II. COULOMB ENERGY AND REMNANT SYMMETRY

On the lattice, minimal Coulomb gauge consists of fixing to the configuration on the gauge orbit maximizing the quantity

$$R=\sum_{\mathbf{x},t}\sum_{k=1}^{3}\mathrm{Tr}[U_k(\mathbf{x},t)]. \tag{2.1}$$

Maximizing R does not fix the gauge completely, since there is still the freedom to perform time-dependent gauge transformations

$$U_k(\mathbf{x},t)\to g(t)U_k(\mathbf{x},t)g^\dagger(t)\quad (k=1,2,3),$$

$$U_0(\mathbf{x},t)\to g(t)U_0(\mathbf{x},t)g^\dagger(t+1). \tag{2.2}$$

To understand the role of this remnant gauge symmetry, consider a state Ψ_q^a obtained by operating on the lattice Yang-Mills vacuum state Ψ_0 (i.e., the ground state of the Coulomb gauge transfer matrix) with a heavy quark operator q^a

$$\Psi_q^a[\mathbf{x};A]=q^a(\mathbf{x})\Psi_0[A]. \tag{2.3}$$

Evolve this state for Euclidean time $T<n_t$, where n_t is the lattice extension in the time direction. Dividing out the vacuum factor $\exp[-E_0T]$ and taking the inner product with $\Psi_q^b[\mathbf{x};A]$, we have

$$\begin{aligned}G^{ba}(T)&=\langle\Psi_q^b|e^{-(H-E_0)T}|\Psi_q^a\rangle\\&=\sum_n\langle\Psi_q^b|\Psi_n\rangle\langle\Psi_n|\Psi_q^a\rangle e^{-(E_n-E_0)T}\\&=\Gamma e^{-m_qT}\langle L^{ba}(\mathbf{x},1,T)\rangle,\end{aligned} \tag{2.4}$$

where index n refers to summation over charged energy eigenstates, $m_q\to\infty$ is the heavy quark mass, Γ is a numerical factor, and $L(\mathbf{x},t_1,t_2)$ is a Wilson line

$$L(\mathbf{x},t_1,t_2)=U_0(\mathbf{x},t_1)U_0(\mathbf{x},t_1+1)\cdots U_0(\mathbf{x},t_2). \tag{2.5}$$

In a confining theory, the excitation energy $\Delta E_n=E_n-E_0$ of any state Ψ_n containing a single static quark must be infinite. It follows that $G^{ba}(T)=0$, and therefore that the expectation value of any timelike Wilson line $L(\mathbf{x},1,T)$ must vanish. Now, under the remnant gauge symmetry, L transforms as

$$L(\mathbf{x},t_1,t_2)\to g(t_1)L(\mathbf{x},t_1,t_2)g^\dagger(t_2+1). \tag{2.6}$$

If $T=t_2-t_1+1$ is less than the lattice extension n_t in the time direction, then not only L but also $\mathrm{Tr}[L]$ is noninvariant under the remnant gauge symmetry.[1] If the remnant symmetry is unbroken, then $\langle L\rangle$ must vanish. The confining phase is therefore a phase of unbroken remnant gauge symmetry; i.e., unbroken remnant symmetry is a necessary condition for confinement.

There are several issues which require some further comment. First of all, why is unbroken remnant symmetry not also a *sufficient* condition for confinement? The answer is that in an unconfined phase, where there exist finite energy states containing a single static charge in an infinite volume, there is still the possibility that these finite energy states have vanishing overlap with Ψ_q^a as defined in Eq. (2.3). Thus $\langle L\rangle=0$ and unbroken remnant symmetry could be found, in principle, also in the absence of confinement. Secondly, it is obviously impossible to insert a single charge in a finite volume with, e.g., periodic boundary conditions; electric field lines starting from the static charge must end on some other charge, regardless of whether or not the theory is in a confining phase. How, then, is this fact reflected in the remnant symmetry-breaking criterion? The same question can be raised in connection with Polyakov lines, and the answer is the same: Strictly speaking, spontaneous symmetry breaking cannot occur in a finite volume, so $\langle L\rangle=0$ always, consistent with the absence of an isolated static charge. Nevertheless (again similar to Polyakov lines), it is possible to construct an order parameter which detects the infinite-volume transition via finite-volume calculations which are subsequently

[1] If $T=n_t$, then L is a Polyakov line, whose trace is invariant under gauge transformations, but noninvariant under global center transformations.

extrapolated to infinite volume.[2] We will construct such an operator below. Finally, there is the question of Elitzur's theorem. Although the remnant symmetry is global on a time slice, it is local in the time direction, and according to the theorem local symmetries cannot break spontaneously. So how could we ever have $\langle L\rangle \neq 0$, even in an infinite volume? The answer is that in fact the average value of $L(\mathbf{x},t,T+t)$ does indeed vanish on an infinite lattice, in accordance with the Elitzur theorem, providing the averaging is done over all spatial $\mathbf{x}$ and all times t. On a time slice, however, the symmetry is only global, and it is possible in any given configuration that the average value of $L(\mathbf{x},t,T+t)$ is finite on an infinite lattice, when averaged over all $\mathbf{x}$ at *fixed* time t. This is what we will mean by the phrase "spontaneous breaking of the remnant symmetry," and it involves no actual violation of the Elitzur theorem.

With these points in mind, we propose to construct an order parameter for remnant symmetry breaking from the timelike link variable averaged over spatial volume at fixed time. Let

$$\tilde{U}(t)=\frac{1}{V_3}\sum_{\mathbf{x}} U_0(\mathbf{x},t), \tag{2.7}$$

where $V_3=n_x n_y n_z$ is the 3-volume of a lattice time slice. If remnant symmetry is unbroken, then $\tilde{U}(t)=0+O(1/V_3^{1/2})$ in any thermalized lattice configuration. The order parameter Q is defined to be, for SU(2),

$$Q=\frac{1}{n_t}\sum_{t=1}^{n_t}\left\langle \sqrt{\frac{1}{2}\mathrm{Tr}[\tilde{U}(t)\tilde{U}^{\dagger}(t)]}\right\rangle. \tag{2.8}$$

Then Q is positive definite on a finite lattice, and on general grounds

$$Q=c+\frac{b}{\sqrt{V_3}}, \quad \text{where} \begin{cases} c=0 & \text{in the symmetric phase,} \\ c>0 & \text{in the broken phase.} \end{cases} \tag{2.9}$$

If Q extrapolates to a nonzero value as $V_3\to\infty$, then the remnant symmetry in Coulomb gauge is spontaneously broken.

Next, we make the connection between unbroken remnant symmetry and the existence of a confining Coulomb potential. We first recall that the Hamiltonian operator in Coulomb gauge has the form $H=H_{\mathrm{glue}}+H_{\mathrm{Coul}}$ where, in the continuum

$$H_{\mathrm{glue}}=\frac{1}{2}\int d^3x(\mathcal{J}^{-1/2}\vec{E}^{\mathrm{tr},a}\mathcal{J}\cdot\vec{E}^{\mathrm{tr},a}\mathcal{J}^{-1/2}+\vec{B}^a\cdot\vec{B}^a),$$

$$H_{\mathrm{Coul}}=\frac{1}{2}\int d^3x d^3y\,\mathcal{J}^{-1/2}\rho^a(x)\mathcal{J}K^{ab}(x,y;A)\rho^b(y)\mathcal{J}^{-1/2},$$

$$K^{ab}(x,y;A)=\left[\frac{1}{\nabla\cdot D(A)}(-\nabla^2)\frac{1}{\nabla\cdot D(A)}\right]^{ab}_{xy},$$

$$\rho^a=\rho^a_q-gf^{abc}A^b_k E^{\mathrm{tr},c}_k,$$

$$\mathcal{J}=\det[-\nabla\cdot D(A)] \tag{2.10}$$

and the factors of $\mathcal{J}$ arise from operator ordering considerations [8]. It is understood that A is identically transverse in Coulomb gauge, $A=A^{\mathrm{tr}}$. Note that $\mathcal{J}$ commutes with all quantities except E^{tr}, and in particular with $\rho_q(x)$ and $K^{ab}(x,y;A)$. The expectation value of $K(x,y;A)$ is the instantaneous piece of the $\langle A_0A_0\rangle$ gluon propagator, i.e.,

$$\langle A^a_0(x)A^b_0(y)\rangle = D(\mathbf{x}-\mathbf{y})\,\delta^{ab}\delta(x_0-y_0)+\text{non-instantaneous},$$

$$D(\mathbf{x}-\mathbf{y})\,\delta^{ab}=\left\langle\left[\frac{1}{\nabla\cdot D[A]}(-\nabla^2)\frac{1}{\nabla\cdot D[A]}\right]^{a,b}_{x,y}\right\rangle, \tag{2.11}$$

as shown in Ref. [9]. We see from these expressions that the Coulomb interaction energy between two charged static sources is given by instantaneous (dressed) one-gluon exchange.

Now consider a physical state in Coulomb gauge containing massive quark-antiquark sources

$$|\Psi_{qq}\rangle=\bar{q}(0)q(R)|\Psi_0\rangle \tag{2.12}$$

which is invariant under the remnant symmetry. The excitation energy is

$$\begin{aligned}\mathcal{E}&=\langle\Psi_{qq}|H|\Psi_{qq}\rangle-\langle\Psi_0|H|\Psi_0\rangle\\ &=V_{\mathrm{Coul}}(R)+E_{se},\end{aligned} \tag{2.13}$$

where E_{se} is an R-independent constant, on the order of the inverse lattice spacing, to be specified below. The R dependence of $\mathcal{E}$ can only come from the expectation value of the nonlocal quark-quark part of the Hamiltonian

$$H_{qq}=\frac{1}{2}\int d^3x d^3y\,\rho^a_q(x)K^{ab}(x,y;A)\rho^b_q(y). \tag{2.14}$$

Thus the R-dependent piece $V_{\mathrm{Coul}}(R)$ can be identified as the Coulomb potential due to these static sources. Moreover, the same kernel $K(x,y;A)$ appears in H_{qq} and in the instantaneous part $D(\mathbf{x})$ of the (dressed) one-gluon propagator $\langle A_0A_0\rangle$. This yields the formula

$$V_{\mathrm{Coul}}(|\mathbf{x}|)+E_{se}=C_r[D(0)-D(\mathbf{x})], \tag{2.15}$$

where C_r [$=3/4$ for the SU(2) gauge group] is the Casimir factor in the fundamental representation.

The correlator of two Wilson lines can be expressed in terms of the Hamiltonian operator and the state Ψ_{qq} as follows:

$$\begin{aligned}G(R,T)&=\left\langle\frac{1}{2}\mathrm{Tr}[L^{\dagger}(\mathbf{x},0,T)L(\mathbf{y},0,T)]\right\rangle\\ &=\langle\Psi_{qq}|e^{-(H-E_0)T}|\Psi_{qq}\rangle,\end{aligned} \tag{2.16}$$

where $R=|\mathbf{x}-\mathbf{y}|$, and L is now the timelike Wilson line in the continuum theory. We have

$$G(R,T)=\sum_n |\langle\Psi_n|\Psi_{qq}\rangle|^2 e^{-\Delta E_n T} \tag{2.17}$$

[2]In the case of spontaneous center symmetry breaking, the accepted order parameter on a finite lattice is the absolute value of the spatial average of Polyakov lines.

and we define the logarithmic derivative

$$V(R,T) = -\frac{d}{dT}\log[G(R,T)]. \tag{2.18}$$

It is easy to see that the Coulomb energy is obtained at $T \to 0$, i.e.,

$$\begin{aligned}\mathcal{E} &= V_{\rm Coul}(R) + E_{se} \\ &= V(R,0).\end{aligned} \tag{2.19}$$

The minimum energy of a state containing two static quark antiquark charges, which in a confining theory would be the energy of the flux tube ground state, is obtained in the opposite $T \to \infty$ limit

$$\begin{aligned}\mathcal{E}_{\rm min} &= V(R) + E'_{se} \\ &= \lim_{T\to\infty} V(R,T)\end{aligned} \tag{2.20}$$

and in this limit $V(R)$ is usual static quark potential.

The idea of using the correlator of timelike Wilson lines to compute the static quark potential, in Coulomb gauge on the lattice, was put forward some years ago by Marinari *et al.* [10]. These authors also noted that the remnant symmetry in Coulomb gauge is unbroken in the confining phase.

We now recall an inequality first pointed out by one of us (D.Z.) in Ref. [11]. With a lattice regularization, E_{se} and E'_{se} are finite constants. In a confining theory, both of these constants are negligible compared to $V(R)$, at sufficiently large R. But since $\mathcal{E}_{\rm min} \leqslant \mathcal{E}$, it follows that

$$V(R) \leqslant V_{\rm Coul}(R) \tag{2.21}$$

asymptotically. The intriguing implication is that if confinement exists at all, then it exists already at the level of dressed one-gluon exchange in Coulomb gauge. But we also see that because the Coulomb potential is only an upper bound on the static potential, a confining Coulomb potential is a necessary but not a sufficient condition for the existence of a confining static quark potential.

On the lattice, the continuum logarithmic derivative in Eq. (2.18) is replaced by

$$V(R,T) = \frac{1}{a}\log\left[\frac{G(R,T)}{G(R,T+a)}\right], \tag{2.22}$$

where a is the lattice spacing. In particular, in lattice units $a=1$,

$$\begin{aligned}V(R,0) &= -\log[G(R,1)] \\ &= -\log\left[\left\langle \frac{1}{2}\mathrm{Tr}[U_0(\mathbf{x},1)U_0^\dagger(\mathbf{y},1)]\right\rangle\right],\end{aligned} \tag{2.23}$$

and at large β, where the lattice logarithmic derivative approximates the continuum, $V(R,0)$ provides an estimate of the Coulomb potential $V_{\rm Coul}(R)$ (up to an additive constant E_{se}).

In Eq. (2.23) the relation between the confining property of the Coulomb potential, and the unbroken realization of remnant symmetry, is manifest. For if $Q \to 0$ at infinite volume, then also

$$\begin{aligned}\lim_{R\to\infty}\langle \mathrm{Tr}[U_0(\mathbf{x},t)U_0^\dagger(\mathbf{y},t)]\rangle &= \lim_{V_3\to\infty}\langle \mathrm{Tr}[\tilde{U}(t)\tilde{U}^\dagger(t)]\rangle \\ &= 0\end{aligned} \tag{2.24}$$

in which case the potential $V(R,0)$ rises to infinity as $R \to \infty$. Conversely, if $Q>0$, then the limit in Eq. (2.24) is finite, and $V(R,0)$ is asymptotically flat. Since $V_{\rm Coul}(R) \approx V(R,0)$ is an upper bound on the static quark potential, we see again that unbroken remnant symmetry is a necessary but not sufficient condition for confinement.

$V(R,T)$ has been computed numerically for pure SU(2) lattice gauge theory at a range of lattice couplings β in Ref. [2]. We recall the essential results.

(1) $V(R,T)$ increases linearly with R at large R and all T, at any coupling.

(2) The associated string tension $\sigma(T)$ converges (from above) to the usual asymptotic string tension σ, at any given β, as T increases.

(3) At weaker couplings, the Coulomb string tension $\sigma_{\rm Coul} \equiv \sigma(0)$ is substantially greater (by about a factor of three) than the asymptotic string tension.

(4) Removing center vortices from lattice configurations (by the de Forcrand–D'Elia procedure [6]) sends $\sigma(T) \to 0$ at all T, including the Coulomb string tension $\sigma_{\rm Coul} \to 0$ at $T=0$.

In the following sections we extend the investigation to models including scalar matter fields, and to SU(2) gauge theory across the high-temperature deconfinement transition.[3] First, however, we would like to remark on the scaling properties of $\sigma_{\rm Coul} \approx \sigma(0)$ in pure SU(2) gauge theory at zero temperature. The ratio of the Coulomb ($T=0$) to the asymptotic ($T\to\infty$) string tensions, reported in Ref. [2], varies somewhat with β in the range ($\beta \in [2.2,2.5]$) of couplings investigated. The ratio $\sigma(0)/\sigma$ tends to rise in this interval, as shown in Fig. 1. However, it is known that σ does not quite conform to the two-loop scaling formula associated with asymptotic freedom, in the range of β we have studied, and it is always possible that scaling sets in at different values of β for different physical quantities. What we find is that when our values for $\sigma(0)$ are divided by the asymptotic freedom expression

$$F(\beta) = \left(\frac{6\pi^2}{11}\beta\right)^{102/121}\exp\left(-\frac{6\pi^2}{11}\beta\right) \tag{2.25}$$

[3]A note on gauge fixing to Coulomb gauge: In this investigation we generate eight random gauge copies of each lattice configuration, and carry out gauge-fixing on each configuration by over-relaxation for 250 iterations. The best copy of eight copies is then chosen, and the over-relaxation procedure is continued until the average value of the gauge-fixed links has changed, in the last 10 iterations, by less than 2×10^{-7}.

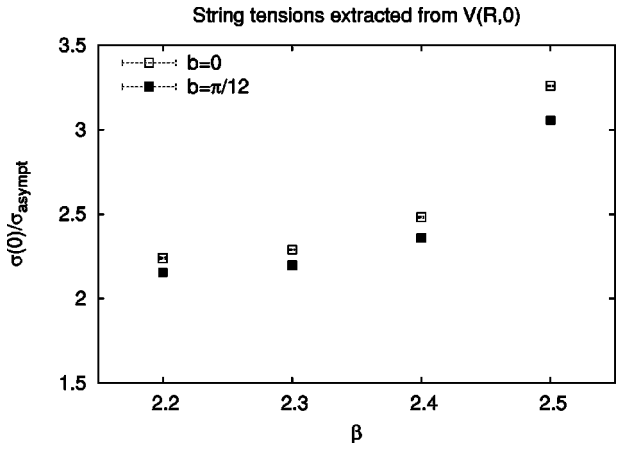

FIG. 1. The ratio $\sigma(0)/\sigma$ at various β, from fits which either include ($b=\pi/12$) or do not include ($b=0$) the Lüscher term. This ratio should equal the ratio of $\sigma_{\mathrm{Coul}}/\sigma$ in the continuum limit. (Data points come from numerical simulations on lattice sizes 16^4, 20^4, 20^4, and 32^4 at $\beta=2.2$, 2.3, 2.4, and 2.5, respectively.)

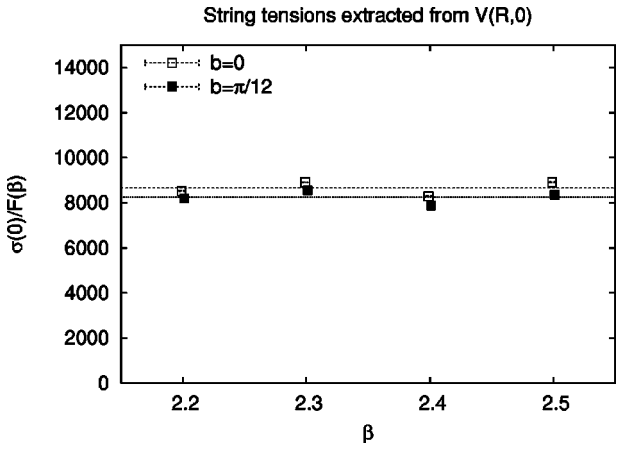

FIG. 2. The ratio $\sigma(0)/F(\beta)$ vs β, again from fits with ($b=\pi/12$) or without ($b=0$) the Lüscher term. $F(\beta)$ is the asymptotic freedom expression given in the text; a constant ratio implies scaling according to the two-loop beta function.

relevant to the SU(2) string tension, the ratio $\sigma(0)/F(\beta)$ is virtually constant as seen in Fig. 2. This fact suggests that scaling according to asymptotic freedom may set in earlier for the Coulomb string tension $\sigma_{\mathrm{Coul}}\approx\sigma(0)$ than for the asymptotic string tension σ.

A. Divergent constant in $D(R)$

We would also like to remark, at this stage, on a subtlety in identifying the color Coulomb potential with dressed one-gluon exchange.[4] The energy expectation value of the static $q\bar{q}$ state, $\mathcal{E}(R)=V_{\mathrm{Coul}}(R)+E_{se}$, is finite for finite quark separation R and finite lattice spacing a; in fact we have calculated this quantity [$=V(R,0)$] numerically. So it might seem natural, from Eq. (2.15), to identify the R-dependent Coulomb interaction energy as $V_{\mathrm{Coul}}(R)=-C_rD(R)$. That cannot be quite right, however. The reason is that $\frac{1}{2}C_rD(0)$ is the energy of an *isolated* quark state in an infinite volume; it is the energy we would extract from the logarithmic time derivative of $G(T)=\langle\frac{1}{2}\,\mathrm{Tr}\,L(0,0,T)\rangle$. In the case of unbroken remnant symmetry we have $G(T)=0$, and therefore $\frac{1}{2}C_rD(0)=\infty$ in an infinite volume, even though the lattice spacing a is nonzero. This infinity, which has a nonperturbative, infrared origin, should not be confused with the usual ultraviolet contribution to the quark self-energy, which is only infinite in the continuum limit. Since $\mathcal{E}(R)$ is finite, the infrared divergence in $C_rD(0)$ must be cancelled, in Eq. (2.15), by a corresponding divergent constant contained in $C_rD(R)$. In other words, only the difference $D(0)-D(R)$ is finite, and $V_{\mathrm{Coul}}(R)$, if finite, differs from $-C_rD(R)$ by an infinite constant. In order that $V_{\mathrm{Coul}}(R)$ and E_{se} be separately finite and well-defined, we may relate them to the gluon propagator with an (arbitrary) subtraction at $R=R_0$ which removes the infrared divergence, i.e.,

$$\begin{aligned} V_{\mathrm{Coul}}(R) &= -C_r[D(R)-D(R_0)] \\ &= \mathcal{E}(R)-\mathcal{E}(R_0), \\ E_{se} &= C_r[D(0)-D(R_0)] \\ &= \mathcal{E}(R_0). \end{aligned} \tag{2.26}$$

Defined in this way, $V_{\mathrm{Coul}}(R)$ crosses zero at the subtraction point, and E_{se} contains the ultraviolet, but not the infrared, contributions to the quark-antiquark self-energies.

As a check of the cancellation (or noncancellation) of infrared divergences, consider a colored state consisting, e.g., of two static quarks, rather than a quark and antiquark. The energy of such a state could be extracted from an LL correlator, which is zero if remnant symmetry is unbroken. The energy is therefore infinite, and according to our previous analysis would be proportional to $D(0)+D(R)$. In this case, the divergent constant in $D(R)$ adds to, rather than subtracts from, the divergent constant in $D(0)$, and the resulting sum is divergent, as it should be. The argument can be readily generalized to baryonic states in SU(N) gauge theories composed of static charges. The energy of a color singlet state, with charges at points $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N$ is obtained from the logarithmic time derivative of the correlator

$$G(\{\mathbf{x}_i\},T)=\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\langle L^{i_1j_1}(\mathbf{x}_1,0,T)\cdots L^{i_Nj_N}(\mathbf{x}_N,0,T)\rangle. \tag{2.27}$$

The order T contributions to $G(\{\mathbf{x}_i\},T)$ are terms proportional to $D(0)$ and $D(\mathbf{x}_m-\mathbf{x}_n)$, $m\neq n$, with differing signs. On the other hand, for a color singlet state, the operator $\epsilon\epsilon\,L\cdots L$ is a T-independent constant in the $x_1=x_2=\cdots=x_N$ coincidence limit. From this it is clear that the propagators completely cancel in the coincidence limit, and any constant terms in the propagators cancel in general. This means that the energy of a color singlet baryonic state is finite. Conversely, the divergent constants do not cancel in color nonsinglet states, so their energies are infinite.

[4]We thank M. Polikarpov for a helpful discussion on this point.

III. SU(2) GAUGE-ADJOINT HIGGS THEORY

The lattice action for SU(2) gauge theory with a Higgs field in the adjoint representation of the gauge group is

$$S=\beta\sum_{\rm plaq}\frac{1}{2}{\rm Tr}[UUU^\dagger U^\dagger] + \frac{\gamma}{4}\sum_{x,\mu}\phi^a(x)\phi^b(x+\hat{\mu}){\rm Tr}[\sigma^a U_\mu(x)\sigma^b U_\mu^\dagger(x)] \tag{3.1}$$

with the radially ''frozen,'' three-component Higgs field ϕ subject to the restriction

$$\sum_{a=1}^{3}\phi^a(x)\phi^a(x)=1. \tag{3.2}$$

This theory was first studied numerically by Brower *et al.* [12]. In addition to SU(2) gauge symmetry, the action is also invariant under global Z_2 center symmetry

$$U_0(\mathbf{x},t_0)\rightarrow -U_0(\mathbf{x},t_0) \tag{3.3}$$

for some choice of $t=t_0$, with all other fields unchanged. The existence of this apparently innocent global symmetry in the action has profound consequences; in its absence the static quark potential is asymptotically flat, and there can be no truly confined phase. The significance of center symmetry to the confinement property in general is reviewed in Ref. [13].

Since the action (3.2) is symmetric under global center transformations, a transition from the Higgs phase to a distinct confinement phase is possible. The Higgs phase is the phase of spontaneously broken center symmetry, while confinement corresponds to the symmetric phase. This division of the β-γ phase diagram into two separate phases was verified numerically long ago, in the Monte Carlo investigation of Ref. [12], which mapped out the approximate location of the transition line.

The Higgs phase of the adjoint Higgs model is often characterized as a spontaneous breaking of the local gauge symmetry, from $SU(2)$ down to $U(1)$. In view of the Elitzur theorem, which states that a local symmetry cannot break spontaneously, this characterization is a little misleading. However, as we have discussed above, there exists in Coulomb gauge a remnant gauge symmetry which is global on a time slice, and which can break spontaneously on a time slice in the sense described in the previous section. We have therefore studied the phase diagram of the adjoint Higgs theory via two observables: (i) the plaquette energy

$$E_p=\left\langle\frac{1}{2}{\rm Tr}[UUU^\dagger U^\dagger]\right\rangle \tag{3.4}$$

and (ii) the remnant symmetry breaking order parameter Q defined in Eq. (2.8). What we find is that the transition lines (Fig. 3) detected by each of these two parameters coincide; the common line location agrees with the earlier results of Brower *et al.* based on the plaquette energy alone. In Fig. 4

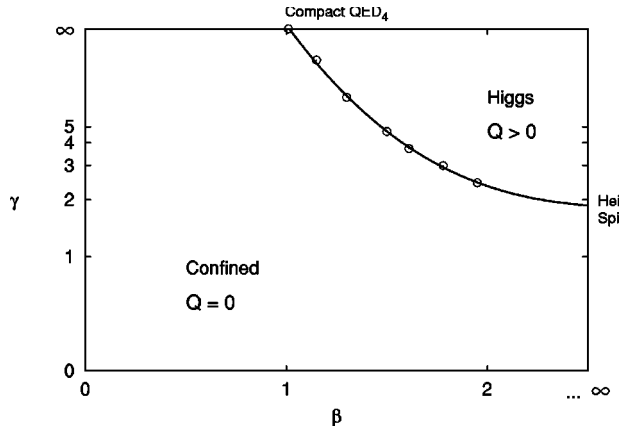

FIG. 3. Phase diagram of the SU(2) adjoint Higgs model. The plaquette energy E_p and the remnant symmetry order parameter Q locate the same transition line between the confined and Higgs phases.

we plot E_p vs β at three values of γ. The existence of a phase transition for the two larger values of γ is clearly visible; there is no transition apparent at the smallest γ value. Figure 5 is the corresponding plot of Q vs β at the same three values of γ. At β,γ values where E_p shows a transition, the transition in Q is even more evident. Conversely, where no transition is seen in E_p, at $\gamma=0.5$, neither is there a transition in Q. Finally, in Fig. 6, Q is plotted against $V_3^{-1/2}$, and we show the extrapolation of Q to a small value (consistent with zero) at infinite volume, for couplings in the confined phase.

IV. HIGH-TEMPERATURE DECONFINEMENT

We have seen that in the SU(2)-adjoint Higgs model, things go much as one might have expected *a priori*: remnant symmetry breaking coincides with Z_2 center symmetry breaking, and in consequence the presence of a confining Coulomb potential is correlated with the presence of a confining static quark potential. One might then guess that remnant and center symmetry breaking always go together. This

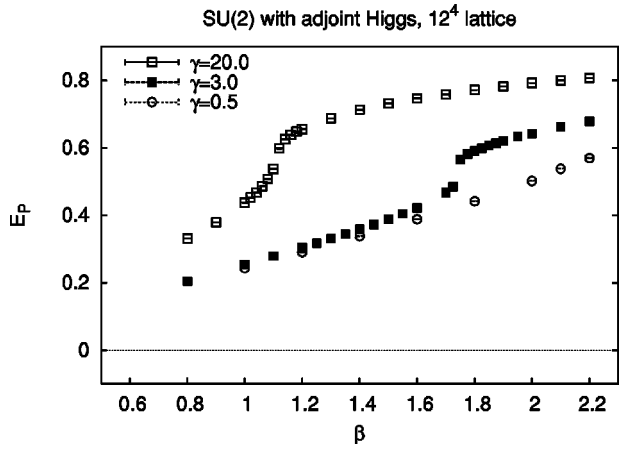

FIG. 4. Plaquette energy E_p vs β at three values of γ in the gauge-adjoint Higgs model.

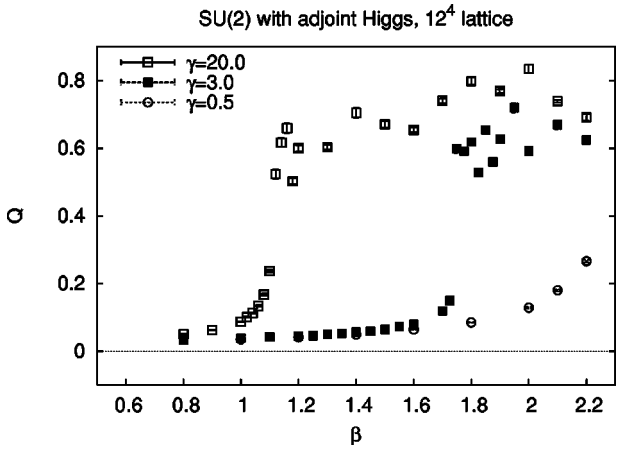

FIG. 5. Remnant symmetry order parameter Q vs β at three values of γ in the gauge-adjoint Higgs model.

appears not to be true, as we have discovered in our investigation of pure SU(2) lattice gauge theory at high temperature.

Monte Carlo simulations of the pure SU(2) gauge theory were carried out on $L^3\times 2$ lattices at $\beta=2.3$, which is inside the deconfined phase. Figure 7 is a plot of Q vs $1/\sqrt{V_3}$, where it seems that Q tends to zero at large volume (although we cannot entirely rule out a small nonzero intercept at $V_3 \rightarrow\infty$). This impression is strengthened by our plot of $V(R,0)$ in Fig. 8, where it is clear that the Coulomb potential goes asymptotically to a straight line, as the lattice volume increases. There is no indication of the screening of the static quark potential (as measured by Polyakov line correlators), which occurs at much smaller distances.

The results for the Coulomb potential in the deconfined phase are not paradoxical; we have already noted that remnant symmetry breaking is a necessary but not sufficient condition for confinement, and that the Coulomb potential is only an upper bound on the static quark potential. Thus it is possible for the Coulomb potential to increase linearly even if the static quark potential is screened, as evidently occurs in the deconfined phase. Nevertheless, this result is a little surprising, and it would be nice to understand it a little better.

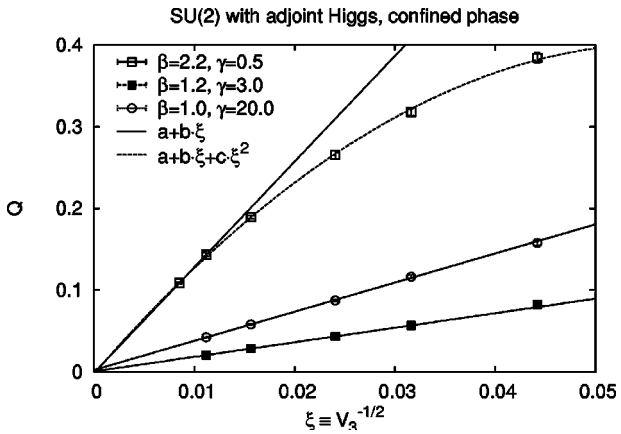

FIG. 6. Plot of Q vs root inverse 3-volume, and extrapolation of Q to infinite volume, at several couplings in the confined phase of the gauge-adjoint Higgs model.

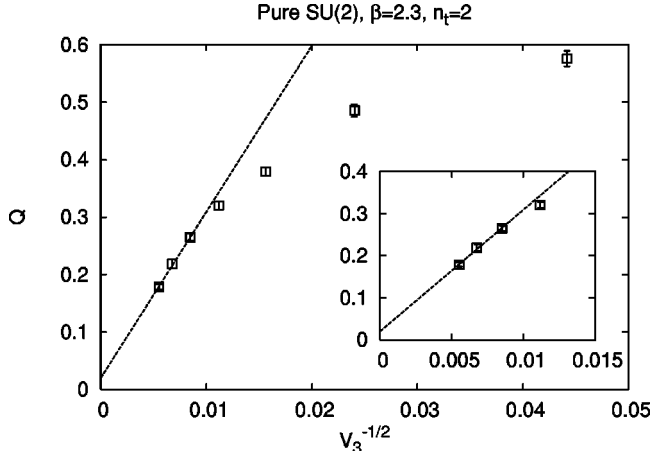

FIG. 7. The Q parameter vs root inverse 3-volume in the high-temperature deconfined phase, pure SU(2) gauge theory at $\beta=2.3$, and $n_t=2$ lattice spacings.

Recall that in the continuum, the instantaneous part of the timelike gluon propagator, which is proportional to the Coulomb interaction energy between static charges, is given by Eq. (2.11). Note that this is the expectation value of an operator which depends only on the space components of the vector potential at a fixed time. On the lattice, this translates into an operator which depends only on spacelike links on a time slice. However, we know that spacelike links, at fixed time, are a confining ensemble, in the sense that spacelike Wilson loops have an area law falloff even in the high-temperature deconfinement phase. If the confining property of the spacelike links on a timeslice is not removed by the deconfinement transition, then it is perhaps less surprising that the confining property of the (latticized) operator in Eq. (2.11), which depends only on spacelike links on a timeslice, survives in the deconfinement regime.

As a check, we apply a procedure that is known to remove the confining properties of lattice configurations. This is the de Forcrand–D'Elia [6] method of center vortex removal.

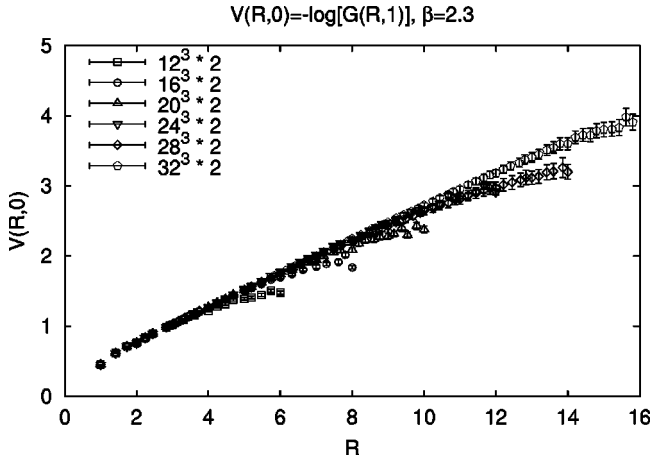

FIG. 8. $V(R,0)$ in the deconfined phase, at $\beta=2.3$ with $n_t=2$ lattice spacings in the time direction, and space volumes ranging from 12^3 to 32^3.

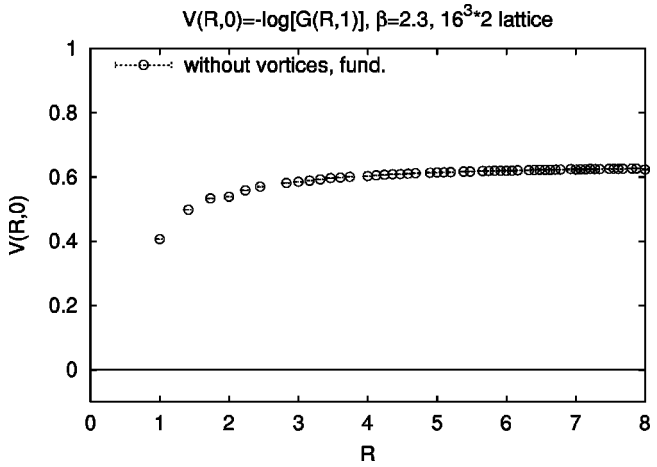

FIG. 9. The effect of vortex removal on $V(R,0)$ in the deconfined phase, $\beta=2.3$ on a $16^3\times 2$ lattice.

The procedure is to first fix a given thermalized lattice configuration to direct maximal center gauge, i.e., the gauge which maximizes

$$R=\sum_{x,\mu}\left(\frac{1}{2}\mathrm{Tr}[U_\mu(x)]\right)^2 \tag{4.1}$$

and carry out center projection

$$Z_\mu(x)=\mathrm{sgn}\,\mathrm{Tr}[U_\mu(x)] \tag{4.2}$$

to locate the vortices. Vortices are then ''removed'' from the original configuration by setting

$$U_\mu(x)\to U'_\mu(x)=Z_\mu(x)U_\mu(x). \tag{4.3}$$

In effect this procedure superimposes a thin Z_2 vortex inside the thick $SU(2)$ center vortices. The effect of the thin vortex is to cancel out the long range influence of the thick vortex on Wilson loops. It was found that this procedure not only removes the Wilson loop area law falloff, but also removes chiral symmetry breaking, and sends every configuration into the zero topological charge sector [6].

Having removed center vortices from the $L^3\times 2$ lattice, thereby removing the area-law falloff of spacelike Wilson loops, we fix the modified configuration to Coulomb gauge, and compute timelike link correlators in order to measure $V(R,0)$. The effect is quite dramatic. It was found in Ref. [2] that vortex removal in pure gauge theory, in the low-temperature confined phase, removes the confinement property of the Coulomb potential. Now we see, from Fig. 9, that vortex removal also removes the confining property of the Coulomb potential in the high-temperature deconfinement phase. This is in accord with the idea that it is the confining property of the ensemble of spacelike links at fixed time (or, perhaps, the percolation of center vortices on any time slice) that is crucial for the confining property of the Coulomb energy.

It is interesting to ask whether there is some connection between the center vortex confinement mechanism, and other proposals based on the Gribov horizon [14]. While this question cannot yet be answered definitively, there is one very intriguing fact that may be relevant: when gauge transformed to the minimal Coulomb gauge, thin-vortex configurations lie on the Gribov horizon, as we now explain.

V. CONFINEMENT SCENARIO IN COULOMB GAUGE AND VORTEX DOMINANCE

In the confined phase [2], and in the deconfined phase, Fig. 8, one sees clear evidence of a linearly rising color-Coulomb potential. There is a simple intuitive scenario in the minimal Coulomb gauge that explains why $V_{\mathrm{Coul}}(R)$ is long range [14]. In minimal Coulomb gauge the gauge-fixed configurations are (three-dimensionally) transverse configurations that lie in the fundamental modular region Λ. In continuum gauge theory, Λ is convex and bounded in every direction [15]. By simple entropy considerations, the population in a bounded region of a high-dimensional space gets concentrated at the boundary. For example, inside a sphere of radius R in a D-dimensional space, the radial density is given by $r^{D-1}dr$ and, for $r\leqslant R$, is highly concentrated near the boundary $r=R$ for D large. With lattice discretization the dimension D of configuration space is proportional to the volume V of the lattice. Moreover on the boundary the Faddeev-Popov operator $M(A)=-\nabla\cdot D(A)$ has a vanishing eigenvalue, as we will see. (At large volume it has a high density of small positive eigenvalues.) This makes the color-Coulomb interaction kernel $K(x,y;A)=M^{-1}(A)(-\nabla^2)M^{-1}(A)|_{xy}$ of long range for typical configurations A that dominate the functional integral. We thus expect that $V_{\mathrm{Coul}}(|\mathbf{x}-\mathbf{y}|)=\langle K(x,y;A)\rangle$ is long range, although this qualitative argument is not precise enough to establish that $V_{\mathrm{Coul}}(R)$ rises linearly at large R, as suggested by the numerical data.

We shall show that vortex dominance, which is strongly supported by the data just presented, is consistent with this simple confinement scenario in minimal Coulomb gauge. More precisely we shall show that when a center configuration (defined below) is gauge transformed to minimal Coulomb gauge it lies on the boundary $\partial\Lambda$ of the fundamental modular region Λ. According to the confinement scenario in minimal Coulomb gauge, the probability measure is dominated by points at or near the boundary $\partial\Lambda$. So center dominance, when translated into the minimal Coulomb gauge, means dominance by a subset of configurations on the boundary $\partial\Lambda$. This is a stronger condition than the confinement scenario in minimal Coulomb gauge, but consistent with it.

Proof of assertion. To simplify the kinematics we give the continuum version of the argument. Numerical gauge fixing to minimal Coulomb gauge corresponds to minimizing on each time slice the functional $F(A)=||A||^2$, with respect to local gauge transformations. Here $||A||^2=\int d^3x|A|^2$ is the square Hilbert norm of A_i^a. At a minimum, which may be relative or absolute, the first variation with respect to infinitesimal gauge transformations $\delta A_i=D_i(A)\omega$, vanishes for all ω, $\delta||A||^2=2[A_i,D_i(A)\omega]=2[A_i,\partial_i\omega]=0$, which gives the Coulomb gauge condition $\partial_i A_i=0$. Moreover at a relative or absolute minimum the second variation with re-

spect to gauge transformations $\delta^2||A||^2=2[D(A)_i\omega,\partial_i\omega]\geqslant 0$ is non-negative for all ω, which is the statement that the Faddeev-Popov operator $M(A)=-\nabla\cdot D(A)$ is non-negative. These two conditions define the Gribov region Ω,

$$\Omega\equiv\{A:\partial_i A_i=0 \text{ and } -\partial_i D_i(A)\geqslant 0\}. \tag{5.1}$$

The fundamental modular region Λ is the set of absolute minima with respect to gauge transformations

$$\Lambda\equiv\{A:||A||\leqslant||{}^gA|| \text{ for all } g\}. \tag{5.2}$$

It is included in the Gribov region $\Lambda\subset\Omega$. In the interior of Ω all eigenvalues of $M(A)$ are strictly positive $\lambda_n>0$ (apart from the trivial null eigenvalue associated with constant gauge transformations $\partial_i\omega=0$), and on the boundary $\partial\Omega$ there is a nontrivial null eigenvector $\partial_i D_i(A)\omega_0=0$, and all other eigenvalues are non-negative.[5]

We call a center configuration any lattice configuration $Z_i(x)$ for which every link variable is a center element $Z_i(x)\in Z$ for every link $(x,\hat{i})$. The only nonzero action excitations of center configurations are thin center vortices. Such configurations are invariant under all global gauge transformations $g^{-1}Z_i(x)g=Z_i(x)$. Now apply an arbitrary local gauge transformation $h(x)$ to the center configuration $Z_i(x)\to V_i(x)=h^{-1}(x)Z_i(x)h(x+\hat{i})$. We shall take $h(x)$ to be the gauge transformation that brings the center configuration into the minimal Coulomb gauge. In general, the transformed configuration $V_i(x)$ is not an element of the center, but it is invariant, $V_i(x)=g'^{-1}(x)V_i(x)g'(x+\hat{i})$, with respect to the gauge transformation $g'(x)=h^{-1}(x)gh(x)$ which, in general, is no longer global.

We give an infinitesimal characterization of the invariance of the configuration $V_i(x)$ under the gauge transformations $g'(x)$. The set of global gauge transformations form the SU(N) Lie group and the $g'(x)=\exp[\omega(x)]$ form a representation of this group. Here $\omega(x)$ is an element of the Lie algebra of SU(N). This algebra has N^2-1 linearly independent elements $\omega^n(x)$, where $n=1,\ldots,N^2-1$, that satisfy $[\omega^l(x),\omega^m(x)]=f^{lmn}\omega^n(x)$. Thus the configuration $V_i(x)=\exp[A_i(x)]$, which is the gauge transform of the center configuration $Z_i(x)$ into the minimal Coulomb gauge, is invariant under local gauge transformations with N^2-1 independent generators ω^n.[6] This is the statement that in continuum notation reads $A_i=A_i+\epsilon D_i(A)\omega^n$, or $D_i(A)\omega^n=0$. It follows that the ω^n also satisfy the weaker condition $\nabla\cdot D(A)\omega^n=0$. Here A is the representative in minimal Coulomb gauge of the center configuration, and as such it lies in the fundamental modular region $A\in\Lambda$ (by definition) that moreover is included in the Gribov region $\Lambda\subset\Omega$, so we have $A\in\Omega$. However, the equation $\nabla\cdot D(A)\omega^n=0$ for $A\in\Omega$ means that A lies on its boundary $\partial\Omega$. With $\Lambda\subset\Omega$ it follows that A lies on the boundary $\partial\Lambda$ of Λ, at a point where the two boundaries touch. We conclude that the gauge transform of a center configuration lies on $\partial\Lambda$, as asserted.[7]

The argument just given also applies to Abelian configurations, namely, a configuration that lies in an Abelian subalgebra of the Lie algebra. Such a configuration is invariant under a global U(1) gauge transformation. When an Abelian configuration is gauge transformed to the minimal Coulomb gauge, it is mapped into a point where the two boundaries $\partial\Lambda$ and $\partial\Omega$ touch. But now the equation $D_i(A)\omega=0$ has, in general, only one linearly independent (nontrivial) solution, instead of N^2-1. Thus Abelian dominance is also compatible with dominance of configurations on the boundary $\partial\Lambda$.

VI. SU(2) GAUGE-FUNDAMENTAL HIGGS THEORY

We now consider SU(2) gauge theory with the radially frozen Higgs field in the fundamental representation. For the SU(2) gauge group, the lattice action can be written in the form [18]

$$S=\beta\sum_{\text{plaq}}\frac{1}{2}\text{Tr}[UUU^\dagger U^\dagger] + \gamma\sum_{x,\mu}\frac{1}{2}\text{Tr}[\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu})] \tag{6.1}$$

with ϕ an SU(2) group-valued field. This theory cannot be truly confining for nonzero γ, since the matter field can screen any charge, and this simply reflects the absence of a nontrivial global center symmetry. However, at sufficiently small γ there exists a "pseudoconfinement" region, where the static potential (as measured by the correlator of Polyakov loops) is linear for some intermediate range of quark separations before the onset of screening. At large γ there is a Higgs region, where the linear potential is completely absent. It was shown many years ago by Fradkin and Shenker [5] that any two points in the Higgs and pseudoconfinement regions can be joined by a path in the β-γ coupling plane that avoids all thermodynamic singularities. Although there exists a line of first-order phase transitions in the β-γ plane, this line has an endpoint and does not divide the diagram into thermodynamically separate phases. (It appears as the solid line in Fig. 12, below.) In accord with the Fradkin-Shenker observation, numerical simulations suggest that there is only one nonconfining phase in the gauge-fundamental Higgs theory.

[5]The Gribov region Ω consists of relative minima that are Gribov copies of the set Λ of absolute minima. Our numerical procedure selects the best Gribov copy obtained from eight random gauge copies; there is no known method for finding the absolute minimum. On theoretical grounds, one expects the sensitivity of our results to the choice of Gribov copy be small [16]. This expectation is susceptible to numerical investigation, as has been done recently for the ghost propagator [17].

[6]The gauge orbit of the center configuration is degenerate and has N^2-1 fewer dimensions than a generic gauge orbit.

[7]It should be noted that the numerical procedure that we have used to remove center vortices is 4 dimensional. The argument given here applies in Coulomb gauge to center configurations defined within 3-dimensional time slices.

In apparent contradiction to this fact, one can make a strong case for the existence of a remnant symmetry-breaking transition at small β. We set $\beta=0$ so the action is

$$S_\phi=\gamma\sum_{x,\mu}\frac{1}{2}\mathrm{Tr}[\,\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu})]. \tag{6.2}$$

We shall show that (i) at large γ there is spontaneous breaking of the remnant gauge symmetry, associated with the short range of $V_{\mathrm{Coul}}(R)$, but (ii) at small γ there is a linear rise of $V_{\mathrm{Coul}}(R)$ at large R. Thus along the line $\beta=0$ in the β-γ plane we expect a transition at some finite value of γ.

(i) First consider the limit $\gamma\to\infty$. The action S_ϕ is a maximum when $U_\mu(x)=\phi(x)\phi^\dagger(x+\hat{\mu})$ holds on each link, and when γ gets large, $U_\mu(x)$ gets frozen at this value. This configuration is a gauge transform of the identity $U_\mu(x)=I$, and when fixed to the minimal Coulomb gauge the spatial components get fixed to the identity $U_i(\mathbf{x},t)=\phi(\mathbf{x},t)\phi^\dagger(\mathbf{x}+\hat{\imath},t)=I$, for all $\mathbf{x}$ and i. Thus $\phi(\mathbf{x},t)$ is independent of $\mathbf{x}$. We write $\phi(\mathbf{x},t)=g(t)$, and we have $U_0(\mathbf{x},t)=g(t)g^\dagger(t+1)$, which is also independent of $\mathbf{x}$. This is the gauge analog of all spins aligned, and we expect spontaneous breaking of the remnant gauge symmetry. Indeed, we have

$$\tilde{U}(t)=\frac{1}{V_3}\sum_{\mathbf{x}}U_0(\mathbf{x},t)=g(t)g^\dagger(t+1), \tag{6.3}$$

and the order parameter Q defined above has the value $Q=1$. This is maximal breaking of the remnant gauge symmetry.

(ii) Now consider small values of γ. We shall calculate the lattice analog of $V_{\mathrm{Coul}}(R)$, namely, $V(R,0)=-\log[G(R,1)]$, to leading nonzero order in γ. The gauge fixing involves only the spatial link variables U_i, and with the action S_ϕ the integration over the U_0 factorizes into a product over link integrals. To evaluate $G(R,1)=\langle\frac{1}{2}\mathrm{Tr}[\,U_0(\mathbf{x},1)U_0^\dagger(\mathbf{y},1)]\rangle$, we first integrate over the $U_0(\mathbf{x},t)$, with the result, to leading order in γ,

$$G(R,1)=\frac{\gamma^2}{16}\left\langle\frac{1}{2}\mathrm{Tr}[\,\phi^\dagger(\mathbf{x},2)\phi(\mathbf{x},1)\phi^\dagger(\mathbf{y},2)\phi(\mathbf{y},1)]\right\rangle. \tag{6.4}$$

There are now 4 unsaturated ϕ fields. For simplicity we suppose that $(\mathbf{x},1)$ and $(\mathbf{y},1)$ are joined by a principle axis, which we take to be the 1 axis, and $R=|\mathbf{x}-\mathbf{y}|$. The leading contribution to the ϕ integration at small γ is obtained by saturating each link on the line that runs from $(\mathbf{x},1)$ to $(\mathbf{y},1)$ by "bringing down" from the exponent S_ϕ the term $\frac{1}{2}\gamma\,\mathrm{Tr}[\,\phi(\mathbf{z},1)U_1(\mathbf{z},1)\phi^\dagger(\mathbf{z}+\hat{1},1)]$, and likewise for the line from $(\mathbf{x},2)$ to $(\mathbf{y},2)$. This gives a factor γ^{2R}. The ϕ integrations are now effected. The remaining integration on the spatial link variables U_i is finite because of the gauge fixing. We cannot evaluate it explicitly, but this last integration does not introduce any further γ dependence. We thus obtain $G(R,1)=\gamma^{2R+2}\times H(R)$. Here $H(R)$ is not known, but it is independent of γ. This gives $V(R,0)=-(2R+2)\log\gamma-\log[H(R)]$. The asymptotic fall off in the correlator $G(R,1)$ should not be more rapid than exponential, so $\lim_{R\to\infty}\log[H(R)]=-cR$. We thus obtain a linear rise at large R in $V(R,0)\sim\sigma_{\mathrm{Coul}}R$, where the "Coulomb" string tension is given by

$$\sigma_{\mathrm{Coul}}=-2\log\gamma+c. \tag{6.5}$$

This is nonzero for small γ. To make this argument rigorous one would have to show that the expansion of $G(R,1)$ in powers of γ converges. However, this calculation does strongly suggest that at $\beta=0$ the remnant symmetry is unbroken for small γ whereas, as we have seen, it is broken at large γ.[8]

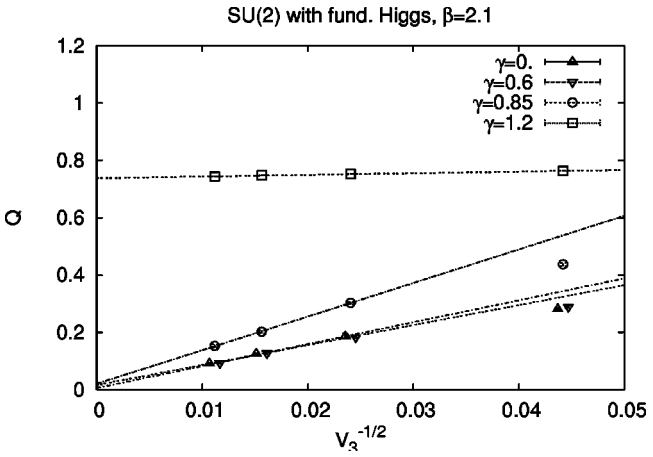

FIG. 10. Plot of Q vs root inverse 3-volume, and extrapolation of Q to infinite volume in the gauge-fundamental Higgs theory at $\beta=2.1$ and various γ, above ($\gamma=1.2$) and below the first-order transition point around $\gamma=0.9$.

Returning to the SU(2) gauge-fundamental Higgs theory, we note that the above calculations at large and small γ are easily understood in terms of the Coulomb gauge and center-dominance scenarios. Indeed, for large γ (and any β), the gauge-fixed configurations are at or near $U_i(x)=I$. This configuration is an interior point of the fundamental modular region Λ. Thus at large γ the coupling to the fundamental Higgs is effective in keeping configurations away from the boundary $\partial\Lambda$, where the thin vortex configurations are to be found, and where the Faddeev-Popov operator is of long range. On the contrary, at small γ the coupling to the Higgs field is ineffective, and entropy leads to dominance of configurations on the boundary $\partial\Lambda$.

Now we turn to numerical simulation. Figure 10 is a calculation of Q vs γ at $\beta=2.1$, where it is known (from the work of Lang *et al.* [18]) that the first-order transition is around $\gamma=0.9$. Below $\gamma=0.9$, Q seems to extrapolate to zero at infinite volume, while above the transition Q extrapo-

[8]One can make a similar calculation of $V(R,0)$ for small β (strong coupling) in pure SU(N) gauge theory. This yields a finite "Coulomb" string tension at small β given by $\sigma_{\mathrm{Coul}}=-\log\beta$. This suggests that at least in the strong-coupling region there is a "Coulomb flux tube" that connects the external sources.

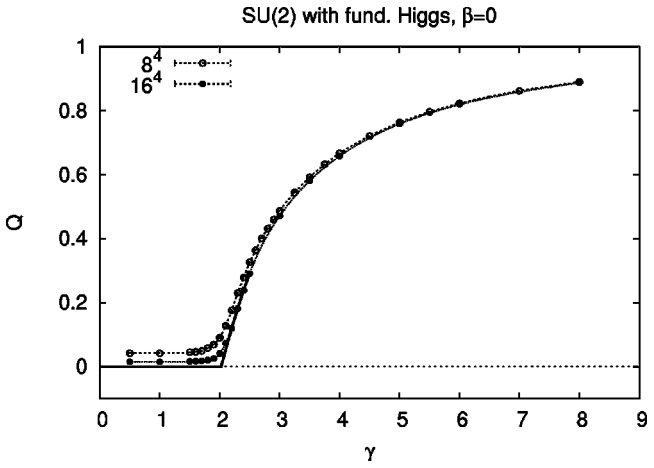

FIG. 11. Q vs γ at $\beta=0$ in the SU(2) fundamental Higgs model, on 8^4 and 16^4 lattices. The solid line is the presumed extrapolation of Q to infinite volume.

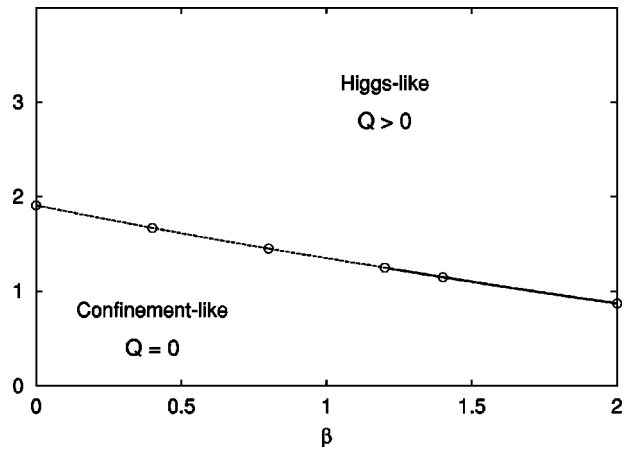

FIG. 12. Phase diagram of the SU(2) fundamental Higgs model. There is a thermodynamic transition and a Q (remnant symmetry-breaking) transition along the solid line, but only a nonthermodynamic transition (Kertész line) along the dashed line.

lates to a nonzero constant. There appears to be an actual discontinuity in Q across the whole line of first-order (thermodynamic) phase transitions in the β-γ coupling plane. At sufficiently small values of β, there is no thermodynamic transition, and we see no discontinuity in Q as a function of γ. What we see instead is that $Q\approx 0$ over a finite range of γ, and then, beyond a critical value $\gamma=\gamma_{\rm cr}$, Q smoothly increases with increasing γ. Our results for Q vs γ at $\beta=0$, on 8^4 and 16^4 lattices, are shown in Fig. 11; the solid line is the presumed extrapolation to infinite volume. If we were dealing with a spin system, and Q were the magnetization, this would clearly represent a second order phase transition. In the present case it is certainly a symmetry-breaking transition, separating a symmetric region with $Q=0$ from a broken-symmetry region of $Q>0$.

On the other hand, despite the existence of a symmetry-breaking transition, there is no thermodynamic transition of any kind at $\beta=0$. At this value of β the free energy can be computed exactly, with the result, in a 4-volume V

$$F(\gamma)=4V\log\left[\frac{2I_1(\gamma)}{\gamma}\right] \tag{6.6}$$

which is perfectly analytic for all $\gamma>0$. Thus we have confirmed the theoretical argument that there must be a remnant symmetry-breaking transition even at small β, but we have also found that this transition is not accompanied (at small β) by a thermodynamic transition, defined as some degree of nonanalyticity in the free energy. Our result for the line of critical couplings of the remnant symmetry-breaking transition is shown in Fig. 12. Along the solid line there is also a thermodynamic (first-order) transition, which is absent along the dashed line.

Our result for the gauge fundamental Higgs system is not entirely new; it was in fact anticipated by Langfeld in Ref. [3], who considered a closely related model in Landau (rather than Coulomb) gauge. In that work the modulus of the space-time averaged Higgs field was used as an order parameter to detect the breaking of remnant symmetry, which in Landau gauge must be both space *and* time independent. In Ref. [4] it was further suggested that the line of remnant symmetry breaking transitions, where it is unaccompanied by a line of thermodynamic transitions, is a Kertész line [19]. A Kertész line is a line of percolation transitions; the original example comes from the Ising model. In the Ising model, in the absence of an external magnetic field, there is a phase transition from a Z_2 symmetric to an ordered phase, and this transition can be expressed, in different variables, as a transition from a percolating phase at low temperature, to a nonpercolating phase at high temperature. In the presence of a magnetic field, the partition function and thermodynamic observables become analytic in temperature; there is no thermodynamic phase transition. Nevertheless, the percolation transition persists, and traces out a Kertész line in the temperature-magnetic field plane, completely separating the phase diagram into two regions. But if there is a Kertész line in the gauge-Higgs coupling plane, the question is what kind of objects are percolating. Based in part on results reported by Bertle and Faber [20], Langfeld [4] proposed that the unbroken remnant symmetry region is a region of percolating center vortices, which cease percolating in the broken symmetry region. There is now very good evidence for a vortex percolation transition of this kind in gauge-fundamental Higgs theory, reported in Ref. [21].[9]

Our findings here support the idea that there is some physical distinction that can be made between the Higgs and the pseudoconfining regions of the gauge-fundamental Higgs phase diagram. In the pseudoconfining region the remnant symmetry is unbroken, the Coulomb potential rises linearly, and center vortices percolate, while the Higgs region is a region of broken symmetry, the Coulomb potential is asymptotically flat, and center vortices do not percolate. This distinction appears to exist despite the fact that the two regions are thermodynamically connected, as demonstrated by Fradkin and Shenker in Ref. [5].

[9]The role of Kertész lines in high-temperature QCD is also discussed by Satz in Ref. [22].

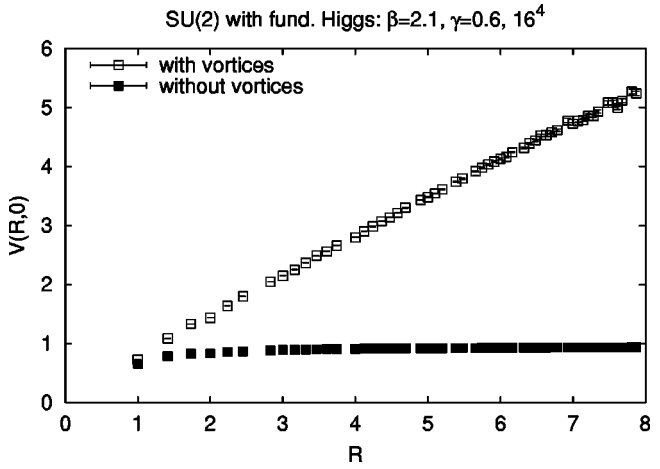

FIG. 13. Gauge-fundamental Higgs theory: effect of vortex removal in the symmetric (pseudoconfinement) phase, $\beta=2.1$, $\gamma=0.6$.

Before leaving the gauge-fundamental Higgs theory, we should ask what are the effects of vortex removal in the symmetric phase. We already know that the Coulomb potential in the fundamental Higgs theory must be confining in the symmetric (pseudoconfinement) phase and screened in the broken phase; the only issue is how the Coulomb potential is affected in each phase if center vortices are removed. In Figs. 13 and 14 we see the Coulomb potential in the symmetric ($\beta=2.1$, $\gamma=0.6$) and broken ($\beta=2.1$, $\gamma=1.2$) phases, respectively, before and after vortex removal. In the symmetric phase, vortex removal by the de Forcrand–D'Elia procedure sends the Coulomb string tension to zero, as in the high-temperature phase of pure gauge theory. Deep in the Higgs phase, on the other hand, the effect of vortex removal is seen to be very minor.

VII. THE ADJOINT-HIGGS MODEL REVISITED

We have understood theoretically why there should be a remnant symmetry-breaking transition at small β in the gauge-fundamental Higgs model, as seen in the numerical data. This raises immediately the question of why there is no corresponding transition in Q, at small β, in the gauge-adjoint Higgs theory. Now in the fundamental-Higgs model, the transition at small β appears to be a percolation transition, as discussed in the previous section. But the absence of a percolation transition at low β in the adjoint-Higgs case is easy to understand. There the Higgs part of the action is insensitive to the existence of thin center vortices, and cannot suppress their condensation at small β regardless of the value of γ. More concretely, at $\beta=0$, the action is invariant with respect to local transformations $U_0(x)\to z(x)U_0(x)$, where $z(x)=\pm 1$, and this immediately implies that $Q=0$ at infinite volume, again regardless of the value of γ.

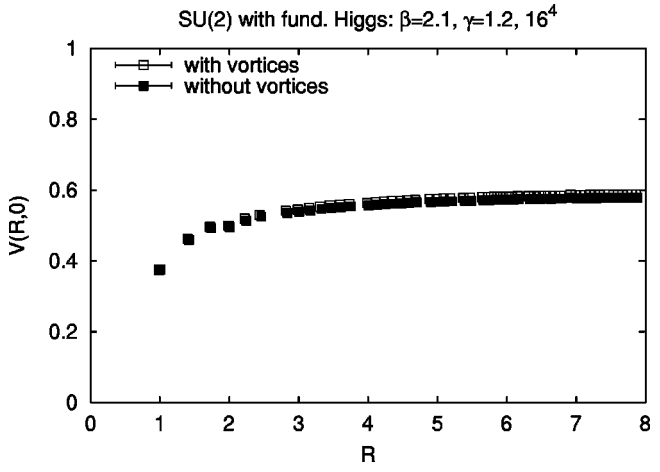

FIG. 14. Gauge-fundamental Higgs theory: effect of vortex removal in the broken (Higgs) phase, $\beta=2.1$, $\gamma=1.2$.

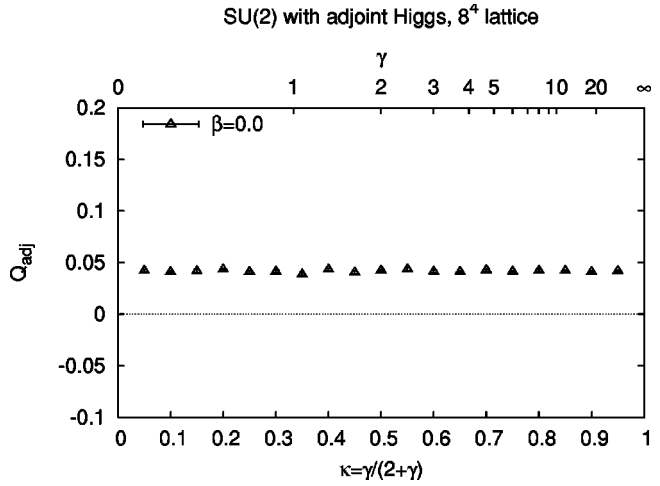

FIG. 15. (Lack of) variation of $Q_{\rm adj}$ with γ at $\beta=0$ in the gauge-adjoint Higgs theory.

On the other hand, if the absence of a remnant symmetry-breaking transition at low β is due to large fluctuations of center elements $z(x)$, then one might still expect breaking, at low β and large γ, of the $\mathrm{SO}(3)=\mathrm{SU}(2)/Z_2$ part of the remnant symmetry group, which is insensitive to $U_0(x)\to z(x)U_0(x)$ fluctuations. The relevant order parameter is

$$Q_{\rm adj}=\frac{1}{n_t}\sum_{t=1}^{n_t}\left\langle\sqrt{\frac{1}{3}\mathrm{Tr}[\tilde{U}_{\rm adj}(t)\tilde{U}^\dagger_{\rm adj}(t)]}\right\rangle, \qquad (7.1)$$

where $\tilde{U}_{\rm adj}(t)$ is the spatial average of timelike links in the adjoint representation

$$\tilde{U}_{\rm adj}(t)=\frac{1}{V_3}\sum_{\mathbf{x}}U_{0,{\rm adj}}(\mathbf{x},t),$$

$$U^{ab}_{0,{\rm adj}}(\mathbf{x},t)=\frac{1}{2}\mathrm{Tr}[\sigma^a U_0(\mathbf{x},t)\sigma^b U_0^\dagger(\mathbf{x},t)]. \qquad (7.2)$$

Rather surprisingly, there appears to be no transition in $Q_{\rm adj}$ either, in the adjoint-Higgs model. As we see in Fig. 15, for data taken on a small 8^4 lattice, there is no sign of any transition for $Q_{\rm adj}$ at finite γ and $\beta=0$. In fact, $Q_{\rm adj}$ is essentially γ independent. Extrapolation of $Q_{\rm adj}$ to infinite volume anywhere in the confined ($Q=0$) phase is consistent

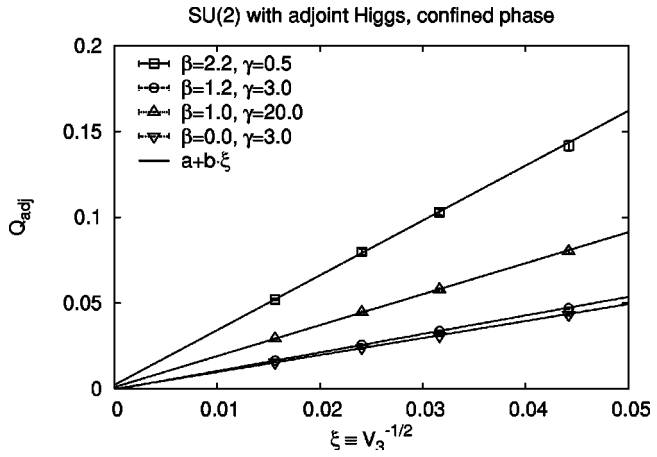

FIG. 16. Plot of $Q_{\rm adj}$ vs root inverse 3-volume, and extrapolation of $Q_{\rm adj}$ to infinite volume, at a variety of couplings in the confined phase, in gauge-adjoint Higgs theory.

with $Q_{\rm adj}=0$, as seen in Fig. 16. Thus the transition line for $Q_{\rm adj}$ in the β-γ coupling plane appears to be the same as for Q in Fig. 3, but not similar to Q in Fig. 12, and this fact calls for an explanation.

The difference in the phase diagrams of Figs. 3 and 12 occurs at small β and large γ. To understand this difference we set $\beta=0$ in the adjoint Higgs action (3.2) so it reads

$$S_\phi=\frac{\gamma}{2}\sum_{x,\mu}\ \phi^a(x)U^{ab}_{\mu,\rm adj}\phi^b(x+\hat{\mu}), \tag{7.3}$$

and we shall evaluate the equal time correlator in minimal Coulomb gauge

$$G_{\rm adj}(\mathbf{x}-\mathbf{y},1)=\left\langle\frac{1}{3}{\rm Tr}[\,U_{0,\rm adj}(\mathbf{x},1)U^\dagger_{0,\rm adj}(\mathbf{y},1)]\right\rangle \tag{7.4}$$

for γ large.

Step 1: Integration over U_0. The gauge fixing does not involve the U_0, so the integrals over the U_0 factorize, with the result on each timelike link

$$\langle U^{ab}_{0,\rm adj}(x)\rangle=\lambda\langle\phi^a(x)\phi^b(x+\hat{0})\rangle,$$

$$\lambda=\frac{\cosh\left(\frac{\gamma}{2}\right)}{\sinh\left(\frac{\gamma}{2}\right)}-\frac{2}{\gamma}. \tag{7.5}$$

Here λ has the limiting value $\lambda=1$ at large γ. The correlator factorizes into a product of expectation values on adjacent time slices

$$G_{\rm adj}(\mathbf{x}-\mathbf{y},1)=\frac{1}{3}\langle\phi^a(\mathbf{x},1)\phi^b(\mathbf{x},2)\phi^b(\mathbf{y},2)\phi^a(\mathbf{y},1)\rangle$$

$$=\frac{1}{3}\langle\phi^a(\mathbf{x},1)\phi^a(\mathbf{y},1)\rangle\langle\phi^b(\mathbf{x},2)\phi^b(\mathbf{y},2)\rangle. \tag{7.6}$$

This gives

$$G_{\rm adj}(\mathbf{x}-\mathbf{y},1)=3C^2(\mathbf{x}-\mathbf{y}), \tag{7.7}$$

where

$$C(\mathbf{x}-\mathbf{y})=\frac{1}{3}\langle\phi^a(\mathbf{x},1)\phi^a(\mathbf{y},1)\rangle \tag{7.8}$$

is the ϕ propagator within a single time slice. It is calculated using the action within a time slice,

$$S_{\rm slice}=\frac{\gamma}{2}\sum_{\mathbf{x},i}\ \phi^a(\mathbf{x})U^{ab}_{i,\rm adj}(\mathbf{x})\phi^b(\mathbf{x}+\hat{\imath}). \tag{7.9}$$

(The argument t that enumerates time slices is suppressed in the following.)

Step 2: Reduction to U(1) theory. To evaluate the ϕ propagator at large γ, we introduce a gauge transformation $g(\mathbf{x})\in{\rm SU}(2)$ that depends on $\phi(\mathbf{x})$ with the property that the group element in the adjoint representation, denoted $\mathcal{R}^{ab}[g(\mathbf{x})]$, is the SO(3) rotation matrix that rotates the 3 direction into the ϕ direction, $\phi^a(\mathbf{x})=\mathcal{R}^{a3}[g(\mathbf{x})]$. The action and correlator are given by

$$S_{\rm slice}=\frac{\gamma}{2}\sum_{\mathbf{x},i}\ \mathcal{R}^{33}[g^{-1}(\mathbf{x})U_i(\mathbf{x})g(\mathbf{x}+\hat{\imath})]$$

$$C(\mathbf{x}-\mathbf{y})=\frac{1}{3}\langle\mathcal{R}^{a3}[g(\mathbf{x})]\mathcal{R}^{a3}[g(\mathbf{y})]\rangle. \tag{7.10}$$

These expressions are invariant under $g(\mathbf{x})\to g(\mathbf{x})h(\mathbf{x})$, where $h(\mathbf{x})$ is an SU(2) element that corresponds to a rotation about the 3 axis. So we may average over $h(\mathbf{x})$ which results in replacing the integral over $\phi(\mathbf{x})$ by an integral over $g(\mathbf{x})\in{\rm SU}(2)$. At large γ the functional integral is dominated by the maximum of the action $S_{\rm slice}$, which occurs where $\mathcal{R}^{33}[g^{-1}(\mathbf{x})U_i(\mathbf{x})g(\mathbf{x}+\hat{\imath})]=1$ holds on each link, namely, where $u_i(\mathbf{x})\equiv g^{-1}(\mathbf{x})U_i(\mathbf{x})g(\mathbf{x}+\hat{\imath})$ is a rotation about the 3 axis. These rotations form the U(1) group. Thus at the maximum of the action the link variables are given by

$$U_i(\mathbf{x})=g(\mathbf{x})u_i(\mathbf{x})g^{-1}(\mathbf{x}+\hat{\imath})$$

$$={}^g u_i(\mathbf{x}), \tag{7.11}$$

where $g(\mathbf{x})\in{\rm SU}(2)$. Thus $U_i(\mathbf{x})$ is an SU(2) gauge transform of $u_i(\mathbf{x})\in{\rm U}(1)$. At large γ the $U_i(x)$ get frozen into this form, and the integral over $U_i(\mathbf{x})\in{\rm SU}(2)$ gets reduced to an integral over Abelian configurations $u_i(\mathbf{x})\in$ U(1). We have noted that we may replace $g(\mathbf{x})$ by $g(\mathbf{x})h(\mathbf{x})$, where $h(\mathbf{x})\in{\rm U}(1)$. We use this freedom to gauge fix the $u_i(\mathbf{x})$ within the U(1) group of rotations about the 3-axis. Naturally we choose the minimal Coulomb gauge of U(1) gauge theory. Thus the integral over the $U_i(\mathbf{x})\in{\rm SU}(2)$ gets replaced by an integral over gauge-fixed configurations $u_i(\mathbf{x})\in{\rm U}(1)$. We designate this set by T. In continuum gauge theory, T is the set of *all* transverse Abelian configurations $A^3_i(\mathbf{x})$. In sharp contrast to the SU(2) case, in Abelian gauge

theory there are no Gribov copies. Different transverse configurations $A_i^3(\mathbf{x})$ are gauge inequivalent, and the set T of gauge-fixed Abelian configurations $A_i^3(\mathbf{x})$ is the unbounded set of *all* transverse Abelian configurations, without restriction to a fundamental modular region. (This is a major difference between Abelian and non-Abelian gauge theories, and is the basis of the confinement scenario in minimal Coulomb gauge.)

Step 3: Integration over the $g(\mathbf{x})$. Although $u \in T$ is completely gauge-fixed by minimizing with respect to local $U(1)$ gauge transformations, it is not gauge fixed by minimizing with respect to local SU(2) gauge transformations, because that is a larger group. But our calculation in minimal Coulomb gauge requires that the SU(2) configuration $U_i(x) = {}^g u_i(\mathbf{x}) \in \Lambda$ be completely gauge fixed with respect to local SU(2) gauge transformations. So for given u, $g(\mathbf{x})$ is the unique SU(2) gauge transformation (modulo global gauge transformations) that accomplishes this gauge fixing. We write $g(\mathbf{x}) = g(\mathbf{x};u)$, and the ϕ propagator is given by

$$C(\mathbf{x}-\mathbf{y}) = \int_T du P(u) \frac{1}{3} \mathcal{R}^{a3}[g(\mathbf{x};u)] \mathcal{R}^{a3}[g(\mathbf{y};u)], \tag{7.12}$$

where $P(u)$ is a positive probability density.

Step 4: Integration over the $u(\mathbf{x})$. We saw in Sec. V that when configurations in an Abelian subgroup are gauge fixed, they get mapped into the boundary $\partial\Lambda$. However, that conclusion is an overstatement which ignores the fact that some U(1) configurations $u \in T$ lie in the interior of Λ (although this happens with probability 0, as we shall see). Since in the continuum limit Λ is a subset of transverse SU(2) configurations $A_i^{a,\mathrm{tr}}(\mathbf{x})$ that is bounded in every direction whereas T is the unbounded set of *all* transverse Abelian configurations $A_i^{3,\mathrm{tr}}(\mathbf{x})$, it follows that some configurations $u \in T$ lie inside Λ and some lie outside Λ. Correspondingly we break up the integral into contributions from $u \in T$ inside Λ, and from $u \in T$ outside Λ,

$$C(\mathbf{x}-\mathbf{y}) = C_{\mathrm{in}}(\mathbf{x}-\mathbf{y}) + C_{\mathrm{out}}(\mathbf{x}-\mathbf{y}),$$

$$C_{\mathrm{in}}(\mathbf{x}-\mathbf{y}) = \int_{T\cap\Lambda} du P(u) \frac{1}{3} \mathcal{R}^{a3}[g(\mathbf{x};u)] \mathcal{R}^{a3}[g(\mathbf{y};u)],$$

$$C_{\mathrm{out}}(\mathbf{x}-\mathbf{y}) = \int_{T-T\cap\Lambda} du P(u) \frac{1}{3} \mathcal{R}^{a3}[g(\mathbf{x};u)] \mathcal{R}^{a3}[g(\mathbf{y};u)]. \tag{7.13}$$

For $u \in T\cap\Lambda$ the unique gauge transformation $g(\mathbf{x},u)$ that brings u inside Λ is the identity transformation

$$g(\mathbf{x},u) = I; \quad u \in T\cap\Lambda. \tag{7.14}$$

This gives $\mathcal{R}^{a3}[g(\mathbf{x};u)] = \mathcal{R}^{a3}[I] = \delta_{a3}$, and we obtain

$$C_{\mathrm{in}}(\mathbf{x}-\mathbf{y}) = \int_{u\in T\cap\Lambda} du P(u) = P_{\mathrm{in}}, \tag{7.15}$$

where P_{in} is that probability that $u \in T$ lies inside Λ. This is independent of $\mathbf{x}$ and $\mathbf{y}$, and corresponds to ordered spins. It resembles the calculation with coupling to fundamental Higgs fields at $\beta = 0$ and γ large, where $P_{\mathrm{in}} = 1$. For u outside Λ it appears that the solution $g(\mathbf{x},u)$ of the spin-glass minimization problem depends in a very irregular and disordered way on u and $\mathbf{x}$, so $C_{\mathrm{out}}(R)$ decays rapidly, $\lim_{R\to\infty} C_{\mathrm{out}}(R) = 0$. With $G_{\mathrm{adj}}(R,1) = 3C^2(R)$ this gives

$$\lim_{R\to\infty} G_{\mathrm{adj}}(R,1) = 3P_{\mathrm{in}}^2. \tag{7.16}$$

Thus, remarkably, a numerical determination of $G_{\mathrm{adj}}(R,1)$ provides a direct measurement of the probability P_{in} that a configuration $u \in T$ lies inside Λ.

The data of Fig. 16 strongly suggest that Q_{adj} extrapolates to 0 at infinite volume. This is the disordered phase, in which $\lim_{R\to\infty} G_{\mathrm{adj}}(R,1) = 0$ holds. This gives

$$\lim_{V\to\infty} P_{\mathrm{in}} = 0. \tag{7.17}$$

We have noted that in continuum gauge theory Λ is bounded in every direction whereas T is unbounded in all directions. According to the simple entropy estimate, $p(r)dr \sim r^{D-1}dr$, where D is the (very high) dimension of the space T of transverse U(1) configurations, the fraction P_{in} of the set T that lies inside the bounded region Λ is negligible, $P_{\mathrm{in}} \to 0$, as in fact the data indicate, and we have also $P_{\mathrm{out}} \to 1$. Moreover an Abelian configuration $u \in T$ that lies outside Λ, gets gauge transformed in the minimal Coulomb gauge into a configuration $U = {}^g u$ that lies on the boundary $\partial\Lambda$, as was shown in Sec. V, so in this instance all the probability lies on the boundary $\partial\Lambda$ and the measure of the interior vanishes. This exemplifies the simple scenario in Coulomb gauge, according to which confinement occurs when the functional integral is dominated by the boundary $\partial\Lambda$.

The absence at $\beta = 0$ of a transition in Q_{adj} as γ increases from 0 to ∞ is now explained. For with coupling to the adjoint Higgs field, the measure of ordered spins that would break the remnant gauge symmetry is $P_{\mathrm{in}} = 0$ at large γ, and the remnant gauge symmetry is preserved. By contrast, with coupling to the fundamental Higgs field for γ large, $U_i(x)$ is a gauge transform of the identity I, as we have seen in Sec. VI. Since I is certainly in Λ, then $P_{\mathrm{in}} = 1$, and the remnant symmetry is maximally broken. The coupling to the fundamental Higgs field at large γ keeps configurations away from the boundary $\partial\Lambda$.

Finally we wish to emphasize that with coupling to the adjoint Higgs field, the numerical result $P_{\mathrm{in}} = 0$ is a direct manifestation of the deep difference between the fundamental modular regions (defined by minimizing $F[A]$ with respect to local gauge transformations) of an Abelian and non-Abelian gauge theory. The fundamental modular region is unbounded in every direction in an Abelian gauge theory, but bounded in every direction in a non-Abelian gauge theory.

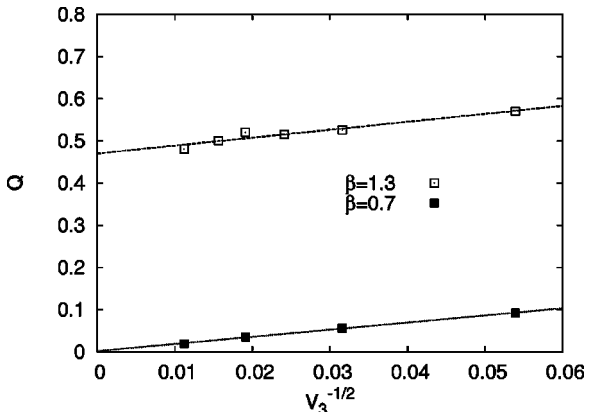

FIG. 17. Plot of Q vs root inverse 3-volume, and extrapolation of Q to infinite volume in QED_4, for $\beta=0.7$ (confining phase) and $\beta=1.3$ (massless phase).

VIII. COMPACT QED

Although in the preceding sections we have considered only the theories with a non-Abelian SU(2) gauge invariance, there is no barrier to studying remnant symmetry breaking in an Abelian model such as compact QED_4. In this model we have confinement at strong couplings and a massless phase at weak couplings, with a transition between the two phases at approximately $\beta=1$. In Fig. 17 we show our results for (the Abelian analogue of) Q at $\beta=0.7$, which is inside the confinement phase, and at $\beta=1.3$, which lies in the massless phase. The results are as expected; Q extrapolates nicely to zero at infinite volume in the confined phase, and appears to extrapolate to a nonzero value in the massless phase. Figure 18 is a plot of $V(R,0)$ vs charge separation R at $\beta=0.7$ on variety of hypercubic lattice volumes L^4. Note that the potential is insensitive to changes in lattice volume, and that deviations from the linear potential, where they are statistically significant, arise from the lack of rotation invariance at strong couplings.

The situation changes drastically in the massless phase at $\beta=1.3$, where our results for $V(R,0)$ are displayed for 5^4, 10^4, and 20^4 lattices in Fig. 19. Although the string tension extracted from $V(R,0)$ appears to be nonzero on each of these lattices (which cannot be correct for the Coulomb potential in the weak-coupling regime), this string tension drops very markedly (by about 1/3) with each doubling of the lattice volume. This volume dependence of the string tension is very different from what we have observed for the non-Abelian theories in various phases. For the Abelian theory, the expected result $\sigma_{\mathrm{Coul}}=0$ is presumably recovered in the large volume limit. In fact, from the value of Q extrapolated to infinite volume, we can estimate that on a large lattice $V(R,0)$ should be bounded, very roughly, by $-\ln(Q^2)\approx 1.6$ at large R. This bound implies that $V(R,0)$ has to level out in large volumes, resulting in $\sigma_{\mathrm{Coul}}=0$.

What we learn from the weak-coupling case is that $V(R,0)$ may have very significant lattice size dependence in theories such as QED_4 in the massless phase, where the correlation length is large (comparable to lattice size) or infinite. It is therefore important to compute $V(R,0)$ on a variety of lattice sizes, and to extrapolate Q to infinite volumes, as we have done in the preceding sections.

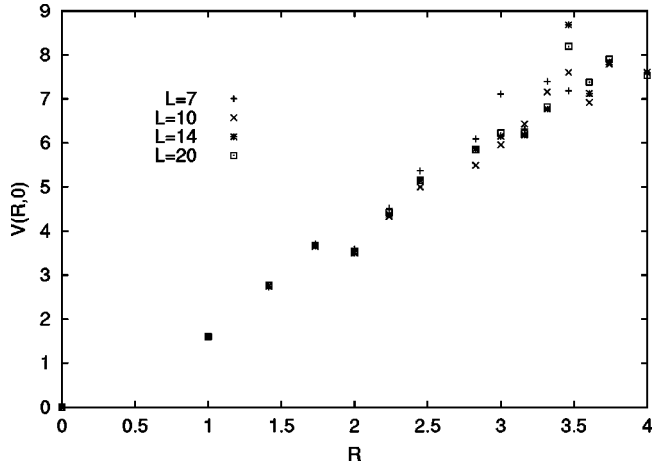

FIG. 18. The potential $V(R,0)$ in QED_4 at $\beta=0.7$, on a variety of L^4 lattice volumes.

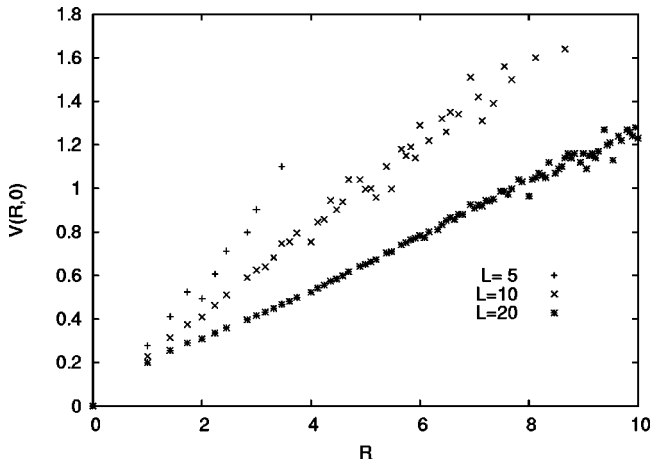

FIG. 19. The potential $V(R,0)$ in QED_4 at $\beta=1.3$, on several L^4 lattice volumes.

IX. TRANSLATION TO TEMPORAL GAUGE AND STRINGLESS STATES

In this section we shall show how to translate back and forth between the minimal Coulomb gauge and the temporal gauge, $A_0=0$, so the measurements reported here have an equivalent description in the temporal gauge. Both of these gauges are compatible with a Hamiltonian formulation and a physical transfer matrix. Moreover the temporal gauge is invariant under all space dependent but time-independent gauge transformations $g(\mathbf{x})$, where $\mathbf{x}$ is a 3-vector, and it may be helpful to express things in a more gauge-invariant way. We shall see that the state obtained here by numerically gauge-fixing to the minimal Coulomb gauge becomes, after translation into temporal gauge, a ''stringless'' state of the type introduced by Lavelle and McMullan [7] that does not involve a Wilson line joining the sources.

A. Temporal gauge

In temporal gauge the continuum Hamiltonian has the canonical form

$$H=\frac{1}{2g^2}\int d^3x(E^2+B^2). \tag{9.1}$$

Here the spatial components $A_i^a(x)$ of the vector potential are canonically conjugate to the color-electric field operator $E_i^a(x)=i[\delta/\delta A_i^a(x)]$, and $B_i^a=\epsilon_{ijk}(\partial_j A_k^a+\frac{1}{2}f^{abc}A_j^bA_k^c)$.

The color-charge density of the static quarks does not appear in H because $A_0=0$. Instead it appears in the generator of local space-dependent gauge transformations

$$G^a(x)=-D^{ac}(A)\cdot E^c(x)+\rho_q^a(x), \tag{9.2}$$

where

$$\rho_q^a(x)=t_1^a\delta(x-x_1)+t_2^a\delta(x-x_2) \tag{9.3}$$

is the color-charge density of a pair of external quarks at x_1 and x_2. The wave functional $\Psi_{\alpha\beta}(A)$ bears the color indices of the external quarks, on which the quark charge vectors act according to

$$(t_1^a\Psi)_{\alpha\beta}=t^a_{\alpha\gamma}\Psi_{\gamma\beta},\quad (t_2^a\Psi)_{\alpha\beta}=-t^{a*}_{\beta\gamma}\Psi_{\alpha\gamma}, \tag{9.4}$$

where the t^a form the basis of an irreducible representation of the Lie algebra of SU(N), $[t^a,t^b]=if^{abc}t^c$. One may verify that $G(\omega)=\int d^3x\omega^a(x)G^a(x)$ generates an infinitesimal gauge transformation

$$\begin{aligned}[G(\omega),A_i^a]&=iD_i^{ac}\omega^c,\\ [G(\omega),E_i^a]&=if^{abc}E_i^b\omega^c,\\ [G(\omega),\rho_q^a]&=if^{abc}\rho_q^b\omega^c.\end{aligned} \tag{9.5}$$

This transformation leaves H invariant, $[G(\omega),H]=0$, so we may choose wave functionals that transform irreducibly under the local gauge group. Physical wave functionals are required to satisfy the subsidiary condition

$$G(x)\Psi=0, \tag{9.6}$$

which is both Gauss' law and the statement that the wave functional is gauge invariant. This condition determines the gauge-transformation properties of the wave functional

$$\Psi({}^gA)=g(x_1)\Psi(A)g^\dagger(x_2), \tag{9.7}$$

where we use matrix notation for the quark color indices, and ${}^gA_i=gA_ig^\dagger+g\partial_ig^\dagger$.

B. From temporal gauge to minimal Coulomb gauge and back

The continuum temporal gauge does not really provide a well-defined quantum mechanics because the inner product for gauge-invariant wave functionals

$$(\Psi_1,\Psi_2)=\int dA\Psi_1^*(A)\Psi_2(A) \tag{9.8}$$

diverges due to the local gauge invariance of the wave-functionals. Gauge-fixing is required to correctly normalize the wave functionals in temporal gauge. We do this by applying the Faddeev-Popov formula to gauge-invariant inner products. For this purpose we parametrize configurations by $A={}^gA^{\rm tr}$, where $A^{\rm tr}$ is the representative of A in the minimal Coulomb gauge, so $A^{\rm tr}\in\Lambda$ is a transverse configuration in the fundamental modular region. The Faddeev-Popov formula gives

$$(\Psi_1,\Psi_2)=\int_\Lambda dA^{\rm tr}\det M(A^{\rm tr})\Psi_1^*(A^{\rm tr})\Psi_2(A^{\rm tr}). \tag{9.9}$$

Here $M(A^{\rm tr})=-\nabla\cdot D(A^{\rm tr})$ is the Faddeev-Popov operator, which is symmetric and positive for $A^{\rm tr}\in\Lambda$. The right hand side is the inner product in minimal Coulomb gauge. Thus, in the 3-dimensional operator formalism, the minimal Coulomb gauge is a gauge-fixing *within* the temporal gauge of the 3-dimensional local gauge invariance, and the wave functional in minimal Coulomb gauge is the restriction of the wave-functional in temporal gauge to the fundamental modular region[10]

$$\Psi_{\rm Coul}(A^{\rm tr})=\Psi(A^{\rm tr}),\quad A^{\rm tr}\in\Lambda. \tag{9.10}$$

Conversely, the gauge invariance of the wave functional in temporal gauge defines the unique extension of the wave functional in minimal Coulomb gauge into a wave functional in temporal gauge. We parametrize an arbitrary configuration by $A={}^gA^{\rm tr}$, where $A^{\rm tr}\in\Lambda$, and $g(x;A)$ and $A^{\rm tr}(x;A)$ depend on the configuration A. The existence and uniqueness of these quantities at the nonperturbative level is assured (with lattice regularization) by the existence of an absolute minimum with respect to gauge transformations of $F_A(g)=||{}^{g^{-1}}A||^2$. From Eq. (9.7) above we obtain, in matrix notation

$$\begin{aligned}\Psi(A)&=\Psi[{}^gA^{\rm tr}]\\ &=g(x_1;A)\Psi[A^{\rm tr}(A)]g^\dagger(x_2;A),\end{aligned} \tag{9.11}$$

which expresses the wave functional in temporal gauge in terms of the wave functional $\Psi(A^{\rm tr})$ in minimal Coulomb gauge. The gauge transformation $g(x;A)$ that is found numerically when gauge fixing to the minimal Coulomb gauge has reappeared in the wave functional in the temporal gauge.

For completeness, we note that a quick way to obtain the Hamiltonian in Coulomb gauge [8] is to apply the Faddeev-Popov formula to the matrix elements of E^2,

[10]To avoid confusion we note that the two conditions $A_0=0$ and $\partial_iA_i=0$ can be imposed at a *fixed* time, which is sufficient for the 3-dimensional operator formalism. They cannot both be maintained for *all* time, and in the 4-dimensional Feynman path-integral formalism in Coulomb gauge one maintains $\partial_iA_i=0$, but $A_0\neq0$.

$$(E_i^a\Psi_1, E_i^a\Psi_2) = \int_\Lambda dA^{\rm tr} \det M(A^{\rm tr})(E_i\Psi_1)^* E_i\Psi_2|_{A=A^{\rm tr}}. \tag{9.12}$$

From this formula the matrix elements of the Hamiltonian in Coulomb gauge are easily found once $E_i\Psi|_{A=A^{\rm tr}}$ is specified. To evaluate $E_i\Psi|_{A=A^{\rm tr}}$ we solve Gauss' law for the longitudinal part of the color-electric field. We write $E_i = E_i^{\rm tr} - \partial_i\phi$, where $\phi^a(x)$ is the color-Coulomb potential operator, and $E_i^{\rm tr} = i(\delta/\delta A^{\rm tr})$ satisfies

$$[E_i^{a,{\rm tr}}(x), A_j^{b,{\rm tr}}(y)] = i[\delta_{ij} - \partial_i\partial_j(\partial^2)^{-1}]\delta(x-y)\delta^{ab}. \tag{9.13}$$

Gauss' law in temporal gauge reads

$$-D_i\partial_i\phi\Psi + A_i \times E_i^{\rm tr}\Psi = \rho_q\Psi, \tag{9.14}$$

where $(X\times Y)^a = f^{abc}X^bY^c$ is the Lie bracket. We solve for the color-Coulomb potential

$$\phi(x)\Psi|_{A=A^{\rm tr}} = [M^{-1}(A^{\rm tr})\rho](x)\Psi(A^{\rm tr}), \tag{9.15}$$

where $A^{\rm tr}\in\Lambda$. Here $M(A^{\rm tr}) = -D(A^{\rm tr})\cdot\partial = -\partial\cdot D(A^{\rm tr})$ is the Faddeev-Popov operator, and

$$\rho \equiv -A_i^{\rm tr}\times E_i^{\rm tr} + \rho_q \tag{9.16}$$

is the total color-charge density of quarks and dynamical gluons. This gives the desired expression

$$E_i\Psi|_{A=A^{\rm tr}} = [E_i^{\rm tr} - \partial_i M^{-1}(A^{\rm tr})\rho]\Psi(A^{\rm tr}), \tag{9.17}$$

which is to be used in Eq. (9.12).

C. Energy in Coulomb gauge is energy in temporal gauge

The quantity we have measured is the expectation value

$$\langle H_{\rm Coul}\rangle - E_0 = V_{\rm Coul}(|\mathbf{x}-\mathbf{y}|) + E_{se} \tag{9.18}$$

in the state with wave-functional in minimal Coulomb gauge

$$\Psi_{\alpha\beta}(A^{\rm tr}) = \frac{1}{\sqrt{2}}\delta_{\alpha\beta}\Phi_0(A^{\rm tr}), \tag{9.19}$$

where $\Phi_0(A^{\rm tr})$ is the vacuum state of pure glue.

We translate this back into temporal gauge. From Eq. (9.11) we obtain

$$\Psi_{\alpha\beta}(A) = \frac{1}{\sqrt{2}}[g(x_1;A)g^\dagger(x_2;A)]_{\alpha\beta}\Phi_0(A), \tag{9.20}$$

where we have used the gauge invariance of the vacuum state of pure glue, $\Phi_0(A^{\rm tr}) = \Phi_0({}^gA^{\rm tr}) = \Phi_0(A)$. Thus the quantity we measure may equivalently be described as the expectation-value of the Hamiltonian in temporal gauge, with this wave functional

$$(\Psi, H\Psi) - E_0 = V_{\rm Coul}(|\mathbf{x}-\mathbf{y}|) + E_{se}. \tag{9.21}$$

D. Stringless states

A wave functional of type (9.20), with $\Phi_0(A)$ an arbitrary gauge-invariant scalar function, has been considered before [7] and was called a ''stringless'' state.[11] The motivation for constructing this state was that it has the correct gauge-transformation properties, which is equivalent to Gauss' law being satisfied exactly. Moreover in QED, with external charges only, the stringless state is the exact wave functional, with $g(x;A) = \exp[i(\nabla^2)^{-1}\nabla\cdot A]$. This leads us to expect that in QCD the stringless state becomes exact at short distance $|\mathbf{x}-\mathbf{y}|$. Thus the stringless state has several attractive properties.

The stringless state was originally constructed using a perturbative expansion for $g(x;A)$. This expansion does not converge for $A^{\rm tr}$ outside the fundamental modular region Λ. However, as we have noted, the existence of $g(x;A)$ at the nonperturbative level is assured by the minimization procedure, and $g(x;A)$ is the gauge transformation that we have found numerically.

Since our numerical data strongly suggest that $V_{\rm Coul}(R)$ rises linearly, it appears from Eq. (9.20) that the ''stringless'' state of quarks manifests a finite string tension $\sigma_{\rm Coul}$ at large separation. Our numerical finding is that this string tension exceeds the standard string tension σ by $\sigma_{\rm Coul}\sim 3\sigma$. This provides a measure of the extent to which the stringless state (9.20) fails to be the exact ground state of a pair of external quarks.

X. CONCLUSIONS

In this article we have shown that the confining property of the color-Coulomb potential is tied to the unbroken realization of a remnant global gauge symmetry in Coulomb gauge. We have studied this type of confinement numerically in SU(2) gauge-Higgs theories, and in pure gauge theory at zero and at finite temperatures. Confinement in the color-Coulomb potential is not identical to confinement in the static quark potential. We have seen that the deconfined phase in pure gauge theory, and the pseudoconfinement region of gauge-fundamental Higgs theory, are instances in which the color-Coulomb potential is asymptotically linear, even though the static quark potential is screened. In terms of symmetries, the point is that center symmetry breaking, spontaneous or explicit, does not necessarily imply remnant symmetry breaking.

The existence of a confining color-Coulomb potential, in cases where the static quark potential is screened, has some

[11] It was called ''stringless'' because in its construction the thin string, $P\exp(\int_{x_1}^{x_2}A_i dx^i)$, was replaced by $g(x_1;A)g^\dagger(x_2;A)$ which transforms in the same way under gauge transformation. It was argued that the thin string has infinite energy, whereas the ''stringless'' state has finite energy (after ultraviolet renormalization) and is a better approximation to the correct hadron state. However, when A lies on a degenerate orbit, ${}^hA = A$ for some $h(x)\neq I$, the parametrization $A = {}^gA^{\rm tr}$ is singular. As a result the stringless wave functional is singular for such configurations, which may raise its energy significantly.

bearing on the question: In what sense does confinement exist in real QCD, with dynamical quarks? The problem is that in gauge theories with matter fields in the fundamental representation, such as real QCD, there is no nontrivial center symmetry, and no possibility of having an asymptotically confining static potential. Further, in gauge theories with a scalar matter field in the fundamental representation, there is no local order parameter that can distinguish between the Higgs and confinement phases, and the Fradkin-Shenker theorem assures us that there is no thermodynamic transition of any kind that can isolate the Higgs phase from a distinct confinement phase.[12] All this suggests that there is no fundamental difference between real QCD and SU(3) gauge theory in a Higgs phase, and it should be possible to interpolate smoothly from the one theory to the other by a continuous change of parameters.

The results reported in this paper suggest otherwise. In gauge theories with matter fields in the fundamental representation, the ''confinement'' phase and the Higgs phase are distinguished by the symmetric or broken realization of a remnant gauge symmetry. Remnant symmetry breaking is not accompanied by nonanalyticity in the free energy, nevertheless the (nonlocal) order parameter Q is directly related to the color-Coulomb potential, which is confining in the $Q=0$ phase. Thus, in real QCD, the gluon propagator in Coulomb gauge is confining. This confining property of (dressed) one-gluon exchange is absent in the Higgs phase of a gauge field theory.

We have also uncovered a strong correlation between the presence of center vortices and the existence of a confining Coulomb potential. Thin center vortices, as pointed out above, lie on the Gribov horizon. In every case where we find a confining Coulomb potential, we also find that removal of center vortices takes the Coulomb string tension to zero. In related work on the SU(2) gauge-fundamental Higgs theory, it was suggested [4], and recently verified [21], that vortices percolate throughout the lattice in the pseudoconfined phase, and do not percolate in the Higgs phase. These phases correspond to regions of unbroken and broken remnant symmetry, which are completely separated by a percolation transition located along a Kertész line.

There are many open questions. Presumably the linear color-Coulomb potential is associated with a flux tube of longitudinal color electric field. If this is really so, does the tube have stringlike properties; i.e., roughening and a Lüscher term? Since $\sigma_{\rm Coul}$ is several times greater than σ, the Coulomb flux tube must be an excited state. By what mechanism is the string tension σ of the minimal energy flux tube reduced below the value $\sigma_{\rm Coul}$ of the Coulomb flux tube? Does this mechanism involve production of constituent gluons, along the lines of the gluon-chain model [23], or is some other process at work? We hope to address some of these issues in a future investigation.

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[12]While it is sometimes suggested that confinement should simply be understood as the condition that asymptotic particle states are all color singlets, this condition is also fulfilled in the Higgs phase of gauge-Higgs theories (cf., e.g., Ref. [13]).

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