Analytical solution of the Gross-Neveu model at finite density

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(Received 26 September 2003; published 18 March 2004)

Recent numerical calculations have shown that the ground state of the Gross-Neveu model at finite density is a crystal. Here, we present the analytical solution of this problem in terms of elliptic functions. The scalar potential is the superpotential of the nonrelativistic Lamé Hamiltonian. This model can also serve as an analytically solvable toy model for a relativistic superconductor in the Larkin-Ovchinnikov-Fulde-Ferrell phase.

DOI: 10.1103/PhysRevD.69.067703 PACS number(s): 11.10.Kk

In this paper we reconsider the simplest variant of the Gross-Neveu (GN) model, a (1+1)-dimensional relativistic field theory with N species of fermions interacting via a quartic self-interaction [1]

$$\mathcal{L}=\overline{\psi}^{(i)}\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi^{(i)}+\frac{1}{2}g^2(\overline{\psi}^{(i)}\psi^{(i)})^2. \tag{1}$$

In a previous work [2], we found that the widely accepted phase diagram of this model in the large N limit [3] needed some revision. The dynamically generated scalar mean field becomes inhomogeneous in a certain region of temperature and chemical potential, a fact which had been overlooked so far. The four-fermion interaction then does not merely lead to mass generation but to formation of a kink-antikink crystal. This in turn reflects the presence of bound baryons in the GN model as can be most clearly seen in the low-density limit. The approach used in [2] was a numerical implementation of the Dirac-Hartree-Fock method (equivalent to the saddle point method in the functional integral approach).

Here, we repeat the same calculation in an analytical manner. We focus on the $T=0$ case and explain how to construct the crystal ground state at any density in closed, analytical form. The finite temperature calculation along similar lines will be the subject of a forthcoming work. Since the results of Ref. [2] are fully confirmed by our new method we refer to this paper for more details, figures and a discussion of the underlying physics. For a more general introduction into the field of (1+1)-dimensional toy models for hot and dense matter, see the review article [4].

We start from the Hartree-Fock Dirac equation

$$\left(\gamma^5\frac{1}{i}\frac{\partial}{\partial x}+\gamma^0 S(x)\right)\psi(x)=\omega\psi(x), \tag{2}$$

choosing the γ matrices as follows:

$$\gamma^0=-\sigma_1,\quad \gamma^1=i\sigma_3,\quad \gamma^5=\gamma^0\gamma^1=-\sigma_2. \tag{3}$$

In terms of the upper and lower spinor components $\phi_{\pm}$ the Dirac equation consists of two coupled equations:

$$\mp\left(\frac{\partial}{\partial x}\pm S\right)\phi_{\pm}=\omega\phi_{\mp}, \tag{4}$$

which can be decoupled by squaring,

$$\left(-\frac{\partial^2}{\partial x^2}\mp\frac{\partial S}{\partial x}+S^2\right)\phi_{\pm}=\omega^2\phi_{\pm}. \tag{5}$$

Note that Eqs. (4), (5) fall precisely into the pattern of supersymmetric (SUSY) quantum mechanics. Let us now make an ansatz for $S(x)$ based on the superpotential of the well-known Lamé potential [5,6]

$$S(x)=A\kappa^2\frac{\mathrm{sn}(Ax|\kappa^2)\mathrm{cn}(Ax|\kappa^2)}{\mathrm{dn}(Ax|\kappa^2)}\equiv A\tilde{S}(Ax). \tag{6}$$

Here, three types of Jacobi elliptic functions with modulus κ appear [7]. The spatial period of $\tilde{S}(Ax)$ is

$$\ell=2\mathbf{K}, \tag{7}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind, $\mathbf{K}(\kappa^2)$ [7]. We shall choose the parameter A in such a way that $S(x)$ has the period a determined by the mean density [2],

$$a=\frac{1}{\rho}=\frac{\pi}{p_f}, \tag{8}$$

hence

$$A=\frac{\ell}{a}=\frac{2p_f\mathbf{K}}{\pi}. \tag{9}$$

The resulting potential still has one free parameter, κ, which determines both its shape and its size; the period is now fixed by the mean density. Denoting Ax by ξ, Eq. (5) for ϕ_+ can then be converted into

$$\left(-\frac{\partial^2}{\partial\xi^2}+2\kappa^2\,\mathrm{sn}^2(\xi|\kappa^2)\right)\phi_+=\mathcal{E}\phi_+ \tag{10}$$

with

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$$\mathcal{E}=\frac{a^2}{\ell^2}\omega^2+\kappa^2, \tag{11}$$

which is recognized as the simplest case of the Lamé equation [5]. The corresponding equation for ϕ_- differs only by a translation of the potential through half a period and thus yields an identical spectrum [6]. We can now simply use all the well-known results for the Lamé potential. For our purpose we found Ref. [8] particularly well suited.

In order to determine the yet unknown parameter κ we shall minimize the ground state energy density

$$E_{\text{g.s.}}=-2\int_{p_f}^{\Lambda/2}\frac{dp}{2\pi}\omega+\frac{1}{2Ng^2a}\int_0^a dx S^2(x)\equiv E_1+E_2. \tag{12}$$

E_1 is the sum over single particle energies over all filled negative energy states regularized by a cutoff. (For simplicity, we consider antimatter by leaving the valence band in the Dirac sea unoccupied [2].) E_2 is the standard correction term for double counting of the interaction energy. Consider E_1 first, transforming the Bloch momenta p and single particle energies ω to the corresponding quantities from the Lamé equation,

$$E_1=-2\frac{\ell^2}{a^2}\int_{k_{\min}}^{k_{\max}}\frac{dk}{2\pi}\sqrt{\mathcal{E}-\kappa^2} \tag{13}$$

where

$$k_{\min}=\frac{\pi}{\ell},\quad k_{\max}=\frac{a\Lambda}{2\ell}. \tag{14}$$

It is actually more convenient to integrate over $\mathcal{E}$, using [8]

$$\frac{dk}{d\mathcal{E}}=\frac{\mathbf{E}/\mathbf{K}+\kappa^2-\mathcal{E}}{2\sqrt{(1-\mathcal{E})(\mathcal{E}-\kappa^2)(1+\kappa^2-\mathcal{E})}}. \tag{15}$$

Here, $\mathbf{E}$ is the complete elliptic integral of the second kind, $\mathbf{E}(\kappa^2)$. We thus have to evaluate

$$E_1=-2\frac{\ell^2}{a^2}\int_{\mathcal{E}_{\min}}^{\mathcal{E}_{\max}}\frac{d\mathcal{E}}{2\pi}\left|\frac{dk}{d\mathcal{E}}\right|\sqrt{\mathcal{E}-\kappa^2} \tag{16}$$

where now the lower limit is the band edge,

$$\mathcal{E}_{\min}=1+\kappa^2, \tag{17}$$

whereas the upper limit can be inferred from Eq. (15) to be

$$\mathcal{E}_{\max}=k_{\max}^2+2\left(1-\frac{\mathbf{E}}{\mathbf{K}}\right) \tag{18}$$

with $k_{\max}$ given in Eq. (14). It is necessary to keep the subleading term here, since the integral over $d\mathcal{E}$ is linearly divergent and the divergent part will be subtracted. Performing the integration in Eq. (16) yields

$$E_1=-\frac{\Lambda^2}{8\pi}+\frac{\ell^2}{4\pi\mathbf{K}a^2}(4\mathbf{E}+(\kappa^2-2)\mathbf{K})+\frac{\ell^2}{2\pi\mathbf{K}a^2}(2\mathbf{E}+(\kappa^2-2)\mathbf{K})\ln\left(\frac{a\Lambda}{\ell\kappa}\right). \tag{19}$$

The term $\sim\Lambda^2$ can be eliminated by subtracting the energy of the trivial vacuum. Now consider the double counting correction E_2, Eq. (12), in the form

$$E_2=\frac{\ell}{2Ng^2a^2}\int_0^\ell d\xi\tilde{S}^2(\xi). \tag{20}$$

Inserting $\tilde{S}$ from Eq. (6) and transforming to the integration variable $s=\text{sn}\,\xi$, the integration can be carried out as follows:

$$\int_0^\ell d\xi\frac{\text{sn}^2\xi\,\text{cn}^2\xi}{\text{dn}^2\xi}=-\frac{2}{\kappa^4}(2\mathbf{E}+(\kappa^2-2)\mathbf{K}). \tag{21}$$

The coupling constant is related to the cutoff via the (vacuum) gap equation [2,9] which reads (in units where the vacuum fermion mass is 1)

$$Ng^2=\frac{\pi}{\ln\Lambda}. \tag{22}$$

Combining Eqs. (20)–(22), we find

$$E_2=-\frac{\ell}{\pi a^2}(2\mathbf{E}+(\kappa^2-2)\mathbf{K})\ln\Lambda. \tag{23}$$

Upon adding E_1 and E_2 and recalling Eq. (7), the logarithmically divergent terms cancel and we obtain the finite, renormalized ground state energy density,

$$E_{\text{ren}}=\frac{p_f^2\mathbf{K}}{\pi^3}(4\mathbf{E}+(\kappa^2-2)\mathbf{K})+\frac{2p_f^2\mathbf{K}}{\pi^3}(2\mathbf{E}+(\kappa^2-2)\mathbf{K})\ln\left(\frac{\pi}{2p_f\kappa\mathbf{K}}\right). \tag{24}$$

Let us minimize this expression with respect to the modulus κ, our variational parameter. This yields the simple condition

$$\kappa=\frac{a}{\ell}=\frac{\pi}{2p_f\mathbf{K}}, \tag{25}$$

a transcendental equation for κ. Eliminating p_f from E_{ren} with the help of this relation, we finally get the following parametric representation of the ground state energy as a function of density (parameter κ),

$$E_{\text{ren}}=\frac{1}{4\pi}+\frac{1}{\pi\kappa^2}\left(\frac{\mathbf{E}}{\mathbf{K}}-\frac{1}{2}\right), \tag{26}$$

$$\frac{p_f}{\pi}=\frac{1}{2\kappa \mathbf{K}}. \tag{27}$$

We also give the scalar potential $S(x)$ corresponding to the optimal value of the modulus κ,

$$S(x)=\kappa\frac{\mathrm{sn}\left(\frac{x}{\kappa}\middle|\kappa^2\right)\mathrm{cn}\left(\frac{x}{\kappa}\middle|\kappa^2\right)}{\mathrm{dn}\left(\frac{x}{\kappa}\middle|\kappa^2\right)}. \tag{28}$$

$S(x)$ interpolates smoothly between widely spaced kinks and antikinks ($\sim\pm\tanh x$) at $\kappa\to 1$ (low-density limit) and the function $(\kappa/2)\sin(2x/\kappa)$ for $\kappa\to 0$ (high-density limit).

All we have done may be viewed as a variational calculation of the ground state of baryonic matter in the GN model. Comparing the results for $S(x)$ or E_{ren} with the numerical results of Ref. [2], we find excellent agreement. This suggests that the scalar potential $S(x)$ is actually self-consistent. One can indeed prove the self-consistency analytically, using the detailed form of the Lamé wave functions [5] and a number of identities for elliptic functions. This proof is given in the Appendix.

Let us now make use of the closed formulas derived above to illustrate certain features of the GN crystal. If we go to the low-or high-density limit, it becomes possible to systematically resolve the transcendental equation relating p_f and κ,

$$\kappa \underset{p_f\to 0}{\approx} 1-8e^{-\pi/p_f}+\frac{32(\pi+p_f)}{p_f}e^{-2\pi/p_f}$$

$$\kappa \underset{p_f\to\infty}{\approx} \frac{1}{p_f}-\frac{1}{4p_f^3}+\frac{3}{64p_f^5}. \tag{29}$$

For the energy as a function of density, one finds

$$E_{\mathrm{ren}} \underset{p_f\to 0}{\approx} -\frac{1}{4\pi}+\frac{2p_f}{\pi^2}+\frac{8p_f}{\pi^2}e^{-\pi/p_f}$$

$$E_{\mathrm{ren}} \underset{p_f\to\infty}{\approx} \frac{p_f^2}{2\pi}-\frac{1}{2^6\pi p_f^2}+\frac{3}{2^{14}\pi p_f^6}. \tag{30}$$

In the low-density limit, the three terms correspond to the vacuum energy density, the contribution from the baryon mass ($\sim\rho M_B$ with $M_B=2/\pi$) and a term describing the repulsive baryon-baryon interaction. At high densities, we can identify the free massless Fermi gas piece, the leading perturbative correction already given in [2] and the next term coming from higher order effects, suggesting fast convergence. It is also possible to determine the Fourier coefficients S_n of $S(x)$ (the variational parameters used in [2]),

$$S(x)=\sum_n S_n e^{i2\pi nx/a},\quad (S_{-n}=S_n^*). \tag{31}$$

Upon using Eq. (16) on p. 912 of Gradshteyn-Ryzhik [10] and correcting a misprint (π^2 should read π on the right hand side), we find the following closed expression (only odd n's appear),

$$iS_n=\frac{2p_f}{\sinh(n\pi\mathbf{K}'/\mathbf{K})}, \tag{32}$$

where

$$\mathbf{K}'=\mathbf{K}(1-\kappa^2). \tag{33}$$

Finally, we wish to point out that the GN model at finite density can also serve as a solvable model for a relativistic, inhomogeneous superconductor. Along the lines described in Ref. [11], one can map the GN Lagrangian onto a ''dual'' Lagrangian which has quark-quark rather than quark-antiquark pairing. All one has to do is redefine quarks into anti-quarks for left-handed quarks only. If one works at nonzero chemical potential, a baryonic chemical potential μ in the GN model corresponds to an ''axial'' chemical potential μ_5 in the dual BCS-type model. Left-handed and right-handed fermions have opposite chemical potentials, hence μ_5 in 1+1 dimensions acts like a magnetic field in 3+1 dimensions. This favors the appearance of the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) phase with spatially varying Cooper pair condensate [12,13]. It is then natural to identify the kink-antikink crystal of the GN model with the LOFF phase of the dual BCS-type model. In this sense, the present study may also be of some use for model studies of relativistic superconductors.

APPENDIX: ANALYTICAL PROOF OF SELF-CONSISTENCY OF $S(x)$

We would like to show that

$$S(x)=-Ng^2\sum_\alpha^{\mathrm{occ}}\ \overline{\psi}_\alpha(x)\psi_\alpha(x), \tag{A1}$$

where the sum runs over all negative energy levels corresponding to the ''upper band'' of the Lamé spectrum (we are again considering antimatter). First, we have to construct normalized spinor solutions of the Dirac equation (2) out of the known solutions of the Lamé equation (10). We write

$$\psi=\mathcal{N}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix} \tag{A2}$$

and choose for ϕ_+ a solution of the 2nd order differential equation (10). We recall that the relations between Dirac variables and Lamé variables are

$$\mathcal{E}=\kappa^2(\omega^2+1),\quad x=\kappa\xi,\quad p=k/\kappa, \tag{A3}$$

where we have used condition (25). Once ϕ_+ is chosen, ϕ_- follows from Eq. (4),

$$\phi_- = -\frac{1}{\kappa\omega}\left(\frac{\partial}{\partial\xi} + \tilde{S}(\xi)\right)\phi_+ . \tag{A4}$$

Let us first compute the normalization factor $\mathcal{N}$. In a continuum normalization, the (spatially averaged) fermion density is normalized to 1 for each level,

$$1 = \frac{1}{a}\int_0^a dx\, \psi^\dagger\psi = 2|\mathcal{N}|^2\frac{1}{\ell}\int_0^\ell d\xi\, |\phi_+|^2 \tag{A5}$$

($|\phi_+|^2$ and $|\phi_-|^2$ give the same contribution). We now insert the solution ϕ_+ taken from the literature in terms of Jacobi functions [5,6,8]

$$\phi_+ = \frac{H(\xi+\alpha)}{\Theta(\xi)}e^{-\xi Z(\alpha)} = \frac{\vartheta_1(v+w,q)}{\vartheta_4(v,q)}e^{-\xi Z(\alpha)} \tag{A6}$$

with

$$v = \frac{\pi\xi}{2\mathbf{K}}, \quad w = \frac{\pi\alpha}{2\mathbf{K}}, \quad q = \mathrm{nome}(\kappa). \tag{A7}$$

For the upper band, $\alpha = \mathrm{i}\eta$. There is a 2nd solution, ϕ_+^*, which will simply be accounted for by a factor of 2 below. For the definitions of the various Jacobi functions, see [7]. Using the following addition theorem for ϑ_1 [14]

$$\vartheta_3^2(0)\,\vartheta_1(x+y)\,\vartheta_1(x-y) = \vartheta_4^2(x)\,\vartheta_2^2(y) - \vartheta_2^2(x)\,\vartheta_4^2(y) \tag{A8}$$

together with standard relations between different Jacobi functions [7], we find

$$|\phi_+|^2 = \mathcal{A}\left(1 - \frac{\mathrm{cn}^2(\xi|\kappa^2)}{\mathrm{cn}^2(\alpha|\kappa^2)}\right), \quad \mathcal{A} = \frac{\vartheta_2^2(w,q)}{\vartheta_3^2(0,q)}. \tag{A9}$$

Now the $d\xi$ integration in Eq. (A5) can be performed with the result

$$1 = \frac{2|\mathcal{N}|^2\mathcal{A}}{\kappa^2\ \mathrm{cn}^2(\alpha|\kappa^2)}\left(\mathrm{dn}^2(\alpha|\kappa^2) - \frac{\mathbf{E}}{\mathbf{K}}\right). \tag{A10}$$

This determines the normalization factor $|\mathcal{N}|^2$. Let us now consider the scalar density. In our representation of the Dirac matrices, Eq. (3),

$$\overline{\psi}\psi = -|\mathcal{N}|^2(\phi_+^*\phi_- + \phi_-^*\phi_+) = \frac{|\mathcal{N}|^2}{\kappa\omega}(\partial_\xi + 2\tilde{S})|\phi_+|^2. \tag{A11}$$

In the 2nd equation, we have used Eq. (A4). Inserting the expression Eq. (A9) and performing some straightforward calculations yields

$$\overline{\psi}\psi = \frac{2\mathcal{A}|\mathcal{N}|^2}{\kappa^3\omega}\frac{\mathrm{dn}^2(\alpha|\kappa^2)}{\mathrm{cn}^2(\alpha|\kappa^2)}\tilde{S}. \tag{A12}$$

With the normalization factor $|\mathcal{N}|^2$ determined above,

$$\overline{\psi}\psi = \frac{1}{\omega}\frac{\mathrm{dn}^2(\alpha\,|\,\kappa^2)}{\mathrm{dn}^2(\alpha\,|\,\kappa^2) - \mathbf{E}/\mathbf{K}}S(x). \tag{A13}$$

In view of the relations [see [8] and Eq. (11)]

$$\mathrm{dn}^2(\alpha|\kappa^2) = \mathcal{E} - \kappa^2, \quad \kappa\omega = \pm\sqrt{\mathcal{E}-\kappa^2}, \tag{A14}$$

we get, for negative energy states,

$$\overline{\psi}\psi = -\kappa\frac{\sqrt{\mathcal{E}-\kappa^2}}{\mathcal{E}-\kappa^2-\mathbf{E}/\mathbf{K}}S(x). \tag{A15}$$

Finally we sum over all filled states. As in the calculation of the ground state energy, we convert the integration over crystal momenta into an integration over $\mathcal{E}$, include a factor of 2 for the twofold degeneracy of the orbits and employ the integration limits Eqs. (17), (18),

$$\sum_\alpha^{\mathrm{occ}} \overline{\psi}_\alpha\psi_\alpha = \frac{2}{\kappa}\int_{\mathcal{E}_{\min}}^{\mathcal{E}_{\max}}\frac{d\mathcal{E}}{2\pi}\left|\frac{dk}{d\mathcal{E}}\right|\overline{\psi}\psi = -\frac{1}{\pi}\ln(\Lambda)S(x). \tag{A16}$$

We have dropped terms of order $1/\Lambda^2$ and higher, but of course no finite terms. Inserting the relation between coupling constant and cutoff from the gap equation, Eq. (22), then reproduces Eq. (A1) and proves the self-consistency of the scalar potential (28).

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