Monopoles in the Higgs phase

David Tong*

Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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We describe new solutions of Yang-Mills-Higgs theories consisting of magnetic monopoles in a phase with fully broken gauge symmetry. Rather than spreading out radially, the magnetic field lines form flux tubes. The solution is topologically stable and, when embedded in $\mathcal{N}=2$ SQCD, preserves 1/4 of the supercharges. From the perspective of the flux tube the monopole appears as a kink. Many monopoles may be threaded onto a single flux tube and placed at arbitrary separation to create a stable, BPS necklace of solitons.

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Should we ever be lucky enough to find a magnetic monopole, one might consider displaying it in the Natural History Museum embedded within a superconductor. The magnetic flux lines would not spread out radially, but instead have the peculiar property of forming a flux tube. Adjoining interactive displays could describe this delightful consequence of the Meissner effect while waxing lyrical about an analogous mechanism in QCD which is responsible for holding us all together. Nearby, the holographic image of a celebrity physicist might explain how similar strings are conjectured to underlie the very fabric of our Universe.

In this paper we shall describe smooth, topologically stable, magnetic monopole solutions with the property described above. Recall that in QED, monopoles are Dirac-like singular affairs, essentially put into the theory by hand. To find smooth solutions, we must turn to $SU(2)$ Yang-Mills theories. When the gauge group is broken to $U(1)$ by an adjoint scalar field, 't Hooft and Polyakov showed that topological considerations guarantee the existence of monopoles [1]. However the theory is in the Coulomb phase and the magnetic flux lines spread out radially. Suppose we attempt to naively break the gauge symmetry further so that the $U(1)$ is also broken at low energies by the Higgs mechanism. The magnetic field lines must now form flux tubes at large distances, but the price we have paid is to lose the topological stability of the configuration which remains, at best, metastable. An exception to this is if $U(1)\to Z_2$ which can be achieved by a second adjoint scalar field. In this case Z_2 strings are supported and the resulting stable monopole-flux tube configuration was discussed by Hindmarsh and Kibble [2]. Monopoles attached to Z_N strings have also been discussed in [4].

Here we shall discuss a slightly different symmetry breaking structure, involving a locking of gauge and flavor symmetries, which supports both $U(1)$ flux tubes of the familiar Nielsen-Olesen form [3] and magnetic monopoles in the manner of 't Hooft–Polyakov [1]. We work with an $\mathcal{N}=2$ supersymmetric theory in $d=3+1$ dimensions with a $U(N)_G$ vector multiplet and $N_f=N$ fundamental hypermultiplets[1] with an $SU(N)_F$ flavor symmetry. The full symmetry group is[2]

$$G=U(N)_G\times SU(N)_F .$$

The bosonic field content of the theory is as follows: the vector multiplet contains a $U(N)_G$ gauge field A_μ, together with a complex adjoint scalar field ϕ; the hypermultiplets contain scalars q_i, $i=1,\dots,N_F$, each of which transforms in the $\mathbf{N}$ representation of $U(N)_G$, and a further N_f scalars $\tilde{q}_i$ transforming in the $\overline{\mathbf{N}}$. The bosonic part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}=&\,\mathrm{Tr}\left(\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2e^2}|\mathcal{D}_\mu\phi|^2\right)+\sum_{i=1}^{N_f}(|D_\mu q_i|^2+|D_\mu\tilde{q}_i|^2)\\&-\mathrm{Tr}\left(\frac{1}{2e^2}[\phi^\dagger,\phi]^2+e^2\left|\sum_{i=1}^{N_f}q_i\tilde{q}_i\right|^2\right.\\&\left.+\frac{e^2}{2}\left(\sum_{i=1}^{N_f}q_iq_i^\dagger-\tilde{q}_i^\dagger\tilde{q}_i-v^2\right)^2\right)\\&-\sum_{i=1}^{N_f}(q_i^\dagger|\phi-m_i|^2q_i+\tilde{q}_i|\phi-m_i|^2\tilde{q}_i^\dagger).\end{aligned}$$

In the above expression we have introduced complex mass parameters m_i and a real FI parameter v^2, each consistent with $\mathcal{N}=2$ supersymmetry. For generic values of these parameters the theory has a unique vacuum state, up to Weyl permutations, given by

$$\phi=\mathrm{diag}(m_i),\quad q_i^a=v\,\delta_i^a,\quad \tilde{q}_i^a=0, \tag{1}$$

[1]This is a minimal choice: the solutions we describe exist for any $N_f\geqslant N$.

[2]The classical theory has a further $SU(2)_R\times U(1)_R$ R-symmetry group, but this will not be responsible for stabilizing any soliton solutions and we shall pay it less attention.

*Email address: dtong@mit.edu

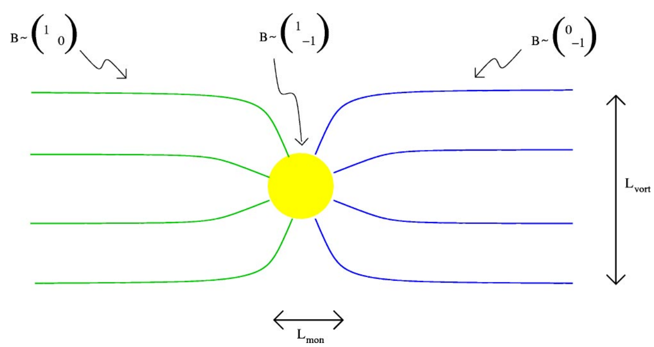

FIG. 1. An impressionistic rendering of the $U(2)$ monopole in the Higgs phase when $L_{\rm vort} \gg L_{\rm mon}$.

where $a=1,\ldots,N$ is the color index. The $U(N)_G$ gauge symmetry is completely broken and the theory lies in a gapped, color-flavor-locked phased.

The pattern of symmetry breaking at intermediate energy scales depends on the relative values of m_i and v^2. For $|m_i - m_j| \gg ev$, the flavor group is explicitly broken by the masses at a higher scale than the spontaneous symmetry breaking induced by the FI parameter,

$$U(N)_G \times SU(N)_F \xrightarrow{m} U(1)_G^N \times U(1)_F^{N-1} \xrightarrow{v} U(1)_{\rm diag}^{N-1}. \quad (2)$$

However, if $ev \gg |m_i - m_j|$, then the spontaneous breaking due to the vacuum expectation value of q occurs at a higher scale than the explicit breaking due to masses,

$$U(N)_G \times SU(N)_F \xrightarrow{v} SU(N)_{\rm diag} \xrightarrow{m} U(1)_{\rm diag}^{N-1}. \quad (3)$$

For both patterns (2) and (3) the symmetry breaking due to the masses supports magnetic monopoles $[\Pi_2(SU(N)/U(1)^{N-1}) = \mathbf{Z}^{N-1}]$ while the symmetry breaking due to the FI parameter breaks a $U(1)$ factor, ensuring the stability of vortices $[\Pi_1(U(1)) = \mathbf{Z}]$. Thus the topology suggests the existence both monopoles and fluxes. We shall now see that indeed the theory admits magnetic monopoles attached to two vortex strings which whisk away their flux (see Fig. 1).

The solutions will turn out not to involve the fields $\tilde{q}$ and we set them to zero at this stage. Moreover, the simplest configurations have $\mathrm{Im}(m_i)=0$ which allows us to also set $\mathrm{Im}(\phi)=0$. In the following ϕ will therefore denote a real adjoint scalar field.[3] Since the flux will leave the monopole in a tube, we must decide in which direction this string will head: we choose the x^3 direction. Restricting to time independent configurations the Hamiltonian reads

$$\begin{aligned}\mathcal{H} &= \frac{1}{2e^2}B_\rho^2 + \frac{1}{2e^2}|\mathcal{D}_\rho \phi|^2 + |\mathcal{D}_\rho q_i|^2 + \frac{e^2}{2}(q_i q_i^\dagger - v^2)^2 \\ &\quad + q_i^\dagger(\phi - m_i)^2 q_i \\ &= \frac{1}{2e^2}(\mathcal{D}_1\phi - B_1)^2 + \frac{1}{2e^2}(\mathcal{D}_2\phi - B_2)^2 \\ &\quad + (\mathcal{D}_3\phi - B_3 - e^2(q_i q_i^\dagger - v^2))^2 + |\mathcal{D}_1 q_i - i\mathcal{D}_2 q_i|^2 \\ &\quad + |\mathcal{D}_3 q_i + (\phi - m_i)q_i|^2 - v^2 B_3 + \frac{1}{e^2}\partial_\rho(\phi B_\rho) \\ &\geq -v^2 B_3 + \frac{1}{e^2}\partial_\rho(\phi B_\rho),\end{aligned} \quad (4)$$

where we have left color indices and traces implicit, summed over the flavor index i, and introduced the spatial index $\rho = 1,2,3$. Both terms in the final line are topological invariants. The first measures the flux carried by vortex strings lying in the x^3 direction; the second measures the magnetic charge carried by a monopole. As we shall see, we can have strings without any need for monopoles, but the presence of a monopole will require two, semi-infinite vortex strings to carry away its flux. In the Coulomb phase, the integral of $\partial \cdot (\phi B)$ is evaluated on the $\mathbf{S}^2_\infty$ boundary. In the present case the monopole's flux does not make it to all points on the boundary and is instead captured by integrals over the two planes $\mathbf{R}^2_\infty$ at $x^3 = \pm\infty$. The Bogomoln'yi equations can be found within the total squares on the second line of Eq. (4) and read

$$B_1 = \mathcal{D}_1\phi, \quad B_2 = \mathcal{D}_2\phi, \quad B_3 = \mathcal{D}_3\phi + e^2\left(\sum_{i=1}^N q_i q_i^\dagger - v^2\right),$$

$$\mathcal{D}_1 q_i = i\mathcal{D}_2 q_i, \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i. \quad (5)$$

A quick glance reveals these to be interesting mix of the monopole and vortex equations. I have not been able to find an explicit solution. Indeed, since no analytic solution exists to the Nielsen-Olesen vortex equations, it seems rather unlikely that the task is simpler in these generalized equations. Nevertheless, we can gain insight into the form of the solution by studying the equation in two different limits.

Let us start by considering the limit $|m_i - m_j| \gg ev$. The equations in the second line of Eq. (5) can be solved simply by $q_i = 0$, while, if we ignore the effect of v^2 for now, the equations in the top line become the familiar Bogomoln'yi equations $B_\rho \approx \mathcal{D}_\rho \phi$ describing a monopole with a non-Abelian core of width $L_{\rm mon} \sim 1/|m_i - m_j|$. For distances $L \gg L_{\rm mon}$, the magnetic field lies primarily within the Cartan subalgebra $U(1)_G^{N-1} \subset SU(N)_G \subset U(N)_G$ and emerges radially from the monopole core. However, this radial behavior cannot continue indefinitely. At scales $L_{vort} \sim 1/ev \gg L_{\rm mon}$, the effect of the Higgs mechanism becomes apparent, damping the magnetic field as can be seen from the third of the Bogomoln'yi equations in Eq. (5). At this point, it becomes

[3]It seems likely that interesting dyonic monopole-flux tube configurations can be built by relaxing this condition to allow $\mathrm{Im}(m_i) \neq 0$.

energetically favorable to set the scalar fields ϕ and q_i to their vacuum expectation values (1) in order to allow the magnetic field B to vanish throughout the bulk. However, the magnetic flux from the monopole has to go somewhere. To see where, note that when ϕ is set to its constant expectation value and A_ρ is restricted to lie in the Cartan subalgebra then the nontrivial equations of Eq. (5) read

$$B_3=e^2\left(\sum_{i=1}^{N} q_i q_i^\dagger - v^2\right), \quad \mathcal{D}_1 q_i = i\mathcal{D}_2 q_i, \tag{6}$$

which are the non-Abelian form of the familiar Abelian vortex equations. They describe a tube of magnetic flux of width $L_{\rm vort}$ lying in the x^3 direction. The string has finite tension $2\pi v^2$, and therefore infinite mass due to its infinite length. This reflects the fact that, like quarks in QCD, monopoles in the Higgs phase do not like to be alone.

For a Dirac monopole in QED, the flux string is expected to depart in only one direction. When this happens, the tension of the string causes the monopole to accelerate and the configuration is unstable. However, for the superconductor example of the opening paragraph this situation is avoided as the Cooper pair condensate has charge 2 which allows for the formation of strings carrying a half quantum of flux [5]. Thus the flux from the monopole may be carried away by two flux tubes of equal tension, leaving in opposite directions. Here we shall see that the solution to Eq. (5) has a similar property where each flux tube now carries a single quantum of flux lying in a different $U(1)\subset U(N)_G$ subgroup. To see this, we turn to the opposite limit $ev \gg |m_i - m_j|$ where the width of the vortex $L_{\rm vort}$ is much smaller than the width of the monopole core $L_{\rm mon}$. There is now no spatial region in which the monopole looks like the usual 't Hooft–Polyakov radial configuration. However, we can make progress by studying the monopole from the perspective of the vortex string. In fact, let us start by considering the situation $m_i=0$, so that the symmetry breaking is simply $G\rightarrow SU(N)_{\rm diag}$. The theory now supports vortex strings, but not monopoles. The vortices satisfy Eqs. (6) and were studied recently in [6] (related systems were examined even more recently in [7]). For a single vortex of unit winding number $({\rm Tr}\int d^2x B_3 = -2\pi)$, it was shown that the surviving $SU(N)_{\rm diag}$ group acts on the soliton resulting in a moduli space $\mathcal{V}_N$ of solutions,

$$\mathcal{V}_N \cong \mathbf{C}\times \mathbf{CP}^{N-1},$$

where $\mathbf{C}$ parametrizes the center of mass of the vortex string in the x^1-x^2 plane, while $\mathbf{CP}^{N-1}$ describes the internal degrees of freedom of the vortex arising from the $SU(N)_{\rm diag}$ action. The Kähler class of $\mathbf{CP}^{N-1}$ is $2\pi/e^2$ [6]. The low-energy dynamics of the vortex string can be described by a $d=(1+1)$-dimensional sigma model with target space $\mathcal{V}_N$. Since the vortex is BPS [9], the low-energy dynamics preserves $\mathcal{N}=(2,2)$ supersymmetry.

How is this picture changed by the introduction of masses m_i? The masses break the $SU(N)_{\rm diag}$ symmetry in the pattern (3), lifting the $\mathbf{CP}^{N-1}$ moduli space. For a vortex of unit winding number there are now N isolated solutions corresponding to an Abelian vortex embedded diagonally in one of the $U(1)\subset U(N)_G$ subgroups. These different solutions are related by discrete $SU(N)_{\rm diag}$ transformations but, as this includes the action of a global symmetry group, are physically distinguishable configurations. From the perspective of the low-energy dynamics, the masses m_i can be thought of as inducing a potential V on $\mathbf{CP}^{N-1}$ with N isolated minima. In fact, the exact form of the potential V can be determined using the techniques described in [8] and is of the form $V\sim k^2$, where k is a Killing vector on $\mathbf{CP}^{N-1}$.

Rather than enter into the details of the $U(N)_G$ theory, here we simply concentrate on the case of $U(2)_G$ gauge group for which the internal vortex moduli space is $\mathbf{CP}^1$. We parametrize $\mathbf{CP}^1$ by a circle fibration over an interval, with $\psi\in[0,2\pi)$ labeling the circle, and $-\pi/e^2\leq r\leq \pi/e^2$ labeling the interval. The circle degenerates at $r=\pm\pi/e^2$ to yield the topology of the sphere. Writing the two mass parameters as $(m_1,m_2)=(m,-m)$, the low-energy internal dynamics of the vortex string is governed by a $d=(1+1)$-dimensional massive sigma model with $\mathbf{CP}^1$ target space,

$$\mathcal{L}_{\rm vort}=\frac{1}{2}H(r)(\partial r)^2+\frac{1}{2}H^{-1}(r)(\partial\psi)^2-2m^2H^{-1}(r),$$

where

$$H(r)=\frac{1}{\pi/e^2+r}+\frac{1}{\pi/e^2-r}.$$

The kinetic terms are those of a sigma model on $\mathbf{CP}^1$ endowed with the round metric, while the potential term is proportional to the length^2 of the ∂_ψ Killing vector on $\mathbf{CP}^1$. As we described above, the masses have lifted the moduli space of vortices, leaving behind two isolated configurations at the minima of the potential $r=\pm\pi/e^2$. These correspond to vortices carrying magnetic flux $B_3\sim{\rm diag}(0,1)$ and $B_3\sim{\rm diag}(1,0)$ respectively. From the perspective of the vortex theory, the existence of two different configurations gives rise to the possibility of a new object: a kink. Such a string would start at $r=-\pi/e^2$ at $x^3\rightarrow-\infty$ and conclude at $r=+\pi/e^2$ as $x^3\rightarrow+\infty$. In fact domain walls in massive $\mathbf{CP}^N$ sigma models of this type have been much studied in the literature, starting in [10]. The solution is simply

$$r=\frac{\pi}{e^2}\tanh(m(x^3-x_0)), \quad \psi={\rm const},$$

where x_0 is the center of mass of the kink along the string. From the perspective of the $d=3+1$ gauge theory, this kink on the vortex world sheet is simply the monopole described by Eqs. (5). To see this, first note that the mass of the domain wall is $4\pi m/e^2$, in agreement with the mass of the monopole calculated from the final term in Eq. (4). Secondly, we can examine the fluxes carried by the vortex string. As $x^3\rightarrow-\infty$, the $U(2)$ magnetic field lies in $B_3\sim{\rm diag}(1,0)$, while for $x^3\rightarrow+\infty$, the magnetic field lies in $B_3\sim{\rm diag}(0,1)$. Taking into account the direction of the flux, we see that the domain wall acts as a magnetic source of the form $B\sim{\rm diag}(1,-1)$. This is precisely the flux emitted by the monopole. Note that the vortex preserves half the original $\mathcal{N}=2$ supersymmetry [9] and the domain wall preserves half the supersymmetry of the vortex theory [10]. The monopole-flux-tube-combo is therefore a 1/4-BPS state in $\mathcal{N}=2$ SQCD. An impressionistic, and not entirely accurate, portrait of the magnetic flux lines is offered in the figure.

It is interesting to note that the original fascination with domain walls in $\mathbf{CP}^1$ sigma-models derived from the observation that they exhibit features reminiscent of magnetic monopoles [10]. Here we provide a simple explanation for this fact: the domain walls *are* magnetic monopoles. The monopoles in question lie in the Higgs phase, and are therefore restricted to sit on a string of flux wherein they appear as domain walls.

For $U(N)_G$ gauge group, the situation is similar. There are now N vacua of the low-energy vortex dynamics, and one can consider domain walls interpolating from the first vacuum $[B\sim\mathrm{diag}(1,0,\ldots,0)]$ to the last $[B\sim\mathrm{diag}(0,\ldots,0,1)]$. Such domain walls were studied in detail in [11]. It was shown that the kinks can be placed at arbitrary separation without experiencing attractive or repulsive forces. From the perspective of monopoles, this corresponds to the fact that [so called $(1,1,\ldots,1)$] monopoles can be threaded on a flux tube and placed arbitrary separation. They may slide along the string at will and are constrained only in that they may not pass each other. This results in a BPS necklace of monopoles, acting like hard beads threaded on a vortex flux tube.

Let us close by recalling two other areas of physics where solutions similar to those discussed above appear. The first sits on a tabletop: the A phase of superfluid ^{3}He supports configurations analogous to a monopole emitting one (or more) vortex strings [5]. This composite object is referred to as a nexus. In the case of ^{3}He, the strings are supported by a global symmetry but similar configurations with gauged vortices are argued to appear in chiral p-wave superconductors [5]. The second application is in the context of cosmology. Configurations of the type discussed here have been invoked as a way to catalyze monopole-anti-monopole annihilation. This could be of interest either in the early Universe to rid us of GUT monopoles [12], or in the current epoch where necklaces of monopoles have been suggested as a source for ultrahigh-energy cosmic rays [13]. It is to be hoped that the existence of the simple Bogomoln'yi equations (5) may be of help in determining the dynamics of solitonic necklaces.

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