Point-to-point description of the bubble wall dynamics in two-vacuum scalar field models

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We develop a new approach to the Lagrangian description of the bubble wall dynamics in the nonlinear Klein-Gordon equation with a two-vacuum potential of general form having a small vacuum energy difference. The approach is based on an ordinary differential equation governing the motion of an arbitrary point of the wall in the second approximation in the vacuum energy difference and inverse bubble radius. The equation is model independent: the concrete shape of the potential affects the constants involved only. We give a detailed derivation of this equation and present the full scheme of our method. As examples, we find some wall solutions for the ϕ^3-ϕ^4 and ϕ^4-ϕ^6 potentials and compare them with solutions obtained by the direct numerical integration of the nonlinear Klein-Gordon equation.

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I. INTRODUCTION

There are many physical systems, the evolution of which is determined by effective potentials with two or more different minima. Under certain conditions in such systems first-order phase transitions are possible, which proceed through bubble nucleation and subsequent expansion of supercritical bubbles of a new phase inside the old one. The two main mechanisms of spontaneous nucleation are commonly known. In particle physics and some cosmological models the bubbles come into being by quantum tunneling through a potential barrier separating different minima of an effective potential [1]. In condensed matter the dynamics of the order parameter field is usually governed by a nonlinear evolution equation of diffusion type. The bubble nucleation is caused there by thermal fluctuations [2]. Recently, yet another mechanism of nucleation has been proposed [3]. It works in nonlinear wave systems and is based on parametric resonance phenomenon and modulational instability of false vacuum oscillations.

In any case, let us suppose that a new-phase bubble has already formed. Qualitatively, the subsequent stage of its expansion is almost the same for different physical systems. In the present paper we investigate the dynamics of the bubble expansion in the simplest classical field model,

$$\phi_{tt} - \Delta\phi + U'(\phi) = 0. \tag{1}$$

Here $U(\phi)$ is a potential depending on a real scalar field ϕ and having the shape depicted in Fig. 1. Such potentials take on great significance in the context of the currently discussed single-field models of open inflation [4]. In particle physics effective potentials can acquire this form if quantum corrections are accounted for [1]. The nonlinear Klein-Gordon equation (1) with potentials having several different minima also appears in condensed matter physics (e.g., when describing nonlinear modes in anharmonic crystals [5], spin waves in ^{3}He [6]) and in nonlinear optics (double sine-Gordon (sG) model of self-induced transparency [6]).

Many exact solutions of Eq. (1) for various potentials are presently known, both in one- and multidimensional cases. However, most solutions obtained in multidimensions possess neither finite energy nor proper spatial symmetry and can be frequently treated as interacting plane solitary waves (see, e.g., Ref. [7]). In the rotationally symmetric case of our interest only a few exact time-dependent bubblelike solutions have been found for potentials of some specific types [8].

In the present work we are concerned with approximate solutions of Eq. (1). Consider a field configuration, the energy of which is concentrated in a transitional layer, a wall, separating two regions with different vacuum states. Suppose the wall's thickness is much less than a characteristic size of the configuration, say, its radius. In this case the evolution of the configuration reduces mainly to the motion of the wall and can be analyzed by asymptotic methods. For the ϕ^4 and sG type potentials having one degenerate vacuum this problem has been considered by many authors (see, e.g., Refs. [9–16]). Clearly, in such models the rotationally symmetric topological defects (i.e., circular strings, cylindrical and

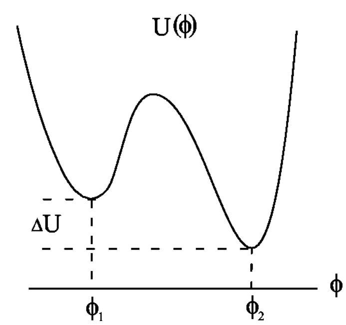

FIG. 1. Typical shape of the potentials considered. The vacuum energy difference ΔU is assumed to be small compared with the barrier height.

spherical domain walls) can only collapse, just due to the surface tension: ‘‘It is like a soap bubble, but with no restoring overpressure inside it’’ [13]. The dynamics of the collapse can be easily investigated in the thin-wall approximation using Nambu’s action [17]. Potentials we are dealing with have at least one local minimum (false vacuum), say, at $\phi=\phi_1$, and the global minimum (true vacuum) at $\phi=\phi_2$, so that $\Delta U = U(\phi_1) - U(\phi_2) > 0$ (see Fig. 1). In such models the rotationally symmetric field configurations usually evolve from nuclei arising in the local first-order phase transitions $\phi_1 \rightarrow \phi_2$. If the nucleus radius R exceeds some critical value R_c, a nucleus will expand without limit in the shape of a new phase bubble. In this process ΔU plays the role of the difference of pressures in and outside the bubble.

The first attempt to describe the bubble expansion in model (1) with a two-vacuum potential has been made by Voloshin *et al.* [18]. These authors have found an approximate expression for the total energy of a bubble and considered, as an example, the potential $U = U_0(\phi) + c_1\phi + c_3\phi^3$ with $U_0(\phi) = (\phi^2 - \eta^2)^2$ and small constants c_1, c_3. An arbitrary perturbed sG potential, $U = 2\sin^2(\phi/2) + W(\phi)$, has been considered by Maslov in Ref. [19]. Having applied the perturbation theory based on the inverse scattering transform, the author derived, in particular, an equation for the radius $R(t)$ of the kinklike bubble in the first order in $1/R$ and $\Delta U \sim W$. The equation for $R(t)$ in a dissipative version of Eq. (1) has been recently obtained by Rotstein and Nepomnyashchy [20], who studied in the first-order approximation the wall motion for the potential $U = U_0(\phi) + c_1\phi$ with a symmetric double-well leading part $U_0(\phi)$ and small c_1.

In the present paper we investigate bubble wall dynamics in the second-order approximation. Moreover, we consider potentials of a general form for which the smallness of ΔU is only required. In particular, we make no special assumptions of the symmetry properties of the leading part of $U(\phi)$.

The paper is organized as follows. Section II is wholly devoted to the detailed derivation of the equation for the bubble ‘‘radius’’ $R(t)$ that has been recently proposed in Ref. [21]. We give a key idea of our analysis and reduce Eq. (1) to the system of the first-order differential equations for a ‘‘phase’’ of the bubble wall. Then we apply the generalized Krylov-Bogoliubov method and derive, to second order in $1/R$ and ΔU, the equation of motion of an arbitrary point $r = R(t)$ lying on the wall. In Sec. III we consider static solutions of the obtained equation describing the structure of the critical bubbles. As examples, ϕ^3-ϕ^4 and ϕ^4-ϕ^6 potentials are examined there. In Sec. IV the dynamics of the bubble expansion is considered. We find the integral of motion of the equation obtained and give the general scheme of our approach. A comparison with results of the direct numerical integration of Eq. (1) is also presented. Some remarks on the validity domain of the approach can be found in Sec. V.

II. DERIVATION OF THE BASIC EQUATION

A. Suggestions and the key idea

Let us consider a bubble of a large radius R evolved from an initial large radius nucleus,

$$\phi(r,0) = f(r), \quad \phi_t(r,0) = g(r), \tag{2}$$

with f and g being appropriate smooth functions. Inside such a bubble the field is practically equal to the true vacuum value ϕ_2, while outside it the field equals ϕ_1. The transition $\phi_1 \rightarrow \phi_2$ occurs in a thin layer of a thickness $l \ll R$.

To investigate the bubble dynamics we construct a substitution $\phi(r,t) = \varphi(\theta)$, where $\theta(r,t)$ is a ‘‘phase.’’ Obviously, there is some freedom of the choice of the dependence $\varphi(\theta)$. At the same time this dependence should reflect adequately the expected shape of the solution.

To choose $\varphi(\theta)$ in the proper way let us take first a potential $U_0(\phi)$ with $\Delta U = 0$. Consider, for simplicity, a one-dimensional case, i.e., a plane domain wall moving along the x axis. The function $\varphi(\theta)$ then obeys the equation

$$\varphi_{\theta\theta} = U_0'(\varphi), \tag{3}$$

where $\theta = (x - X_0 - vt)(1 - v^2)^{-1/2}$. This equation describes the shape of the wall moving with a constant velocity v in the positive direction of the x axis. It can be integrated, at least through a quadrature, with the boundary conditions $\varphi(\theta = -\infty) = \phi_2$, $\varphi(\theta = +\infty) = \phi_1$.

Consider now the two-vacuum potential $U(\phi)$ depicted in Fig. 1. When the wall goes through a fixed spatial point, the field passes from state ϕ_1 to state ϕ_2, losing the energy ΔU. At this point the phase θ changes from $+\infty$ at $t = -\infty$ to $-\infty$ at $t = +\infty$. To describe the transition in terms of $\varphi(\theta)$ we add to the left-hand side of Eq. (3), as the simplest possibility, an effective friction term that compensates for the contribution appearing in the right-hand side due to ΔU. Namely, we require the function $\varphi(\theta)$ to satisfy the equation

$$\varphi_{\theta\theta} + \gamma\varphi_\theta = U'(\varphi) \tag{4}$$

with the boundary conditions

$$\varphi(\theta = -\infty) = \phi_2, \quad \varphi(\theta = +\infty) = \phi_1, \tag{5}$$

where γ is a positive constant to be found. This is a key idea of our analysis.

Physically, Eq. (4) describes the motion with friction of a mechanical particle in the potential $-U(\varphi)$. The particle starts at $\varphi = \phi_2$ when $\theta = -\infty$ and stops due to the friction at $\varphi = \phi_1$ when $\theta = +\infty$. Mathematically, this equation and the imposed boundary conditions (5) are an eigenvalue problem determining uniquely $\varphi(\theta)$ and the effective friction coefficient γ. As it follows from (4),

$$\gamma = \Delta U \Big/ \int_{-\infty}^{\infty} \varphi_\theta^2 d\theta, \tag{6}$$

so that γ is small for small ΔU. For some potentials the problem (4) and (5) has exact analytical solutions (see examples in Sec. III). Otherwise, it can be easily solved by a simple perturbation technique or numerically. In so doing, it is useful to take into account the asymptotic behavior of $\varphi(\theta)$ given by

$$\varphi(\theta)_{\theta\to\pm\infty}\sim\phi_{1,2}\pm e^{\mp\kappa_\pm(\theta-\theta_\pm)} + \frac{U'''(\phi_{1,2})}{2\kappa_\pm(3\kappa_\pm\mp\gamma)}e^{\mp2\kappa_\pm(\theta-\theta_\pm)}\pm\dots, \quad (7)$$

where $\theta_\pm$ are some constants,

$$\kappa_\pm=\pm\frac{\gamma}{2}+\left(\frac{\gamma^2}{4}+U''(\phi_{1,2})\right)^{1/2}. \quad (8)$$

It should be stressed that postulating Eq. (4) we add no dissipation to the system (1), but define the dependence $\varphi(\theta)$ only.

Of course, from (4) and (5) it follows that $\phi_1<\varphi(\theta)<\phi_2$ $(|\theta|<\infty)$. Thus, using $\varphi(\theta)$ so defined one cannot describe, e.g., the vacuum oscillations around ϕ_2 arising behind the moving wall under specific initial conditions. In such events $\varphi(\theta)$ will describe correctly the region of the wall only, which is quite sufficient for our purposes.

It is of interest that Eq. (4) was applied long before for description of travelling waves in chemical and biological reaction-diffusion systems [22]. In particular, it appears in the theory of flame propagation [23] where φ has a sense of normalized temperature or concentration. In this connection the existence and uniqueness theorem for solution of problem (4) and (5) has been proved in Ref. [24].

Suppose problem (4) and (5) is solved. Assuming the rotational symmetry we substitute $\phi=\varphi(\theta)$ into (1) and use (4) to eliminate $dU/d\phi$. The result can be written as the system

$$k_r+p_t+\frac{\varphi_{\theta\theta}}{\varphi_\theta}(k^2-p^2-1)=\gamma\left(1-\frac{n-1}{\rho}k\right), \quad (9)$$

$$\theta_r=k, \quad (10)$$

$$\theta_t=-p, \quad (11)$$

where n is the number of spatial dimensions, and we set $1/r=\gamma/\rho$. Note that ρ can be treated as an additional dependent variable satisfying the equations $\rho_r=\gamma$ and $\rho_t=0$. We assume that $\rho\gtrsim1$. As we will see below the latter implies the bubble's radius is greater than or of the order of the critical one. The velocity of the wall is defined in the ordinary way as $v=p/k$.

B. The asymptotic expansion procedure

Recall that we consider potentials with small ΔU, so that γ in Eq. (9) is also small. We make use of this fact to construct asymptotic expansions.

With $\gamma=0$ the system (9)–(11) has the exact solution $k=(1-v^2)^{-1/2}$, $p=v(1-v^2)^{-1/2}$, $\theta=(r-R_0-vt)(1-v^2)^{-1/2}$, where v and R_0 are constants. When $0<\gamma\ll1$ the quantities k, p, and, of course, ρ are considered to be slow functions of r, t, compared with the phase θ. It is assumed that fast variations of k and p are small, of the order of γ. To separate the slow motions from the fast ones we apply the generalized Krylov-Bogoliubov method [25]. The essence is that the slowly varying parts of k and p are treated as new variables. We will seek the corresponding transformation in the form

$$k=\bar{k}+\sum_{i=1}^{\infty}\gamma^iK_i(\bar{k},\bar{p},\bar{\rho},\theta), \quad (12)$$

$$p=\bar{p}+\sum_{i=1}^{\infty}\gamma^iP_i(\bar{k},\bar{p},\bar{\rho},\theta), \quad (13)$$

provided that

$$\bar{k}_r=\sum_{i=1}^{\infty}\gamma^iA_i(\bar{k},\bar{p},\bar{\rho}), \quad (14)$$

$$\bar{k}_t=\sum_{i=1}^{\infty}\gamma^iB_i(\bar{k},\bar{p},\bar{\rho}), \quad (15)$$

$$\bar{p}_r=\sum_{i=1}^{\infty}\gamma^iC_i(\bar{k},\bar{p},\bar{\rho}), \quad (16)$$

$$\bar{p}_t=\sum_{i=1}^{\infty}\gamma^iD_i(\bar{k},\bar{p},\bar{\rho}). \quad (17)$$

Thus, $\bar{k}$, $\bar{p}$, and $\bar{\rho}=\rho=\gamma r$ are new, slow variables, while θ is the fast variable.

Differentiating Eqs. (12) and (13) and making use of Eqs. (10)–(17), one finds

$$k_r=\sum_{i=1}^{\infty}\gamma^i\sum_{j=0}^{i}(A_{i-j}K_{j\,\bar{k}}+C_{i-j}K_{j\,\bar{p}}+\delta^1_{i-j}K_{j\,\bar{\rho}}+K_{i-j}K_{j\,\theta}), \quad (18)$$

$$k_t=\sum_{i=1}^{\infty}\gamma^i\sum_{j=0}^{i}(B_{i-j}K_{j\,\bar{k}}+D_{i-j}K_{j\,\bar{p}}-P_{i-j}K_{j\,\theta}), \quad (19)$$

$$p_r=\sum_{i=1}^{\infty}\gamma^i\sum_{j=0}^{i}(A_{i-j}P_{j\,\bar{k}}+C_{i-j}P_{j\,\bar{p}}+\delta^1_{i-j}P_{j\,\bar{\rho}}+K_{i-j}P_{j\,\theta}), \quad (20)$$

$$p_t=\sum_{i=1}^{\infty}\gamma^i\sum_{j=0}^{i}(B_{i-j}P_{j\,\bar{k}}+D_{i-j}P_{j\,\bar{p}}-P_{i-j}P_{j\,\theta}). \quad (21)$$

In Eqs. (18)–(21) we set $K_0=\bar{k}$, $P_0=\bar{p}$, $A_0=B_0=C_0=D_0=0$, the indices $\bar{k}$, $\bar{p}$, $\bar{\rho}$, θ mean hereinafter the derivatives with respect to the corresponding variables, and δ^i_j is the Kroneker symbol.

Now we substitute expansions (12), (13), (18), and (21) into Eq. (9) and equate coefficients of the corresponding powers of γ. In zeroth order this immediately gives

$$\bar{k}^2-\bar{p}^2=1. \quad (22)$$

In the next orders we arrive at the equation

$$(\partial/\partial\theta+2\varphi_{\theta\theta}/\varphi_\theta)y_i=A_i+D_i+h_i \quad (i=1,2,\dots), \quad (23)$$

where, by definition,

$$y_i(\bar{k},\bar{p},\bar{\rho},\theta)=\frac{1}{2}\sum_{j=0}^{i}(P_{i-j}P_j-K_{i-j}K_j), \tag{24}$$

$$\begin{aligned} h_i(\bar{k},\bar{p},\bar{\rho},\theta)=&\sum_{j=1}^{i-1}(A_{i-j}K_{j\bar{k}}+B_{i-j}P_{j\bar{k}}+C_{i-j}K_{j\bar{p}}\\ &+D_{i-j}P_{j\bar{p}})\\ &+K_{i-1\,\bar{\rho}}+(n-1)K_{i-1}/\bar{\rho}-\delta_i^1. \end{aligned} \tag{25}$$

Note that in Eq. (25) the summation is up to and including $j=i-1$, so that h_i involves the quantities determined from the lower-order equations. Thus we have

$$y_1=\bar{p}P_1-\bar{k}K_1, \tag{26}$$

$$h_1=(n-1)\bar{k}/\bar{\rho}-1, \tag{27}$$

$$y_2=\bar{p}P_2-\bar{k}K_2+(P_1^2-K_1^2)/2, \tag{28}$$

$$\begin{aligned} h_2=&A_1K_{1\bar{k}}+B_1P_{1\bar{k}}+C_1K_{1\bar{p}}+D_1P_{1\bar{p}}+K_{1\bar{\rho}}\\ &+(n-1)K_1/\bar{\rho},\ \dots \end{aligned} \tag{29}$$

Consider Eq. (23). The corresponding homogeneous equation has the solution φ_θ^{-2}. This function grows exponentially when $|\theta|\to\infty$. We require that a solution of Eq. (23) does not contain the exponentially growing terms. Such solution is

$$y_i=\frac{1}{\varphi_\theta^2}\int_{-\infty}^{\theta}(A_i+D_i+h_i)\varphi_\theta^2 d\theta \tag{30}$$

provided that the orthogonality condition

$$\int_{-\infty}^{\infty}(A_i+D_i+h_i)\varphi_\theta^2 d\theta=0 \tag{31}$$

is fulfilled.

The expansions (12)–(17) must be supplemented with compatibility conditions. The first one follows from Eqs. (10) and (11) and reads $k_t+p_r=0$. Substituting here the expressions (19) and (20) we arrive at the equation

$$(\bar{p}K_i-\bar{k}P_i)_\theta=B_i+C_i+q_i \quad (i=1,2,\dots), \tag{32}$$

where the quantities

$$\begin{aligned} q_i&(\bar{k},\bar{p},\bar{\rho},\theta)\\ &=\sum_{j=1}^{i-1}(A_{i-j}P_{j\bar{k}}+B_{i-j}K_{j\bar{k}}+C_{i-j}P_{j\bar{p}}+D_{i-j}K_{j\bar{p}}\\ &\quad+\delta_{i-j}^1P_{j\bar{\rho}}+K_{i-j}P_{j\,\theta}-P_{i-j}K_{j\,\theta}) \end{aligned} \tag{33}$$

are determined from the lower-order equations. Thus,

$$(\bar{p}K_1-\bar{k}P_1)_\theta=B_1+C_1, \tag{34}$$

$$\begin{aligned} (\bar{p}K_2-\bar{k}P_2)_\theta=&B_2+C_2+A_1P_{1\bar{k}}+B_1K_{1\bar{k}}+C_1P_{1\bar{p}}\\ &+D_1K_{1\bar{p}}+P_{1\bar{\rho}}+K_1P_{1\,\theta}-P_1K_{1\,\theta},\ \dots \end{aligned} \tag{35}$$

The next two compatibility conditions, $\bar{k}_{rt}=\bar{k}_{tr}$ and $\bar{p}_{rt}=\bar{p}_{tr}$, place restrictions on the functions A_i, B_i, C_i and D_i in Eqs. (14)–(17). In view of the constraint (22) we can take into account only one of the two conditions, e.g., the first one. This gives

$$\sum_{j=1}^{i-1}(A_{i-j\,\bar{k}}B_j+A_{i-j\,\bar{p}}D_j-B_{i-j\,\bar{k}}A_j-B_{i-j\,\bar{p}}C_j)=B_{i-1\,\bar{\rho}}$$

$$(i=2,3,\dots). \tag{36}$$

On the other hand, differentiation of Eq. (22) with the use of Eqs. (14)–(17) leads to the relations

$$\bar{k}A_i=\bar{p}C_i, \quad \bar{k}B_i=\bar{p}D_i. \tag{37}$$

Hence, introducing the functions

$$\chi_i(\bar{k},\bar{p},\bar{\rho})=A_i+D_i, \quad \psi_i(\bar{k},\bar{p},\bar{\rho})=B_i+C_i, \tag{38}$$

we can write

$$\begin{aligned} A_i&=\bar{p}(\bar{k}\psi_i-\bar{p}\chi_i), \quad B_i=-\bar{p}(\bar{p}\psi_i-\bar{k}\chi_i),\\ C_i&=\bar{k}(\bar{k}\psi_i-\bar{p}\chi_i), \quad D_i=-\bar{k}(\bar{p}\psi_i-\bar{k}\chi_i). \end{aligned} \tag{39}$$

Substituting these expressions into Eq. (36) and using (22) we obtain

$$\begin{aligned} \sum_{j=1}^{i-1}&[\bar{k}(\psi_{i-j\,\bar{p}}\chi_j-\psi_{i-j}\chi_{j\,\bar{p}})+\bar{p}(\psi_{i-j\,\bar{k}}\chi_j-\psi_{i-j}\chi_{j\,\bar{k}})\\ &+\psi_{i-j}\psi_j-\chi_{i-j}\chi_j]\\ &=\bar{k}\chi_{i-1\,\bar{\rho}}-\bar{p}\psi_{i-1\,\bar{\rho}} \quad (i=2,3,\dots). \end{aligned} \tag{40}$$

Note that, in view of (31),

$$\chi_i=-\int_{-\infty}^{\infty}h_i\varphi_\theta^2 d\theta \Big/ \int_{-\infty}^{\infty}\varphi_\theta^2 d\theta. \tag{41}$$

At given χ_i the condition (40) can be treated as a system of differential constraints for the functions ψ_i. It can be resolved, in particular, by the choice $\psi_i=(\bar{k}/\bar{p})\chi_i$.

C. The second-order approximation

In principle, the above procedure permits us to find the expansions (12)–(17) in any order in γ. Here we will obtain the first- and the second-order terms of the expansions.

Since h_1 does not depend on θ [see Eq. (27)], from (41) one finds

$$\chi_1 = -h_1 = 1-(n-1)\overline{k}/\overline{\rho}. \tag{42}$$

This implies that $y_1=0$, i.e., $\overline{k}K_1=\overline{p}P_1$ [see Eqs. (30) and (26)]. With this relation Eq. (34) becomes $K_{1\,\theta}=-\overline{p}\psi_1$. Thus, we obtain

$$K_1=-\overline{p}(\xi_1+\psi_1\theta), \quad P_1=-\overline{k}(\xi_1+\psi_1\theta), \tag{43}$$

where $\xi_1(\overline{k},\overline{p},\overline{\rho})$ is an arbitrary function.

Now let us calculate h_2, χ_2, and y_2. The formulas (29), (39), (43) result in

$$h_2=-M-N\theta, \tag{44}$$

where

$$M(\overline{k},\overline{p},\overline{\rho})=\psi_1\xi_1+\chi_1(\overline{k}\xi_{1\,\overline{p}}+\overline{p}\xi_{1\,\overline{k}})+\overline{p}\left(\xi_{1\,\overline{\rho}}+\frac{n-1}{\overline{\rho}}\xi_1\right), \tag{45}$$

$$N(\overline{k},\overline{p},\overline{\rho})=\psi_1^2+\chi_1(\overline{k}\psi_{1\,\overline{p}}+\overline{p}\psi_{1\,\overline{k}})+\overline{p}\left(\psi_{1\,\overline{\rho}}+\frac{n-1}{\overline{\rho}}\psi_1\right). \tag{46}$$

Expression (46) can be simplified with the help of Eq. (40). Indeed, when $i=2$ Eq. (40) reads

$$\chi_1(\overline{k}\psi_{1\,\overline{p}}+\overline{p}\psi_{1\,\overline{k}})-\overline{p}\psi_1\chi_{1\,\overline{k}}+\psi_1^2-\chi_1^2=\overline{k}\chi_{1\,\overline{\rho}}-\overline{p}\psi_{1\,\overline{\rho}}. \tag{47}$$

Making use of Eqs. (47) and (42) we finally obtain

$$N=1-2\frac{n-1}{\overline{\rho}}\overline{k}+\frac{n(n-1)}{\overline{\rho}^2}\overline{k}^2. \tag{48}$$

Substitution of the expression (44) into (41) gives

$$\chi_2=M+\langle\theta\rangle N, \tag{49}$$

where we introduce the mean value of the phase

$$\langle\theta\rangle=\int_{-\infty}^{\infty}\theta\varphi_\theta^2 d\theta \bigg/ \int_{-\infty}^{\infty}\varphi_\theta^2 d\theta. \tag{50}$$

With (44) and (49) the formula (30) becomes

$$y_2=\frac{N}{\varphi_\theta^2}\int_{-\infty}^{\theta}(\langle\theta\rangle-\theta)\varphi_\theta^2 d\theta. \tag{51}$$

We next calculate q_2, K_2, and P_2. Formula (33) with (39) and (43) results in

$$q_2=-Q-S\theta, \tag{52}$$

where

$$Q(\overline{k},\overline{p},\overline{\rho})=\xi_1\chi_1+\psi_1(\overline{k}\xi_{1\,\overline{p}}+\overline{p}\xi_{1\,\overline{k}})+\overline{k}\xi_{1\,\overline{\rho}}, \tag{53}$$

$$S(\overline{k},\overline{p},\overline{\rho})=\psi_1\chi_1+\psi_1(\overline{k}\psi_{1\,\overline{p}}+\overline{p}\psi_{1\,\overline{k}})+\overline{k}\psi_{1\,\overline{\rho}}. \tag{54}$$

The integration of Eq. (35) then gives

$$\overline{p}K_2-\overline{k}P_2=\xi_2+\psi_2\theta+\int q_2 d\theta, \tag{55}$$

where $\xi_2(\overline{k},\overline{p},\overline{\rho})$ is an arbitrary function. On the other hand, from Eqs. (28) and (43) it follows that

$$\overline{k}K_2-\overline{p}P_2=\frac{1}{2}(\xi_1+\psi_1\theta)^2-y_2, \tag{56}$$

where y_2 is given by Eq. (51). From (55) and (56) we immediately find

$$K_2=\frac{1}{2}\overline{k}\xi_1^2-\overline{p}\xi_2+(\overline{k}\xi_1\psi_1-\overline{p}\psi_2+\overline{p}Q)\theta + \frac{1}{2}(\overline{k}\psi_1^2+\overline{p}S)\theta^2-\overline{k}y_2, \tag{57}$$

$$P_2=\frac{1}{2}\overline{p}\xi_1^2-\overline{k}\xi_2+(\overline{p}\xi_1\psi_1-\overline{k}\psi_2+\overline{k}Q)\theta + \frac{1}{2}(\overline{p}\psi_1^2+\overline{k}S)\theta^2-\overline{p}y_2.$$

We give this result for completeness sake only and do not use it below. Equations (57) become necessary when considering the bubble wall dynamics in the third approximation and up. In the next subsection we will obtain the equation describing the dynamics of the wall in the second approximation.

D. The basic equation

Let us define, in accordance with Eqs. (10) and (11), the velocity field and the averaged velocity field, respectively, as

$$v=p/k, \quad \overline{v}=\overline{p}/\overline{k}. \tag{58}$$

In view of (22), it follows that

$$\overline{k}=(1-\overline{v}^2)^{-1/2}, \quad \overline{p}=\overline{v}(1-\overline{v}^2)^{-1/2}. \tag{59}$$

Then we have

$$\begin{aligned} v=\frac{\overline{p}+\gamma P_1+\gamma^2 P_2+O(\gamma^3)}{\overline{k}+\gamma K_1+\gamma^2 K_2+O(\gamma^3)}&=\overline{v}-\gamma(1-\overline{v}^2)(\overline{p}K_1-\overline{k}P_1) \\ &+\gamma^2(1-\overline{v}^2)^{3/2}[K_1(\overline{p}K_1-\overline{k}P_1)-\overline{k}(\overline{p}K_2-\overline{k}P_2)] \\ &+O(\gamma^3), \end{aligned} \tag{60}$$

where K_1, P_1, and $\overline{p}K_2-\overline{k}P_2$ are determined by Eqs. (43) and (55).

Now let us calculate the total derivative of v. Differentiation of Eq. (60) with the use of Eqs. (10)–(17), (39), and (59) gives

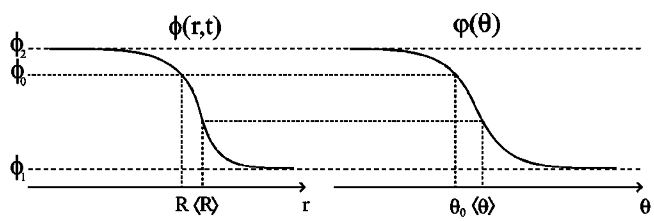

FIG. 2. A point-to-point description of a wall in terms of the phase variable θ.

$$v_t+vv_r=\overline{v_t}+\overline{vv_r}-\gamma^2(1-\overline{v^2})^{3/2}\{\chi_1[\overline{p}(\xi_{1\overline{k}}+\psi_{1\overline{k}}\theta)+\overline{k}(\xi_{1\overline{p}}+\psi_{1\overline{p}}\theta)]+\overline{p}(\xi_{1\overline{\rho}}+\psi_{1\overline{\rho}}\theta)-(\xi_1+\psi_1\theta)\times(3\overline{v}\chi_1-\psi_1)\}+O(\gamma^3), \tag{61}$$

where

$$\overline{v_t}+\overline{vv_r}=(1-\overline{v^2})^{3/2}(\gamma\chi_1+\gamma^2\chi_2)+O(\gamma^3). \tag{62}$$

Note that the functions ξ_2, ψ_2, and q_2 do not appear in this approximation. Next we insert the formulas (42) and (49) for χ_1 and χ_2, and express $\overline{v}$ in terms of v as

$$\overline{v}=v+\gamma(1-v^2)(\xi_1+\psi_1\theta)+O(\gamma^2) \tag{63}$$

[see Eq. (60)]. Using the formulas (45), (46), and (48) one can see that the terms involving ξ_1 and ψ_1 cancel out. As a result, the right-hand side of Eq. (61) is significantly simplified. Replacing $\overline{\rho}$ by γr we arrive at the equation

$$v_t+vv_r=\gamma(1-v^2)^{3/2}-\frac{n-1}{r}(1-v^2)+(\langle\theta\rangle-\theta)(1-v^2)^{1/2}\times\left[\gamma^2(1-v^2)-2\gamma\frac{n-1}{r}(1-v^2)^{1/2}+\frac{n(n-1)}{r^2}\right]. \tag{64}$$

Thus, in the second approximation in γ and $1/r\lesssim\gamma$, we have reduced the system (9)–(11) for the phase θ to the equation of the hydrodynamic type, which, however, still involves θ explicitly. The last step is as follows. Define $R(t)$ as a coordinate of a point on the wall where the field ϕ takes a value ϕ_0, and, with the known $\varphi(\theta)$, define θ_0 as a phase of this point, i.e.,

$$\phi(R(t),t)=\phi_0=\varphi(\theta_0)\quad(\phi_1<\phi_0<\phi_2) \tag{65}$$

(see Fig. 2). Setting $r=R(t)$, $v(R,t)=\dot{R}$, $dv/dt=\ddot{R}$, $\theta=\theta_0=\text{const}$, from (64) we finally obtain [21]

$$\ddot{R}=\gamma(1-\dot{R}^2)^{3/2}-\frac{n-1}{R}(1-\dot{R}^2)+(\langle\theta\rangle-\theta_0)\left[\gamma^2(1-\dot{R}^2)^{3/2}-2\gamma\frac{n-1}{R}(1-\dot{R}^2)+\frac{n(n-1)}{R^2}(1-\dot{R}^2)^{1/2}\right]. \tag{66}$$

This is the basic equation of our approach. It has the unified form for any two-vacuum models (1) with small ΔU: the concrete shape of $U(\phi)$ affects the constants γ, $\langle\theta\rangle$, and θ_0 only. Recall that these constants are determined by Eqs. (6), (50), and (65), in accordance with the solution of the eigenvalue problem (4) and (5) which is assumed to be known. The initial conditions for Eq. (66) follow from (2) and read

$$f(R(0))=\phi_0,\quad \dot{R}(0)=-\left(\frac{g}{f_r}\right)_{r=R(0)}. \tag{67}$$

The first and the second terms in the right-hand side of Eq. (66) are of first order in $\gamma\sim\Delta U$ and $1/R\lesssim\gamma$. They have the same form as those obtained in Refs. [19,20]. The first term is the accelerating force caused by the energy releasing in the course of the phase transition. The second term of the opposite sign describes the action of the surface tension force tending to collapse the bubble. The balance of these two forces gives the value of the effective critical radius of the bubble,

$$R_c=\frac{n-1}{\gamma}, \tag{68}$$

thus measured for a mean point of the bubble wall where $\theta_0=\langle\theta\rangle$.

The expression in the square brackets of Eq. (66) consists of second-order terms and is positive. With the factor $\langle\theta\rangle-\theta_0$ it describes, in particular, the increasing of the slope of the wall in the course of the bubble expansion.

The value ϕ_0, and hence θ_0, can be treated as a Lagrangian label of a point R on the wall. Thus, Eq. (66) gives the complete point-to-point description of both the motion of the wall as a whole and its structure.

III. STATIC SOLUTION: THE STRUCTURE OF THE CRITICAL BUBBLE WALL

The static solution will be immediately derived from Eq. (66) if we just set there $\dot{R}=0$, $\ddot{R}=0$. The result can be written as

$$\langle\theta\rangle-\theta_0=\gamma^{-1}\frac{\dfrac{R}{R_c}\left(1-\dfrac{R}{R_c}\right)}{\left(\dfrac{R}{R_c}\right)^2-2\dfrac{R}{R_c}+\dfrac{n}{n-1}}, \tag{69}$$

where we used the definition (68) of the critical radius R_c. This equation gives the dependence $\theta_0(R)$ valid for $R\gtrsim R_c$. Knowing the solution $\{\varphi(\theta),\gamma\}$ of the eigenvalue problem (4) and (5), we thus obtain the structure of the wall of the critical bubble as follows:

$$\phi_c(r)=\{[\varphi(\theta)]_{\theta=\theta_0(R)}\}_{R=r}. \tag{70}$$

Let us consider two examples. The first one is the ϕ^3-ϕ^4 theory,

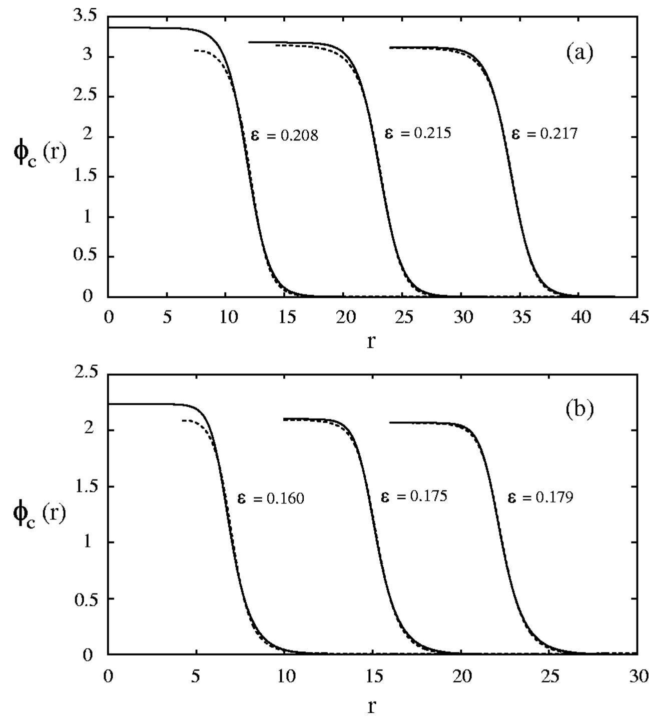

FIG. 3. The wall profiles of the critical spherical ($n=3$) bubbles obtained by our method (dashed lines) and by the numerical integration of Eq. (1) (solid lines). The calculations were performed for the potentials (71) (upper panel) and (75) (lower panel) with different values of ε. It is seen the larger the bubble radius the better our solutions approximate the true ones.

$$U(\phi)=\frac{1}{2}\phi^2-\frac{1}{3}\phi^3+\frac{\varepsilon}{4}\phi^4 \quad \left(0<\varepsilon<\frac{2}{9}\right). \tag{71}$$

The solution of the eigenvalue problem (4) and (5) is

$$\varphi(\theta)=\phi_2\{1+\exp[\phi_2(\varepsilon/2)^{1/2}(\theta-\theta^*)]\}^{-1}, \tag{72}$$

$$\gamma=(8\varepsilon)^{-1/2}[3(1-4\varepsilon)^{1/2}-1] \tag{73}$$

(see, e.g., Ref. [23]), where

$$\phi_2=(2\varepsilon)^{-1}[1+(1-4\varepsilon)^{1/2}], \quad \theta^*=\langle\theta\rangle. \tag{74}$$

Using (73) we first obtain R_c by Eq. (68). Then we take $\theta_0(R)$ from Eq. (69) and insert $\theta=\theta_0(r)$ into (72), in accordance with the formula (70). The resulting profiles for several values of ε are presented in Fig. 3(a).

The second example is the ϕ^4-ϕ^6 theory,

$$U(\phi)=\frac{1}{2}\phi^2-\frac{1}{4}\phi^4+\frac{\varepsilon}{6}\phi^6 \quad \left(0<\varepsilon<\frac{3}{16}\right). \tag{75}$$

In this case the solution of the eigenvalue problem (4) and (5) reads

$$\varphi(\theta)=\phi_2\{1+\exp[\phi_2^2(4\varepsilon/3)^{1/2}(\theta-\theta^*)]\}^{-1/2}, \tag{76}$$

$$\gamma=(3\varepsilon)^{-1/2}[2(1-4\varepsilon)^{1/2}-1], \tag{77}$$

where

$$\phi_2=(2\varepsilon)^{-1/2}[1+(1-4\varepsilon)^{1/2}]^{1/2},$$
$$\theta^*=\langle\theta\rangle-(4\varepsilon/3)^{-1/2}\phi_2^{-2}. \tag{78}$$

The critical radius and the profile of the wall result from Eqs. (68) and (70) in the same way. The comparison graphs are shown in Fig. 3(b). We thus conclude that our description of the critical bubble wall is in a good agreement with the results of the direct numerical integration of Eq. (1).

IV. INTEGRAL OF MOTION AND GENERAL SCHEME OF THE APPROACH

Consider now time-dependent solutions. At first, let us try to find an integral of motion of Eq. (66). The standard procedure is as follows. Assuming $\dot{R}^2=F(R)$, from Eq. (66) one obtains the first-order differential equation for $F(R)$. Its solution involves an arbitrary constant, say E, which is the required integral of motion. In this way we find

$$E=-\gamma(1+\gamma\delta)\frac{R(1-\dot{R}^2)^{1/2}-u_+(\delta)}{R(1-\dot{R}^2)^{1/2}-u_-(\delta)}R^{n\sigma(\delta)}, \tag{79}$$

where

$$u_\pm(\delta)=\frac{n}{2\gamma(1+\gamma\delta)}\left[1+\frac{2(n-1)}{n}\gamma\delta\pm\sigma(\delta)\right], \tag{80}$$

$$\sigma(\delta)=\left[1-\frac{4(n-1)}{n^2}\gamma^2\delta^2\right]^{1/2}, \quad \delta=\langle\theta\rangle-\theta_0. \tag{81}$$

Note that the expression (79) is, in fact, the total energy of the bubble. Indeed, in the first approximation

$$E\approx-\gamma R^n+n(1-\dot{R}^2)^{-1/2}R^{n-1}. \tag{82}$$

It is seen that the first term on the right-hand side of Eq. (82) is the volume energy of the n-dimensional bubble, while the second term is its surface energy.

From Eq. (79) we immediately obtain the velocity squared of a point on the wall having the ‘‘coordinate’’ δ,

$$\dot{R}^2=1-\left\{\frac{Eu_-(\delta)+\gamma(1+\gamma\delta)u_+(\delta)R^{n\sigma(\delta)}}{R[E+\gamma(1+\gamma\delta)R^{n\sigma(\delta)}]}\right\}^2. \tag{83}$$

The solution of this equation is expressed through a quadrature. In one-dimensional case it can be evaluated in explicit form. Indeed, for $n=1$ we have

$$u_\pm(\delta)=\frac{1\pm1}{2\gamma(1+\gamma\delta)}, \quad \sigma=1, \tag{84}$$

so that the integration yields

$$R(t)=R(0)+\{[1+\gamma^2(1+\gamma\delta)^2(t+t_0)^2]^{1/2}$$
$$-[1-\dot{R}^2(0)]^{-1/2}\}/\{\gamma(1+\gamma\delta)\}, \tag{85}$$

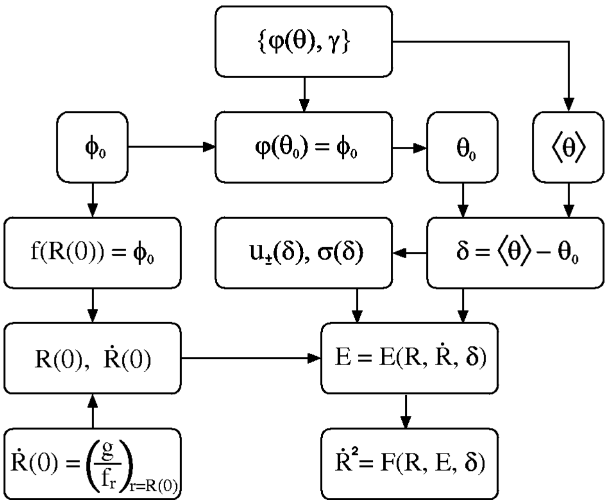

FIG. 4. The procedure of finding the trajectory family $R(t,\theta_0)$.

where

$$t_0 = \frac{\dot{R}(0)}{\gamma(1+\gamma\delta)[1-\dot{R}^2(0)]^{1/2}}. \tag{86}$$

Now we can describe the general scheme of our approach. Suppose we need to solve the initial problem (2). First of all, we should find the solution $\{\varphi(\theta),\gamma\}$ of the eigenvalue problem (4) and (5). Then we fix a point on the wall by the choice of a value ϕ_0 (see Fig. 2). The initial coordinate $R(0)$ and the velocity $\dot{R}(0)$ of the point are found from the formulas (67). The function $\varphi(\theta)$ determines the values of $\langle\theta\rangle$ and θ_0 by Eqs. (50) and (65), respectively. Knowing $\delta=\langle\theta\rangle-\theta_0$ we then calculate the values of $u_{\pm}(\delta)$ and $\sigma(\delta)$ by the formulas (80) and (81), and thus obtain the value E of the integral of motion (79). The trajectory of the point considered is found by the integration of Eq. (83). The described scheme is presented in Fig. 4.

Of course, this procedure should be performed for each point of the wall. As a result, we will have a family of trajectories parametrized by θ_0, i.e., $R=R(t,\theta_0)$. This implies that $\theta_0=\theta_0(R,t)$. Hence, the transition to the Eulerian variables is given by

$$\phi(r,t)=\{[\varphi(\theta)]_{\theta=\theta_0(R,t)}\}_{R=r}. \tag{87}$$

In Fig. 5 the dynamics of the bubble expansion in the ϕ^4-ϕ^6 theory (75) is presented. As an initial state, a perturbed critical bubble was chosen. One can compare the wall profile curves obtained by the direct numerical integration of Eq. (1) with the curves obtained from formula (87) with the use of the solution of Eq. (66) with and without the second-order terms. It is seen that taking into account just first two terms on the right-hand side of Eq. (66) leads to the correct description of the motion for the mean point $\langle R\rangle$ only, i.e., for $\theta_0=\langle\theta\rangle$, while the addition of the second-order terms results in the wall profile, which is in good agreement with the numerical solution of Eq. (1).

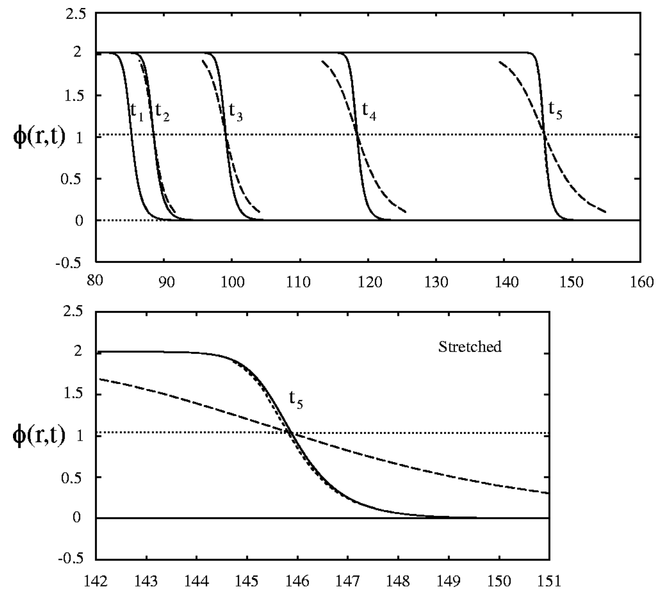

FIG. 5. The expansion dynamics of the initially perturbed critical bubble for the ϕ^4-ϕ^6 potential (75). In calculations we chose $\varepsilon=0.185$, $n=3$ (spherical symmetry). Solid lines represent the numerical solutions of Eq. (1) at $t_1=0$, $t_2=45$, $t_3=90$, $t_4=135$, $t_5=180$. Dashed lines denote the corresponding solutions obtained from Eq. (66) taken without the second-order terms. Short dashed lines result from the solutions of Eq. (66) if the second-order terms are accounted for.

V. CONCLUDING REMARKS

So, we have reduced the problem of solving the nonlinear Klein-Gordon equation (1) to the eigenvalue problem (4) and (5) and to the equation (66) governing the motion of each point of the wall. The solution of the eigenvalue problem can be easily obtained numerically or even analytically. Equation (66) is integrated through the quadrature. The resulting formula (87) describes the dynamics of the bubble expansion. Our approach is applicable to the potentials with small energy difference $\Delta U\sim\gamma$ and is valid for the bubbles of large radii $\langle R\rangle\gtrsim R_c\sim\gamma^{-1}$.

It should be noted, however, that Eq. (66), being a consequence of Eq. (64), has some features that are typical for the hydrodynamic description. Thus our numerical experiments have shown that some initial states can give rise to the intersection of the trajectories $R(t,\theta_0)$ of the different points of the wall. This implies that at some moment the hydrodynamic singularity arises and the wall breaks, acquiring a triple-valued profile. In this case the moment of the breaking limits naturally the validity of our approach in time. It can be shown that in the one-dimensional case [see Eq. (85)] the sufficient condition for the absence of the breaking of the central part of the wall is given by

$$\frac{l_0\Delta U}{\gamma(\Delta\phi)^2}\gtrsim 1, \tag{88}$$

where $\Delta\phi=\phi_2-\phi_1$ and l_0 is the initial value of the wall's thickness l. It means that if the wall is gently sloping no points of the wall lying in the region $|\langle R\rangle-R|<l/2$ will have

finite time to become equal in the coordinate. We examined the condition (88) in many numerical experiments. It turned out that it works not only for plane walls, but for cylindrical and spherical walls as well.

Note that the fulfillment of the condition (88) does not exclude the breaking at the ''tails'' of the wall, where ϕ approaches $\phi_{1,2}$. Moreover, in formally solving Eq. (66) this breaking always arises, independently of the condition (88), before the possible breaking of the central part of the wall. For sufficiently small γ the tail breaking takes place very far from the wall and becomes practically invisible, as in Fig. 5. It is easy to see, however, that the tail breaking is physically meaningless, because it arises beyond the validity domain of our approach. Indeed, in the tail regions $\gamma|\langle\theta\rangle-\theta_0|\gtrsim 1$ (see Fig. 2), so that the second-order terms in Eq. (66) become of the first order, and, hence, this equation is no longer valid. Therefore, the validity of the approach in time is limited solely by central breaking that can arise only if the condition (88) is violated. If the condition (88) is fulfilled, the difference between the exact and approximate solution at the central part of the wall will be small and slowly varying, since the velocities of all points in both solutions level off in tending to unity (see Fig. 5).

In summary, we have presented a point-to-point description of bubble wall dynamics consisting of the solution of the equation of motion for each point of the wall and subsequent reconstruction of the wall profile. As examples, we have found the wall profiles of critical bubbles for the ϕ^3-ϕ^4 and ϕ^4-ϕ^6 potentials. Also, we have considered the expansion dynamics of a perturbed critical bubble. The results have turned out to be in good agreement with numerical solutions of the starting nonlinear Klein-Gordon equation.

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