

Chiral symmetry breaking and pions in nonsupersymmetric gauge/gravity duals

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We study gravity duals of large N nonsupersymmetric gauge theories with matter in the fundamental representation by introducing a D7-brane probe into deformed AdS backgrounds. In particular, we consider a D7-brane probe in both the AdS Schwarzschild black hole solution and in the background found by Constable and Myers, which involves a nonconstant dilaton and S^5 radius. Both these backgrounds exhibit confinement of fundamental matter and a discrete glueball and meson spectrum. We numerically compute the $\bar{\Psi}\Psi$ condensate and meson spectrum associated with these backgrounds. In the AdS–black-hole background, a quark-bilinear condensate develops only at a nonzero quark mass. We speculate on the existence of a third order phase transition at a critical quark mass where the D7 embedding undergoes a geometric transition. In the Constable-Myers background, we find a chiral symmetry breaking condensate as well as the associated Goldstone boson in the limit of small quark mass. The existence of the condensate ensures that the D7-brane never reaches the naked singularity at the origin of the deformed AdS space.

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I. INTRODUCTION

The discovery of the AdS conformal field theory (CFT) correspondence [1–3] has led to promising new ideas for studying strong coupling phenomena in large N gauge theories. Generalizations of the correspondence in which conformal symmetry and supersymmetry are broken are potentially useful for describing realistic quantum field theories. In particular, it is hoped that methods based on gauge-gravity duality will eventually be applicable to QCD. The simplest generalizations involve deforming AdS by the inclusion of relevant operators [4]. These geometries are asymptotically AdS, with the deformations interpreted as renormalization group (RG) flow from a superconformal gauge theory in the ultraviolet to a QCD-like theory in the infrared. Moreover a number of nonsupersymmetric ten-dimensional geometries of this or related form have been found [5–9] and have been shown to describe confining gauge dynamics. There have been interesting calculations of the glueball spectrum in three- and four-dimensional QCD by solving classical supergravity equations in various deformed AdS geometries [10–20].

A difficulty with describing QCD in this way arises due to the asymptotic freedom of QCD. The vanishing of the 't Hooft coupling in the UV requires the dual geometry to be infinitely curved in the region corresponding to the UV. In this case classical supergravity is insufficient and one needs to use full string theory. Formulating string theory in the relevant backgrounds has thus far proven difficult. The existing glueball calculations involve geometries with small curvature that return asymptotically to AdS (the field theory

returns to the strongly coupled $\mathcal{N}=4$ theory in the UV), and are in the same coupling regime as strong coupling lattice calculations far from the continuum limit. There is nevertheless optimism that the glueball calculations are fairly accurate, based on comparisons with lattice data [21–23].

In this paper we shall discuss progress on a further problem that must be solved to study QCD using gravity, namely the inclusion of particles in the fundamental representation of the gauge group, i.e. quarks. Since AdS geometries arise as near-horizon limits for coincident branes, the dual field theories have only adjoint matter. The same is true for deformations of AdS. It turns out that fundamental representations can be introduced by including appropriately embedded probe branes. Examples of this [24–27] were found by embedding a probe brane on an AdS_d subspace of the full AdS_D geometry, where $d < D$ and the boundary of AdS_d is part of the boundary of the AdS_D . Such embeddings give rise to a dual field theory with “quarks” that are not free to move in all spatial directions. These “defect” conformal field theories are interesting for a variety of reasons, for instance for localizing gravity [24] or for their mathematical properties [28], but are somewhat far from QCD.

In order to obtain fundamental fields in four space-time dimensions, Karch and Katz [29,30] consider a configuration in which a D7-brane in $\text{AdS}_5 \times S^5$ fills the AdS_5 and wraps an S^3 inside S^5 . This configuration is dual to a four-dimensional $\mathcal{N}=2$ Yang-Mills theory describing open strings in the presence of one D7-brane and N D3-branes sharing three spatial directions. The degrees of freedom are those of the $\mathcal{N}=4$ super-Yang-Mills theory, coupled to an $\mathcal{N}=2$ hypermultiplet with fields in the fundamental representation of $\text{SU}(N)$. The latter arise from strings stretched between the D7 and D3 branes. When the D7 and D3 branes are separated, the fundamental matter becomes massive and the dual description involves a probe D7 on which the induced metric is only asymptotically $\text{AdS}_5 \times S^3$. In this case there is a discrete spectrum of mesons. This spectrum has been computed

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(exactly) at large 't Hooft coupling [31] using an approach analogous to the glueball calculations in deformed AdS backgrounds. The novel feature here is that the “quark” bound states are described by the scalar fields in the Dirac-Born-Infeld action of the D7 brane probe.

In view of a gravity description of Yang-Mills theory with confined quarks, it is natural to attempt to generalize these calculations to probes of deformed AdS spaces. For instance in Ref. [32], a way to embed D7-branes in the Klebanov-Strassler (KS) background [33] was found, following the suggestion of Ref. [29]. Moreover in Ref. [32], the spectrum of mesons dual to fluctuations of the D7-brane probe in the KS geometry was calculated. The underlying theory is an $\mathcal{N}=1$ gauge theory with massive chiral superfields in the fundamental representation. Related work may also be found in Ref. [34].

One of the most important features of QCD dynamics though is chiral symmetry breaking by a quark condensate, but since this is forbidden by unbroken supersymmetry,¹ these constructions do not let us address this issue. In the present paper we attempt to come somewhat closer to QCD by considering the embedding of D7-branes in two nonsupersymmetric backgrounds that exhibit confinement. Although neither of these backgrounds corresponds exactly to QCD since they contain more degrees of freedom than just gluons and quarks, we might nevertheless expect chiral symmetry breaking behavior. The quark mass m and the quark condensate expectation value c are given by the UV asymptotic behavior of the solutions to the supergravity equations of motion in the standard holographic way (see Ref. [35] for an example of this methodology). In the $\mathcal{N}=2$ supersymmetric scenario of Ref. [29] with a D7 probe in standard AdS space, we show that there cannot be any regular solution that has $c \neq 0$; the supersymmetric theory does not allow a quark condensate. We then find that, for the deformed AdS backgrounds we consider, there are regular solutions with $c \neq 0$. The case $c \neq 0$ with $m=0$ corresponds to spontaneous chiral symmetry breaking.

The first supergravity background we consider is the Schwarzschild black hole in $\text{AdS}_5 \times S^5$. In the absence of 7-branes, this background is dual to strongly coupled $\mathcal{N}=4$ super-Yang-Mills theory at finite temperature and is in the same universality class as three-dimensional pure QCD [5]. Glueball spectra in this case were computed in Refs. [10,12]. We introduce D7-branes into this background and compute the quark condensate as a function of the bare quark mass, as well as the meson spectrum. This background is dual to the finite temperature version of the $\mathcal{N}=2$ super-Yang-Mills theory considered in Refs. [29,31]. The finite temperature $\mathcal{N}=2$ theory is not in the same universality class as three-dimensional QCD with light quarks since the antiperiodic boundary conditions for fermions in the Euclidean time di-

rection give a nonzero mass to the quarks upon reduction to three dimensions, even if the hypermultiplet mass of the underlying $\mathcal{N}=2$ theory vanishes. In fact these quarks decouple if one takes the temperature to infinity in order to obtain a truly three-dimensional theory. Nevertheless at finite temperature the geometry describes an interesting four-dimensional strongly coupled gauge configuration with quarks. The meson spectrum we obtain has a mass gap of order of the glueball mass. Furthermore we find that the $\Psi\Psi$ condensate vanishes for zero hypermultiplet mass, such that there is no spontaneous violation of parity in three dimensions or chiral symmetry in four dimensions. However for $m \neq 0$ we find a condensate c which at first grows linearly with m , and then shrinks back towards zero. Increasing m further, the D7 embedding undergoes a geometric transition at a critical mass m_c . At sufficiently large $m \gg m_c$ the condensate is negligible and the spectrum matches smoothly with the one found in Ref. [31] for the $\mathcal{N}=2$ theory. We speculate that the geometric transition corresponds to a third order phase transition in the dual gauge theory, at which dc/dm is discontinuous.

The second nonsupersymmetric background which we consider was found by Constable and Myers [9]. This background is asymptotically $\text{AdS}_5 \times S^5$ but has a nonconstant dilaton and S^5 radius. In the field theory an operator of dimension 4 with zero R-charge has been introduced. This deformation does not give mass to the adjoint fermions and scalars of the underlying $\mathcal{N}=4$ theory but does leave a nonsupersymmetric gauge background. Furthermore, unlike the AdS black-hole background, the geometry has a naked singularity. Nevertheless, in a certain parameter range, this background gives an area law for the Wilson loop and a discrete spectrum of glueballs with a mass gap.

We obtain numerical solutions for the D7-brane equations of motion in the Constable-Myers background with asymptotic behavior determined by a quark mass m and chiral condensate c . We compute the condensate c as a function of the quark mass m subject to a regularity constraint. Remarkably, our results are not sensitive to the singular behavior of the metric in the IR. For a given mass there are two regular solutions of which the physical, lowest action solution corresponds to the D7 brane “ending” before reaching the curvature singularity. Of course the D7-brane does not really end; however the S^3 about which it is wrapped contracts to zero size, similarly to the scenario discussed in Ref. [29]. In our case the screening of the singularity is related to the existence of the condensate. Furthermore we find numerical evidence for a nonzero condensate in the limit $m \rightarrow 0$. This corresponds to spontaneous breaking of the $U(1)$ chiral symmetry that is nonanomalous in the large N limit [36] (for a review see Ref. [37]).

We also compute the meson spectrum by studying classical fluctuations about the D7-embedding. For zero quark mass, the meson spectrum contains a massless mode, as it must due to the spontaneous chiral symmetry breaking. Note that since the spontaneously broken axial symmetry is $U(1)$ for a single D7-brane, the associated Goldstone mode is a close cousin of the η' of QCD, which is a Goldstone boson

¹A quark bilinear $\Psi\tilde{\Psi}$, where Ψ and $\tilde{\Psi}$ are fermionic components of chiral superfields $Q=q+\theta\Psi\cdots$, $\tilde{Q}=\tilde{q}+\theta\tilde{\Psi}+\cdots$, can be written as a SUSY variation of another operator (it is an F term of the composite operator $\tilde{Q}Q$).

in the large N limit. In principle, stringy corrections to supergravity should give the η' a mass. We briefly comment on generalizations to the case of more than one flavor or, equivalently, more than one D7-brane. Moreover we give a holographic version of the Goldstone theorem.

The main message of this paper is that nonsupersymmetric gravity duals of gauge theories dynamically generate quark condensates and can break chiral symmetries. We stress that the physical interpretation of naked singularities is a delicate issue, for instance in the light of the analysis of Ref. [38]. This applies in particular to the discussion of light quarks and mesons. It is therefore an important part of our analysis that in the presence of a condensate the physical solutions to the supergravity and Dirac-Born-Infeld equations of motion never reach the singularity in the IR. Of course it would be interesting to understand this mechanism further and to see if it occurs in other supergravity backgrounds as well.

The organization of this paper is follows. In Sec. II we review some of the previous results from the study of D7 probes in the AdS/CFT correspondence. In Sec. III we consider D7-branes in the AdS Schwarzschild black hole background. In Sec. IV we study D7-brane probes in the Constable-Myers background [9]. In Sec. V we conclude and present some open problems.

II. ADS/CFT DUALITY FOR AN $\mathcal{N}=2$ GAUGE THEORY WITH FUNDAMENTAL MATTER

It was first observed in Ref. [29] that one can obtain a holographic dual of a four-dimensional Yang-Mills theory with fundamental matter by taking a near-horizon limit of a system of intersecting D3 and D7 branes. We first review some of the features of this duality, and then test numerical techniques that we will use later to study deformations of this duality.

Consider a stack of N D3-branes spanning the directions 1,2,3 and another stack of N_f D7-branes spanning the directions 1,2,3,4,5,6,7. The low-energy dynamics of open strings in this setting is described by a $\mathcal{N}=2$ super-Yang-Mills theory. This theory contains the degrees of freedom of the $\mathcal{N}=4$ theory, namely an $\mathcal{N}=2$ vector multiplet and an $\mathcal{N}=2$ adjoint hypermultiplet, as well as N_f $\mathcal{N}=2$ hypermultiplets in the fundamental representation of $SU(N)$. The theory is conformal in the limit $N \rightarrow \infty$ with N_f fixed. There is an $SU(2) \times U(1)$ R-symmetry. The $U(1)$ R-symmetry acts as a chiral rotation on the ‘‘quarks,’’ which are the fermionic components of the $\mathcal{N}=2$ hypermultiplet composed of fundamental and antifundamental chiral superfields Q and \tilde{Q} . This symmetry also acts as a phase rotation on the scalar component of one of the adjoint chiral superfields. When the D7-branes are separated from the D3-branes in the two mutually transverse directions X^8 and X^9 , the fields Q, \tilde{Q} become massive, explicitly breaking the $U(1)$ R-symmetry and conformal invariance. As shown in Ref. [29], the $\mathcal{N}=2$ theory as well as its renormalization group flow have an elegant holographic description.

This holographic description is obtained as follows. In the limit of large N at fixed but large 't Hooft coupling

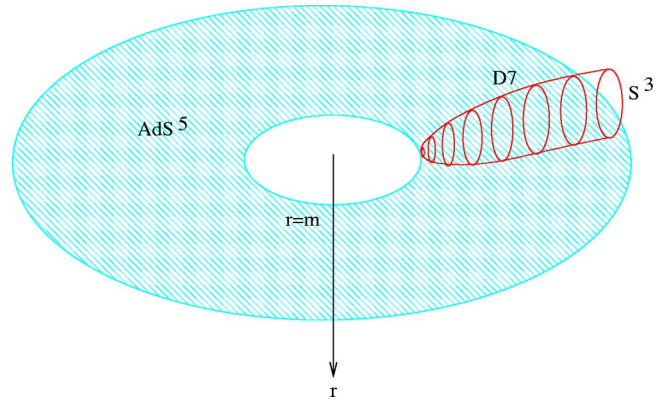


FIG. 1. The D7 embedding in $AdS_5 \times S^5$.

$\lambda = g^2 N \gg 1$, the D3-branes may be replaced with their near-horizon $AdS_5 \times S^5$ geometry. Since the number N_f of D7-branes is finite, their back-reaction on the geometry can be effectively ignored. For determining the D7 probe brane embedding in $AdS_5 \times S^5$, let us consider the D3-brane metric in the form

$$ds^2 = f(r)^{-1/2} (-dt^2 + d\vec{x}^2) + f(r)^{1/2} d\vec{y}^2, \quad (1)$$

where $\vec{x} = (X^1, X^2, X^3)$, $\vec{y} = (X^4, \dots, X^9)$, $r^2 \equiv \vec{y}^2$, and

$$f(r) = 1 + \frac{R^4}{r^4}. \quad (2)$$

As usual, the $AdS_5 \times S^5$ geometry is obtained by dropping the 1 in $f(r)$, which is suitable in the near-horizon region $r/R \ll 1$.

For massless flavors, the D7 brane embedding in the D3-metric (1) is given by $y^5 = y^6 = 0$ (corresponding to $X^8 = X^9 = 0$). In the $AdS_5 \times S^5$ geometry, the induced metric on the D7-brane is $AdS_5 \times S^3$. The D7-brane fills AdS_5 , while wrapping a great three-sphere of the S^5 . The isometries of the $AdS_5 \times S^5$ metric that preserve the embedding correspond to the conformal group and R-symmetries of the $\mathcal{N}=2$ gauge theory. The conformal group $SO(2,4)$ is the isometry group of AdS_5 , while the $SU(2) \times U(1)$ R-symmetry corresponds to the rotations of the S^3 inside S^5 and rotations of the y^5, y^6 coordinates.

The holographic description for massive flavors is found by considering the D7-brane embedding $y^5 = X^8 = 0$, $y^6 = X^9 = m$. In this case the D7-geometry is still $AdS_5 \times S^3$ in the $r \rightarrow \infty$ region corresponding to the ultraviolet. However as one decreases r , the S^3 of the induced geometry on the D7-brane contracts to zero size at $r = m$. This is possible because the S^3 is contractible within the S^5 of the full ten-dimensional geometry. The D7-brane ‘‘ends’’ at the value of r at which the S^3 collapses, meaning that it does not fill all of AdS_5 , but only a region outside a core of radius $r = m$. Note that although the D7-brane ends at $r = m$, the D7-geometry is perfectly smooth, as is illustrated in Fig. 1. In the massive case, the conformal and $U(1)$ symmetries are broken, and the D7 embedding is no longer invariant under the corresponding isometries.

A. Testing numerical methods

In the subsequent sections we will numerically compute condensates and meson spectra in deformations of the duality discussed above. Therefore we first test these numerical techniques against some exact results in the undeformed case.

It will be convenient to write the transverse $d\tilde{y}^2$ part of the metric (1) in the following way:

$$S_{D7} = -T_7 \int d^8 \xi \sqrt{-\det(P[G_{ab}])} = -T_7 \int d^8 \xi \epsilon_3 \rho^3 \sqrt{1 + \frac{g^{ab}}{\rho^2 + y_5^2 + y_6^2} (\partial_a y_5 \partial_b y_5 + \partial_a y_6 \partial_b y_6)}, \quad (4)$$

where g_{ab} is the induced metric on the D7 brane and ϵ_3 is the determinant factor from the three-sphere. The ground state configuration of the D7 brane is given by the solution of the Euler-Lagrange equation with dependence only on the ρ variable. In this case the equations of motion become

$$\frac{d}{d\rho} \left[\frac{\rho^3}{\sqrt{1 + (dy_6/d\rho)^2}} \frac{dy_6}{d\rho} \right] = 0, \quad (5)$$

where we consider solutions with $y_5 = 0$ only. Recall that the $U(1)$ R-symmetry corresponds to rotations in the y^5, y^6 plane.

The equations of motion have asymptotic ($\rho \rightarrow \infty$) solutions of the form

$$y_6 = m + \frac{c}{\rho^2}. \quad (6)$$

The identification of these constants as field theory operators requires a coordinate transformation because the scalar kinetic term is not of the usual canonical AdS form. Transforming to the coordinates of Ref. [29] in which the kinetic term has canonical form, we see that m has dimension 1 and c has dimension 3. The scalars are then identified [35] with the quark mass m_q and condensate $\langle \bar{q}q \rangle$, respectively, in agreement with the usual AdS/CFT dictionary.

Note that $y_6(\rho) = m$ is an exact solution of the equations of motion, corresponding to the embedding [29] reviewed above. On the other hand there should be something ill-behaved about the solutions with nonzero c , since a quark condensate is forbidden by supersymmetry. In Fig. 2 we plot numerical solutions of the equations of motion (obtained by a shooting technique using MATHEMATICA) for solutions with nonzero c , and find that they are divergent. The divergence of these solutions is not, by itself, pathological because the variable y_6 is just the location of the D7-brane. However the AdS radius $r^2 = y_6(\rho)^2 + \rho^2$ is not monotonically increasing as a function of ρ for the divergent solutions. This means that these solutions have no interpretation as a renormalization

$$d\tilde{y}^2 = d\rho^2 + \rho^2 d\Omega_3^2 + dy_5^2 + dy_6^2, \quad (3)$$

where $d\Omega_3^2$ is a three-sphere metric. To study the implications of the classical D7 probe dynamics for the dual field theory, we now evaluate the scalar contributions to the Dirac-Born-Infeld (DBI) action for the D7 brane in the $AdS_5 \times S^5$ background. We work in static gauge where the world volume coordinates of the brane are identified with the spacetime coordinates by $\xi^a \sim t, x_1, x_2, x_3, y_1, \dots, y_4$. The DBI action is then

group flow, or as a vacuum of the dual field theory. As expected, the mass only solution is the only well-behaved solution.

The other exact result that we wish to test numerically is the meson spectrum. In order to find the states with zero spin on the S^3 , one looks for normalizable solutions of the equations of motion of the form

$$y_6 + iy_5 = m + \delta(\rho) e^{ikx}, \quad M^2 = -k^2 \quad (7)$$

where one linearizes in the small fluctuation $\delta(\rho)$. The linearized equation of motion is

$$\partial_\rho^2 \delta(\rho) + \frac{3}{\rho} \partial_\rho \delta(\rho) + \frac{M^2}{(\rho^2 + m^2)^2} \delta(\rho) = 0. \quad (8)$$

This was solved exactly in Ref. [31], where it was shown that the solutions can be written as hypergeometric functions

$$\delta(\rho) = \frac{A}{(\rho^2 + 1)^{n+1}} F(-n-1, -n; 2, -\rho^2) \quad (9)$$

with A a constant. The exact mass spectrum is then given by

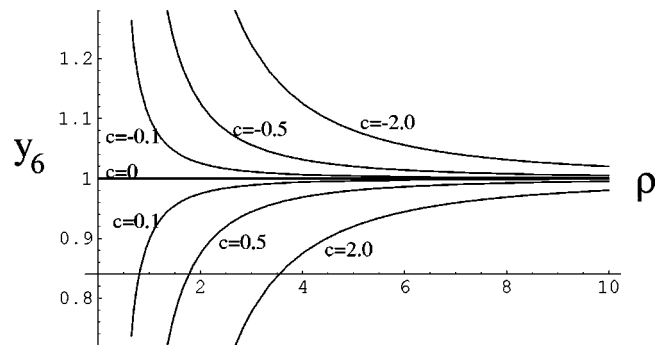


FIG. 2. Numerical solutions of the equations of motion in AdS space showing that in the presence of a condensate the solutions are divergent asymptotically. The regular solution is the mass-only solution.

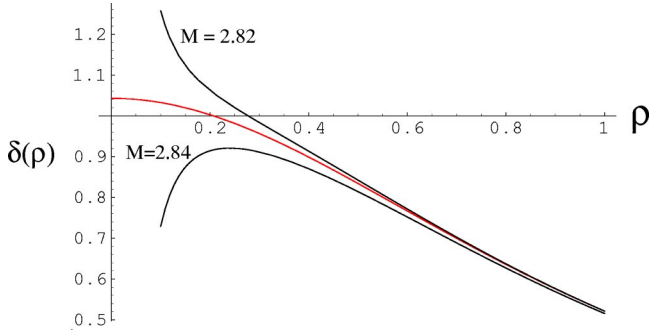


FIG. 3. Numerical solutions of the meson equation of motion for different values of M showing the identification of the first bound state mass. The exact regular solution is plotted between the two numerical flows.

$$M = 2m\sqrt{(n+1)(n+2)}, \quad n=0,1,2,\dots \quad (10)$$

We are interested in whether we can reproduce this result numerically, via a shooting technique. The equation of motion can be solved numerically subject to boundary conditions $\delta(\rho) \sim c/\rho^2$ at large ρ indicating that the meson is a quark bilinear of dimension 3 in the UV. Solutions of the equation must be regular at all ρ so the allowed M^2 solutions can be found by tuning to these regular forms. The result (10) is easily reproduced to 2 significant figures. We show an example of the method in action in Fig. 3.

III. THE ADS-SCHWARZSCHILD SOLUTION

A. The background

We now move on to study quark condensates and mesons in a nonsupersymmetric deformation of the AdS/CFT correspondence and study the AdS-Schwarzschild black hole solution. This geometry is dual to the $\mathcal{N}=4$ gauge theory at finite temperature [5], which is in the same universality class as pure three-dimensional QCD.

The Euclidean AdS-Schwarzschild solution is given by

$$ds^2 = K(r)d\tau^2 + \frac{dr^2}{K(r)} + r^2 dx_{\parallel}^2 + d\Omega_3^2, \quad (11)$$

where

$$K(r) = r^2 - \frac{b^4}{r^2}. \quad (12)$$

This space-time is smooth and complete if τ is periodic with period πb . Note that the S^1 parametrized by τ collapses at $r=b$. The fact that the geometry “ends” at $r=b$ is responsible for the existence of an area law for the Wilson loop and a mass gap in the dual field theory (see Ref. [5]). The period of τ is equivalent to the temperature in the dual $\mathcal{N}=4$ gauge theory. The parameter b sets the scale of the deformation and for convenience in the numerical work below we shall set it equal to 1. At finite temperature, the fermions have antiperiodic boundary conditions in the Euclidean time direction and become massive upon dimensional reduction to three dimen-

sions. The adjoint scalars also become massive at one loop. Thus in the high-temperature limit, the adjoint fermions and scalars decouple, leaving pure three-dimensional QCD.

We now introduce a D7-brane into this background, which corresponds to the addition of matter in the fundamental representation. The dual gauge theory is the $\mathcal{N}=2$ gauge theory of Karch and Katz at finite temperature. Note that the fermions in the fundamental representation also have antiperiodic boundary conditions in the Euclidean time direction. Thus these also decouple in the high-temperature limit, as do the fundamental scalars that get masses at one loop, leaving pure QCD₃ as before. Thus in this particular case we are not interested in the high-temperature limit, but only the region accessible to supergravity and Dirac-Born-Infeld theory. Although the dual field theory cannot be viewed as a three-dimensional gauge theory with light quarks, it is nevertheless a four-dimensional nonsupersymmetric gauge theory with confined degrees of freedom in the fundamental representation. This provides an interesting, if exotic, setting to compute quark condensates and meson spectra using Dirac-Born-Infeld theory. The Constable-Myers background that we will consider later turns out to have more realistic properties.

B. Embedding of a D7 brane

To embed a D7 brane in the AdS black-hole background it is useful to recast the metric (11) to a form with an explicit flat 6-plane. To this end, we change variables from r to w , such that

$$\frac{dw}{w} \equiv \frac{rdr}{(r^4 - b^4)^{1/2}}, \quad (13)$$

which is solved by

$$2w^2 = r^2 + \sqrt{r^4 - b^4}. \quad (14)$$

The metric is then

$$ds^2 = \left(w^2 + \frac{b^4}{4w^2} \right) d\vec{x}^2 + \frac{(4w^4 - b^4)^2}{4w^2(4w^4 + b^4)} dt^2 + \frac{1}{w^2} \left(\sum_{i=1}^6 dw_i^2 \right), \quad (15)$$

where $\sum_i dw_i^2 = dw^2 + w^2 d\Omega_5^2$, which for reasons of convenience will also be written as $d\rho^2 + \rho^2 d\Omega_3^2 + dw_5^2 + dw_6^2$ where $d\Omega_3^2$ is the unit three-sphere metric. The AdS black hole geometry asymptotically approaches $\text{AdS}_5 \times S^5$ at large w . Here the background becomes supersymmetric, and the D7 embedding should approach that discussed in Ref. [29]. The asymptotic solution has the form $w_6 = m, w_5 = 0$, where m should be interpreted as a bare quark mass. To take the deformation into account, we will consider a more general ansatz for the embedding of the form $w_6 = w_6(\rho), w_5 = 0$,

with the function $w_6(\rho)$ to be determined numerically. The DBI action for the orthogonal directions w_5, w_6 is

$$S_{D7} = -\mu_7 \int d^8\xi \epsilon_3 \mathcal{G}(\rho, w_5, w_6) \times \left(1 + \frac{g^{ab}}{(\rho^2 + w_5^2 + w_6^2)} \partial_a w_5 \partial_b w_5 + \frac{g^{ab}}{(\rho^2 + w_5^2 + w_6^2)} \partial_a w_6 \partial_b w_6 \right)^{1/2}, \quad (16)$$

where the determinant of the metric is given by

$$\mathcal{G}(\rho, w_5, w_6) = \sqrt{\frac{g_{tt} g_{xx}^3 \rho^6}{(\rho^2 + w_5^2 + w_6^2)^4}} = \rho^3 \frac{(4(\rho^2 + w_5^2 + w_6^2)^2 + b^4)(4(\rho^2 + w_5^2 + w_6^2)^2 - b^4)}{16(\rho^2 + w_5^2 + w_6^2)^4}. \quad (17)$$

With the ansatz $w_5 = 0$ and $w_6 = w_6(\rho)$, the equation of motion becomes

$$\frac{d}{d\rho} \left[\mathcal{G}(\rho, w_6) \sqrt{\frac{1}{1 + (dw_6/d\rho)^2}} \frac{dw_6}{d\rho} \right] - \sqrt{1 + \left(\frac{dw_6}{d\rho}\right)^2} \frac{b^8 \rho^3 w_6}{2(\rho^2 + w_6^2)^5} = 0. \quad (18)$$

The solutions of this equation determine the induced metric on the D7 brane which is given by

$$ds^2 = \left(\tilde{w}^2 + \frac{b^4}{4\tilde{w}^2} \right) d\vec{x}^2 + \frac{(4\tilde{w}^4 - b^4)^2}{4\tilde{w}^2(4\tilde{w}^4 + b^4)} dt^2 + \frac{1 + (\partial_\rho w_6)^2}{\tilde{w}^2} d\rho^2 + \frac{\rho^2}{\tilde{w}^2} d\Omega_3^2, \quad (19)$$

with $\tilde{w}^2 = \rho^2 + w_6^2(\rho)$. The D7 brane metric becomes $\text{AdS}_5 \times S^3$ for $\rho \gg b, m$.

C. Karch-Katz solutions versus condensate solutions

Before computing the explicit D7-brane solutions, we remark that there are several possibilities for the topology of the D7-brane embedding, which we illustrate in Fig. 4. The UV asymptotic (large ρ) solution, where the geometry returns to $\text{AdS}_5 \times S^5$, is of the form

$$w_6(\rho) \sim m + \frac{c}{\rho^2}. \quad (20)$$

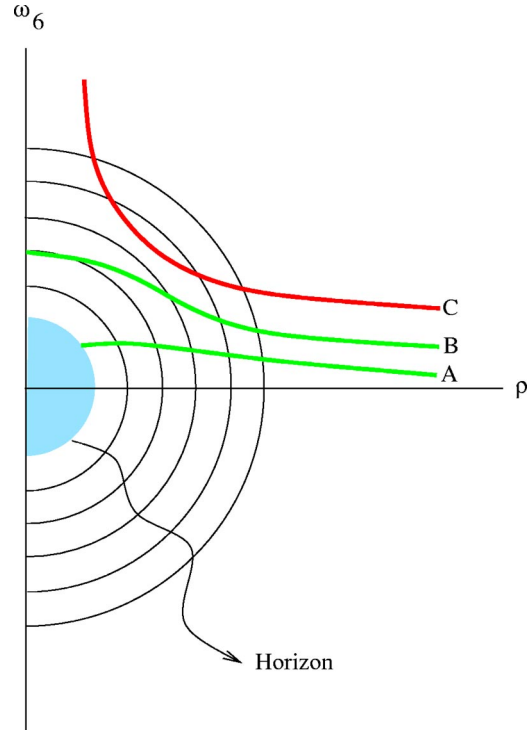


FIG. 4. Different possibilities for solutions of the D7-brane equations of motion. The semicircles are lines of constant r , which should be interpreted as a scale in the dual Yang-Mills theory. The curves of type A, B have an interpretations as an RG flow, while the curve C does not.

The parameters m and c have the interpretation as a quark mass and bilinear quark condensate respectively, as discussed below Eq. (6). These parameters can be taken as the boundary conditions for the second order differential equation (18), which we solve using a numerical shooting technique. Of course the physical solutions should not have arbitrary m and c . The condition that we use to identify physical solutions is that the D7-brane embedding should have an interpretation as a RG flow. This implies for instance that if one slices the D7-brane geometry at a fixed value of r^2 , or equivalently at a fixed value of $w_6^2 + \rho^2$, one should obtain at most one copy of the geometry $R^4 \times S^3$. In other words, $r^2 = \rho^2 + w_6(\rho)^2$ should be a monotonically increasing function of ρ . This is certainly not the case for divergent solutions. Such solutions are not in correspondence with a vacuum of the dual gauge theory and are discarded.

There are then two possible forms of regular solutions. The geometry in which the D7-brane is embedded has the boundary topology $R^3 \times S^1 \times S^5$, which contains the D7-brane boundary $R^3 \times S^1 \times S^3$. Recall that the S^3 is contractible within S^5 . Furthermore the S^1 is contractible within the bulk geometry and shrinks to zero as one approaches the horizon $r = b$. The D7-brane may either “end” at some $r > b$ if the S^3 collapses, or it may continue all the way to the horizon where the S^1 collapses but the S^3 has finite size. In other words, the D7-topology may be either $R^3 \times B^4 \times S^1$ or $R^3 \times S^3 \times B^2$. The former is the type found in Ref. [29]. As one might expect, this topology occurs when the quark mass m is sufficiently large compared to b . In this case, the S^3 of

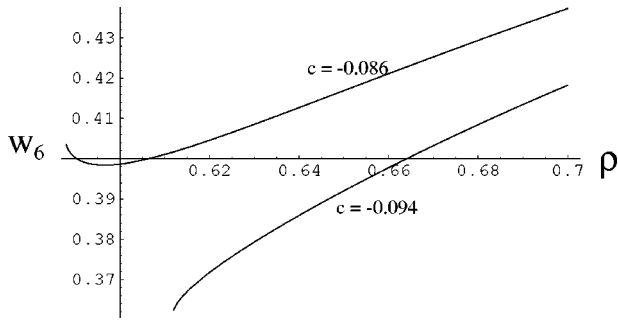


FIG. 5. An example (for $m=0.6$) of the different flow behaviors around the regular (physical) solution.

the D7-brane contracts to zero size in the asymptotic region where the deformation of $AdS_5 \times S^5$ is negligible. We shall find the other topology for sufficiently small m .

For any chosen value of m , we find only a discrete choice of c that gives a regular solution that can be interpreted as an RG flow. In Fig. 5 we show sample numerical flows used to identify a regular solution.

For the regular solutions the D7-brane either ends at the horizon,

$$w_6^2 + \rho^2 = \frac{1}{2} b^2, \tag{21}$$

at which the S^1 collapses, or ends at a point outside the horizon,

$$\rho = 0, w_6^2 \geq \frac{1}{2} b^2, \tag{22}$$

at which the S^3 collapses [see (19)]. Both types of solutions are illustrated in Fig. 6 for several choices of m . We choose units such that $b=1$. We refer to solutions with collapsing S^3 as Karch-Katz solutions, and solutions with collapsing S^1 as “condensate” solutions, for reasons that will become apparent shortly. Note that the boundary between the Karch-Katz and condensate solutions is at a critical value of the mass $m = m_{crit}$ such that $w_6(\rho=0) = \sqrt{1/2}b$. In this case both the S^1 and S^3 collapse simultaneously. Numerically, we find $m_{crit} \approx 0.92$.

There is an exact solution of the equation of motion $w_6=0$ that is regular and corresponds to $m=c=0$. Thus

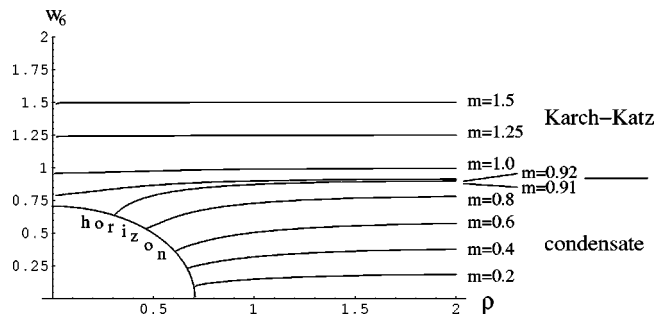


FIG. 6. Two classes of regular solutions in the AdS black hole background.

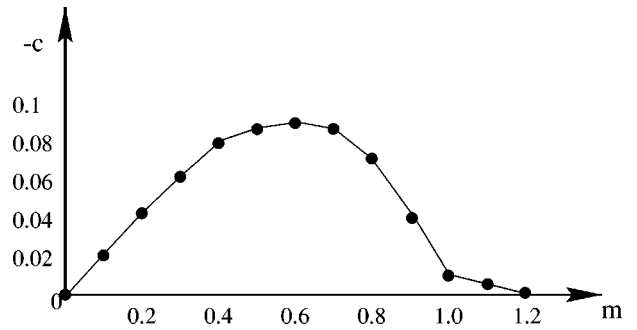


FIG. 7. A plot of the parameter c vs m for the regular solutions in the AdS Schwarzschild background. The linear fit between points is just to guide the eye.

there is no condensate when the quarks are massless (from the four-dimensional point of view). This should not be disappointing, since the theory is not in the same universality class as QCD_3 with light quarks. The quarks obtain a mass of order of the temperature that will tend to suppress the formation of a condensate. For nonzero m we obtain the solutions numerically. The dependence of the condensate on the mass is illustrated in Fig. 7. We find that as m increases, the condensate c initially increases and then decreases again. At sufficiently large m , the condensate becomes negligible, which is to be expected as the D7-brane ends in the region where the deformation of AdS space is small. Recall that there is no condensate in the Yang-Mills theory with unbroken $\mathcal{N}=2$ supersymmetry described by D7-branes in undeformed AdS space.

Since the D7-brane topology changes as one crosses m_{crit} , one might expect a phase transition to occur at this point.² Looking at Fig. 7, there does not appear to be any discontinuity in $c(m)$ at $m = m_{crit}$. It is still possible that there is a third order transition at this point, corresponding to a discontinuity in the slope dc/dm . Of course the numerical results must be refined significantly to find evidence for such a transition.

D. Meson spectrum

The meson spectrum can be found by solving the linearized equations of motion for small fluctuations about the D7-embeddings found above. Let us consider the fluctuations of the variable w_5 about the embedding, which has $w_5=0$. We take

$$w_5 = f(\rho) \sin(k \cdot \vec{x}). \tag{23}$$

The linearized (in w_5) equation of motion is

²For $m > m_{crit}$ there is the interesting possibility of introducing an even spin structure on the S^1 of the D7-brane, since this S^1 is no longer contractible on the D7. If this is a sensible (i.e. stable) background, it would correspond to a different field theory, in which the fundamental fermions are periodic on S^1 and do not have a Kaluza-Klein mass. We have not analyzed this possibility.

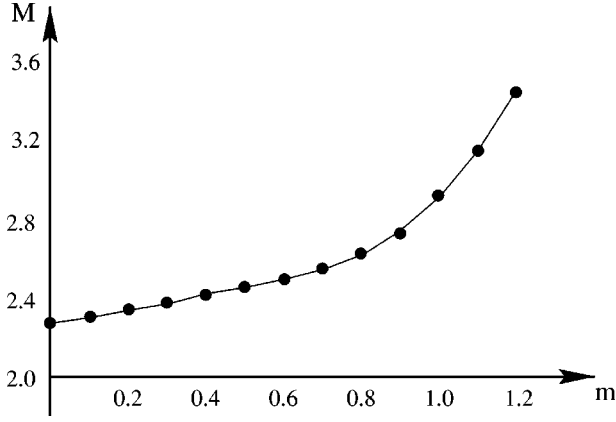


FIG. 8. A plot of the w_5 meson mass vs m in the AdS Schwarzschild background. The linear fit between points is just to guide the eye.

$$\begin{aligned}
& \frac{d}{d\rho} \left[\mathcal{G}(\rho, w_6) \sqrt{\frac{1}{1 + (dw_6/d\rho)^2}} \frac{df(\rho)}{d\rho} \right. \\
& + \mathcal{G}(\rho, w_6) \sqrt{\frac{1}{1 + (dw_6/d\rho)^2}} \left(\frac{4}{4(\rho^2 + w_6^2)^2 + b^4} \right) \\
& \left. \times M^2 f(\rho) - \sqrt{1 + \left(\frac{dw_6}{d\rho}\right)^2} \frac{b^8 \rho^3 f(\rho)}{2(\rho^2 + w_6^2)^5} \right] = 0.
\end{aligned} \tag{24}$$

where $M^2 = \vec{k}^2$. The allowed values of \vec{k}^2 are determined by requiring the solution to be normalizable and regular. Note that if the $U(1)$ symmetry, which rotates w_5 and w_6 , were spontaneously broken by an embedding of the asymptotic form $w_6 \sim c/\rho^2$ with nonzero c , there would be a massless state in the spectrum associated with w_5 fluctuations. This is not the case in the present setting, since the condensate is only nonzero for nonzero quark mass m . Instead we find a mass gap in the meson spectrum. We have computed the meson spectrum by solving (24) by a numerical shooting technique. As in the Karch-Katz geometry, we seek regular solutions for w_5 that are asymptotically of the form c/ρ^2 in the presence of the background w_6 solution. The results for the meson masses are plotted in Fig. 8. Of course the meson mass gap here can be largely attributed to the Kaluza-Klein masses of the constituent quarks, which are of the same order as the temperature ($T \sim \pi$ in units with $b = 1$).

Thus we have seen that, while the thermal gauge background allows a quark condensate when it does not spontaneously break any symmetries, there is no chiral (parity) symmetry breaking at zero quark mass. The meson spectrum reflects this by having a mass gap—the fermions have an induced mass from the presence of finite temperature. In the subsequent discussion we will consider another background that admits light constituent quarks and has properties much closer to QCD.

IV. THE CONSTABLE-MYERS DEFORMATION

A. The background

We consider the nonsupersymmetric deformed AdS geometry originally constructed in Ref. [9]. This geometry corresponds to the $\mathcal{N}=4$ super-Yang-Mills theory deformed by the presence of a vacuum expectation value for an R-singlet operator with dimension four (such as $\text{tr} F^{\mu\nu} F_{\mu\nu}$). The supergravity background has a dilaton and S^5 volume factor depending on the radial direction. In a certain parameter range, this background implies an area law for the Wilson loop and a mass gap in the glueball spectrum. Whether the geometry, which has a naked singularity, actually describes the stable nonsupersymmetric vacuum of a field theory is not well understood [9]. This is not so important from our point of view though since the geometry is a well defined gravity description of a nonsupersymmetric gauge configuration. We can just ask about the behavior of quarks in that background.

The geometry in *Einstein frame* is given by

$$\begin{aligned}
ds^2 = & H^{-1/2} \left(1 + \frac{2b^4}{r^4} \right)^{\delta/4} dx_4^2 + H^{1/2} \left(1 + \frac{2b^4}{r^4} \right)^{(2-\delta)/4} \\
& \times \frac{r^2}{(1 + b^4/r^4)^{1/2}} \left[\frac{r^6}{(r^4 + b^4)^2} dr^2 + d\Omega_5^2 \right],
\end{aligned} \tag{25}$$

where

$$H = \left(1 + \frac{2b^4}{r^4} \right)^\delta - 1 \tag{26}$$

and with the dilaton and four-form given by

$$e^{2\phi} = e^{2\phi_0} \left(1 + \frac{2b^4}{r^4} \right)^\Delta, \quad C_{(4)} = -\frac{1}{4} H^{-1} dt \wedge dx \wedge dy \wedge dz. \tag{27}$$

The parameter b corresponds to the vacuum expectation value (vev) of the dimension 4 operator. The parameters Δ and δ are constrained by

$$\Delta^2 + \delta^2 = 10. \tag{28}$$

Asymptotically the AdS curvature is given by $L^4 = 2\delta b^4$, so it makes sense to set (with $L = 1$)

$$\delta = \frac{1}{2b^4}. \tag{29}$$

b is the only free parameter in the geometry and its value sets the scale of the conformal symmetry and supersymmetry breaking. We will again numerically set it equal to 1 below.

To embed a D7 brane in this background it will again be convenient to recast the metric in a form containing an explicit flat 6-plane. To this end, we change variables from r to w , such that

$$\frac{dw}{w} \equiv \frac{r^2 d(r^2)}{2(r^4 + b^4)}, \tag{30}$$

which is solved by

$$\ln(w/w_0)^4 = \ln(r^4 + b^4) \quad (31)$$

or

$$(w/w_0)^4 = r^4 + b^4. \quad (32)$$

So for the case of $b=0$ we should set the integration constant $w_0=1$. The full metric is now

$$ds^2 = H^{-1/2} \left(\frac{w^4 + b^4}{w^4 - b^4} \right)^{\delta/4} dx_4^2 + H^{1/2} \left(\frac{w^4 + b^4}{w^4 - b^4} \right)^{(2-\delta)/4} \frac{w^4 - b^4}{w^4} \sum_{i=1}^6 dw_i^2, \quad (33)$$

where

$$H = \left(\frac{w^4 + b^4}{w^4 - b^4} \right)^{\delta} - 1 \quad (34)$$

and the dilaton and four-form become

$$e^{2\phi} = e^{2\phi_0} \left(\frac{w^4 + b^4}{w^4 - b^4} \right)^{\Delta}, \quad C_{(4)} = -\frac{1}{4} H^{-1} dt \wedge dx \wedge dy \wedge dz. \quad (35)$$

We now consider the D7-brane-action in the static gauge with world-volume coordinates identified with the four Minkowski coordinates—denoted by x_4 —and with $w_{1,2,3,4}$. The transverse fluctuations are parametrized by w_5 and w_6 . It is again convenient to define a coordinate ρ such that $\sum_{i=1}^4 dw_i^2 = d\rho^2 + \rho^2 d\Omega_3^2$. The DBI action in Einstein frame

$$S_{D7} = -T_7 \int d^8 \xi e^{\phi} \sqrt{-\det(P[G_{ab}])} \quad (36)$$

can then be written as

$$S_{D7} = -T_7 \int d^8 \xi \epsilon_3 e^{\phi} \mathcal{G}(\rho, w_5, w_6) \times (1 + g^{ab} g_{55} \partial_a w_5 \partial_b w_5 + g^{ab} g_{66} \partial_a w_6 \partial_b w_6)^{1/2}, \quad (37)$$

where

$$\mathcal{G}(\rho, w_5, w_6) = \rho^3 \frac{[(\rho^2 + w_5^2 + w_6^2)^2 + b^4][(\rho^2 + w_5^2 + w_6^2) - b^4]}{(\rho^2 + w_5^2 + w_6^2)^4}. \quad (38)$$

We again look for classical solutions to the equations of motion of the form $w_6 = w_6(\rho), w_5 = 0$, that define the ground state. They satisfy

$$\frac{d}{d\rho} \left[\frac{e^{\phi} \mathcal{G}(\rho, w_6)}{\sqrt{1 + (\partial_{\rho} w_6)^2}} (\partial_{\rho} w_6) \right] - \sqrt{1 + (\partial_{\rho} w_6)^2} \frac{d}{dw_6} [e^{\phi} \mathcal{G}(\rho, w_6)] = 0. \quad (39)$$

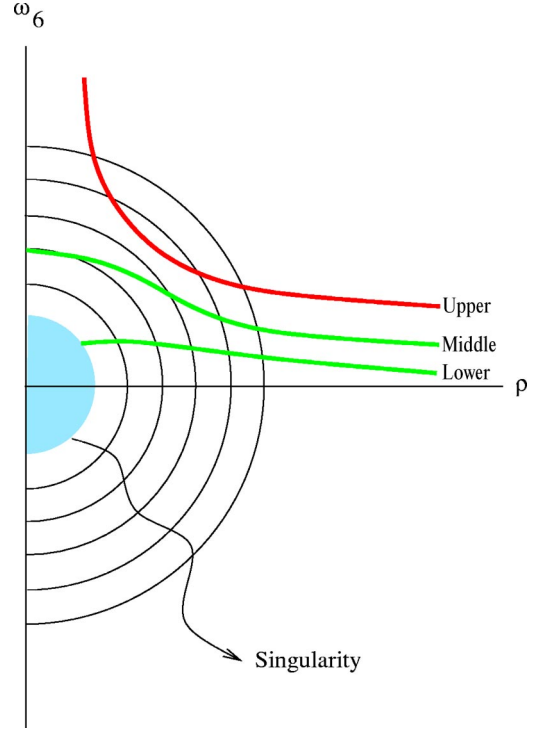


FIG. 9. Different possibilities for solutions of the D7-brane equations of motion. The semicircles are lines of constant r , which should be interpreted as a scale in the dual Yang-Mills theory. The “upper” curve cannot be interpreted as an RG flow. The other curves have an RG flow interpretation; however, the infrared (small r) region of the “lower” curve is outside the range of validity of DBI/supergravity.

The last term in the above is a “potential”-like term that is evaluated to be

$$\frac{d}{dw_6} [e^{\phi} \mathcal{G}(\rho, w_6)] = \frac{4b^4 \rho^3 w_6}{(\rho^2 + w_6^2)^5} \left(\frac{(\rho^2 + w_6^2)^2 + b^4}{(\rho^2 + w_6^2)^2 - b^4} \right)^{\Delta/2} \times [2b^4 - \Delta(\rho^2 + w_6^2)^2]. \quad (40)$$

We now consider numerical solutions with the asymptotic behavior $w_6 \sim m + c/\rho^2$, and find the physical solutions by imposing a regularity constraint as discussed in the preceding section. Note that unlike the Euclidean AdS black hole, the Constable-Myers background has a naked singularity at $r=0$ or $\rho^2 + w_6^2 = b^2$. Thus there are two possibilities for a solution with an interpretation as an RG flow, which are as follows. Either the D7-brane terminates at a value of r away from the naked singularity via a collapse of the S^3 , or the D7-brane goes all the way to the singularity. In the latter case we would have little control over the physics without a better understanding of string theory in such highly curved backgrounds. Different possibilities for solutions of the D7-brane equations of motion are illustrated in Fig. 9.

Fortunately something remarkable happens. For positive values of m we find that there is a discrete regular solution for each value of the mass that terminates at $w \geq 1.3$ before reaching the singularity at $w=1$. Some of these regular solutions are plotted in Fig. 10. For $m=0$, the solution

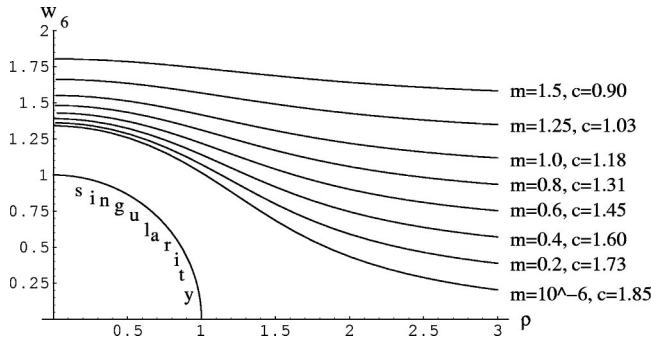


FIG. 10. Regular solutions in the Constable-Myers background.

$w_6=0$ is exact, which naively seems to indicate the absence of a chiral condensate ($c=0$). However, this solution reaches the singularity, and therefore cannot be trusted. On the other hand for a very small but nonzero mass, the regular solutions require a nonvanishing c and terminate before reaching the singularity. The numerical evidence (see Fig. 10) suggests that there is a nonzero condensate in the limit $m \rightarrow 0$. In other words the geometry spontaneously breaks the $U(1)$ chiral symmetry. We plot c as a function of m in Fig. 11. This seems to be analogous to the situation in field theory, in which the path integral *formally* gives no spontaneous symmetry breaking, which is found only in the limit that a small explicit symmetry breaking parameter is taken to zero.

We can study this phenomenon further in the deep infra-red, in particular in view of gaining further understanding of the behavior of the solutions shown in Fig. 10. Consider (39) as $\rho \rightarrow 0$ with $w_6 \neq 0$ —the dilaton becomes ρ independent whilst $\mathcal{G} \sim \rho^3$. Thus the potential term vanishes as ρ^3 whilst the derivative piece contains a term that behaves like $\rho^2 \partial_\rho w_6$ and dominates. Clearly there is a solution where w_6 is just a constant. This is the regular behavior we are numerically tuning to. It is now easy to find the flows in reverse by setting the infrared constant value of w_6 and numerically solving out to the UV. This method ensures that the flow is always regular. We have checked that the asymptotic values of the condensate as a function of mass match our previous computation at the level of a percent, showing that the numerics are under control.

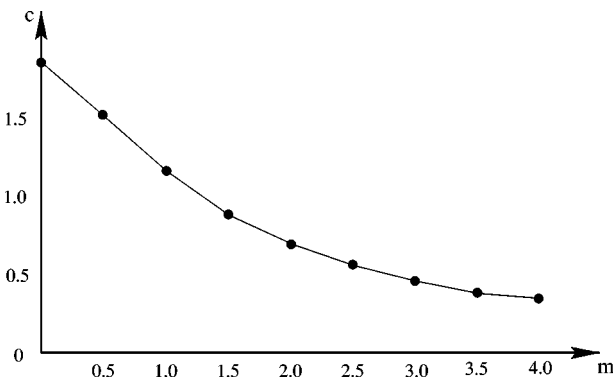


FIG. 11. A plot of the condensate parameter c vs quark mass m for the regular solutions of the equation of motion in the Constable-Myers background.

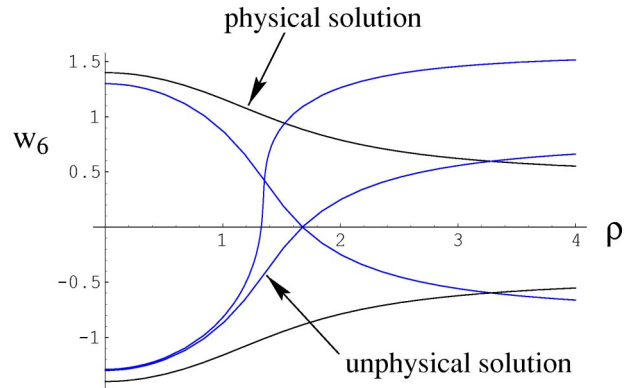


FIG. 12. Regular D7 embedding solutions in the Constable-Myers geometry that lie close to the singularity in the infra-red.

We now realize though that there are infrared solutions where $w_6(\rho=0) < 1.3$. These flows lie close to the singularity at $w_6(\rho=0) = 1$, which we had hoped to exclude. Solving these flows numerically we find that they flow to negative masses in the ultraviolet. We show these flows in Fig. 12. Since there is a $w_6 \rightarrow -w_6$ symmetry of the solution though this means there is a second regular flow for each positive mass which in the infrared flows to negative values, also shown in Fig. 12. The flows that begin closer to the singularity in the infrared flow out to larger masses in the ultraviolet. This strongly suggests these flows are not physical. When the quarks have a large mass, relative to the scale of the deformation, we do not expect the infrared dynamics to have a large influence on the physics. Thus the flows shown in Fig. 10 that match onto the Karch-Katz type solution for large mass are the expected physical solutions. To find some analytic support for this conjecture we have calculated the action of two of the solutions for each mass value. The action is formally infinite if we let the flow cover the whole space. To see the difference in action we have calculated the contribution for $0 < \rho < 3$, only, which covers the infrared part of the solution (varying the upper limit does not change the conclusions). We plot the action of the two solutions versus the quark mass in Fig. 13, from which it can

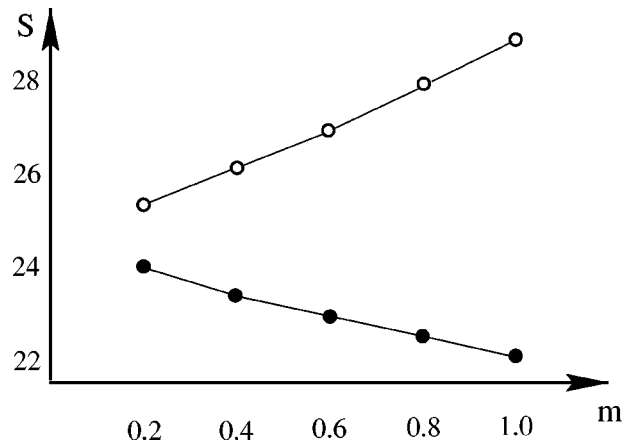


FIG. 13. A plot of action vs mass for the two regular D7 embedding solutions in the Constable-Myers geometry. The higher action solutions correspond to the flows that end at $|w_6| < 1.3$.

be seen that the action for the solution lying closer to the singularity is larger. Therefore the corresponding solution is not relevant for the physics.

In a certain sense, the condensate screens the probe physics from the naked singularity. In the limit of small explicit symmetry breaking parameter m , any solution (or vacuum) that did not break the $U(1)$ symmetry would have $w_6 \rightarrow 0$ for all ρ . If this were the case, the solution would reach the singularity (see Fig. 9). The screening of the singularity is reminiscent of the *enhçon* [39] found in $\mathcal{N}=2$ gravity duals—an important part of that analysis was understanding that the singularity of the geometry was screened from the physics of a D3 brane probe that led to an understanding of how the singularity could be removed. It is possible we are seeing hints of something similar, if more complicated, here, although at this stage we cannot see how to remove the singular behavior.

B. Large N Goldstone boson (η')

Since there is chiral symmetry breaking via a condensate in the $m \rightarrow 0$ limit, we also expect there to be a Goldstone boson in the meson spectrum. Such a Goldstone mode must exist as a solution to the DBI equation of motion, as the following holographic version of the Goldstone theorem shows. Assume a D7-embedding with $w_5 = 0$ and $w_6 \sim c/\rho^2$ asymptotically. A small $U(1)$ rotation $\exp(i\epsilon)$ of $w_5 + iw_6$ generates a solution that is a normalizable small fluctuation about this background, with w_6 unchanged [to order (ϵ^2)] and $w_5 = \epsilon c/\rho^2$. Thus a small fluctuation with w_6 unchanged and $w_5 = \epsilon(c/\rho^2)\sin(k \cdot x)$ is a normalizable solution of the *linearized* equations of motion provided $k^2 = 0$. In other words there must be a Goldstone boson associated with w_5 fluctuations. Note that if the embedding were asymptotically $w_6 \sim m + c(m)/\rho^2$ for nonzero m , a $U(1)$ rotation of $w_5 + iw_6$ would still generate another solution. However, this solution is no longer a *normalizable* small fluctuation about the original embedding—asymptotically the mass will acquire a different phase moving us to a different theory. Thus if the mass is kept fixed asymptotically one does not find a massless particle in the spectrum, which reflects the explicit symmetry breaking by the quark mass m .

Note that the $U(1)$ chiral symmetry is nonanomalous only in the limit $N \rightarrow \infty$. The Goldstone boson discussed above is analogous to the η' in QCD, which becomes a Goldstone boson in the large N limit [36]. Its masslessness in this setting should not be a surprise, since the regime of validity of supergravity corresponds to large N in the dual field theory. We expect that finite N stringy effects will give the η' a nonzero mass.

In the solutions discussed where w_6 has a background value, fluctuations in w_5 should contain the Goldstone mode. Let us turn to the numerical study of these fluctuations in the background of the w_6 solutions we have obtained above. The linearized equation of motion for small fluctuations of the form $w_5 = f(\rho)\sin(k \cdot x)$, with x the four Minkowski coordinates, is

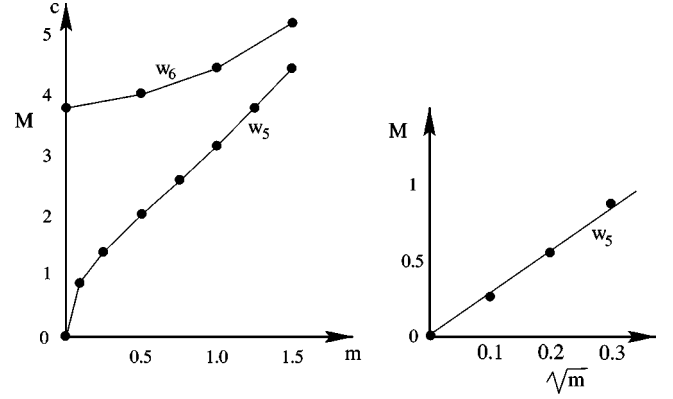


FIG. 14. A plot of the w_5 and w_6 meson mass vs quark mass m associated with the fluctuations about the regular solutions of the equation of motion for the Constable-Myers flow. The Goldstone mass is also plotted vs \sqrt{m} with a linear fit.

$$\begin{aligned}
 & \frac{d}{d\rho} \left[\frac{e^\phi \mathcal{G}(\rho, w_6)}{\sqrt{1 + (\partial_\rho w_6)^2}} \partial_\rho f(\rho) \right] \\
 & + M^2 \frac{e^\phi \mathcal{G}(\rho, w_6)}{\sqrt{1 + (\partial_\rho w_6)^2}} H \left(\frac{(\rho^2 + w_6^2)^2 + b^4}{(\rho^2 + w_6^2)^2 - b^4} \right)^{(1-\delta)/2} \\
 & \times \frac{(\rho^2 + w_6^2)^2 - b^4}{(\rho^2 + w_6^2)^2} f(\rho) \\
 & - \sqrt{1 + (\partial_\rho w_6)^2} \frac{4b^4 \rho^3}{(\rho^2 + w_6^2)^5} \left(\frac{(\rho^2 + w_6^2)^2 + b^4}{(\rho^2 + w_6^2)^2 - b^4} \right)^{\Delta/2} \\
 & \times [2b^4 - \Delta(\rho^2 + w_6^2)^2] f(\rho) = 0. \tag{41}
 \end{aligned}$$

The meson mass as a function of quark mass for the regular solutions for w_5 are plotted in Fig. 14. The meson mass indeed falls to zero as the quark mass is taken to zero providing further evidence of chiral symmetry breaking.

At small m , the mass associated with the w_5 fluctuations scales like \sqrt{m} . This is consistent with field theory expectations. A well known field theory argument for this scaling is as follows. The low-energy effective Lagrangian depends on a field η' where $\exp(i\eta'/f)$ parametrizes the vacuum manifold and transforms by a phase under chiral $U(1)$ rotations. A quark mass term transforms by the same phase under $U(1)$ rotations, and thus breaks the $U(1)$ explicitly. A chiral Lagrangian consistent with this breaking has a term $\mu^3 \text{Re}[m \exp(i\eta'/f)]$ where μ is some parameter with dimensions of mass. For real m , expanding this term to quadratic order gives a mass term $(\mu^3/f^2)m\eta'^2$. It would be very interesting to demonstrate this scaling with m analytically in the DBI/supergravity setting, along with other low-energy “theorems.”

For comparison it is interesting to study w_6 fluctuations as well, which we expect to have a mass gap. Analytically linearizing the w_6 equation of motion is straightforward but the result is unrevealingly messy. Since we must eventually solve the equation numerically, we can use a simple numeri-

cal trick to obtain the solutions. We solve Eq. (39) for w_6 above but write $w_6 = w_6^0 + \delta w_6(r)$, where numerically we enforce δw_6 to be very small relative to the background configuration w_6^0 . With this ansatz we retain the field equation in its nonlinear form, but it is numerically equivalent to standard linearization. We must also add a term to the left-hand side of (39) that takes into account the x dependence of δw_6 . This dependence takes the same form as that in the linearized w_5 equation (41), i.e. $\delta w_6 = h(\rho) \sin(k \cdot x)$. The extra term to be added to (39) is

$$\Delta V = M^2 \frac{e^{\phi} \mathcal{G}(\rho, w_6)}{\sqrt{1 + (\partial_\rho w_6)^2}} H \left(\frac{(\rho^2 + w_6^2)^2 + b^4}{(\rho^2 + w_6^2)^2 - b^4} \right)^{(1-\delta)/2} \times \frac{(\rho^2 + w_6^2)^2 - b^4}{(\rho^2 + w_6^2)^2} \delta w_6. \quad (42)$$

The numerical solutions for the w_6 fluctuations are plotted in Fig. 14. The w_6 fluctuations have a mass gap, as expected since they are transverse to the vacuum manifold.

C. Pions

Thus far we have found a particle in the spectrum that is analogous to the η' in the large N limit of QCD where it becomes a Goldstone boson. In order to obtain true pions one must have a non-Abelian flavor symmetry. Unfortunately in the background that we consider, taking $N_f > 1$ D7-branes does not give rise to a $U(N_f)_L \times U(N_f)_R$ chiral symmetry. Instead one gets only diagonal $U(N_f)$ times an axial $U(1)$. The reason is that the theory contains a coupling $\tilde{\psi}_i X \psi_i$, where i is a flavor index and X is an adjoint scalar without any flavor indices. The coupling to X explicitly breaks the $U(N_f)_L \times U(N_f)_R$ chiral symmetry to the diagonal subgroup, but preserves an axial $U(1)$ which acts as

$$\psi_i \rightarrow e^{i\theta} \psi_i, \quad \tilde{\psi}_i \rightarrow e^{i\theta} \tilde{\psi}_i, \quad X \rightarrow e^{-2i\theta} X. \quad (43)$$

Thus a $\tilde{\psi}_i \psi_i$ condensate will only give rise to one Goldstone boson, even if $N_f > 1$. If X were massive, there would be an approximate $U(N_f)_L \times U(N_f)_R$ symmetry at low energies, but this is not the case in the Constable-Myers background.

Note that $N_f > 1$ coincident D7-branes may be embedded in the same way as a single D7-brane. In this case there are N_f independent solutions to the linearized equations of motion for small fluctuations about this embedding, corresponding to fluctuations of the diagonal entries in a diagonal $N_f \times N_f$ matrix. These fluctuations would naively give rise to at least N_f Goldstone bosons, rather than one. These extra states will not remain massless though since the interaction with the scalar fields that breaks the symmetry will induce a mass.

Nevertheless we can still make a rough comparison between our η' and QCD pions. In a two-flavor large N QCD model where the quarks are degenerate one would expect four degenerate Goldstone bosons. As N is decreased, instanton effects will enter to raise the η' mass. However, since we are at large N , the mass formula for our Goldstone boson is

that applicable to the pions. It is therefore amusing to compare the Goldstone mass we predict to that of the QCD pion. There is considerable uncertainty in matching the strong coupling scale of our theory to that of QCD. In QCD the bare up or down quark mass is roughly 0.01Λ and the pion mass of order 0.5Λ (it is of course hard to know precisely what value one should pick for Λ). The comparison to our theory is a little hard to make but if we assume that $\Lambda \approx b = 1$, then for this quark mass we find $m_\pi \approx 0.25\Lambda$. The gravity dual is correctly predicting the pion mass at the level of a factor of two. Of course we cannot expect a perfect match given the additional degrees of freedom in the deformed $\mathcal{N}=4$ theory relative to real QCD.

V. CONCLUSIONS AND OPEN QUESTIONS

We have studied two nonsupersymmetric gravity backgrounds with embedded D7-brane probes, corresponding to Yang-Mills theories with confined fundamental matter. The AdS Schwarzschild black-hole background, which in the presence of the probe describes an $\mathcal{N}=2$ Yang-Mills theory at finite temperature, exhibits interesting behavior such as a bilinear quark condensate and a geometric transition, which may correspond to a third order transition in the gauge theory. The D7-brane embedding into the Constable-Myers background is more QCD-like, showing a chiral condensate at small quark mass as well as the accompanying pion (or large N η').

In the Constable-Myers background with a D7-brane, there is a spontaneously broken $U(1)$ axial symmetry. A closer approximation to (large N) QCD would require a spontaneously broken $U(N_f)_L \times U(N_f)_R$ chiral symmetry with $N_f=2$ or $N_f=3$. Unfortunately, simply adding D7-branes does not accomplish this in the Constable-Myers background. Assuming that one were able to find a background with the full chiral symmetry, it would be very interesting to obtain a holographic interpretation of low-energy current algebra theorems and make predictions for chiral-Lagrangian parameters based on the non-Abelian Dirac-Born-Infeld action. One challenge would be to obtain such terms as the Wess-Zumino-Witten term, $\int d^5x \text{tr}(d\Sigma^\dagger)^5$. Note that this term requires an auxiliary fifth dimension, which appears naturally in the holographic context.

It is clearly important to look at more physical geometries. We chose these backgrounds because they are particularly simple; the S^5 is left invariant and hence embedding the D7 is straightforward and the RG flow depends only on the radial direction. More complicated geometries, such as the Yang-Mills* geometry [8], that include mass terms for the adjoint scalars and fermions of $\mathcal{N}=4$, have extra dependence on the angles of the S^5 and the resulting equations of motion are much less tractable.

In conclusion the results presented here represent another success for the AdS/CFT correspondence. The results suggest that gravity duals of nonsupersymmetric gauge theories may induce chiral symmetry breaking if light quarks are introduced, just as is observed in QCD. This opens up the possibility of studying the light meson sector of QCD using these new techniques.

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