

Metric perturbation approach to gravitational waves in isotropic cosmologies

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Gravitational waves in isotropic cosmologies were recently studied using the gauge-invariant approach of Ellis and Bruni. We now construct the linearized metric perturbations of the background Robertson-Walker space-time which reproduce the results obtained in that study. The analysis carried out here also facilitates an easy comparison with Bardeen.

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I. INTRODUCTION

In a recent paper [1] the gauge-invariant and covariant approach of Ellis and Bruni [2] was used to examine shear-free gravitational waves propagating through isotropic cosmologies. In this approach the waves are modeled as small perturbations of the Robertson-Walker space-time. The presence of the waves is found to perturb the shear and also more notably to introduce anisotropic stress into the Universe. Other basic gauge-invariant quantities, for example, the vorticity and energy flow, remain unchanged by the presence of gravitational radiation.

Our purpose here is to construct the metric perturbations of the Robertson-Walker space-time which give rise to the perturbations of the anisotropic stress and shear found in [1]. The difficulty is that we wish to derive gauge-invariant perturbations and there is no way *a priori* to identify which terms in the perturbed metric are pure gauge terms without carrying out a lengthy calculation. In the process of studying the perturbed metric we identify the gauge terms and without loss of generality we then put these terms equal to zero.

The paper is organized as follows: In Sec. II we introduce the notation used and give some important equations. The unperturbed Robertson-Walker space-time is described in Sec. III. In Sec. IV we summarize the results of the gauge-invariant and covariant study of gravitational radiation carried out in [1]. The perturbed metric is introduced in Sec. V. Also in this section and in Sec. VI we demonstrate how the perturbed metric leads to the required gauge-invariant perturbations of the shear and anisotropic stress. The Ricci tensor components of the metric are listed in Appendix A and in Appendix B we briefly outline the calculation involved in identifying those variables that are responsible for the presence of gauge terms. The paper ends with a discussion in which our results are compared with those of Bardeen [3].

II. NOTATION AND BASIC EQUATIONS

Throughout this paper we use the notation and sign conventions of [4]. We are concerned with a four-dimensional space-time manifold with metric tensor components g_{ab} in a local coordinate system $\{x^a\}$ and a preferred congruence of world lines tangent to a time-like vector field with compo-

nents u^a and $u^a u_a = -1$. With respect to this 4-velocity field the symmetric energy-momentum-stress tensor T^{ab} can be decomposed as

$$T^{ab} = \mu u^a u^b + p h^{ab} + q^a u^b + q^b u^a + \pi^{ab}, \quad (2.1)$$

where

$$h^{ab} = g^{ab} + u^a u^b \quad (2.2)$$

is the projection tensor and

$$q^a u_a = 0, \quad \pi^{ab} u_a = 0, \quad \pi^a_a = 0, \quad (2.3)$$

with $\pi^{ab} = \pi^{ba}$. Here μ is the matter energy density measured by the observer with 4-velocity u^a , p is the isotropic pressure, q^a is the energy flow relative to u^a (for example, heat flow) and π^{ab} is the trace-free anisotropic stress (due to processes such as viscosity).

We indicate covariant differentiation with a semicolon, partial differentiation by a comma and covariant differentiation in the direction of u^a by a dot. Also, as usual square brackets denote skew symmetrization, round brackets denote symmetrization and a definition is indicated by a colon followed by an equality sign. Thus the 4-acceleration of the time-like congruence is

$$\dot{u}^a = u^a_{;b} u^b, \quad (2.4)$$

and $u_{a;b}$ can be decomposed into

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta h_{ab} - \dot{u}_a u_b, \quad (2.5)$$

where

$$\omega_{ab} := u_{[a;b]} + \dot{u}_{[a} u_{b]} \quad (2.6)$$

is the vorticity tensor of the congruence,

$$\sigma_{ab} := u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{1}{3} \theta h_{ab} \quad (2.7)$$

is the shear tensor of the congruence and

$$\theta := u^a_{;a} \quad (2.8)$$

is the expansion (or contraction) of the congruence.

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We shall make use of the Ricci identities

$$u_{a;dc} - u_{a;cd} = R_{abcd}u^b, \quad (2.9)$$

where R_{abcd} is the Riemann curvature tensor but for the problem at hand the key equations are Einstein's field equations

$$R_{ab} - \frac{1}{2}g_{ab}R = T_{ab}. \quad (2.10)$$

Here $R_{ab} := R^c{}_{abc}$ are the components of the Ricci tensor, $R := R^c{}_c$ is the Ricci scalar and we have absorbed the coupling constant into the energy-momentum-stress tensor. Noting that $R = -T$ ($:= T^a{}_a$) and using Eq. (2.1) the field equations can be decomposed into

$$\begin{aligned} R_{ab}u^a u^b &= \frac{1}{2}(\mu + 3p), \\ R_{ab}u^a h^b{}_c &= -q_c, \\ R_{ab}h^a{}_c h^b{}_d &= \frac{1}{2}(\mu - p)h_{cd} + \pi_{cd}. \end{aligned} \quad (2.11)$$

It is in this form that we shall use Eq. (2.10) in later sections.

III. THE BACKGROUND SPACE-TIME

We choose as the unperturbed (background) space-time a Robertson-Walker space-time with line element

$$ds^2 = R^2(t) \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\left(1 + \frac{k}{4}r^2\right)^2} - dt^2, \quad (3.1)$$

where $R(t)$ is the scale factor, $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $k = 0, \pm 1$ is the Gaussian curvature of the space-like hypersurfaces $t = \text{const}$. The world lines of the fluid particles are the integral curves of the vector field $u^a \partial/\partial x^a = \partial/\partial t$ (thus $u^a = \delta^a_4$ since we shall label the coordinates $x^1 = y, x^2 = z, x^3 = x, x^4 = t$). The background energy-momentum-stress tensor is Eq. (2.1) specialized to a perfect fluid (by setting $q^a = 0 = \pi^{ab}$) with proper density

$$\mu = 3\frac{\dot{R}^2}{R^2} + 3\frac{k}{R^2}, \quad (3.2)$$

and isotropic pressure

$$p = -\frac{\dot{R}^2}{R^2} - 2\frac{\ddot{R}}{R} - \frac{k}{R^2}. \quad (3.3)$$

We find it convenient to put the line element given above in the following forms:

$$ds^2 = R^2(t)\{dx^2 + p_0^{-2}f^2(dy^2 + dz^2)\} - dt^2, \quad (3.4)$$

with $p_0 = 1 + (K/4)(y^2 + z^2)$, $K = \text{const}$, $f = f(x)$. We identify three distinct cases.

Case 1. We have

if $k = +1$ then $K = +1$ and $f(x) = \sin x$.

Noting that the transformation $x \rightarrow \pi/2 - x$ does not affect the *form* of the line element (3.4) we see that in this case $f(x)$ could equivalently be written $f(x) = \cos x$.

Case 2. We have

$$\text{if } k = 0 \text{ then } \begin{cases} K = 0 & \text{and } f(x) = 1, \\ & \text{or} \\ K = +1 & \text{and } f(x) = x. \end{cases}$$

Case 3. We have

$$\text{if } k = -1 \text{ then } \begin{cases} K = -1 & \text{and } f(x) = \cosh x, \\ & \text{or} \\ K = 0 & \text{and } f(x) = \frac{1}{2}e^x, \\ & \text{or} \\ K = +1 & \text{and } f(x) = \sinh x. \end{cases}$$

The form of the line-element (3.4) is also invariant under the transformation $x \rightarrow -x$ so when $K = 0$ in case 3 we could instead write $f(x) = \frac{1}{2}e^{-x}$. For a detailed explanation why these cases arise see, for example, Eqs. (5.3)–(5.19) in [1]. In space-times with line-elements (3.4) the hypersurfaces

$$\phi(x^a) := x - T(t) = \text{const}, \quad (3.5)$$

where $dT/dt = R^{-1}$ are null hypersurfaces. The expansion of the null geodesic generators of these surfaces is

$$\frac{1}{2}\phi^a{}_{;a} = \frac{f'}{R^2 f} + \frac{\dot{R}}{R^2}, \quad (3.6)$$

where $f' = df/dx$, $\dot{R} = dR/dt$. Using Eq. (3.5) we can show that

$$2\phi_{,a;b} = \xi_a \phi_{,b} + \xi_b \phi_{,a} + \phi_{,d}{}^{;d} g_{ab}, \quad (3.7)$$

where

$$\xi_a = -\frac{f'}{f}\phi_{,a} + R\phi_{,d}{}^{;d}u_a. \quad (3.8)$$

It follows from Eq. (3.7) that $\phi_{,a}$ is shear-free [5].

Finally in this section we note that for convenience we have used the same coordinate labels $\{y, z, x, t\}$ for all the special cases included in Eq. (3.4). Clearly the ranges of some of these coordinates will vary from case to case and within cases 2 and 3. For example, in case 2 $x \in (-\infty, +\infty)$ if $K = 0$ but $x \in [0, +\infty)$ and is a radial polar coordinate if $K = +1$. The shear-free null hypersurfaces (3.5) will also be different in the different cases. This can be seen by examining the intersections of these null hypersurfaces with the space-like hypersurfaces $t = \text{const}$.

Case 1. The intersection is a 2-sphere.

Case 2. If $K = +1$ the intersection is a 2-sphere and if $K = 0$ the intersection is a 2-plane. Thus it is obvious that Eq. (3.5) describes two different families of shear-free null hypersurfaces that can occur in an open, spatially flat universe.

Case 3. In this case the intersection of Eq. (3.5) with the $t = \text{const}$ hypersurfaces is always a 2-space of constant curvature. The curvature of this 2-space is given by K which takes values $0, \pm 1$. So we have three different families of shear-free null hypersurfaces in a $k = -1$ universe. We refer the reader to [6] for a geometrical explanation for the existence of these subcases.

IV. GAUGE-INVARIANT AND COVARIANT APPROACH TO GRAVITATIONAL WAVES

In a recent paper [1] we used the gauge-invariant and covariant approach of Ellis and Bruni [2] to construct gravitational wave perturbations of the Robertson-Walker spacetimes described in the previous section. This involves working in a general local coordinate system with gauge-invariant small quantities which by their nature vanish in the background, rather than small perturbations of the background metric. For isotropic space-times the Ellis and Bruni variables are σ_{ab} , \dot{u}^a , ω_{ab} , $X_a = h_a^b \mu_{,b}$, $Y_a = h_a^b p_{,b}$, $Z_a = h_a^b \theta_{,b}$, π_{ab} , q_a and the ‘‘electric’’ and ‘‘magnetic’’ parts of the Weyl tensor, with components C_{abcd} , given, respectively, by

$$E_{ab} = C_{apbq} u^p u^q, \quad H_{ab} = {}^* C_{apbq} u^p u^q. \quad (4.1)$$

Here ${}^* C_{apbq} = \frac{1}{2} \eta_{ap}{}^{rs} C_{rsbq}$ is the dual of the Weyl tensor (the left and right duals being equal), $\eta_{abcd} = \sqrt{-g} \epsilon_{abcd}$ where $g = \det(g_{ab})$ and ϵ_{abcd} is the Levi-Civita permutation symbol. However, we found that it is tensor quantities that describe gravitational wave perturbations. Thus for this problem the important Ellis and Bruni variables are σ_{ab} , π_{ab} , E_{ab} , H_{ab} and we can set all other gauge-invariant variables equal to zero. The equations satisfied by these variables are obtained by projections in the direction u^a and orthogonal to u^a of the Ricci identities, the equations of motion and the energy conservation equation contained in $T^{ab}{}_{;b} = 0$ and the Bianchi identities written in the form

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{g} g^{c[a} R^{b]}. \quad (4.2)$$

To keep this section to a reasonable length we shall not list all of the equations (they are given in Eqs. (2.14)–(2.25) in [1]). We note here that from the projections of the Ricci identities (after setting $\dot{u}^a = 0 = \omega_{ab}$) we find

$$E_{ab} = \frac{1}{2} \pi_{ab} + \frac{2}{3} \sigma^2 h_{ab} - \frac{2}{3} \theta \sigma_{ab} - \sigma_{af} \sigma^f{}_b - h_a^f h_b^g \dot{\sigma}_{fg}, \quad (4.3)$$

and

$$H_{ab} = -h_a^f h_b^g \sigma_{(f}{}^{g;c} \eta_{s)fgc} u^f. \quad (4.4)$$

Thus these variables are derived from π_{ab} and σ_{ab} .

We now assume that the perturbed shear and anisotropic stress have the following form:

$$\sigma_{ab} = s_{ab} F(\phi), \quad \pi_{ab} = \Pi_{ab} F(\phi), \quad (4.5)$$

where F is an arbitrary real-valued function of its argument $\phi(x^a)$. We emphasize that at this point $\phi(x^a)$ is arbitrary and not that defined in Eq. (3.5). This idea of introducing arbitrary functions into solutions of Einstein’s equations describing gravitational waves goes back to work by Trautman [7] and the above form for the gauge-invariant variables was introduced by Hogan and Ellis [8]. Substituting (4.5) into the linearized versions of the equations satisfied by these variables and noting that s_{ab} and Π_{ab} are trace-free and orthogonal to u_a with respect to the background metric we find that [1]

$$g^{ab} \phi_{,a} \phi_{,b} = 0, \quad s^{ab} \phi_{,b} = 0, \quad \Pi^{ab} \phi_{,b} = 0, \quad (4.6)$$

with g_{ab} here the background metric, and

$$s^{ab}{}_{|b} = 0, \quad \Pi^{ab}{}_{|b} = 0, \quad (4.7)$$

where for clarity we have used a stroke to denote covariant differentiation with respect to the background metric. We also discover (see [1]) the following wave equation for s_{ab} :

$$\begin{aligned} s^{ab|d}{}_{|d} - \frac{2}{3} \theta \dot{s}^{ab} - \left(\frac{1}{3} \dot{\theta} + \frac{4}{9} \theta^2 \right) s^{ab} + \left(p - \frac{1}{3} \mu \right) s^{ab} \\ = -\dot{\Pi}^{ab} - \frac{2}{3} \theta \Pi^{ab}, \end{aligned} \quad (4.8)$$

and a propagation equation for s_{ab} along the null geodesics tangent to $\phi^{,d}$, namely,

$$s'_{ib} + \left(\frac{1}{2} \phi^{,d}{}_{|d} - \frac{1}{3} \theta \dot{\phi} \right) s_{ib} = -\frac{1}{2} \dot{\phi} \Pi_{ib}, \quad (4.9)$$

where $s'_{ib} := s_{ib|d} \phi^{,d}$ and $\dot{\phi} = \phi_{,a} u^a$. The internal consistencies of these equations were checked in [1]. The ‘‘electric’’ and ‘‘magnetic’’ parts of the Weyl tensor are now given by [1]

$$E_{ab} = \left(\frac{1}{2} \Pi_{ab} - \dot{s}_{ab} - \frac{2}{3} \theta s_{ab} \right) F - \dot{\phi} s_{ab} F', \quad (4.10)$$

and

$$H_{ab} = -s_{(a}{}^{p|c} \eta_{b)fpc} u^f F - s_{(a}{}^p \eta_{b)fpc} u^f \phi^{,c} F', \quad (4.11)$$

where $F' = \partial F / \partial \phi$. These equations are easily checked by substituting (4.5) into (4.3) and (4.4).

We wish to construct pure gravitational wave perturbations, i.e. having pure type N perturbed Weyl tensor in the Petrov classification. It is shown in [1] that on account of (4.6) the F' parts of E_{ab} and H_{ab} above are type N with degenerate principal null direction $\phi^{,a}$. Then if we also require the F parts of E_{ab} and H_{ab} to be type N the perturbations we have constructed describe pure gravitational waves with propagation direction $\phi^{,a}$ in the Robertson-Walker

background and the histories of the wave fronts are the null hypersurfaces $\phi(x^a)=\text{const}$. Making use of the following null tetrad, $k_a = -\dot{\phi}^{-1}\phi_{,a}$, $l_a = u_a - \frac{1}{2}k_a$ and m_a, \bar{m}_a a complex covariant vector field and its complex conjugate, chosen so they are null ($m^a m_a = 0 = \bar{m}^a \bar{m}_a$), are orthogonal to k^a and l^a and satisfy $m^a \bar{m}_a = 1$. We find that a simple way to ensure that the F parts of E_{ab} and H_{ab} are type N is to require the null hypersurfaces $\phi(x^a)=\text{const}$ to satisfy (see [1])

$$\phi_{,b|c}\bar{m}^b l^c = 0, \quad (4.12)$$

and

$$\phi_{,a|b}m^a m^b = 0. \quad (4.13)$$

To exhibit explicit examples we specialize to the case $\phi = x - T(t)$ with $T(t)$ introduced in Eq. (3.5). Then the null tetrad described above is given by the 1-forms

$$k_a dx^a = R dx - dt, \quad l_a dx^a = -\frac{1}{2}(R dx + dt),$$

$$m_a dx^a = \frac{1}{\sqrt{2}} R p_0^{-1} f(dy + iz), \quad (4.14)$$

and it is straightforward to check that Eqs. (4.12) and (4.13) are satisfied. Since s^{ab} and Π^{ab} are trace-free and orthogonal to u^a and ϕ^a , they each have only two independent components. These components are $s^{22} = -s^{11} = \hat{\alpha}(y, z, x, t)$, $s^{12} = s^{21} = \hat{\beta}(y, z, x, t)$ and $\Pi^{22} = -\Pi^{11} = A(y, z, x, t)$, $\Pi^{12} = \Pi^{21} = B(y, z, x, t)$ where we have labeled the coordinates $x^1 = y, x^2 = z, x^3 = x, x^4 = t$. Now we can write

$$s^{ab} = \bar{s} m^a m^b + s \bar{m}^a \bar{m}^b, \quad (4.15)$$

with

$$\bar{s} = -R^2 p_0^{-2} f^2(\hat{\alpha} + i\hat{\beta}), \quad (4.16)$$

and

$$\Pi^{ab} = \bar{\Pi} m^a m^b + \Pi \bar{m}^a \bar{m}^b, \quad (4.17)$$

with

$$\bar{\Pi} = -R^2 p_0^{-2} f^2(A + iB). \quad (4.18)$$

It follows from (4.7) that $\hat{\alpha}$, $\hat{\beta}$ and A, B must satisfy the Cauchy-Riemann equations

$$\frac{\partial}{\partial y}(p_0^{-4}\hat{\alpha}) - \frac{\partial}{\partial z}(p_0^{-4}\hat{\beta}) = 0, \quad (4.19)$$

$$\frac{\partial}{\partial y}(p_0^{-4}\hat{\beta}) + \frac{\partial}{\partial z}(p_0^{-4}\hat{\alpha}) = 0 \quad (4.20)$$

and

$$\frac{\partial}{\partial y}(p_0^{-4}A) - \frac{\partial}{\partial z}(p_0^{-4}B) = 0, \quad (4.21)$$

$$\frac{\partial}{\partial y}(p_0^{-4}B) + \frac{\partial}{\partial z}(p_0^{-4}A) = 0. \quad (4.22)$$

If we define $\mathcal{G} = p_0^{-4} f^3 R^3 (\hat{\alpha} + i\hat{\beta})$ and note that $f = f(x)$, $R = R(t)$ and $\hat{\alpha}$, $\hat{\beta}$ satisfy Eqs. (4.19) and (4.20) then \mathcal{G} is an analytic function of $\zeta := y + iz$. We can now rewrite Eq. (4.16) as

$$\bar{s} = -R^{-1} p_0^2 f^{-1} \mathcal{G}(\zeta, x, t). \quad (4.23)$$

From the propagation equation (4.9) we find

$$\bar{\Pi} = -2R^{-2} p_0^2 f^{-1} (D\mathcal{G} + \dot{R}\mathcal{G}), \quad (4.24)$$

where D is given by $D = \partial/\partial x + R\partial/\partial t = \partial/\partial x + \partial/\partial T$ and the dot indicates differentiation with respect to t . As a consequence of this and (4.18) $A + iB$ is analytic in ζ and so Eqs. (4.21) and (4.22) are automatically satisfied. Replacing s^{ab} by Eqs. (4.15) and (4.23) and Π^{ab} by Eqs. (4.17) and (4.24) the wave equation (4.8) simplifies to

$$D^2 \mathcal{G} + k\mathcal{G} = 0, \quad (4.25)$$

with $k = 0, \pm 1$ labelling the Robertson-Walker backgrounds with line elements of the form (3.4). The solutions of these three differential equations are as follows: for $k = 0$,

$$\mathcal{G}(\zeta, x, t) = a(\zeta, x - T)(x + T) + b(\zeta, x - T), \quad (4.26)$$

for $k = +1$,

$$\mathcal{G}(\zeta, x, t) = a(\zeta, x - T) \sin\left(\frac{x + T}{2}\right) + b(\zeta, x - T) \cos\left(\frac{x + T}{2}\right), \quad (4.27)$$

and for $k = -1$,

$$\mathcal{G}(\zeta, x, t) = a(\zeta, x - T) \sinh\left(\frac{x + T}{2}\right) + b(\zeta, x - T) \cosh\left(\frac{x + T}{2}\right), \quad (4.28)$$

where in each case $a(\zeta, x - T)$, $b(\zeta, x - T)$ are arbitrary functions. Using the identity $x + T = 2x - (x - T)$ (and some simple trigonometric and hyperbolic relations), we can rewrite Eq. (4.26) in the form

$$\mathcal{G}(\zeta, x, t) = h_1(\zeta, x - T) + x h_2(\zeta, x - T), \quad (4.29)$$

with h_1, h_2 arbitrary, Eq. (4.27) as

$$\mathcal{G}(\zeta, x, t) = h_3(\zeta, x - T) \sin x + h_4(\zeta, x - T) \cos x, \quad (4.30)$$

with h_3, h_4 arbitrary and Eq. (4.28) as

$$\mathcal{G}(\zeta, x, t) = h_5(\zeta, x - T) \sinh x + h_6(\zeta, x - T) \cosh x, \quad (4.31)$$

with h_5, h_6 arbitrary. In addition Eq. (4.31) can be set in the form

$$\mathcal{G}(\zeta, x, t) = h_7(\zeta, x - T)e^x + h_8(\zeta, x - T)e^{-x}. \quad (4.32)$$

When these results are derived from metric perturbations in Sec. V the expressions (4.29)–(4.32) will be more useful for comparison purposes than the equivalent expressions (4.26)–(4.28).

The “electric” and “magnetic” parts of the Weyl tensor [Eqs. (4.10) and (4.11), respectively] are now calculated and we find that they can be written compactly as [1]

$$E^{ab} + iH^{ab} = -2R^{-2}p_0^2 f^{-1} \frac{\partial}{\partial x} (\mathcal{G}F) m^a m^b. \quad (4.33)$$

Here \mathcal{G} is given by Eqs. (4.26)–(4.28) [or equivalently Eqs. (4.29)–(4.32)] and $F = F(x - T)$ so that $F' = \partial F / \partial x$, $p_0 = 1 + (K/4)(y^2 + z^2)$, $f = f(x)$ described in the previous section and $R(t)$ is the scale factor. It follows from Eqs. (4.5), (4.15), (4.17), (4.23) and (4.24) that to find σ_{ab} , π_{ab} from s_{ab} and Π_{ab} we simply replace \mathcal{G} by $\mathcal{G}F$. This does not affect Eqs. (4.24) and (4.25) since $DF = 0$. Conversely with $F = F(x - T)$ and \mathcal{G} given by Eqs. (4.26)–(4.28) F can be absorbed into \mathcal{G} .

V. THE PERTURBED METRIC

We now exhibit a line element which (i) can be viewed as a perturbation of the space-time line-element (3.4) and (ii) produces the same explicit perturbations described in the gauge-invariant formalism of the previous section. We first introduce a pair of null coordinates,

$$u = \frac{1}{\sqrt{2}}[x - T(t)], \quad v = \frac{1}{\sqrt{2}}[x + T(t)], \quad (5.1)$$

with $T(t)$ introduced after (3.5). Writing

$$R(t(T)) \equiv \Omega(T) = \Omega(v - u), \quad (5.2)$$

the line element (3.4) written in terms of u and v reads

$$ds^2 = \Omega^2 p_0^{-2} f^2 (dy^2 + dz^2) + 2\Omega^2 dudv, \quad (5.3)$$

where now $f = f(u + v)$. The coordinates y, z, u, v are such that the surfaces $u = \text{const}$, $v = \text{const}$ are two families of intersecting null hypersurfaces. The general form of a line element in a coordinate system based upon two families of intersecting null hypersurfaces is given in [9]. For our purposes we write this as

$$ds^2 = b^2 h_{AB} (dx^A + a_1^A du + a_2^A dv) (dx^B + a_1^B du + a_2^B dv) + 2cdudv, \quad (5.4)$$

where A, B take values (1,2) [$h_{AB}(y, z, u, v)$] is a unimodular 2×2 symmetric matrix, $(x^1, x^2) = (y, z)$ and a_1^A, a_2^A, b, c are six functions of y, z, u, v . It is convenient to use the following parametrization [10] of (h_{AB}) :

$$(h_{AB}) = \begin{pmatrix} e^{2\alpha} \cosh 2\beta & \sinh 2\beta \\ \sinh 2\beta & e^{-2\alpha} \cosh 2\beta \end{pmatrix}. \quad (5.5)$$

Here α, β are taken to be small of first order. With (h_{AB}) given by Eq. (5.5) it is easy to check that, working to first order, Eq. (5.4) can be written

$$\begin{aligned} ds^2 = & b^2 [(1 + \alpha)dy + \beta dz + \{a_1^1(1 + \alpha) + a_1^2\beta\}du \\ & + \{a_2^1(1 + \alpha) + a_2^2\beta\}dv]^2 \\ & + b^2 [\beta dy + (1 - \alpha)dz + \{a_1^1\beta + a_1^2(1 - \alpha)\}du \\ & + \{a_2^1\beta + a_2^2(1 - \alpha)\}dv]^2 + 2cdudv. \end{aligned} \quad (5.6)$$

The background space-time is obtained from this by setting

$$a_1^A = 0, \quad a_2^A = 0, \quad b = p_0^{-1} \Omega f, \quad c = \Omega^2, \quad \alpha = 0, \quad \beta = 0. \quad (5.7)$$

For the perturbed space-time that we require we find that b, c retain their background values, and we can put $a_1^A = 0 = a_2^A$. These latter quantities actually play the role of gauge terms (see Sec. VII and Appendix B for an illustration of this). Every shear-free system of gravitational waves involves an arbitrary analytic function [11] and we now have two real functions α, β available to provide the real and imaginary parts of this analytic function. Also we find that the 4-velocity u^a , the isotropic pressure p and the matter-energy density μ take their background values (these are given in Sec. III).

To demonstrate that this space-time does indeed describe the perturbations of Sec. IV we shall work on the tetrad given via the 1-forms

$$\begin{aligned} \theta^1 &= p_0^{-1} f \Omega \{(1 + \alpha)dy + \beta dz\}, \\ \theta^2 &= p_0^{-1} f \Omega \{\beta dy + (1 - \alpha)dz\}, \\ \theta^3 &= \Omega du, \\ \theta^4 &= \Omega dv, \end{aligned} \quad (5.8)$$

with $p_0 = 1 + (K/4)(y^2 + z^2)$ as in (3.4). We note that with respect to this tetrad the line element is now

$$ds^2 = (\theta^1)^2 + (\theta^2)^2 + 2\theta^3\theta^4 = g_{ab}\theta^a\theta^b, \quad (5.9)$$

thus defining the tetrad components g_{ab} of the metric tensor. The tetrad components of the matter 4-velocity are given via the 1-form

$$u_a \theta^a = \frac{1}{\sqrt{2}} (\theta^3 - \theta^4). \quad (5.10)$$

Since we wish to reproduce the linear perturbations of the previous section we shall discard any terms which are second order or smaller in α and β . Our first step is to calculate the Ricci rotation coefficients and the Ricci tensor components. This results in a lengthy list of equations which for conve-

nience we give in Appendix A. We now use the Ricci tensor components and (5.10) in the field equations given by Eqs. (2.11). Noting that

$$\Omega' = \frac{1}{\sqrt{2}}R\dot{R}, \quad \Omega'' = \frac{1}{2}R^2\ddot{R} + \frac{1}{2}R\dot{R}^2, \quad (5.11)$$

where the prime denotes differentiation with respect to v and using Eqs. (3.2) and (3.3) it is easily checked that the first of Eqs. (2.11) is identically satisfied. The second equation in (2.11) yields

$$q_1 = \frac{p_0^3}{\sqrt{2}}f^{-1}\Omega^{-2}\{(p_0^{-2}\alpha)_{yu} + (p_0^{-2}\beta)_{zu} - (p_0^{-2}\alpha)_{yv} - (p_0^{-2}\beta)_{zv}\}, \quad (5.12)$$

$$q_2 = \frac{p_0^3}{\sqrt{2}}f^{-1}\Omega^{-2}\{(p_0^{-2}\alpha)_{zv} - (p_0^{-2}\beta)_{yv} - (p_0^{-2}\alpha)_{zu} + (p_0^{-2}\beta)_{yu}\}, \quad (5.13)$$

$$q_3 = 0, \quad (5.14)$$

$$q_4 = 0. \quad (5.15)$$

The subscripts y, z, u, v here indicate partial differentiation with respect to these variables. We recall that in the covariant approach we found that $q_a \equiv 0$. With α, β chosen so they satisfy the Cauchy-Riemann equations in the form:

$$(p_0^{-2}\alpha)_y + (p_0^{-2}\beta)_z = 0, \quad (5.16)$$

$$(p_0^{-2}\alpha)_z - (p_0^{-2}\beta)_y = 0, \quad (5.17)$$

it follows that $q_a \equiv 0$ as required. For later use we note that as a result of these equations $p_0^{-2}(\alpha - i\beta)$ is analytic in $\zeta = y + iz$. With $q_a = 0$ the last of Eqs. (2.11) can be rewritten as

$$R_{ab} = \mu u_a u_b + p h_{ab} + \pi_{ab} - \frac{1}{2}(3p - \mu)g_{ab}. \quad (5.18)$$

With the Ricci tensor components given by (A.11)–(A.20) it follows from this and Eqs. (3.2), (3.3), (5.9)–(5.11), (5.16), (5.17) that $\pi_{ab} = 0$ except for π_{11} , π_{22} and π_{12} with

$$\pi_{11} = \sqrt{2}R^{-2}\dot{R}(\alpha_v - \alpha_u) - 2R^{-2}f^{-1}f'(\alpha_v + \alpha_u) - 2R^{-2}\alpha_{vu}, \quad (5.19)$$

$$\pi_{22} = -\sqrt{2}R^{-2}\dot{R}(\alpha_v - \alpha_u) + 2R^{-2}f^{-1}f'(\alpha_v + \alpha_u) + 2R^{-2}\alpha_{vu}, \quad (5.20)$$

$$\pi_{12} = \sqrt{2}R^{-2}\dot{R}(\beta_v - \beta_u) - 2R^{-2}f^{-1}f'(\beta_v + \beta_u) - 2R^{-2}\beta_{vu}. \quad (5.21)$$

We have made use of

$$2f'' = -kf, \quad 2(f')^2 + kf^2 = K, \quad (5.22)$$

to simplify these equations. We note that the prime here denotes differentiation with respect to $v = (1/\sqrt{2})(x + T)$. Similar equations to these appear in [1] [Eq. (5.41)]. In [1] the prime indicates differentiation with respect to x and hence the factors of 2 in Eqs. (5.22) do not appear there.

Now in terms of the background null tetrad described by Eq. (4.14) we can write the coordinate components of the (small) anisotropic stress tensor as

$$\pi^{ij} = \bar{\pi}m^i m^j + \pi \bar{m}^i \bar{m}^j. \quad (5.23)$$

Using Eqs. (4.14) and (5.8) (with $\alpha = \beta = 0$ since π_{ab} is a first order quantity) we find

$$\bar{\pi} = \frac{1}{2}(\pi_{11} - \pi_{22}) - i\pi_{12}. \quad (5.24)$$

Substituting from Eqs. (5.19)–(5.21) above yields

$$\begin{aligned} \bar{\pi} = & -2p_0^2 f^{-1} R^{-2} [p_0^{-2} f(\alpha - i\beta)_{uv} \\ & + p_0^{-2} f' \{ \alpha_v + \alpha_u - i(\beta_v + \beta_u) \}] \\ & - \sqrt{2} R^{-2} \dot{R} \{ \alpha_u - \alpha_v - i(\beta_u - \beta_v) \}. \end{aligned} \quad (5.25)$$

In the previous section we worked with Π_{ab} and $\bar{\Pi}$ which we expressed in terms of an analytic function \mathcal{G} . But $\pi_{ab} = \Pi_{ab} F$ and as indicated following Eq. (4.33) F can be absorbed into \mathcal{G} . Hence, in order to make contact with the gauge-invariant description $\bar{\pi}$ here must satisfy the same equation as $\bar{\Pi}$ and thus we require

$$\bar{\pi} = -2p_0^2 f^{-1} R^{-2} \{ D\mathcal{G} + \dot{R}\mathcal{G} \}, \quad (5.26)$$

with $D = \partial/\partial x + R\partial/\partial t = \sqrt{2}\partial/\partial v$ for some analytic function \mathcal{G} . Taking

$$\mathcal{G} = \frac{1}{\sqrt{2}} p_0^{-2} f \{ \alpha_u - \alpha_v - i(\beta_u - \beta_v) \}, \quad (5.27)$$

we find that it is indeed possible to write $\bar{\pi}$ in this form provided we choose α, β to satisfy the following:

$$\text{if } f' = 0 \text{ then } \alpha_{vv} = 0, \beta_{vv} = 0; \quad (5.28)$$

$$\text{if } f' \neq 0 \text{ then } \alpha_v = 0, \beta_v = 0. \quad (5.29)$$

We note that the first of these conditions corresponds to the case $k=0, K=0$ described following Eq. (3.4). We now assume that these conditions hold. As a consequence of these and Eq. (5.22) it immediately follows that \mathcal{G} given by (5.27) satisfies the wave equation (4.25). Also noting that $f=f(x)$ and using the Cauchy-Riemann equations (5.16)–(5.17) we see that as before \mathcal{G} is an analytic function of $\zeta = y + iz$.

We now turn our attention to the shear. In a similar fashion to the anisotropic stress the coordinate components of the (small) shear tensor can be written in the form

$$\sigma_{ij} = \bar{s}_p m_i m_j + s_p \bar{m}_i \bar{m}_j, \quad (5.30)$$

where m_i, \bar{m}_j are given by Eq. (4.14) and in terms of the Ricci rotation coefficients

$$\begin{aligned} \bar{s}_p = \frac{1}{2\sqrt{2}} \{ & (Y_{141} - Y_{242} - Y_{131} + Y_{232}) \\ & + i(Y_{132} + Y_{231} - Y_{142} - Y_{241}) \}. \end{aligned} \quad (5.31)$$

Evaluating this using the Ricci rotation coefficients given in Appendix A we find

$$\bar{s}_p = \frac{1}{\sqrt{2}} R^{-1} \{ (\alpha - i\beta)_v - (\alpha - i\beta)_u \} = -p_0^{-2} f^{-1} R^{-1} \mathcal{G}, \quad (5.32)$$

with \mathcal{G} as before. Taking into account that we can absorb F into \mathcal{G} and that $\bar{s}_p = \bar{s}F$ [with \bar{s} defined by Eq. (4.15)] we see that the perturbations we have produced here also satisfy Eq. (4.23). Thus we have shown that *the perturbations described by the metric (5.6) take the same form as those found by the covariant approach*. In the next section we shall illustrate that they also satisfy the wave equation (4.8) and the propagation equation (4.9).

For the remainder of this section we compare the explicit \mathcal{G} found here with the solutions of the wave equation found in [1] and listed in Eqs. (4.26)–(4.28) [or equivalently (4.29)–(4.32)]. *We first examine the case when $k=0$* . There are two subcases to consider here: (i) $K=0$ and $f(x)=1$, (ii) $K=+1$ and $f(x)=x$. When $K=0$, $p_0=1$ and Eq. (5.27) reads

$$\mathcal{G} = \frac{1}{\sqrt{2}} \{ (\alpha - i\beta)_u - (\alpha - i\beta)_v \}. \quad (5.33)$$

Since $f'=0$ in this case we have $\alpha_{vv}=0=\beta_{vv}$. Thus in addition to $\alpha - i\beta$ being analytic in ζ this complex-valued function is also linear in v . Hence we can write

$$\mathcal{G}(\zeta, x, t) = a_1(\zeta, x - T)(x + T) + a_2(\zeta, x - T), \quad (5.34)$$

where a_1, a_2 are arbitrary (analytic) functions of their arguments. When $K=+1$, $f(x)=0$ and from (5.29) we have $\alpha_v=0=\beta_v$. Therefore the function $p_0^{-2}(\alpha - i\beta)$ is analytic in ζ and independent of $v=(x+T)/\sqrt{2}$, i.e. it depends only on ζ and $u=(x-T)/\sqrt{2}$, and we can write (5.27) in the form

$$\mathcal{G} = x a_3(\zeta, x - T), \quad (5.35)$$

with a_3 an arbitrary analytic function. Using the identity $x + T = 2x - (x - T)$ as in Sec. IV we can rewrite (5.34) in the form (4.29). Then (5.35) is the special case of (4.29) corresponding to $h_1(\zeta, x - T) \equiv 0$. *Thus in the case $k=0$ there are two independent expressions for $\mathcal{G}(\zeta, x, t)$ which are given in the form of a superposition in (4.29)*. This arises because (4.29) is obtained by solving the linear wave equation (4.25) with $k=0$ and in general this equation is insensitive to the allowable values of $K=0, \pm 1$.

We now look at the solution when $k=+1$. There is only one case to consider here, $K=+1$ with $f(x)=\sin x$ or equivalently (see Sec. III) $f(x)=\cos x$. Again we have $p_0^{-2}(\alpha - i\beta)$ analytic in ζ and independent of v so (5.27) can now be written

$$\mathcal{G} = a_4(\zeta, x - T) \sin x \quad \text{or} \quad \mathcal{G} = a_5(\zeta, x - T) \cos x, \quad (5.36)$$

where a_4, a_5 are arbitrary functions. *The two equations in (5.36) are equivalent to (4.30)*.

Finally when $k=-1$ there are three subcases to look at corresponding to $K=0, \pm 1$. In all cases $f'(x) \neq 0$ and so we have $p_0^{-2}(\alpha - i\beta)$ independent of v and \mathcal{G} has the form

$$\mathcal{G} = f(x) a_6(\zeta, x - T), \quad (5.37)$$

where a_6 is an arbitrary analytic function. When $K=0$ we have $f(x) = \frac{1}{2}e^x$ or equivalently $f(x) = \frac{1}{2}e^{-x}$ and so *in this case (5.37) agrees with (4.32)*. When $K=+1$, $f(x) = \sinh x$ and now (5.37) *agrees with (4.31)* when $h_6(\zeta, x - T) \equiv 0$. When $K=-1$, $f(x) = \cosh x$ and (5.37) *agrees with (4.31)* when $h_5(\zeta, x - T) \equiv 0$. This case $k=-1$ is a good illustration of the insensitivity of the expressions (4.31) and (4.32) to the values of K .

Thus all of the solutions found here are identical to the solutions found using the gauge-invariant and covariant approach to perturbations in [1].

VI. PROPERTIES OF THE SHEAR AND ANISOTROPIC STRESS

In the previous section we exhibited a perturbation of the Robertson-Walker background line element (3.4) that produced perturbations in the shear and anisotropic stress tensors which satisfied some of the equations found using the gauge-invariant and covariant approaches of Sec. IV. We now show that these perturbations satisfy the remaining equations, namely, that the anisotropic stress and shear tensors are trace-free, orthogonal to u^a , divergence-free with respect to the background metric and also satisfy the wave equation (4.8) and propagation equation (4.9). To do this we shall, in this section, work in coordinate components (in the local coordinates y, z, x, t) instead of the tetrad components we have used up to this point. In terms of this local coordinate system we can write the line element (5.6) (with $a_1^A = a_2^A = 0$, $b = p_0^{-1} \Omega f$, $c = \Omega^2$) in the form

$$ds^2 = \hat{g}_{ab} dx^a dx^b + 2 \gamma_{ab} dx^a dx^b := g_{ab} dx^a dx^b, \quad (6.1)$$

where $\hat{g}_{ab} = \text{diag}\{p_0^{-2} f^2 \Omega^2, p_0^{-2} f^2 \Omega^2, \Omega^2, -1\}$ is the metric of the background space-time and

$$\gamma_{ab} = p_0^{-2} f^2 \Omega^2 \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.2)$$

is the perturbation. Clearly γ_{ab} is trace-free and orthogonal to $u^a = \delta_4^a$ (with $\hat{g}_{ab}u^a u^b = -1$). The non-vanishing Christoffel symbols of the background metric tensor given via the line element (3.4) are

$$\begin{aligned}\hat{\Gamma}_{11}^1 &= -\hat{\Gamma}_{22}^1 = \hat{\Gamma}_{12}^2 = -\frac{1}{2}Kp_0^{-1}y, \\ \hat{\Gamma}_{12}^1 &= -\hat{\Gamma}_{11}^2 = \hat{\Gamma}_{22}^2 = -\frac{1}{2}Kp_0^{-1}z, \\ \hat{\Gamma}_{14}^1 &= \hat{\Gamma}_{24}^2 = \hat{\Gamma}_{34}^3 = \Omega^{-1}\Omega_t, \\ \hat{\Gamma}_{11}^4 &= \hat{\Gamma}_{22}^4 = p_0^{-2}f^2\Omega\Omega_t, \\ \hat{\Gamma}_{11}^3 &= \hat{\Gamma}_{22}^3 = -p_0^{-2}ff_x, \\ \hat{\Gamma}_{13}^1 &= \hat{\Gamma}_{23}^2 = f^{-1}f_x, \\ \hat{\Gamma}_{33}^4 &= \Omega\Omega_t.\end{aligned}\quad (6.3)$$

We have used the hat here to emphasize that these are background Christoffel symbols and we shall continue to use this notation to denote background quantities for the remainder of this section. Using these and the Cauchy-Riemann equations (5.16)–(5.17) it is a simple exercise to show that γ_{ab} defined above is divergence-free.

In order to show that π^{ab} is also divergence-free we first write it in terms of γ_{ab} . We define the perturbation of the Christoffel symbols to be $\delta\Gamma_{bd}^a := \Gamma_{bd}^a - \hat{\Gamma}_{bd}^a$. Noting that $g^{ab} = \hat{g}^{ab} - \gamma^{ab}$ (here $\gamma^{ab} = \hat{g}^{ac}\hat{g}^{bf}\gamma_{cf}$ and we are neglecting second order small quantities) it is easily derived from the definition of the Christoffel symbols that

$$\delta\Gamma_{bd}^a = \frac{1}{2}(\gamma_{b|d}^a + \gamma_{d|b}^a - \hat{g}^{af}\gamma_{bd|f}), \quad (6.4)$$

where as usual the stroke indicates differentiation with respect to the background metric. Now γ^{ab} is divergence-free and thus we can see from this equation that $\delta\Gamma_{ba}^a = 0$. In general the components of the Ricci tensor of a perturbed metric can be written in the form

$$R_{bd} = \hat{R}_{bd} + (\delta\Gamma_{bd}^a)_{|a} - (\delta\Gamma_{ba}^a)_{|d}. \quad (6.5)$$

For the problem at hand we have

$$\hat{R}_{bd} = \mu u_b u_d + p \hat{h}_{bd} - \frac{1}{2}(3p - \mu)\hat{g}_{bd}. \quad (6.6)$$

Substituting for R_{bd} and \hat{R}_{bd} from Eqs. (5.18) and (6.6), respectively, in Eq. (6.5) yields

$$(\delta\Gamma_{bd}^a)_{|a} = \pi_{bd} + \frac{1}{2}(\mu - p)\gamma_{bd}. \quad (6.7)$$

Taking the divergence of Eq. (6.4) and using this equation we arrive at

$$\gamma_{b|da}^a + \gamma_{d|ba}^a - \hat{g}^{af}\gamma_{bd|fa} = 2\pi_{bd} + (\mu - p)\gamma_{bd}. \quad (6.8)$$

Next making use of the Ricci identities

$$\gamma_{ab|dc} - \gamma_{ab|cd} = R_{afcd}\gamma_b^f - R_{fbcd}\gamma_a^f, \quad (6.9)$$

and recalling that γ^{ab} is divergenceless and orthogonal to u^a we find (since $C_{abcd} = 0$ in the background)

$$\gamma_{a|dc}^c = \frac{3}{2}\gamma_d^c R_{cd} + \frac{1}{2}\gamma_d^c R_{ac} - \frac{1}{2}g_{ad}R_{fc}\gamma^{fc} - \frac{1}{6}R\gamma_{ad}. \quad (6.10)$$

With $R = \mu - 3p$ and R_{ab} given by Eq. (5.18) this equation allows us to write (since we are concerned here with first-order terms only)

$$\gamma_{b|da}^a + \gamma_{d|ba}^a = \left(\frac{5}{3}\mu - p\right)\gamma_{bd}, \quad (6.11)$$

and hence Eq. (6.8) now becomes

$$\hat{g}^{af}\gamma_{bd|fa} - \frac{2}{3}\mu\gamma_{bd} = -2\pi_{bd}. \quad (6.12)$$

It is easy to see from this that π_{ab} is trace-free and orthogonal to u^a . Starting with this equation we shall now prove that π_{ab} is indeed divergence-free. That this is necessary to fully make contact with the gauge-invariant and covariant approach of Sec. IV follows from the fact that in this case we wrote $\pi_{ab} = \Pi_{ab}F$ with $\Pi^{ab}|_b = 0$, $F = F(x - T)$ and $\Pi^{ab} = 0$ except for Π^{11} , Π^{22} and Π^{12} and therefore $\Pi^{ab}|_b = 0$ is equivalent to $\pi^{ab}|_b = 0$ in this case. First making use of the Ricci identities for a tensor of type (3,0), Eq. (2.1) and Eq. (2.10), we can write

$$\gamma^{ab|d}|_{db} = (\gamma^{ab}|_b)^d|_d + \left(\frac{7}{6}\mu - \frac{1}{2}p\right)\gamma^{ad}|_d = 0. \quad (6.13)$$

Also since for the perturbed space-time we are considering here the matter density μ retains its background value we have $h_c^b\mu_{,b} = 0$ from which it follows that

$$\mu_{,b} = -\dot{\mu}u_b. \quad (6.14)$$

As a consequence of these last two equations we find, on taking the divergence of Eq. (6.12), that

$$\pi^{ab}|_b = 0, \quad (6.15)$$

as required.

We shall now examine the properties of the shear σ_{ab} . As with the anisotropic stress above it is necessary to express this in terms of γ^{ab} . This is easily done using the definition of the covariant derivative of u_a :

$$u_{a;b} := -\Gamma_{ab}^c u_c = -\hat{\Gamma}_{ab}^c u_c - \delta\Gamma_{ab}^c u_c. \quad (6.16)$$

We remind the reader that the semicolon here indicates covariant differentiation with respect to the perturbed metric (background plus a small perturbation) while a stroke de-

notes covariant differentiation with respect to the background metric. In the background Robertson-Walker space-time the shear, vorticity and the 4-acceleration all vanish and so Eq. (2.5) specializes to

$$\hat{\Gamma}_{ab}^c u_c := u_{a|b} = \frac{1}{3} \theta \hat{h}_{ab} \quad (6.17)$$

in this case. Making use of this equation and Eq. (6.4) in Eq. (6.16) it follows, on account of $h_{ab} = \hat{h}_{ab} + \gamma_{ab}$ (since for the problem at hand u^a is unperturbed), that

$$u_{a;b} = \frac{1}{3} \theta h_{ab} + \frac{1}{2} \dot{\gamma}_{ab}. \quad (6.18)$$

Here and for the remainder of this section a dot indicates covariant differentiation with respect to the background metric in the direction of u^a . Recalling that the 4-acceleration is zero in the background Robertson-Walker space-time (i.e. $u_{a|b} u^b = 0$) it is trivial to see from the latter equation that the 4-acceleration in the perturbed space-time also vanishes. We also note that this equation is symmetric in (a, b) and thus it is clear from Eq. (2.6) that, as in the covariant approach, the vorticity tensor vanishes in the perturbed space-time. Now equating Eqs. (6.18) and (2.5) with the 4-acceleration and vorticity tensor both zero we arrive at a simple relationship between σ_{ab} and γ_{ab} , namely,

$$\sigma_{ab} = \frac{1}{2} \dot{\gamma}_{ab}. \quad (6.19)$$

Using this and the properties of γ_{ab} it is straightforward to check that σ_{ab} is trace-free and orthogonal to u^a . However, further calculation is necessary to show that it is also divergence-free (this is required for similar reasons to those given above while discussing the anisotropic stress). First using the Ricci identities given in Eq. (6.9) and noting that $C_{abcd} = 0$ we calculate

$$\dot{\gamma}^{ab}|_b = (\gamma^{ab}|_b) \cdot + \frac{3}{2} \gamma^{af} \hat{R}_{fc} u^c - \frac{1}{2} \hat{R}_{bf} \gamma^{bf} u^a. \quad (6.20)$$

Replacing \hat{R}_{ab} here by the right-hand side of Eq. (6.6) and keeping in mind that $\gamma^{ab}|_b = 0$, $\gamma^{ab} u_b = 0$ leads to

$$\dot{\gamma}^{ab}|_b = 0, \quad (6.21)$$

and therefore as a result of Eq. (6.19) $\sigma^{ab}|_b = 0$.

At this point all that remains to fully make contact with the gauge-invariant and covariant description of gravitational wave perturbations outlined in Sec. IV is to reconstruct the wave equation (4.8) and the propagation equation (4.9). This is done as follows: Using the Ricci identities the covariant derivative in the direction of u^a of Eq. (6.12) can be written as

$$\begin{aligned} -2 \dot{\pi}^{ab} &= \gamma^{ab|d} |_{cd} u^c - \frac{1}{3} \theta \left(p + \frac{1}{3} \mu \right) \gamma^{ab} + \frac{1}{2} (\mu + 3p) \dot{\gamma}^{ab} \\ &\quad - \frac{2}{3} \dot{\mu} \gamma^{ab} - \frac{2}{3} \mu \dot{\gamma}^{ab}. \end{aligned} \quad (6.22)$$

Also with $h_c^b \mu_{,b} = 0$, $h_c^b p_{,b} = 0$, and $h_c^b \theta_{,b} = 0$ we find, again using the Ricci identities and Eq. (6.12), that

$$\begin{aligned} \gamma^{ab|d} |_{cd} u^c &= (\dot{\gamma}^{ab})|_{cd} - \frac{2}{3} \theta \left(\frac{1}{6} \mu + \frac{1}{2} p \right) \gamma^{ab} - \frac{1}{3} \theta^2 \dot{\gamma}^{ab} \\ &\quad - \frac{2}{3} \theta \left(\frac{2}{3} \mu \gamma^{ab} - 2 \pi^{ab} \right) - \frac{2}{3} \theta \dot{\gamma}^{ab}. \end{aligned} \quad (6.23)$$

Entering this into Eq. (6.22) and replacing $\dot{\gamma}^{ab}$ by $2\sigma^{ab}$ we arrive at

$$\begin{aligned} -\dot{\pi}^{ab} - \frac{2}{3} \theta \pi^{ab} &= \sigma^{ab|d} |_{cd} - \frac{2}{3} \theta \dot{\sigma}^{ab} - \left(\frac{1}{3} \dot{\theta} + \frac{4}{9} \theta^2 \right) \sigma^{ab} \\ &\quad + \left(p - \frac{1}{3} \mu \right) \sigma^{ab}. \end{aligned} \quad (6.24)$$

We have made use of the background values of $\dot{\theta}$ and $\dot{\mu}$ to write the equation in this form. The background value of $\dot{\theta}$ is

$$\dot{\theta} = -\frac{1}{3} \theta^2 - \frac{1}{2} (\mu + 3p) \quad (6.25)$$

which is obtained by specializing Raychaudhuri's equation to the background (i.e. setting q_a , π_{ab} , \dot{u}_a , σ_{ab} and ω_{ab} all equal to zero) and the background value of $\dot{\mu}$ is given by

$$\dot{\mu} = -\theta(\mu + p). \quad (6.26)$$

This is found by specializing to the background the projections along and orthogonal to u^a of the conservation equation $T^{ab}|_b = 0$ [see, for example, Eqs. (2.20) and (2.21) in [1]]. Both the wave equation and the propagation equation are actually contained in Eq. (6.24). To confirm this we again put

$$\sigma_{ab} = s_{ab} F(\phi), \quad \pi_{ab} = \Pi_{ab} F(\phi), \quad (6.27)$$

where $F(\phi)$ is an arbitrary function of its argument $\phi = x - T(t)$. In the covariant approach we found $s^{ab} \phi_{,b} = 0$ and $\Pi^{ab} \phi_{,b} = 0$. This is also true here since $\phi_{,b} = (0, 0, 1, -R^{-1})$ and we have $\pi^{3b} = 0$, $\pi^{4b} = 0$, $\sigma^{3b} = 0$ and $\sigma^{4b} = 0$. In addition since the hypersurfaces $\phi(x^a) = \text{const}$ are null we have $\phi^{,d} \phi_{,d} = 0$. Thus we can write

$$\dot{\pi}^{ab} = \dot{\Pi}^{ab} F + \dot{\phi} \Pi^{ab} F', \quad (6.28)$$

$$\dot{\sigma}^{ab} = \dot{s}^{ab} F + \dot{\phi} s^{ab} F', \quad (6.29)$$

and

$$\sigma^{ab|d} |_{cd} = s^{ab|d} |_{cd} F + (2s^{ab|d} \phi_{,d} + s^{ab} \phi^{,d} |_{cd}) F', \quad (6.30)$$

where $F' = dF/d\phi$. Substituting these expressions for σ^{ab} and π^{ab} into Eq. (6.24) and equating the F and F' parts separately yields the required wave equation (4.8) and propagation equation (4.9).

VII. DISCUSSION

We have shown in Secs. V and VI that the perturbations of the background Robertson-Walker space-time derived here from metric perturbations are exactly the same as those obtained using the covariant approach. Thus the metric (5.6) with $a_1^A=0$, $a_2^A=0$ and α , β chosen to satisfy the Cauchy-Riemann equations (5.16)–(5.17) is indeed that which we set out to find. We mentioned earlier that the functions a_1^A , a_2^A play the role of gauge terms. That this is true is seen by repeating the calculation of \mathcal{G} with $a_1^A \neq 0$, $a_2^A \neq 0$. To save repetition here this calculation is outlined briefly in Appendix B. The result is that a_1^A , a_2^A do not appear in the required analytic function \mathcal{G} , i.e. that which satisfies Eq. (5.26). Thus since all gauge invariant perturbations can be written in terms of this \mathcal{G} we conclude that a_1^A , a_2^A are pure gauge terms which we can set equal to zero without loss of generality.

Metric perturbations of Robertson-Walker space-times, which can be viewed as describing gravitational radiation, have also been studied by Bardeen [3] in an important paper. In this study the background space-time is taken to be a Robertson-Walker space-time with line element

$$ds^2 = \Omega^2(T) \{-dT^2 + {}^3g_{\alpha\beta} dx^\alpha dx^\beta\}. \quad (7.1)$$

Here the greek indices take values 1, 2, 3 and ${}^3g_{\alpha\beta}$ is the metric tensor for a three-space of constant curvature. Comparing this to Eq. (3.4) we see that our background space-time also has this form if we take ${}^3g_{\alpha\beta} = (p_0^{-2} f^2, p_0^{-2} f^2, 1)$ and label the coordinates $x^1 = y, x^2 = z, x^3 = x$. The method used in [3] involves separating the time-dependent and spatial-dependent parts of the perturbations. Now for us the important coordinates are $u = [x - T(t)]/\sqrt{2}$, $v = [x + T(t)]/\sqrt{2}$ and there is no natural way to carry out this separation. Thus it is not possible to directly compare the results found here with those of [3]. However, there are some obvious similarities and differences between the results and we shall briefly comment on these now. One point of agreement is that gravitational radiation is described by tensor perturbations only. Specifically in our case gravitational waves are described by perturbations in the shear and anisotropic stress tensors. The perturbed space-time in [3] is given by

$$ds^2 = -\Omega^2 dT^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad (7.2)$$

where

$$g_{\alpha\beta} = \Omega^2 [{}^3g_{\alpha\beta} + 2H_T^{(2)}(T) Q_{\alpha\beta}^{(2)}(x^\mu)] \quad (7.3)$$

and $Q_{\alpha\beta}^{(2)}$ is a divergenceless trace-free tensor. This bears a strong resemblance to our perturbed space-time described by (6.1) where γ_{ab} given in (6.2) is also divergenceless and trace-free. However, it is clear from (6.2) that, in effect, our small metric perturbations γ_{ab} are expressible in the form of

a 2×2 matrix whereas $(Q_{\alpha\beta}^{(2)})$ is a 3×3 matrix. In addition, γ_{ab} satisfies the inhomogeneous wave equation (6.12) while $Q_{\alpha\beta}^{(2)}$ satisfies the homogeneous wave equation [3]

$$Q^{(2)\alpha\beta;\gamma}_{;\gamma} + k_0^2 Q^{(2)\alpha\beta} = 0, \quad (7.4)$$

where k_0 is a constant.

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APPENDIX A: THE RICCI TENSOR COMPONENTS

In this section we give the Ricci tensor components [on the tetrad given by Eqs. (5.8)] for the metric defined by Eqs. (5.8) and (5.9). In the calculation of the Ricci tensor components we use $\partial\Omega/\partial u = -\partial\Omega/\partial v$ and $\partial f/\partial u = \partial f/\partial v$ to simplify equations. Also for convenience we shall use subscripts y, z, u, v to indicate partial derivatives with respect to these variables and a prime to denote partial differentiation with respect to v . Following the Cartan method to find the Ricci tensor components we first find the non-zero Ricci rotation coefficients to be

$$\begin{aligned} Y_{121} = & -\frac{1}{2}\Omega^{-1}f^{-1}(1+\alpha)Kz + \Omega^{-1}f^{-1}p_0\alpha_z \\ & + \frac{1}{2}\Omega^{-1}f^{-1}K\beta y - \Omega^{-1}f^{-1}p_0\beta_y, \end{aligned} \quad (A1)$$

$$Y_{131} = -\Omega^{-2}\Omega' + \Omega^{-1}f^{-1}f' + \Omega^{-1}\alpha_u, \quad (A2)$$

$$Y_{141} = \Omega^{-2}\Omega' + \Omega^{-1}f^{-1}f' + \Omega^{-1}\alpha_v, \quad (A3)$$

$$\begin{aligned} Y_{212} = & \frac{1}{2}\Omega^{-1}f^{-1}K\beta z - \Omega^{-1}f^{-1}p_0\beta_z \\ & - \frac{1}{2}\Omega^{-1}f^{-1}(1-\alpha)Ky - \Omega^{-1}f^{-1}p_0\alpha_y, \end{aligned} \quad (A4)$$

$$Y_{232} = -\Omega^{-2}\Omega' + \Omega^{-1}f^{-1}f' - \Omega^{-1}\alpha_u, \quad (A5)$$

$$Y_{242} = \Omega^{-2}\Omega' + \Omega^{-1}f^{-1}f' - \Omega^{-1}\alpha_v, \quad (A6)$$

$$Y_{343} = \Omega^{-2}\Omega', \quad (A7)$$

$$Y_{434} = -\Omega^{-2}\Omega', \quad (A8)$$

$$Y_{231} = Y_{132} = \Omega^{-1}\beta_u, \quad (A9)$$

$$Y_{142} = Y_{241} = \Omega^{-1}\beta_v. \quad (A10)$$

We note that in this calculation we have discarded any terms which are second order or smaller in α, β . Using these coefficients we obtain the Ricci tensor components:

$$R_{13} = p_0^3 f^{-1} \Omega^{-2} \{ (p_0^{-2} \alpha)_{yu} + (p_0^{-2} \beta)_{zu} \}, \quad (\text{A11})$$

$$R_{23} = -p_0^3 f^{-1} \Omega^{-2} \{ (p_0^{-2} \alpha)_{zu} - (p_0^{-2} \beta)_{yu} \}, \quad (\text{A12})$$

$$R_{33} = 4\Omega^{-4} \Omega'^2 - 2\Omega^{-3} \Omega'' - 2f^{-1} f'' \Omega^{-2}, \quad (\text{A13})$$

$$R_{44} = 4\Omega^{-4} \Omega'^2 - 2\Omega^{-3} \Omega'' - 2f^{-1} f'' \Omega^{-2}, \quad (\text{A14})$$

$$R_{34} = 4\Omega^{-3} \Omega'' - 2\Omega^{-4} \Omega'^2 - 2f^{-1} f'' \Omega^{-2}, \quad (\text{A15})$$

$$R_{14} = p_0^3 f^{-1} \Omega^{-2} \{ (p_0^{-2} \alpha)_{yv} + (p_0^{-2} \beta)_{zv} \}, \quad (\text{A16})$$

$$R_{24} = -p_0^3 f^{-1} \Omega^{-2} \{ (p_0^{-2} \alpha)_{zv} - (p_0^{-2} \beta)_{yv} \}, \quad (\text{A17})$$

$$R_{12} = 2\Omega^{-3} \Omega' (\beta_v - \beta_u) - 2\Omega^{-2} \beta_{uv} - 2f^{-1} f' \Omega^{-2} (\beta_u + \beta_v), \quad (\text{A18})$$

$$R_{11} = 2\Omega^{-4} \Omega'^2 - 2f^{-1} f'' \Omega^{-2} + 2\Omega^{-3} \Omega'' - 2\Omega^{-2} f^{-2} f'^2 + \Omega^{-2} f^{-2} K + 2\Omega^{-3} \Omega' (\alpha_v - \alpha_u) - 2\Omega^{-2} f^{-1} f' (\alpha_v + \alpha_u) - 2\Omega^{-2} \alpha_{uv} + \Omega^{-2} f^{-2} \{ p_0^4 (p_0^{-2} \alpha)_{yy} - p_0^4 (p_0^{-2} \alpha)_{zz} - Kz p_0^3 (p_0^{-2} \alpha)_z + Ky p_0^3 (p_0^{-2} \alpha)_y \} + \Omega^{-2} f^{-2} \{ 2p_0^4 (p_0^{-2} \beta)_{yz} + Kz p_0^3 (p_0^{-2} \beta)_y + Ky p_0^3 (p_0^{-2} \beta)_z \}, \quad (\text{A19})$$

$$R_{22} = 2\Omega^{-4} \Omega'^2 - 2f^{-1} f'' \Omega^{-2} + 2\Omega^{-3} \Omega'' - 2\Omega^{-2} f^{-2} f'^2 + \Omega^{-2} f^{-2} K - 2\Omega^{-3} \Omega' (\alpha_v - \alpha_u) + 2\Omega^{-2} f^{-1} f' (\alpha_v + \alpha_u) + 2\Omega^{-2} \alpha_{uv} + \Omega^{-2} f^{-2} \{ p_0^4 (p_0^{-2} \alpha)_{yy} - p_0^4 (p_0^{-2} \alpha)_{zz} - Kz p_0^3 (p_0^{-2} \alpha)_z + Ky p_0^3 (p_0^{-2} \alpha)_y \}$$

$$+ \Omega^{-2} f^{-2} \{ 2p_0^4 (p_0^{-2} \beta)_{yz} + Kz p_0^3 (p_0^{-2} \beta)_y + Ky p_0^3 (p_0^{-2} \beta)_z \}. \quad (\text{A20})$$

APPENDIX B: THE EXISTENCE OF GAUGE TERMS IF a_1^A, A_2^A ARE NON-ZERO

In this appendix we demonstrate that a_1^A, a_2^A appearing in Eq. (5.6) are pure gauge terms. For clarity we shall consider only cases when $f' \neq 0$. When $a_1^A \neq 0, a_2^A \neq 0$ the line element (5.6) with b, c given in (5.7) can be written in the form

$$ds^2 = 2\Omega^2 dudv + p_0^{-2} f^2 \Omega^2 \{ (1 + \alpha) dy + \beta dz + Adu + Pdv \}^2 + p_0^{-2} f^2 \Omega^2 \{ \beta dy + (1 - \alpha) dz + Bdu + Qdv \}^2, \quad (\text{B1})$$

where

$$\begin{aligned} A &= a_1^A e^\alpha \cosh \beta + a_2^A e^{-\alpha} \sinh \beta, \\ B &= a_1^A e^\alpha \sinh \beta + a_2^A e^{-\alpha} \cosh \beta, \\ P &= a_2^A e^\alpha \cosh \beta + a_1^A e^{-\alpha} \sinh \beta, \\ Q &= a_2^A e^\alpha \sinh \beta + a_1^A e^{-\alpha} \cosh \beta. \end{aligned} \quad (\text{B2})$$

We find it convenient to work on the following tetrad:

$$\begin{aligned} \theta^1 &= p_0^{-1} f \Omega \{ (1 + \alpha) dy + \beta dz + Adu + Pdv \}, \\ \theta^2 &= p_0^{-1} f \Omega \{ \beta dy + (1 - \alpha) dz + Bdu + Qdv \}, \\ \theta^3 &= \Omega du, \quad \theta^4 = \Omega dv. \end{aligned} \quad (\text{B3})$$

As before our first step is to calculate the Ricci tensor components. In this case they are found to be

$$R_{13} = 2p_0^{-1} f' \Omega^{-2} (P_u - A_v) - p_0^{-1} f \Omega^{-3} \Omega' (P_u - A_v) + \frac{1}{2} p_0^{-1} f \Omega^{-2} (P_u - A_v)_u + p_0^{-3} f^{-1} \Omega^{-2} \{ (p_0^{-2} \alpha)_{yu} + (p_0^{-2} \beta)_{zu} \} - \frac{1}{2} p_0^3 f^{-1} \Omega^{-2} \left(B_z K p_0^{-3} y - \frac{1}{2} B K^2 p_0^{-4} yz - p_0^{-2} B_{yz} \right) - \frac{1}{2} p_0^3 f^{-1} \Omega^{-2} \left(p_0^{-2} A_{zz} - p_0^{-3} K A_{zz} + A K p_0^{-3} - \frac{1}{2} A K^2 p_0^{-4} y^2 \right), \quad (\text{B4})$$

$$R_{23} = 2p_0^{-1} f' \Omega^{-2} (Q_u - B_v) - p_0^{-1} f \Omega^{-3} \Omega' (Q_u - B_v) + \frac{1}{2} p_0^{-1} f \Omega^{-2} (Q_u - B_v)_u - p_0^{-3} f^{-1} \Omega^{-2} \{ (p_0^{-2} \alpha)_{zu} - (p_0^{-2} \beta)_{yu} \} - \frac{1}{2} p_0^3 f^{-1} \Omega^{-2} \left(A_y K p_0^{-3} z - \frac{1}{2} A K^2 p_0^{-4} yz - p_0^{-2} A_{yz} \right) - \frac{1}{2} p_0^3 f^{-1} \Omega^{-2} \left(p_0^{-2} B_{yy} - p_0^{-3} K B_{yz} + B K p_0^{-3} - \frac{1}{2} B K^2 p_0^{-4} z^2 \right), \quad (\text{B5})$$

$$\begin{aligned}
R_{33} = & 4\Omega^{-4}\Omega'^2 - 2\Omega^{-3}\Omega'' - 2\Omega^{-2}f^{-1}f'' + p_0^2\Omega^{-2}\{(p_0^{-2}A)_{yu} + (p_0^{-2}B)_{zu}\} \\
& + 2\Omega^{-2}f^{-1}f'p_0^2\{(p_0^{-2}A)_y + (p_0^{-2}B)_z\},
\end{aligned} \tag{B6}$$

$$\begin{aligned}
R_{44} = & 4\Omega^{-4}\Omega'^2 - 2\Omega^{-3}\Omega'' - 2\Omega^{-2}f^{-1}f'' + p_0^{-2}\Omega^{-2}\{(p_0^{-2}P)_{yv} + (p_0^{-2}Q)_{zv}\} \\
& + 2\Omega^{-2}f^{-1}f'p_0^2\{(p_0^{-2}Q)_z + (p_0^{-2}P)_y\},
\end{aligned} \tag{B7}$$

$$\begin{aligned}
R_{34} = & 4\Omega^{-3}\Omega'' - 2\Omega^{-4}\Omega'^2 - 2\Omega^{-2}f^{-1}f'' + p_0^2\Omega^{-3}\Omega'\{(p_0^{-2}A)_y + (p_0^{-2}B)_z\} \\
& - p_0^2\Omega^{-3}\Omega'\{(p_0^{-2}P)_y + (p_0^{-2}Q)_z\} + \frac{1}{2}p_0^2\Omega^{-2}\{(p_0^{-2}A)_{yv} + (p_0^{-2}B)_{zv}\} \\
& + \frac{1}{2}p_0^2\Omega^{-2}\{(p_0^{-2}P)_{yu} + (p_0^{-2}Q)_{zu}\} + p_0^2f^{-1}f'\Omega^{-2}\{(p_0^{-2}A)_y + (p_0^{-2}B)_z\} \\
& + p_0^2f^{-1}f'\Omega^{-2}\{(p_0^{-2}P)_y + (p_0^{-2}Q)_z\},
\end{aligned} \tag{B8}$$

$$\begin{aligned}
R_{14} = & p_0^3f^{-1}\Omega^{-2}\{(p_0^{-2}\alpha)_{yv} + (p_0^{-2}\beta)_{zv}\} - p_0^{-1}f\Omega^{-3}\Omega'(P_u - A_v) \\
& - 2p_0^{-1}f'\Omega^{-2}(P_u - A_v) - \frac{1}{2}p_0^{-1}f\Omega^{-2}(P_u - A_v)_v \\
& - \frac{1}{2}p_0^3f^{-1}\Omega^{-2}\left\{p_0^{-2}P_{zz} - P_zKp_0^{-3}z + PKp_0^{-3} - \frac{1}{2}PK^2p_0^{-4}y^2\right\} \\
& - \frac{1}{2}p_0^3f^{-1}\Omega^{-2}\left\{Q_zKp_0^{-3}y - p_0^{-2}Q_{yz} - \frac{1}{2}QK^2p_0^{-4}yz\right\},
\end{aligned} \tag{B9}$$

$$\begin{aligned}
R_{24} = & -p_0^3f^{-1}\Omega^{-2}\{(p_0^{-2}\alpha)_{zv} - (p_0^{-2}\beta)_{yv}\} - p_0^{-1}f\Omega^{-3}\Omega'(Q_u - B_v) \\
& - 2p_0^{-1}f'\Omega^{-2}(Q_u - B_v) - \frac{1}{2}p_0^{-1}f\Omega^{-2}(Q_u - B_v)_v \\
& - \frac{1}{2}p_0^3f^{-1}\Omega^{-2}\left\{p_0^{-2}Q_{yy} - Q_yKp_0^{-3}y + QKp_0^{-3} - \frac{1}{2}QK^2p_0^{-4}z^2\right\} \\
& - \frac{1}{2}p_0^3f^{-1}\Omega^{-2}\left\{P_yKp_0^{-3}z - p_0^{-2}P_{yz} - \frac{1}{2}PK^2p_0^{-4}yz\right\},
\end{aligned} \tag{B10}$$

$$\begin{aligned}
R_{11} = & \Omega^{-2}f^{-2}\{p_0^4(p_0^{-2}\alpha)_{zz} - p_0^4(p_0^{-2}\alpha)_{yy} + Kzp_0^3(p_0^{-2}\alpha)_z - Ky p_0^3(p_0^{-2}\alpha)_y\} \\
& + \Omega^{-2}f^{-2}\{2p_0^4(p_0^{-2}\beta)_{yz} + Kzp_0^3(p_0^{-2}\beta)_y + Ky p_0^3(p_0^{-2}\beta)_z\} \\
& + \Omega^{-2}f^{-2}K + (p_0^2\Omega^{-3}\Omega' + p_0^2f^{-1}f'\Omega^{-2})\{3p_0^{-2}A_y - 2AKp_0^{-3}y - 2BKp_0^{-3}z + p_0^{-2}B_z\} \\
& - (p_0^2\Omega^{-3}\Omega' - p_0^2f^{-1}f'\Omega^{-2})\{3p_0^{-2}P_y - 2PKp_0^{-3}y - 2QKp_0^{-3}z + p_0^{-2}Q_z\} \\
& - \frac{1}{2}p_0^{-1}\Omega^{-2}Kz(B_v + Q_u) + \Omega^{-2}\left(A_{yv} - \frac{1}{2}A_vKp_0^{-1}y + P_{uy} - \frac{1}{2}P_uKp_0^{-1}y\right) \\
& - 2\Omega^{-2}\alpha_{uv} - 2\Omega^{-2}f^{-2}f'^2 + 2\Omega^{-4}\Omega'^2 - 2\Omega^{-2}f^{-1}f'' + 2\Omega^{-3}\Omega'' \\
& - 2\Omega^{-3}\Omega'(\alpha_u - \alpha_v) - 2\Omega^{-2}f^{-1}f'(\alpha_v + \alpha_u),
\end{aligned} \tag{B11}$$

$$\begin{aligned}
R_{22} = & \Omega^{-2} f^{-2} \{ p_0^4 (p_0^{-2} \alpha)_{zz} - p_0^4 (p_0^{-2} \alpha)_{yy} + Kz p_0^3 (p_0^{-2} \alpha)_z - Ky p_0^3 (p_0^{-2} \alpha)_y \} \\
& + \Omega^{-2} f^{-2} \{ 2p_0^4 (p_0^{-2} \beta)_{yz} + Ky p_0^3 (p_0^{-2} \beta)_z + Kz p_0^3 (p_0^{-2} \beta)_y \} + \Omega^{-2} f^{-2} K \\
& + (p_0^2 \Omega^{-3} \Omega' + p_0^2 f^{-1} f' \Omega^{-2}) \{ 3p_0^{-2} B_z - 2BK p_0^{-3} z - 2AK p_0^{-3} y + p_0^{-2} A_y \} \\
& - (p_0^2 \Omega^{-3} \Omega' - p_0^2 f^{-1} f' \Omega^{-2}) \{ 3p_0^{-2} Q_z - 2QK p_0^{-3} z - 2PK p_0^{-3} y + p_0^{-2} P_y \} \\
& - \frac{1}{2} p_0^{-1} \Omega^{-2} Ky (A_v + P_u) + \Omega^{-2} \left(B_{zv} - \frac{1}{2} B_v K p_0^{-1} z + Q_{uz} - \frac{1}{2} Q_u K p_0^{-1} z \right) \\
& + 2\Omega^{-2} \alpha_{uv} - 2\Omega^{-2} f^{-2} f'^2 + 2\Omega^{-4} \Omega'^2 - 2\Omega^{-2} f^{-1} f'' + 2\Omega^{-3} \Omega'' \\
& + 2\Omega^{-3} \Omega' (\alpha_u - \alpha_v) + 2\Omega^{-2} f^{-1} f' (\alpha_v + \alpha_u), \tag{B12}
\end{aligned}$$

$$\begin{aligned}
R_{12} = & \Omega^{-3} \Omega' (A_z + B_y) - \Omega^{-3} \Omega' (P_z + Q_y) + 2\Omega^{-3} \Omega' (\beta_v - \beta_u) \\
& - 2\Omega^{-2} \beta_{uv} - 2\Omega^{-2} f^{-1} f' (\beta_u + \beta_v) + \Omega^{-2} f^{-1} f' (A_z + B_y) \\
& + \Omega^{-2} f^{-1} f' (Q_y + P_z) + \frac{1}{2} \Omega^{-2} (A_{zv} + B_{yv}) + \frac{1}{2} \Omega^{-2} (P_{zu} + Q_{yu}). \tag{B13}
\end{aligned}$$

Here the subscripts y, z, u, v indicate partial differentiation with respect to these variables, differentiation with respect to v is denoted by a prime and K is the constant introduced in Eq. (3.4). Using the above Ricci tensor components and Eqs. (3.2), (3.3), (5.11) it is easily checked that the first of the field equations (2.11) is satisfied provided we choose A, B, P, Q to satisfy the Cauchy-Riemann equations

$$(p_0^{-2} A)_z = (p_0^{-2} B)_y, \quad (p_0^{-2} A)_y = -(p_0^{-2} B)_z, \tag{B14}$$

$$(p_0^{-2} P)_z = (p_0^{-2} Q)_y, \quad (p_0^{-2} P)_y = -(p_0^{-2} Q)_z. \tag{B15}$$

Next with A, B, P, Q satisfying these equations we find from the remaining two equations in (2.11) that the conditions

$$p_0^{-4} AK = (p_0^{-2} \alpha)_{yu} + (p_0^{-2} \beta)_{zu}, \tag{B16}$$

$$p_0^{-4} PK = (p_0^{-2} \alpha)_{yv} + (p_0^{-2} \beta)_{zv}, \tag{B17}$$

$$p_0^{-4} BK = -(p_0^{-2} \alpha)_{zu} + (p_0^{-2} \beta)_{yu}, \tag{B18}$$

$$p_0^{-4} QK = -(p_0^{-2} \alpha)_{zv} + (p_0^{-2} \beta)_{yv}, \tag{B19}$$

$$P_u = A_v, \quad Q_u = B_v, \tag{B20}$$

are sufficient to have $q_a \equiv 0$ and $\pi_{ab} = 0$ except for

$$\begin{aligned}
\pi_{11} = & \left(\frac{1}{\sqrt{2}} R^{-2} \dot{R} p_0^2 + R^{-2} f^{-1} f' p_0^2 \right) \{ 3p_0^{-2} A_y - 2AK p_0^{-3} y - 2BK p_0^{-3} z + p_0^{-2} B_z \} \\
& - \left(\frac{1}{\sqrt{2}} R^{-2} \dot{R} p_0^2 - R^{-2} f^{-1} f' p_0^2 \right) \{ 3p_0^{-2} P_y - 2PK p_0^{-3} y - 2QK p_0^{-3} z + p_0^{-2} Q_z \} \\
& - \frac{1}{2} R^{-2} K p_0^{-1} z (B_v + Q_u) + R^{-2} \left(A_{yv} - \frac{1}{2} A_v K p_0^{-1} y + P_{uy} - \frac{1}{2} P_u K p_0^{-1} y \right) \\
& + R^{-2} f^2 \{ p_0^4 (p_0^{-2} \alpha)_{zz} - p_0^4 (p_0^{-2} \alpha)_{yy} + Kz p_0^3 (p_0^{-2} \alpha)_z - Ky p_0^3 (p_0^{-2} \alpha)_y \} \\
& + R^{-2} f^2 \{ 2p_0^4 (p_0^{-2} \beta)_{yz} + Kz p_0^3 (p_0^{-2} \beta)_y + Ky p_0^3 (p_0^{-2} \beta)_z \} - 2R^{-2} \alpha_{vu} \\
& + \frac{2}{\sqrt{2}} R^{-2} \dot{R} (\alpha_v - \alpha_u) - 2R^{-2} f^{-1} f' (\alpha_v + \alpha_u), \tag{B21}
\end{aligned}$$

$$\begin{aligned}
\pi_{22} = & \left(\frac{1}{\sqrt{2}} R^{-2} \dot{R} p_0^2 + R^{-2} f^{-1} f' p_0^2 \right) \{ 3 p_0^{-2} B_z - 2 B K p_0^{-3} z - 2 A K p_0^{-3} y + p_0^{-2} A_y \} \\
& - \left(\frac{1}{\sqrt{2}} R^{-2} \dot{R} p_0^2 - R^{-2} f^{-1} f' p_0^2 \right) \{ 3 p_0^{-2} Q_z - 2 Q K p_0^{-3} z - 2 P K p_0^{-3} y + p_0^{-2} P_y \} \\
& - \frac{1}{2} R^{-2} K p_0^{-1} z (A_v + P_u) + R^{-2} \left(B_{zv} - \frac{1}{2} B_v K p_0^{-1} z + P_{uz} - \frac{1}{2} Q_u K p_0^{-1} z \right) \\
& + R^{-2} f^2 \{ p_0^4 (p_0^{-2} \alpha)_{zz} - p_0^4 (p_0^{-2} \alpha)_{yy} + K z p_0^3 (p_0^{-2} \alpha)_z - K y p_0^3 (p_0^{-2} \alpha)_y \} \\
& + R^{-2} f^2 \{ 2 p_0^4 (p_0^{-2} \beta)_{yz} + K z p_0^3 (p_0^{-2} \beta)_y + K y p_0^3 (p_0^{-2} \beta)_z \} \\
& + 2 R^{-2} \alpha_{vu} + \frac{2}{\sqrt{2}} R^{-2} \dot{R} (\alpha_u - \alpha_v) + 2 R^{-2} f^{-1} f' (\alpha_v + \alpha_u), \tag{B22}
\end{aligned}$$

$$\begin{aligned}
\pi_{12} = & \frac{1}{\sqrt{2}} R^{-2} \dot{R} (A_z + B_y - P_z - Q_y) + R^{-2} f^{-1} f' (A_z + B_y + P_z + Q_y) + \frac{1}{2} R^{-2} (A_{zv} + B_{yv} + P_{zu} + Q_{yu}) \\
& + \sqrt{2} R^{-2} \dot{R} (\beta_v - \beta_u) - 2 R^{-2} \beta_{uv} - 2 R^{-2} f^{-1} f' (\beta_u + \beta_v). \tag{B23}
\end{aligned}$$

Following the procedure described in Sec. V we now construct $\bar{\pi} = (\pi_{11} - \pi_{22} - 2i\pi_{12})/2$. Using the conditions above to cancel terms we arrive at

$$\begin{aligned}
\bar{\pi} = & \sqrt{2} R^{-2} \dot{R} \{ \alpha_v - \alpha_u - i(\beta_v - \beta_u) \} \\
& - 2 R^{-2} f^{-1} f' \{ \alpha_v + \alpha_u - i(\beta_v + \beta_u) \} \\
& + \frac{1}{\sqrt{2}} R^{-2} \dot{R} \{ A_y - B_z + Q_z - P_y - i(A_z + B_y - P_z - Q_y) \} \\
& - 2 R^{-2} (\alpha_{uv} - i\beta_{uv}) \\
& + R^{-2} f^{-1} f' \{ A_y - B_z + P_y - Q_z - i(A_z + B_y + P_z + Q_y) \} \\
& + \frac{1}{2} R^{-2} \{ A_{yv} - B_{zv} + P_{yu} - Q_{zu} \\
& - i(A_{zv} + B_{yv} + P_{zu} + Q_{yu}) \}. \tag{B24}
\end{aligned}$$

We want to write $\bar{\pi}$ in the form given in Eq. (5.26) for some analytic function \mathcal{G} . Before we try to do this we note that on account of the conditions $P_u = A_v$, $Q_u = B_v$ we can write

$$A = F_u, \quad P = F_v, \tag{B25}$$

$$B = G_u, \quad Q = G_v, \tag{B26}$$

for some functions $F(y, z, u, v)$, $G(y, z, u, v)$ which satisfy the Cauchy-Riemann equations

$$(p_0^{-2} F)_y = -(p_0^{-2} G)_z, \quad (p_0^{-2} F)_z = (p_0^{-2} G)_y. \tag{B27}$$

Substituting these into Eq. (B24) gives

$$\begin{aligned}
\bar{\pi} = & \frac{1}{\sqrt{2}} R^{-2} \dot{R} \{ F_{uy} - G_{uz} + G_{vz} - F_{vy} \\
& - i(F_{uz} + G_{uy} - F_{vz} - G_{vy}) \} \\
& + R^{-2} f^{-1} f' \{ F_{uy} - G_{uz} + F_{vy} - G_{vz} \\
& - i(F_{uz} + G_{uy} + F_{vz} + G_{vy}) \} \\
& + R^{-2} \{ F_{uyv} - G_{uvz} - i(F_{uvz} + G_{uvz}) \} \\
& + \sqrt{2} R^{-2} \dot{R} \{ \alpha_v - \alpha_u - i(\beta_v - \beta_u) \} \\
& - 2 R^{-2} f^{-1} f' \{ \alpha_v + \alpha_u - i(\beta_v + \beta_u) \} \\
& - 2 R^{-2} (\alpha_{uv} - i\beta_{uv}). \tag{B28}
\end{aligned}$$

In order to write $\bar{\pi}$ in the required form we choose

$$\begin{aligned}
\mathcal{G} = & \frac{1}{2\sqrt{2}} p_0^{-2} f \{ G_{uz} - F_{uy} - G_{vz} + F_{vy} \\
& + i(F_{uz} + G_{uy} - F_{vz} - G_{vy}) \} \\
& - \frac{1}{\sqrt{2}} p_0^{-2} f \{ \alpha_v - \alpha_u + i(\beta_u - \beta_v) \}. \tag{B29}
\end{aligned}$$

Noting that $D = \sqrt{2} \partial_v$ we find that this \mathcal{G} does satisfy Eq. (5.26) with $\bar{\pi}$ given by (B28) and $f' \neq 0$ provided α , β take the following form:

$$\alpha = \frac{1}{2} (F_y - G_z) + q(y, z, u), \quad \beta = \frac{1}{2} (F_z + G_y) + r(y, z, u). \tag{B30}$$

Here q, r satisfy the Cauchy-Riemann equations

$$(p_0^{-2}q)_y = -(p_0^{-2}r)_z, \quad (p_0^{-2}q)_z = (p_0^{-2}r)_y. \quad (\text{B31})$$

We remark that this is the first time we have made use of the fact that $f' \neq 0$. If $f' = 0$ then α, β have a different form to that given in the last equation. Thus we emphasize that the analysis which follows does not apply if $f' = 0$. When α, β are given by these equations it is straightforward to check using the various Cauchy-Riemann equations that π_{ab} is trace-free and the conditions (B.16)–(B.19) are identically satisfied. Substituting the above expressions for α, β into Eq. (B29) yields

$$\mathcal{G} = \frac{1}{\sqrt{2}} p_0^{-2} f(q_u + ir_u). \quad (\text{B32})$$

With k given by the first of Eqs. (5.22) it is trivial to show that \mathcal{G} satisfies the wave equation (4.25). Now A, B, P, Q do not appear on the right-hand side of Eq. (B32) and hence a_1^A, a_2^A do not contribute to \mathcal{G} . Thus since the perturbed shear and anisotropic stress can both be written in terms of \mathcal{G} we conclude that a_1^A, a_2^A are pure gauge terms.

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