Analytical approach to chiral symmetry breaking in Minkowski space

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The mass gap equation for spontaneous chiral symmetry breaking is studied directly in Minkowski space. In hadronic physics, spontaneous chiral symmetry breaking is crucial to generate a constituent mass for the quarks, and to produce the partially conserved axial vector current theorems, including a small mass for the pion. Here a class of finite kernels is used, expanded in Yukawa interactions. The Schwinger-Dyson equation is solved with an analytical approach. This improves the state of the art of solving the mass gap equation, which is usually solved with the equal-time approximation or with the Euclidean approximation. The mapping from Euclidean space to Minkowski space is also illustrated.

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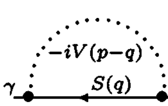
I. INTRODUCTION

Here I solve directly in Minkowski space the mass gap equation for spontaneous chiral symmetry breaking (SχSB). SχSB was introduced by the original work of Nambu and Jona-Lasinio [1], and it is presently accepted to occur in hadronic physics, where it generates a constituent mass for the quarks. SχSB also implies the partially conserved axial vector current theorems [2], including a small mass for the pion. In the literature the mass gap equation is usually solved either in equal time [3] or in Euclidean space [4], in order to avoid the poles and complex quantities which are expected in full Minkowski calculations. Recently the scientific community has been exploring different approaches to Minkowski space [5]. An exact solution in Minkowski space will test the quality of the approximate solutions. Moreover a solution in Minkowski space opens wider applications. For instance, the Bethe-Salpeter amplitudes can be computed on the mass shell momentum $p^2 = M^2 > 0$, and at the same token it is possible to boost the hadrons to any convenient frame. In this paper an analytical approach is applied to finite and analytic kernels. For simplicity the infrared or ultraviolet divergences are not addressed here. Although a complete solution can only be achieved numerically, an interesting insight is nevertheless presented by the analytical approach.

In the Schwinger-Dyson formalism truncated at the rainbow level, the mass gap equation is

$$\mathcal{S}^{-1}(p) = \mathcal{S}_0^{-1}(p) - \gamma$$


$$\gamma,$$

$$S(p) = \frac{i}{A(p^2)\not{p} - B(p^2) + i\epsilon}.$$

$$S(p) = \frac{i}{A(p^2)\not{p} - B(p^2) + i\epsilon}. \tag{1}$$

For the quark-quark interaction I consider the class of kernels that can be decomposed in a class of Yukawa potentials, which is finite both in the ultraviolet and infrared limits

$$-iV(p-q) = -i\sum_i \frac{\alpha_i 4\pi}{(p-q)^2 - \lambda_i^2 + i\epsilon}. \tag{2}$$

To ensure that the kernels are finite in the ultraviolet limit of large $(p-q)^2$ I assume that $\Sigma_i \alpha_i = 0$, since this implies that the kernel vanishes at least as $[(p-q)^2]^{-2}$. In the infrared limit the kernels are finite when the λ_i are finite, or when infrared cancellations occur. The class of kernels defined in Eq. (2) is quite general, for instance it includes the Coulomb interaction regularized by a Pauli-Villars term. It also includes the Fourier transform of the confining linear potential

$$\mathrm{FT}(|\mathbf{r}|e^{-\lambda|\mathbf{r}|}) = \frac{d^2}{d\lambda^2}\frac{4\pi}{\mathbf{k}^2 + \lambda^2}, \tag{3}$$

and of the linear potential with a negative infinite shift

$$\mathrm{FT}(-e^{-\lambda|\mathbf{r}|}) = \frac{d}{d\lambda}\frac{4\pi}{\mathbf{k}^2 + \lambda^2}, \tag{4}$$

which are both obtained in the limit of vanishing λ. Using finite differences, the derivatives in λ can be decomposed in the class of potentials of Eq. (2). The class of kernels addressed in this paper is not only general, it is also covariant, analytical, and forward propagating, and this is convenient to study the mass gap equation in Minkowski space.

Chiral symmetry breaking occurs when a nonvanishing mass is dynamically generated. For clarity let us also assume that $A \simeq 1$ in Eq. (1). This approximation is qualitatively acceptable when the kernel is finite [3,4]. Computing A is not difficult, but it obscures the result of the paper. Then the mass gap equation is a single nonlinear and integral equation for $B = M$,

$$M(p^2) = m_0 + \int_{-\infty}^{\infty} \frac{i d^4 q}{(2\pi)^4} V(p-q) \frac{M(q^2)}{q^2 - M^2(q^2) + i\epsilon}, \tag{5}$$

where all Dirac and color algebraic factors are absorbed in the coupling constants α_i of the potential $V(p-q)$. The chiral limit of vanishing current quark mass (the mass in the free quark propagator) $m_0 \simeq 0$, is also assumed. This is particularly interesting because it applies to the physics of the quarks u and d. It is clear that Eq. (5) then has a trivial solution $M(p^2)=0$. The problem that this paper addresses is the other possible solutions of Eq. (5), with the kernel of Eq. (2). The technical difficulties reside in the multiple integral with poles and complex quantities and in the nonlinearity of the self-consistent equation.

In Sec. II standard approximations are applied to the rainbow Schwinger-Dyson, and the need to perform a calculation in Minkowski space is motivated. In Sec. III the integral in the mass gap equation is computed analytically in the case where the quark mass is assumed to be constant. The solution of the Minkowski nonlinear integral mass gap equation is addressed in Sec. IV. Finally in Sec. V the results and conclusion are presented.

II. USING STANDARD APPROXIMATIONS

In the literature the mass gap equation is usually solved either in equal time [3] or in Euclidean space [4]. In the equal time approximation, the Lorentz invariance is lost. The space and time components of physical constants, say f_π^s or f_π^t may differ [6]. Moreover it is not clear how to boost the hadrons outside the center-of-mass frame. In the Euclidean approximation it is not clear if a simple Wick rotation $p_0 \to i p_4$ is exact because there may exist poles in the path ot the p_0 axis. Moreover it is very hard to rotate back to Minkowski with the inverse rotation $i\, p_4 \to p_0$ when only a numerical expression of the dynamical mass is known. This is connected to the topological problem of discretizing a curved surface. Although these two methods are approximate, they are used in the literature because they are fully consistent with dynamical symmetry breaking, providing the same approximation is also used in the bound state equations [2]. At the same token these methods avoid the technical problem of addressing poles and complex quantities, which are expected in Minkowski space. By providing a solution of the mass gap equation, both Lorentz invariant and in Minkowski space, both the equal time and the Euclidean time approximations may be better understood. Here the approximate methods for solving the Schwinger-Dyson equation are reviewed.

In the instantaneous or equal time approximation, it is assumed that the dependence in p_0 and in q_0 is irrelevant in the kernel of the mass gap equation. The angular integral of the Coulomb potential with neglected time component is

$$\int_{-1}^{1} d\omega \frac{2\pi \mathbf{q}^2 \alpha_i}{\mathbf{p}^2+\mathbf{q}^2-2|\mathbf{p}||\mathbf{q}|\omega+\lambda_i^2} = \frac{2\pi \mathbf{q}^2 \alpha_i}{-2|\mathbf{p}||\mathbf{q}|} \log\left[\frac{(|\mathbf{p}|-|\mathbf{q}|)^2+\lambda_i^2}{(|\mathbf{p}|+|\mathbf{q}|)^2+\lambda_i^2}\right], \tag{6}$$

where the notation for quadrivectors and for trivectors is $p=(p_0,\mathbf{p})$. In what concerns the mass, a solution exists where the mass is independent of q_0, and the integral in q_0 is trivial,

$$\int_{-\infty}^{\infty} dq_0 \frac{i}{q_0^2-[\mathbf{q}^2+M^2(\mathbf{q}^2)-i\epsilon]} = \frac{\pi}{\sqrt{\mathbf{q}^2+M^2(\mathbf{q}^2)}} \tag{7}$$

and the mass gap equation in the instantaneous approximation is

$$M(\mathbf{p}^2) = \int_0^\infty d|\mathbf{q}| \sum_i \frac{-\alpha_i|\mathbf{q}|}{4\pi|\mathbf{p}|} \log\left[\frac{(|\mathbf{p}|-|\mathbf{q}|)^2+\lambda_i^2}{(|\mathbf{p}|+|\mathbf{q}|)^2+\lambda_i^2}\right] \times \frac{M(\mathbf{q}^2)}{\sqrt{\mathbf{q}^2+M^2(\mathbf{q}^2)}}. \tag{8}$$

This is a one-dimensional nonlinear integral equation. It has no singularities and it is solvable numerically [3].

In the Euclidean approximation, it is assumed that the time component of the momentum can be replaced $p_0 \to i p_4$ or $-p_0^2 \to p_4^2$. A convenient angular description of the variables is

$$\begin{cases} p_1 = p_E \sin\phi \sin\theta \sin\eta, \\ p_2 = p_E \cos\phi \sin\theta \sin\eta, \\ p_3 = p_E \cos\theta \sin\eta, \\ p_4 = p_E \cos\eta, \end{cases} \tag{9}$$

where the Euclidean momentum is $p_E^2 = -p^2 = p_1^2+p_2^2+p_3^2+p_4^2$. In this case the boson exchange potential of Eq. (2) is finite for all momenta. Covariance allows the choice of the external momentum parallel to the fourth axis, and three angular integrals can be performed,

$$\begin{aligned} &\frac{-4\pi\alpha q_E^3}{2q_E}\int_0^\pi d\eta \frac{\sin^2\eta}{p_E^2+q_E^2-2p_E q_E \cos\eta+\lambda^2} \\ &= 2\pi^2\alpha q^2 \frac{p^2+q^2-\lambda^2+\sqrt{(p^2+q^2-\lambda^2)^2-4p^2q^2}}{4p^2q^2} \\ &= \frac{\pi^2\alpha}{4p^2}\left[\sqrt{-(p+q)^2+\lambda^2}-\sqrt{-(p-q)^2+\lambda^2}\right]^2 \end{aligned} \tag{10}$$

which is a function of the real and positive variable $-q^2 = q_E^2$. Again the mass gap equation is reduced to a one-dimensional nonlinear integral equation

$$M(p^2) = \int_0^\infty d(-q^2) \sum_i \frac{\alpha_i}{2\pi p^2}\left[p^2+q^2-\lambda^2 + \sqrt{(p^2+q^2-\lambda^2)^2-4p^2q^2}\right] \frac{M(q^2)}{-q^2+M^2(q^2)} \tag{11}$$

which has no singularities and is solvable numerically [4].

A third perspective, differing from the equal-time approximation because it includes the retardation in the kernel and differing from the Euclidean approximation because it addresses timelike momenta, is provided by the coupled channel approach. Here the approximation consists in only considering the positive energy poles of the propagators. For simplicity I consider here a single Yukawa term

$$\frac{1}{q^2-M^2+i\epsilon}\rightarrow\frac{1}{q_0-H_f+i\epsilon}\frac{1}{2H_f},$$

$$\frac{1}{(p-q)^2-\lambda_i^2+i\epsilon}\rightarrow\frac{1}{E-q_0-H_b+i\epsilon}\frac{1}{2H_b},$$

$$H_f(\mathbf{q})=\sqrt{\mathbf{q}^2+M^2},$$

$$H_b(\mathbf{p-q})=\sqrt{(\mathbf{p-q})^2+\lambda_i^2} \tag{12}$$

then the integral in q_0 produces

$$\int\frac{dq_0}{2\pi}\frac{i}{q_0-H_f+i\epsilon}\frac{i}{E-q_0-H_b+i\epsilon}=\frac{i}{E-H_f-H_b+i\epsilon}. \tag{13}$$

The $1/2H_f$, $1/2H_b$, and the fermion mass M can be included in the boson creation vertex

$$\Gamma^\dagger=\sqrt{\frac{M}{4H_fH_b}}, \tag{14}$$

then the mass gap equation is equivalent to a coupled channel Hamiltonian equation, where the one fermion channel is coupled to the fermion plus boson channel by the boson creation and annihilation vertices

$$\begin{bmatrix} H_f-i\epsilon-E & \Gamma \\ \Gamma^\dagger & H_f+H_b-i\epsilon-E\end{bmatrix}\begin{pmatrix}\phi_f\\ \phi_{f,b}\end{pmatrix}=0. \tag{15}$$

Reducing the equation by substitution of the fermion plus boson wavefunction, the secular equation is

$$0=\left[E-H_f(p)-\int\frac{d^3p}{(2\pi)^3}\Gamma(p,q)\times\frac{1}{E-H_f(q)-H_b(p-q)+i\epsilon}\Gamma^\dagger(p,q)\right]\phi(p)=0, \tag{16}$$

where it is clear that when $E>M+\lambda_i$, the threshold for a boson production is open. This produces a cut in the function $M(\mathbf{p})$, with an imaginary component for the mass. Actually more cuts appear when the energy is further increased, and the system couples to 1 fermion and 2 bosons, 1 fermion and 3 bosons, etc. Therefore cusps and imaginary components are expected to appear when $E=M+\lambda_i$, $E=M+2\lambda_i$, etc.

When the mass gap equation is solved in the Minkowski space, then all the features of the three different approaches of Eqs. (8), (11), and (16) are expected to appear. Nontrivial solutions of the mass gap equation, and imaginary masses above thresholds are expected. Moreover It will be interesting to study the effect of the negative energy components on eigenvalues.

III. ONE LOOP ANALYTICAL CALCULATION

As a first approach to the Minkowski integral, I assume a real constant mass M and compute analytically the integral of Eq. (5). This can be regarded a one loop approximation to the mass gap equation, in the sense that the correct solution can be obtained iteratively, in an infinite loop calculation. The analytical result may also be eventually used to remove poles form the numerical iterative program. In this sense I choose to perform the multiple integrals in the same ordering that may be used in a numerical integration. A first attempt to start by an analytical integral of the three angular integrals (both trigonometric and hyperbolic) was abandoned because the result was indeterminate. Therefore the two angular integrations of the three dimensional space are first performed, in order to simplify the kernel. Finally the double integration in the temporal q_0 and in the spatial q are performed.

The main aim of this section is the covariant study of the positive p^2 case which was not accessible to the approximate approaches. Covariance allows one to consider the $p=(p^0,\mathbf{0})$ case. Analyticity can be used later, to continue this function to a p^2 negative case. The two dimensional angular integrals are trivial. The integral in q_0 is also directly computed, using the residue theorem

$$I(p_0,0)=\int d^3\mathbf{q}dq_0\sum_i\frac{\alpha_i}{(q_0-p_0)^2-\mathbf{q}^2-\lambda_i^2+i\epsilon}\frac{i\mathbf{q}^2}{q_0^2-\mathbf{q}^2-M^2+i\epsilon}$$
$$=\int d|\mathbf{q}|\sum_i\frac{-\pi4\pi\mathbf{q}^2\alpha_i}{2\sqrt{\mathbf{q}^2+\lambda_i^2-i\epsilon}\sqrt{\mathbf{q}^2+M^2-i\epsilon}}$$
$$\times\left(\frac{1}{p_0+\sqrt{\mathbf{q}^2+\lambda_i^2-i\epsilon}+\sqrt{\mathbf{q}^2+M^2-i\epsilon}}+\frac{1}{-p_0+\sqrt{\mathbf{q}^2+\lambda_i^2-i\epsilon}+\sqrt{\mathbf{q}^2+M^2-i\epsilon}}\right). \tag{17}$$

The integral in $q=|\vec{q}|$ can also be performed analytically. It is interesting to remark that the threshold for an imaginary contribution appears when $p_0 > M+\lambda_i$. Indeed the pole appears at the root ρ of

$$p_0=\sqrt{\mathbf{q}^2+M^2}+\sqrt{\mathbf{q}^2+\lambda_i^2}\Rightarrow|\mathbf{q}|=\rho, \tag{18}$$

$$\rho=\sqrt{\frac{(p_0^2-M^2-\lambda_i^2)^2-4M^2\lambda_i^2}{4p_0^2}+i\epsilon},$$

where ρ coincides with the mass shell momentum above threshold. A simple algebraic simplification transforms the integral into

$$I=\int d|\mathbf{q}|\sum_I \alpha_i \frac{-\pi}{4p_0^2}\frac{4\pi\mathbf{q}^2}{\mathbf{q}^2-\rho^2} \times\left(\frac{M^2+p_0^2-\lambda_i^2}{\sqrt{M^2+\mathbf{q}^2}}+\frac{-M^2+p_0^2+\lambda_i^2}{\sqrt{\mathbf{q}^2+\lambda_i^2}}\right), \tag{19}$$

where the pole ρ defined in Eq. (19) is real when the threshold opens, at $p_0 > M+\lambda_i$. ρ is also real when $p_0 < |M-\lambda_i|$, below the pseudothreshold

$$\rho=\sqrt{\frac{[p_0^2-(M+\lambda_i)^2][p_0^2-(M-\lambda_i)^2]}{4p_0^2}+i\epsilon}, \tag{20}$$

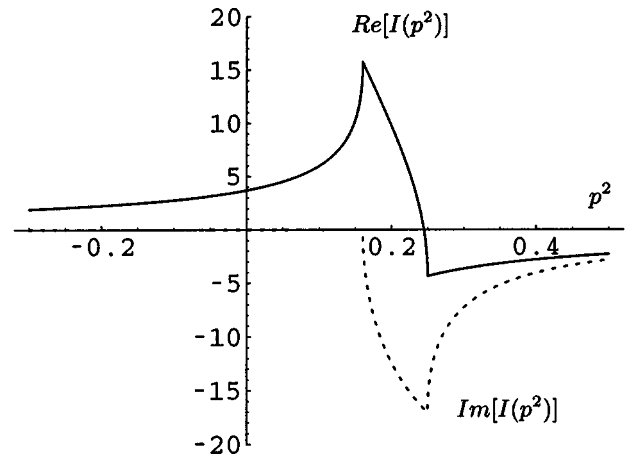


FIG. 1. Real and imaginary part of the integral $I(p^2)$ that dynamically generates quark mass. Here the parameters $M=0.3$ GeV, $\lambda_1=0.1$ GeV and $\lambda_2=0.2$ GeV, $\alpha_1=1$, and $\alpha_2=-1$ are used. The cusps occur precisely where the thresholds open, at $p^2=M+\lambda_1$ and at $p^2=M+\lambda_2$.

however, in this case the residue vanishes, therefore I is only expected to possess an imaginary component above the threshold for the fermion-boson production.

After cumbersome calculations, where the ultraviolet divergent terms cancel because they are proportional to $\Sigma_i\alpha_i$, the integral $I(p_0,0)$ can finally be reduced to the exact form

$$I(p^2)=\sum_i \frac{\alpha_i\pi^2}{2p^2}\Bigg[(-M^2+p^2+\lambda_i^2)\log\frac{M^2}{\lambda_i^2}+(-M^2+p^2+\lambda_i^2)\left(\frac{\rho}{\sqrt{\rho^2+\lambda_i^2}}\left\{\log\left[i\left(1-\frac{\sqrt{\rho^2+\lambda_i^2}}{\rho}\right)\right]\right.\right.$$
$$\left.\left.-\log\left[i\left(1+\frac{\sqrt{\rho^2+\lambda_i^2}}{\rho}\right)\right]\right\}\right)+(M^2+p_0^2-\lambda_i^2)\frac{\rho}{\sqrt{\rho^2+M^2}}\left\{\log\left[i\left(1-\frac{\sqrt{\rho^2+M^2}}{\rho}\right)\right]-\log\left[i\left(1+\frac{\sqrt{\rho^2+M^2}}{\rho}\right)\right]\right\}\Bigg), \tag{21}$$

where ρ is defined in Eq. (19). The result of Eq. (21) is not only correct for time-like momentum, it also applies to negative p_0^2, where the integrals of Eq. (17) remain correct. The integral $I(p^2)$ is depicted in Fig. 1. Figures 1 and 2 are obtained with two Yukawa terms.

There are three particular cases of Eq. (17) that can be tested independently. The imaginary part can be directly computed with the residue theorem applied to Eq. (19). When $p_0 > M+\lambda_i$ the imaginary part is

$$\mathrm{Im}[I(p_0,0)]=\sum_i -\alpha_i\pi^3\frac{\sqrt{(p_0^2-M^2-\lambda_i^2)^2-4M^2\lambda_i^2}}{p_0^2}. \tag{22}$$

Another important particular case is the matching point, between the timelike momentum and the spacelike momentum, of $p^2=p_0^2=0$. The integral of Eq. (17) is then easy to compute:

$$I(0,0)=\sum_i -\alpha_i\pi^2\frac{M^2\log(M^2)-\lambda_i^2\log(\lambda_i^2)}{M^2-\lambda_i^2}. \tag{23}$$

Finally in the spacelike case $p=(0,\mathbf{p})$ can be considered. This implies that the poles are on the correct quadrants to enable a trivial Wick rotation. I can also use the angular integrals already performed in Eq. (10). The resulting integral can be computed analytically,

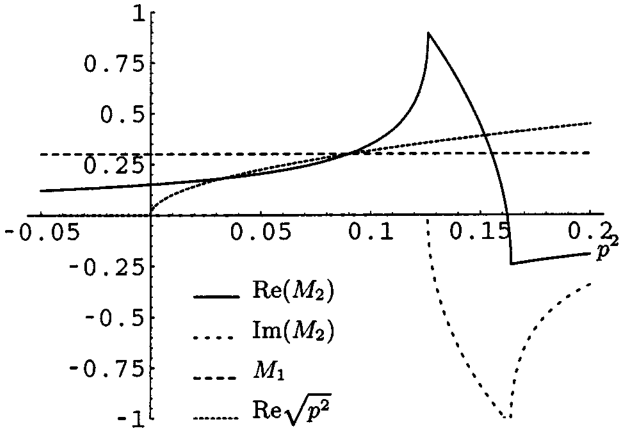


FIG. 2. The initial mass M_1, the one loop mass $M_2(p^2)$, and $\sqrt{p^2}$ are illustrated. They all coincide at 0.3 GeV, and the parameters and $\alpha_1=-\alpha_2=36$ are adjusted accordingly. The parameters are $\lambda_1=0.055$ GeV and $\lambda_2=0.105$ GeV. This case also includes a second solution of the mass gap equation $M^2=p^2=(0.18\text{ GeV})^2$.

$$
\begin{aligned}
I(0,\mathbf{p})&\\
=\sum_i \frac{-\alpha_i\pi^2}{2\mathbf{p}^2}&\Bigg\{(M^2+\mathbf{p}^2-\lambda_i^2)\log\left(\frac{\lambda_i^2}{M^2}\right)\\
&+\sqrt{(\mathbf{p}^2+M^2+\lambda_i^2)^2-4M^2\lambda_i^2}\\
&\times\log\left[\frac{M^2+\mathbf{p}^2+\lambda_i^2+\sqrt{(M^2+\mathbf{p}^2+\lambda_i^2)^2-4M^2\lambda_i^2}}{M^2+\mathbf{p}^2+\lambda_i^2-\sqrt{(M^2+\mathbf{p}^2+\lambda_i^2)^2-4M^2\lambda_i^2}}\right]\Bigg\}.
\end{aligned}
\tag{24}
$$

These three particular cases comply with Eq. (21).

IV. APPROXIMATE ANALYTICAL SOLUTION

Here the mass gap equation is solved in an analytical one loop approximation. The standard method to solve the mass gap equation (5) is the iterative method, where one starts by an initial educated guess for the mass $M_1(p^2)$. I consider a constant M_1, which allows the analytical computation of the integral in Eq. (5), with the techniques used in Sec. III. This produces the next mass in the iterative series

$$M_1\rightarrow M_2(p^2)=\frac{4\pi}{(2\pi)^4}M_1 I(p^2), \tag{25}$$

where the integral $I(p^2)$ is defined in Eq. (21). To find an exact solution to the mass gap equation, one would then need to continue this iterative process, computing again the integral with the function $M_2(p^2)\rightarrow M_3(p^2)$ and so on until the method converges. However these further iterative steps would probably need a numerical computation, since the function $M_2(p^2)$ is already a complicated one. Therefore, in this analytical approach, I choose to stop at the second step of the iteration. To minimize the error of this one loop computation, I demand that the mass $M_2(p^2)$ coincides with the mass M_1 at the mass shell momentum $M_2(p^2)=p^2$,

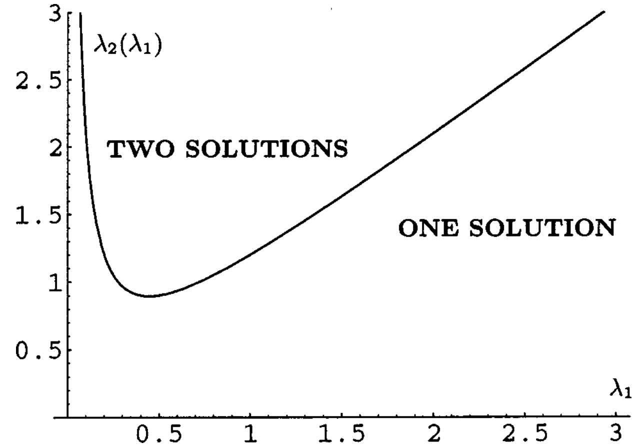


FIG. 3. Phases with one and two solutions of the mass gap equation (5).

$$M_1=M_2(p^2)=p^2. \tag{26}$$

This is equivalent to assume that the mass dependence of the integral in Eq. (5) is dominated by the pole neighborhood. The quark mass, dynamically generated in the mass gap equation, is expected to coincide with the constituent quark mass. The constituent quark mass is estimated in the quark model, where it is a crucial parameter to produce the hadronic spectrum, and where it is of the order of 0.3 GeV. Therefore the coupling constants α_i of the quark interaction (2) are adjusted to reproduce $M_2=0.3$ GeV. An example of this is illustrated in Fig. 2.

I now discuss the different scenarios for the parameters λ_i of the potential. I start with the case with three Yukawa terms. In the particular case of in Eq. (3), this corresponds to the linear potential, which vanishes in the infrared limit. It occurs that in this case the generated quark mass $M_2(p^2)$ is quite small close to the origin $p^2\simeq 0$. It is then very difficult to arrive at a quark mass of 0.3 GeV. This may be related to the infrared cancellation

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{4\pi}{\mathbf{k}^2+\lambda_1^2}-2\frac{4\pi}{\mathbf{k}^2+\lambda_2^2}+\frac{4\pi}{\mathbf{k}^2+\lambda_3^2}\right)=0 \tag{27}$$

which occurs when the λ_i are equally spaced, $\lambda_3-\lambda_2=\lambda_2-\lambda_1$. Moreover the structure of the potential is quite complicated above threshold, with large cusps. Therefore this class of models is abandoned.

Next the simpler case of two Yukawa terms is studied. This is related to the first derivative of Eq. (4). In this case it is easy to adjust the strength parameters $\alpha_1=-\alpha_2$ to produce a quark mass of 0.3 GeV at the pole position of $p^2=0.3\text{ GeV}^2$. This seems to agree with the infrared finite result

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{4\pi}{\mathbf{k}^2+\lambda_1^2}-\frac{4\pi}{\mathbf{k}^2+\lambda_2^2}\right)=\frac{\lambda_1-\lambda_2}{4\pi}. \tag{28}$$

Nevertheless, in this simple case of two Yukawa terms, there are two different scenarios. Depending on the steepness of the function $M^2(p^2)$, it intercepts the function p^2 either at a single point or in two points. From a numerical exploration one concludes that the two scenarios are separated by the line $\lambda_1(\lambda_2-\lambda_1)\simeq 0.2\ \mathrm{GeV}^2$. This line is depicted in Fig. 3, in the parameter space $\lambda_2\lambda_1$. There is a single solution when

$$\lambda_1(\lambda_2-\lambda_1)\geqslant 0.2\ \mathrm{GeV}^2. \tag{29}$$

For instance, this includes the case of a large λ_2, which acts as an ultraviolet Pauli-Villars cutoff, say $\lambda_1=0.3$ GeV, $\lambda_2=3$ GeV. When λ_1 and λ_2 are quite large, the solution $M(p^2)$ of the mass gap equation is very smooth, and the instantaneous or the Euclidean approaches constitute very good approximations to the actual Minkowski solution. It is also clear that in the case of a Pauli-Villars regularization $A\simeq 1$ is not acceptable. A would need to be computed, nevertheless its computation could be performed with the techniques presented in this paper.

One also concludes that there is a double solution when

$$\lambda_1(\lambda_2-\lambda_1)\leqslant 0.2\ \mathrm{GeV}^2. \tag{30}$$

This includes the case of small and similar λ_i. This case is related to the first derivative of Eq. (4). In the limit of vanishing parameters λ_i, Eq. (4) corresponds to a linear potential with a negative and infinite constant shift

$$-\frac{\sigma}{\lambda}e^{-\lambda|\mathbf{r}|}\simeq\frac{-\sigma}{\lambda}+\sigma\mathbf{r}. \tag{31}$$

Comparing with the potential defined in Eq. (2) the string tension is $\sigma=\alpha_1(\lambda_2^2-\lambda_1^2)/2$. Assuming a string constant of the order of $0.14\ \mathrm{GeV}^2$ estimated from the quark spectrum, one arrives at the following parameters:

$$\alpha(\lambda_2-\lambda_1)=1.8\ \ \mathrm{GeV},$$
$$\frac{\lambda_1+\lambda_2}{2}=0.08\ \ \mathrm{GeV}, \tag{32}$$

and this implies that the potential also includes a negative constant shift of the order of -1.7 GeV. An example of the generated masses is depicted in Fig. 2.

To proceed iterating Eq. (5), say at two loop order or more, or to compute the quark condensate $\langle\bar{\psi}\psi\rangle$, the analytical integral becomes quite complicated. A numerical study is probably necessary and this is not addressed in this paper. Nevertheless the qualitative changes to the one loop computation can be anticipated. In what concerns the positive p^2, it is expected that new channels will open whenever $p^2>(M+n\lambda_1+m\lambda_2)^2$, where m and n are positive integers. This will affect the imaginary part of the mass. In what concerns the behavior of the mass $M(p^2)$ for very large positive or negative p^2, an inspection of Eq. (5) shows that

$$M(p^2)\rightarrow\frac{4\pi}{g}\frac{\alpha_1\lambda_2-\alpha_2\lambda_1}{(p^2)^2}\langle\bar{\psi}\psi\rangle,$$
$$\langle\bar{\psi}\psi\rangle=g\int_{-\infty}^{\infty}\frac{id^4q}{(2\pi)^4}\frac{M(q^2)}{q^2-M^2(q^2)+i\epsilon}, \tag{33}$$

and therefore the mass should vanish proportionally to $1/(p^2)^2$. In the one loop approximation the mass only vanishes as $1/p^2$, see Fig. 1. Therefore the large momentum behavior of the generated quark mass is expected to improve in the next iterations.

V. RESULTS AND CONCLUSION

I address the technically challenging problem of solving the nonlinear integral Schwinger-Dyson equation in full Minkowski space. An analytical approach is followed, and approximate but analytical expressions for the quark mass are obtained.

I find that the quark mass exhibits a branch cut above the threshold for boson creation, including an imaginary component. In the case of a linear potential the mass gap equation is expected to have at least two solutions, and this agrees with Ref. [7].

The analytical continuation of a the numerical Euclidean space solution (with negative p^2) into the full Minkowski has been studied in the literature [5]. Here I verify that this analytical continuation is not uniquely defined, even when a dense set of points is known in the spacelike $p^2<0$ sector. In particular, Eq. (5) includes at least three vanishing imaginary numbers $-i\epsilon$, summed, respectively, to the masses λ_1,λ_2, and M. For external spacelike momenta $p^2<0$, these $i\epsilon$ are irrelevant. However for timelike momenta $p^2>0$ there are possible branch cuts both above threshold $p^2>(M+\lambda_i)^2$ and below the pseudothreshold $0<p^2<(M-\lambda_i)^2$, and the integral depends on the sign of each of the three vanishing ϵ. Only one integral, with all the three $\epsilon>0$ is causally correct [8]. For instance, in Fig. 1 the correct imaginary part of the mass is negative, but a continuation with a positive imaginary would also be analytically possible. As another example, a naive extension of the Euclidean Eq. (25) to $p^2>0$ fails to coincides with the full Minkowski solution (21) for some timelike momenta.

The next step of this study will consist in applying numerical methods to proceed with the study of the mass gap equation. A numerical integration seems to be necessary to compute the quark condensate $\langle\bar{\psi}\psi\rangle$. Moreover the exact numerical solution of the mass gap equation (5) may also be attempted. With the analytical method of this paper, it may be possible to subtract the poles from the integrand and to compute their contribution analytically. Another interesting method may be the fully numerical MonteCarlo method [9]. These numerical methods will be applied elsewhere.

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[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

[2] R. Delbourgo and M.D. Scadron, J. Phys. G **5**, 1621 (1979); P. Bicudo, Phys. Rev. C **67**, 035201 (2003); F. Llanes-Estrada and P. Bicudo, Phys. Rev. D **68**, 094014 (2003); A. Holl, A. Krassnigg, and C.D. Roberts, nucl-th/0311033; A. Krassnigg and C.D. Roberts, nucl-th/0308039.

[3] A. Le Yaouanc, L. Oliver, O. Pene, and J-C. Raynal, Phys. Rev. D **29**, 1233 (1984); A. Le Yaouanc, L. Oliver, S. Ono, O. Pene, and J-C. Raynal, *ibid.* **31**, 137 (1985); S. Adler and A.C. Davis, Nucl. Phys. **B244**, 469 (1984); P. Bicudo and J.E. Ribeiro, Phys. Rev. D **42**, 1611 (1990); P. Bicudo, J.E. Ribeiro, and J. Rodrigues, Phys. Rev. C **52**, 2144 (1995); F.J. Llanes-Estrada and S.R. Cotanch, Phys. Rev. Lett. **84**, 1102 (2000); A.P. Szczepaniak and E.S. Swanson, *ibid.* **87**, 072001 (2001); A.P. Szczepaniak and E.S. Swanson, Phys. Rev. D **65**, 025012 (2002).

[4] J. Praschifka, R.T. Cahill, and C.D. Roberts, Int. J. Mod. Phys. A **4**, 4929 (1989); C.D. Roberts and B.H.J. McKellar, Phys. Rev. D **41**, 672 (1990); Y.B. Dai, C.S. Huang, and D.S. Liu, *ibid.* **43**, 1717 (1991); L. von Smekal, P.A. Amundsen, and R. Alkofer, Nucl. Phys. **A529**, 633 (1991).

[5] F.T. Hawes, K. Kusaka, and A.G. Williams, hep-ph/9411238; K.M. Maung, C.A. Hill, M.T. Hill, and G. DeRise, hep-ph/0302228; V. Sauli, J. High Energy Phys. **02**, 001 (2003); V. Sauli and J. Adam, Nucl. Phys. **A689**, 467 (2001); R. Alkofer, W. Detmold, C.S. Fischer, and P. Maris, hep-ph/0309077; R. Alkofer, W. Detmold, C.S. Fischer, and P. Maris, hep-ph/0309078; E. Ruiz Arriola and W. Broniowski, Phys. Rev. D **67**, 074021 (2003).

[6] A. Le Yaouanc, L. Oliver, S. Ono, O. Pene, and J. C. Raynal, Phys. Rev. D **31**, 137 (1985); P. Bicudo, hep-ph/9905345.

[7] P. Bicudo, J.E. Ribeiro, and A.V. Nefediev, Phys. Rev. D **65**, 085026 (2002); A.V. Nefediev and J.E.F. Ribeiro, *ibid.* **67**, 034028 (2003); P. Bicudo and A.V. Nefediev, *ibid.* **68**, 065021 (2003); P.J.A. Bicudo and A.V. Nefediev, Phys. Lett. B **573**, 131 (2003); A.A. Osipov and B. Hiller, hep-th/0307035; B. Hiller and A.A. Osipov, hep-ph/0301025.

[8] F. Kleefeld, hep-th/0312027.

[9] D.E. Soper, Phys. Rev. Lett. **81**, 2638 (1998).