Casimir force on a piston

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We consider a massless scalar field obeying Dirichlet boundary conditions on the walls of a two-dimensional $L\times b$ rectangular box, divided by a movable partition (piston) into two compartments of dimensions $a\times b$ and $(L-a)\times b$. We compute the Casimir force on the piston in the limit $L\to\infty$. Regardless of the value of a/b, the piston is attracted to the nearest end of the box. Asymptotic expressions for the Casimir force on the piston are derived for $a\ll b$ and $a\gg b$.

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I. INTRODUCTION

In 1948 Casimir predicted a remarkable macroscopic quantum effect: two conducting and neutral parallel plates should attract each other due to the disturbance of the vacuum of the electromagnetic field caused by their presence [1] (for a general review on the Casimir effect, see Ref. [2]). Inspired by that result, a few years later Casimir suggested that the zero-point pressure of the electromagnetic field might yield the stresses postulated by Poincaré in order to explain the stability of the electron [3]. Boyer, however, showed that the Casimir force for a conducting spherical shell is repulsive [4], thus invalidating Casimir's model for the electron.

Boyer's result brought attention to the fact that the attractive or repulsive character of the Casimir force depends on the geometry of the configuration. This has been investigated in detail for fields (scalar or electromagnetic) confined in a d-dimensional rectangular box [5–11]. Let us consider, for instance, a massless scalar field subject to Dirichlet boundary conditions at the walls of the two-dimensional box $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$. The vacuum energy is formally given by ($\hbar=c=1$)

$$E_0(a,b)=\frac{1}{2}\sum_{j,k=1}^{\infty}\omega_{jk}, \quad \omega_{jk}=\sqrt{\left(\frac{j\pi}{a}\right)^2+\left(\frac{k\pi}{b}\right)^2}. \tag{1}$$

One can perform the summation using analytic regularization (AR); the result is (see Appendix A)

$$E_{0,\mathrm{AR}}(a,b)=-\frac{ab}{32\pi}Z_2(a,b;3)+\frac{\pi}{48}\left(\frac{1}{a}+\frac{1}{b}\right), \tag{2}$$

where Z_2 is an Epstein zeta function [12]. An analysis of Eq. (2) shows that the sign of the Casimir tension $T=-\partial E_{0,\mathrm{AR}}/\partial A$ (where $A=ab$ is the area of the box) depends on the ratio b/a: it is positive if $1\leqslant b/a<2.74$ and negative if $b/a>2.74$ [2].

There are, however, at least two reasons for which one should be suspicious of the use of Eq. (2) as the basis for such an analysis. First, it does not take into account the contribution to the vacuum energy from the region outside the box, which, in principle, also depends on its dimensions. (This problem was discussed recently in [13], but the solution proposed there is incomplete.) Second, its finiteness is an artifact of the AR scheme: more than just regularizing integrals or sums, it also does a certain amount of renormalization by automatically subtracting power-law divergences (in this respect, AR is similar to dimensional regularization [14]). This is precisely what happens here. If one regularizes the sum over modes in Eq. (1) with a smooth cutoff function and performs the sum using the Abel-Plana formula, one obtains (see Appendix B)

$$E_{0,\mathrm{cutoff}}(a,b)=C_1(\Lambda)ab+C_2(\Lambda)(a+b)+E_{0,\mathrm{AR}}(a,b), \tag{3}$$

with $C_1(\Lambda)\sim\Lambda^3$ and $C_2(\Lambda)\sim\Lambda^2$ as $\Lambda\to\infty$. (We have discarded terms that vanish in that limit.)

The difference between $E_{0,\mathrm{cutoff}}$ and $E_{0,\mathrm{AR}}$ would be harmless if the first two terms on the right-hand side (rhs) of Eq. (3) could be absorbed into counterterms. Let us forget for a moment the problem of neglecting the exterior modes, and examine this question. The first term has the form $\epsilon_0 ab$, where ϵ_0 is the energy density of the vacuum in the absence of the box. It can be cancelled by a "cosmological constant" counterterm, a constant added to the Hamiltonian density in order to make the vacuum energy in free space equal to zero. The problem lies in the second term: being proportional to the perimeter of the box, it may be interpreted as (part of) the self-energy of its walls. Such a term *cannot* be eliminated by a renormalization of the parameters of the theory [15,16]. (This problem also occurs in the parallel plates configuration. In that case, however, it can be ignored if one is interested only in the force between the plates, for their self-energies do not depend on the distance between them. In the present case, the dismissal of the self-energy of the box walls could be justified if perimeter-preserving deformations are the only ones allowed.)

In this work we shall examine a slightly different system in which both problems can be ignored. Instead of the box discussed above, we shall consider a box of dimensions $L\times b$ divided by a movable partition, or piston, into two compartments, A and B, of dimensions $a\times b$ and $(L-a)\times b$, respectively (see Fig. 1). If one is interested—as we are—in computing the Casimir force *on the piston*, then the contribution to the vacuum energy from the region outside the box

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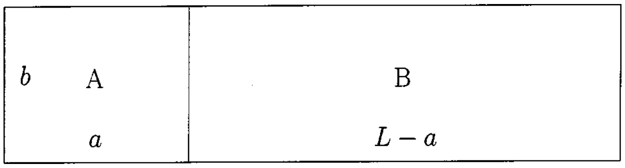

FIG. 1. Two-dimensional $L\times b$ rectangular box. A movable partition (a piston) divides it into two compartments, A and B, of dimensions $a\times b$ and $(L-a)\times b$, respectively. We shall assume that $a,b \ll L\to\infty$.

can be ignored, as it is not affected by the position of the piston. In addition, as will be shown below, the divergent terms in the Casimir energy are naturally eliminated when one computes the force on the piston. We shall compute this force in the limit $L\to\infty$ and show that it pulls the piston to the nearest end of the box regardless of the value of the ratio a/b. We shall also derive asymptotic expressions for the force for $a\ll b$ and $a\gg b$.

II. CASIMIR FORCE

The total energy of the vacuum for the system described in the previous paragraph (and depicted in Fig. 1) can be written as the sum of three terms:

$$E_0 = E_0^{\mathrm{A}} + E_0^{\mathrm{B}} + E_0^{\mathrm{out}}. \tag{4}$$

Using the cutoff regularization discussed in Appendix B, the first two terms are given by $E_0^{\mathrm{A}} = E_{0,\mathrm{cutoff}}(a,b)$ and $E_0^{\mathrm{B}} = E_{0,\mathrm{cutoff}}(L-a,b)$ [see Eq. (3)], so Eq. (4) becomes

$$\begin{aligned} E_0 = {} & E_{0,\mathrm{AR}}(a,b) + E_{0,\mathrm{AR}}(L-a,b) + C_1(\Lambda)Lb \\ & + C_2(\Lambda)(L+2b) + E_0^{\mathrm{out}}. \end{aligned} \tag{5}$$

The Casimir force on the piston is given by $-\partial E_0/\partial a$. Since the last three terms on the rhs of Eq. (5) do not depend on the position of the piston, we obtain the following result for the Casimir force on it:

$$F = -\frac{\partial}{\partial a}[E_{0,\mathrm{AR}}(a,b) + E_{0,\mathrm{AR}}(L-a,b)]. \tag{6}$$

As anticipated, although the total vacuum energy contains divergent terms and a term (E_0^{out}) that one does not know how to compute, the Casimir force on the piston is finite and can be computed exactly.

The result one obtains for the force inserting (2) into Eq. (6) is not very illuminating, so, before we actually compute F, let us derive an alternative expression for $E_{0,\mathrm{AR}}(a,b)$. In order to do that, it is convenient to define the auxiliary function

$$S(m,a;s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=-\infty}^{\infty}\left[\left(\frac{m}{\pi}\right)^2 + \left(\frac{n}{a}\right)^2\right]^{-s/2} \qquad [\mathrm{Re}(s)>1]. \tag{7}$$

Its analytic continuation to the complex s plane (with simple poles at $s=1,-1,-3,\ldots$) is given by [7]

$$S(m,a;s) = \frac{a m^{1-s}}{\pi^{(1-s)/2}}\left[\Gamma\left(\frac{s-1}{2}\right) + 4\sum_{n=1}^{\infty}\frac{K_{(1-s)/2}(2nma)}{(nma)^{(1-s)/2}}\right], \tag{8}$$

where $K_\nu(z)$ is the modified Bessel function. Equations (7) and (8) allow us to reexpress the Epstein zeta function that appears in Eq. (2) as [17]

$$\begin{aligned} Z_2(a,b;3) &= \sum_{j,k=-\infty}^{\infty}{}' (j^2a^2+k^2b^2)^{-3/2} \\ &= \sum_{j=-\infty}^{\infty}{}' \sum_{k=-\infty}^{\infty} (j^2a^2+k^2b^2)^{-3/2} + \sum_{k=-\infty}^{\infty}{}' (k^2b^2)^{-3/2} \\ &= \frac{2\pi^{3/2}}{\Gamma(3/2)}\sum_{j=1}^{\infty} S(\pi j a, 1/b;3) + \frac{2\zeta(3)}{b^3} \\ &= \frac{2\pi^2}{3a^2b} + \frac{16\pi}{ab^2}\sum_{j,k=1}^{\infty}\frac{k}{j}K_1\left(2\pi jk\frac{a}{b}\right) + \frac{2\zeta(3)}{b^3}. \end{aligned} \tag{9}$$

Inserting this result into Eq. (2) yields

$$E_{0,\mathrm{AR}}(a,b) = \frac{\pi}{48b} - \frac{\zeta(3)a}{16\pi b^2} - \frac{1}{2b}\sum_{j,k=1}^{\infty}\frac{k}{j}K_1\left(2\pi jk\frac{a}{b}\right). \tag{10}$$

Inserting Eq. (10) and the corresponding expression for $E_{0,\mathrm{AR}}(L-a,b)$ into Eq. (6) and taking the limit $L\to\infty$ we obtain the following expression for the Casimir force on the piston:

$$\lim_{L\to\infty} F = \frac{\pi}{b^2}\sum_{j,k=1}^{\infty} k^2 K_1'\left(2\pi jk\frac{a}{b}\right), \tag{11}$$

where $K_1'(x) = dK_1(x)/dx$. Since $K_1(x)$ is a monotonic decreasing function of x, it follows from Eq. (11) that $F<0$ for all (positive) values of a/b; in other words, the piston is attracted to the nearest end of the cavity.

It is easy to obtain an asymptotic expression for F valid for $a\gg b$: since $K_1(x)\sim\sqrt{\pi/2x}\exp(-x)$ for large x, one may retain only the term with $j=k=1$ in Eq. (11), thus obtaining

$$F \sim -\frac{\pi}{2}(ab^3)^{-1/2}\exp\left(-\frac{2\pi a}{b}\right) \quad (a\gg b). \tag{12}$$

This result has the same form as the asymptotic expression of the Casimir force between two plates in *one dimension* in the case of a scalar field with mass $m=\pi/b$ [7]. This fact has a simple physical interpretation: when $a\gg b$ the system becomes quasi-one-dimensional, with the field acquiring an effective mass equal to the energy gap $\Delta=\pi/b$ due to the confinement in the transverse direction.

In order to obtain an approximation to F valid for $a \ll b$, we note that $E_{0,\mathrm{AR}}(a,b)=E_{0,\mathrm{AR}}(b,a)$, so that we can replace Eq. (10) by

$$E_{0,\mathrm{AR}}(a,b)=\frac{\pi}{48a}-\frac{\zeta(3)b}{16\pi a^2}-\frac{1}{2a}\sum_{j,k=1}^{\infty}\frac{k}{j}K_1\left(2\pi jk\frac{b}{a}\right). \tag{13}$$

If, on the other hand, we still express $E_{0,\mathrm{AR}}(L-a,b)$ in Eq. (6) according to Eq. (10), we obtain an alternative expression for the force on the piston (in the limit $L\to\infty$):

$$F=-\frac{\zeta(3)b}{8\pi a^3}+\frac{\pi}{48a^2}-\frac{\zeta(3)}{16\pi b^2}+\frac{\pi b}{a^3}\sum_{j,k=1}^{\infty}k^2K_0\left(2\pi jk\frac{b}{a}\right). \tag{14}$$

The last term in Eq. (14) is exponentially suppressed when $a \ll b$, so in this case we have

$$F\sim-\frac{\zeta(3)b}{8\pi a^3}+\frac{\pi}{48a^2}-\frac{\zeta(3)}{16\pi b^2}\quad(a\ll b). \tag{15}$$

If one divides both sides of Eq. (15) by b, the first term on its rhs correctly reproduces the Casimir tension between two infinite parallel lines a distance a apart [7]. The other two terms are subdominant for $a \ll b$, and yield finite size corrections to that result.

III. CONCLUSION

We argued in this work that the knowledge of the vacuum energy *inside* a rectangular cavity is not enough for one to calculate the Casimir force on its faces. Two ingredients are missing in such a calculation: the knowledge of the contribution to the vacuum energy from the region outside the cavity, and the proper handling of divergent terms in the regularized expression of the vacuum energy. We then considered a slightly different type of cavity, namely, a rectangular box divided by a piston into two rectangular compartments. In this case, if one is interested only in the Casimir force on the piston, those ingredients *can* be neglected. In addition, the force-on-the-piston problem has two attractive features: (i) it is a simple generalization of the single-cavity problem, for which results are already available in the literature [5–11], and (ii) from the experimental point of view, it is simpler to construct a cavity with a piston than a variable-size rectangular cavity. Results for the electromagnetic field in a three-dimensional rectangular cavity with a piston will be presented elsewhere.

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APPENDIX A

Let us evaluate the divergent sum over modes in Eq. (1) using analytic regularization. We start with the function

$$\mathcal{E}(a,b;s):=\frac{\pi}{2}\sum_{j,k=1}^{\infty}\left[\left(\frac{j}{a}\right)^2+\left(\frac{k}{b}\right)^2\right]^{-s/2}, \tag{A1}$$

which is defined for $\mathrm{Re}(s)>2$. As we shall see, its analytic continuation to the complex s plane is well defined at $s=-1$, so we can define the analytically regularized Casimir energy as $E_{0,\mathrm{AR}}(a,b)=\mathcal{E}(a,b;-1)$.

In order to obtain the analytic continuation of $\mathcal{E}(a,b;s)$ it is convenient to rewrite Eq. (A1) as [17]

$$\begin{aligned}\mathcal{E}(a,b;s)&=\frac{\pi}{8}\sum_{j=-\infty}^{\infty}{}'\sum_{k=-\infty}^{\infty}{}'\left[\left(\frac{j}{a}\right)^2+\left(\frac{k}{b}\right)^2\right]^{-s/2}\\&=\frac{\pi}{8}\left\{\sum_{j,k=-\infty}^{\infty}{}'\left[\left(\frac{j}{a}\right)^2+\left(\frac{k}{b}\right)^2\right]^{-s/2}\right.\\&\qquad\left.-\sum_{j=-\infty}^{\infty}{}'\left(\frac{|j|}{a}\right)^{-s}-\sum_{k=-\infty}^{\infty}{}'\left(\frac{|k|}{b}\right)^{-s}\right\}\\&=\frac{\pi}{8}Z_2\left(\frac{1}{a},\frac{1}{b};s\right)-\frac{\pi}{4}\zeta(s)(a^s+b^s),\end{aligned} \tag{A2}$$

where $Z_p(a_1,\dots,a_p;s)$ and $\zeta(s)$ denote the Epstein and Riemann zeta functions, respectively. Applying the reflection formulas [7]

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\pi^{(s-1)/2}\zeta(1-s), \tag{A3}$$

$$\begin{aligned}a_1\cdots a_p\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}Z_p(a_1,\dots,a_p;s)\\=\Gamma\left(\frac{p-s}{2}\right)\pi^{(s-p)/2}Z_p(1/a_1,\dots,1/a_p;p-s)\end{aligned} \tag{A4}$$

to Eq. (A2) and taking $s=-1$ we obtain Eq. (2).

APPENDIX B

In this appendix we derive Eq. (3) using the Abel-Plana summation formula [18],

$$\begin{aligned}\sum_{n=0}^{\infty}F(n)&=\frac{1}{2}F(0)+\int_0^{\infty}F(t)dt\\&\quad+i\lim_{\varepsilon\to0^+}\int_0^{\infty}\frac{F(\varepsilon+it)-F(\varepsilon-it)}{e^{2\pi t}-1}dt.\end{aligned} \tag{B1}$$

$F(z)$ is an analytic function in the right half-plane, going to zero sufficiently fast as $|z|\to\infty$, $|\arg(z)|<\pi/2$.

In order to apply the Abel-Plana formula to the series (1), we have to introduce a smooth cutoff function $D_\Lambda(z,w)$:

$$E_{0,\mathrm{cutoff}}(a,b)=\frac{\pi}{2}\sum_{j,k=1}^{\infty}\left(\frac{j^2}{a^2}+\frac{k^2}{b^2}\right)^{1/2}D_\Lambda\left(\frac{j}{a},\frac{k}{b}\right)$$
$$=:\frac{\pi}{2}\sum_{j=1}^{\infty}S_j. \tag{B2}$$

The function $D_\Lambda(z,w)$ must satisfy the following conditions in the region $\mathrm{Re}(z),\mathrm{Re}(w)\geqslant 0$: (i) it is analytic in both variables; (ii) it is real for z and w real; (iii) it vanishes sufficiently fast for $|z|,|w|\to\infty$ (so that the regularized series is absolutely convergent); (iv) it is symmetric, i.e., $D_\Lambda(z,w)=D_\Lambda(w,z)$, and (v) $\lim_{\Lambda\to\infty}D_\Lambda(z,w)=1$. An example of such a function is given by $D_\Lambda(z,w)=d_\Lambda(z)d_\Lambda(w)$, with $d_\Lambda(z)=[1+(z+1)^2/\Lambda^2]^{-2}$.

Applying formula (B1) to the series S_j in (B2), we can rewrite each of them as a sum of three terms, namely,

$$S_j^{(1)}=-\frac{j}{2a}D_\Lambda\left(\frac{j}{a},0\right), \tag{B3}$$

$$S_j^{(2)}=\int_0^\infty du\left(\frac{j^2}{a^2}+\frac{u^2}{b^2}\right)^{1/2}D_\Lambda\left(\frac{j}{a},\frac{u}{b}\right), \tag{B4}$$

$$S_j^{(3)}=i\lim_{\varepsilon\to 0^+}\int_0^\infty\frac{du}{e^{2\pi u}-1}\left[\left(\frac{j^2}{a^2}-\frac{u^2}{b^2}+i\varepsilon\right)^{1/2}\right.$$
$$\left.\times D_\Lambda\left(\frac{j}{a},\frac{\varepsilon+iu}{b}\right)-\text{c.c.}\right]. \tag{B5}$$

Applying (B1) to $\Sigma_j S_j^{(1)}$ yields

$$\sum_{j=1}^{\infty}S_j^{(1)}=\int_0^\infty dv\left[-\frac{v}{2a}D_\Lambda\left(\frac{v}{a},0\right)\right]$$
$$+\frac{1}{a}\int_0^\infty\frac{v\,dv}{e^{2\pi v}-1}\left[D_\Lambda\left(\frac{iv}{a},0\right)+\text{c.c.}\right]. \tag{B6}$$

Changing the variable of integration in the first integral to $t=v/a$ and taking the limit $\Lambda\to\infty$ in the second one, we obtain

$$\sum_{j=1}^{\infty}S_j^{(1)}=-\frac{a}{2}\int_0^\infty dt\,tD_\Lambda(t,0)+\frac{1}{24a}. \tag{B7}$$

Similarly, application of Eq. (B1) to $\Sigma_j S_j^{(2)}$ yields

$$\sum_{j=1}^{\infty}S_j^{(2)}=-\frac{b}{2}\int_0^\infty dt\,tD_\Lambda(0,t)+ab\int_0^\infty du\int_0^\infty dv\,(u^2+v^2)^{1/2}$$
$$\times D_\Lambda(u,v)+I_\Lambda(a,b), \tag{B8}$$

where

$$I_\Lambda(a,b)=iab\lim_{\varepsilon\to 0^+}\int_0^\infty du\int_0^\infty\frac{dv}{e^{2\pi av}-1}[(u^2-v^2+i\varepsilon)^{1/2}$$
$$\times D_\Lambda(\varepsilon+iv,u)-\text{c.c.}]. \tag{B9}$$

Taking the limit $\Lambda\to\infty$ in the integral above we obtain

$$\lim_{\Lambda\to\infty}I_\Lambda(a,b)=-2ab\int_0^\infty du\int_u^\infty\frac{dv}{e^{2\pi av}-1}(v^2-u^2)^{1/2}$$
$$=-2ab\int_0^\infty\frac{dv}{e^{2\pi av}-1}\int_0^v du(v^2-u^2)^{1/2}$$
$$=-\frac{\pi ab}{2}\int_0^\infty\frac{v^2dv}{e^{2\pi av}-1}=-\frac{\zeta(3)b}{8\pi^2a^2}. \tag{B10}$$

Finally, let us take the limit $\Lambda\to\infty$ in Eq. (B5). Changing the variable of integration u to $t=au/jb$, we obtain

$$\lim_{\Lambda\to\infty}S_j^{(3)}=-\frac{2j^2b}{a^2}\int_1^\infty\frac{dt}{e^{2\pi jbt/a}-1}(t^2-1)^{1/2}$$
$$=-\frac{2j^2b}{a^2}\sum_{k=1}^{\infty}\int_1^\infty dt(t^2-1)^{1/2}e^{-2\pi kjbt/a}$$
$$=-\frac{1}{\pi a}\sum_{k=1}^{\infty}\frac{j}{k}K_1\left(2\pi kj\frac{b}{a}\right). \tag{B11}$$

Collecting all pieces together we obtain Eq. (3), with $E_{0,\mathrm{AR}}(a,b)$ in the form given by Eq. (13) and

$$C_1(\Lambda)=\frac{\pi}{2}\int_0^\infty du\int_0^\infty dv(u^2+v^2)^{1/2}D_\Lambda(u,v), \tag{B12}$$

$$C_2(\Lambda)=-\pi\int_0^\infty dt\,tD_\Lambda(t,0). \tag{B13}$$

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[12] The Epstein zeta function $Z_p(a_1,\dots,a_p;s)$ is defined as $Z_p(a_1,\dots,a_p;s)$

$$:= \sum_{n_1,\dots,n_p=-\infty}^{\infty}{}' [(n_1 a_1)^2+\cdots+(n_p a_p)^2]^{-s/2}.$$

The meaning of the primed sum is explained in [17].

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