

Observer dependence for the phonon content of the sound field living on the effective curved space-time background of a Bose-Einstein condensate

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We demonstrate that the ambiguity of the particle content for quantum fields in a generally curved space-time can be experimentally investigated in an ultracold gas of atoms forming a Bose-Einstein condensate. We explicitly evaluate the response of a suitable condensed matter detector, an “atomic quantum dot,” which can be tuned to measure time intervals associated with different effective acoustic space-times. It is found that the detector response related to laboratory, “adiabatic,” and de Sitter time intervals is finite in time and nonstationary, vanishing, and thermal, respectively.

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I. INTRODUCTION

The feat of reproducing certain features of the physics of relativistic quantum fields on curved space-time backgrounds [1,2], in the laboratory, now seems closer than ever before. Because of the realization of the fact that phonons or generally “relativistic” quasiparticles, propagating in a nonrelativistic background fluid, experience an effective curved space-time [3–7], various exotic phenomena of the physics of classical as well as quantum fields in the curved space-times of gravity are getting within the reach of laboratory scale experiments [8].

The particle content of a quantum field state in curved or flat space-time depends on the motional state of the observer. A manifestation of this observer dependence is the Unruh-Davies effect, which consists in the fact that a constantly accelerated detector in the Minkowski vacuum responds as if it were placed in a thermal bath with temperature proportional to its acceleration [1,2]. This effect has eluded observation so far: The value of the Unruh temperature $T_{\text{Unruh}} = [\hbar/(2\pi k_{\text{B}}c_L)]a = 4 \text{ K} \times a/[10^{20}g_{\oplus}]$, where a is the acceleration of the detector in Minkowski space (g_{\oplus} is the gravity acceleration on the surface of the Earth) and c_L the speed of light, makes it obvious that an observation of the effect is a less than trivial undertaking. Proposals for a measurement with ultraintense short pulses of electromagnetic radiation have been put forward in, e.g., Refs. [9,10].

In the following, we shall argue that the observer dependence of the particle content of a quantum field state in curved space-time, related to the familiar nonuniqueness of canonical field quantization in Riemannian spaces [11], can be experimentally demonstrated in the readily available ultralow temperature condensed matter system Bose-Einstein condensate (BEC). More specifically, we argue that the Gibbons-Hawking effect [12], a curved space-time generalization of the Unruh-Davies effect, can be observed in an expanding BEC. To explicitly show that particle detection

depends on the detector setting, we construct a condensed matter detector tuned to time intervals in effective laboratory and de Sitter space-times, respectively [13]. We show that the detector response is strongly different in the two cases, and associated with the corresponding effective space-time. Furthermore, we describe a system of space-time coordinates in which there is no particle detection whatsoever taking place, which we will call the “adiabatic” basis, and which simply corresponds to a detector at rest in the (conformal) Minkowski vacuum.

II. HYDRODYNAMICS IN AN EXPANDING CIGAR-SHAPED BEC

It recently became apparent that among the most suitable systems for the simulation of quantum phenomena in effective space-times are Bose-Einstein condensates (BECs) [14–18]. They offer the primary advantages of dissipation-free superflow, high controllability, with atomic precision, of the physical parameters involved, and the accessibility of ultralow temperatures [19,20]. Even more importantly in the context under consideration here, the theory of phononic quasiparticles in the spatially and temporally inhomogeneous BEC is kinematically identical to that of a massless scalar field propagating on the background of curved space-time in $D+1$ dimensions.

It was shown by Unruh that the action of the phase fluctuations Φ in a moving inhomogeneous superfluid may be written in the form [3,4] (we set $\hbar = m = 1$, where m is the mass of a superfluid constituent particle):

$$S = \int d^{D+1}x \frac{1}{2g} \left[- \left(\frac{\partial}{\partial t} \Phi - \mathbf{v} \cdot \nabla \Phi \right)^2 + c^2 (\nabla \Phi)^2 \right] \\ \equiv \frac{1}{2} \int d^{D+1}x \sqrt{-g} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi. \quad (1)$$

Here, $\mathbf{v}(\mathbf{x}, t)$ is the superfluid background velocity, $c(\mathbf{x}, t) = \sqrt{g\rho_0}(\mathbf{x}, t)$ is the velocity of sound, where g is a constant

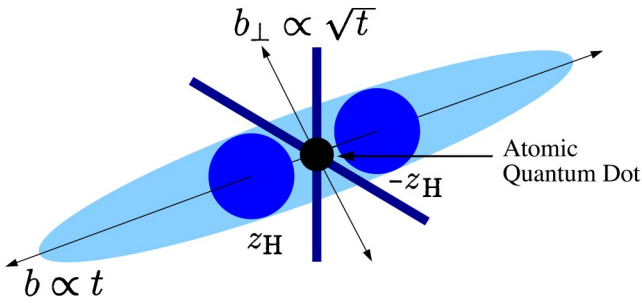


FIG. 1. Expansion of a cigar-shaped Bose-Einstein condensate. The stationary horizon surfaces are located at $\pm z_H$, respectively. The thick dark lines represent lasers creating an optical potential well in the center.

describing the interaction between the constituent particles in the superfluid ($1/g$ is the compressibility of the fluid), and $\rho_0(\mathbf{x}, t)$ is the background density. In the second line of (1), the conventional hydrodynamic action is identified with the action of a minimally coupled scalar field in an effective curved space-time. Furthermore, the velocity potential of the sound perturbations in the BEC satisfies the canonical commutation relations of a relativistic scalar field [14]. We therefore have the exact mapping, on the level of kinematics, of the equation of motion for phononic quasiparticles in a non-relativistic superfluid, to quantized massless scalar fields propagating on a curved space-time background with local Lorentz invariance.

To model curved space-times, we will consider the evolution of the BEC if we change the harmonic trapping frequencies with time. For a description of the expansion (or contraction) of a BEC, the so-called scaling solution approach is conventionally used [21]. One starts from a cigar-shaped BEC containing a large number of constituent particles, i.e., which is in the so-called Thomas-Fermi (TF) limit [22]. According to Ref. [21], the evolution of a Bose-condensed atom cloud under temporal variation of the trapping frequencies $\omega_{\parallel}(t)$ and $\omega_{\perp}(t)$ (in the axial and radial directions, respectively) can then be described by the following solution for the condensate wave function

$$\Psi = \frac{\Psi_{\text{TF}}}{b_{\perp} \sqrt{b}} \exp \left[-i \int g \rho_0(\mathbf{x} = \mathbf{0}, t) dt + i \frac{\dot{b} z^2}{2b} + i \frac{\dot{b}_{\perp} r^2}{2b_{\perp}} \right]. \quad (2)$$

Here, b_{\perp} and b are the scaling parameters describing the condensate evolution in the radial (\hat{r}) and axial (\hat{z}) directions (cf. Fig. 1). The initial ($b = b_{\perp} = 1$) mean-field condensate density is given by the usual TF expression

$$|\Psi_{\text{TF}}|^2 = \rho_{\text{TF}}(r, z) = \rho_m \left(1 - \frac{r^2}{R_{\perp}^2} - \frac{z^2}{R_{\parallel}^2} \right). \quad (3)$$

Here, ρ_m is the maximum density (in the center of the cloud) and the squared initial TF radii are $R_{\parallel}^2 = 2\mu/\omega_{\parallel}^2$ and $R_{\perp}^2 = 2\mu/\omega_{\perp}^2$. The initial chemical potential $\mu = \rho_m g$, where $g = 4\pi a_s$, and a_s is the scattering length characterizing

atomic collisions in the (dilute) BEC. In our cylindrical 3D trap, we have for the initial central density

$$\rho_m = \left(\frac{6N\omega_{\perp}^2\omega_{\parallel}}{\sqrt{8}\pi g^{3/2}} \right)^{2/5}.$$

The condition that the TF approximation be valid implies that $\mu \gg \omega_{\parallel}, \omega_{\perp}$. The solution (2) of the Gross-Pitaevskii mean-field equations becomes exact in this TF limit, independent of the ratio $\omega_{\parallel}/\omega_{\perp}$. However, the solution becomes exact also in the limit that $\omega_{\perp}/\omega_{\parallel} \rightarrow 0$, independent of the validity of the TF limit, the system then acquiring an effectively two-dimensional character [23,24]. We will see below that in the opposite limit of $\omega_{\parallel}/\omega_{\perp} \rightarrow 0$ there is an ‘‘adiabatic basis’’ in which no axial excitations are created during the expansion, i.e., with respect to that basis there are, in particular, no unstable solutions possible, implying the stability of the expanding gas against perturbations.

According to (2), the condensate density evolves as

$$\rho_0(r, z, t) = \frac{\rho_{\text{TF}}(r^2/b_{\perp}^2, z^2/b^2)}{b_{\perp}^2 b}, \quad (4)$$

and the superfluid velocity

$$\mathbf{v} = \frac{\dot{b}_{\perp}}{b_{\perp}} r \mathbf{e}_r + \frac{\dot{b}}{b} z \mathbf{e}_z \quad (5)$$

is the gradient of the condensate phase in Eq. (2). It increases linearly to the axial and radial boundaries of the condensate.

The excitations in the limit $\omega_{\parallel}/\omega_{\perp} \rightarrow 0$ were first studied in Ref. [25]. The description of the modes is based on an adiabatic separation for the axial and longitudinal variables of the phase fluctuation field:

$$\Phi(r, z, t) = \sum_n \phi_n(r) \chi_n(z, t), \quad (6)$$

where $\phi_n(r)$ is the radial wave function characterized by the quantum number n (we consider only zero angular momentum modes). The above ansatz incorporates the fact that for strongly elongated traps the dynamics of the condensate motion separates into a fast radial motion and a slow axial motion, which are essentially independent. The $\chi_n(z, t)$ are the mode functions for traveling wave solutions in the z direction (plane waves for a condensate at rest read $\chi_n \propto \exp[-i\epsilon_{n,k}t + kz]$). The radial motion is assumed to be ‘‘stiff’’ such that the radial part is effectively time independent, because the radial time scale for adjustment of the density distribution after a perturbation is much less than the axial oscillation time scales of interest. The ansatz (6) works independent of the ratio of healing length and radial size of the BEC cigar. In the limit that the healing length is much less than the radial size, TF wave functions are used, in the opposite limit, a Gaussian ansatz for the radial part of the wave function $\phi_n(r)$ is appropriate.

For axial excitations characterized by a wavelength $\lambda = 2\pi/k$ exceeding the radial size R_{\perp} of the condensate, we have $kR_{\perp} \ll 1$, and the dispersion relation reads, in the TF limit for the radial wave function [25]

$$\begin{aligned}\epsilon_{n,k}^2 &= 2\omega_{\perp}^2 n(n+1) + \frac{\omega_{\perp}^2}{4}(kR_{\perp})^2 \\ &= 2\omega_{\perp}^2 n(n+1) + c_0^2 k^2,\end{aligned}\quad (7)$$

where $c_0 = \sqrt{\mu/2}$. Observe that the central speed of sound c_0 is reduced by a factor $\sqrt{2}$ from the well-established value $\sqrt{g\rho_m} = \sqrt{\mu}$ for an infinitely extended liquid [25].

Equations (6) and (7) can be generalized for an expanding condensate. Substituting in Eq. (6) the rescaled radial wave function $\phi_n \equiv \phi_n(r/b_{\perp})$, and inserting the result into the action (1), integrating over the radial coordinates, we find the following effective action for the axial modes of a given radial quantum number n :

$$\begin{aligned}S_n &= \int dt dz \frac{b_{\perp}^2 C_n(z)}{2g} [-(\dot{\chi}_n - v_z \partial_z)^2 + \bar{c}_n^2(z) (\partial_z \chi_n)^2 \\ &\quad + M_n^2(z) \chi_n^2],\end{aligned}\quad (8)$$

where the common ‘‘conformal’’ factor $C_n(z)$ is given by

$$b_{\perp}^2 C_n(z) = \int_{r < r_m} d^2 r \phi_n^2. \quad (9)$$

The integration limits are fixed by the z dependent radial size of the cigar $r_m^2 = R_{\perp}^2 b_{\perp}^2 (1 - z^2/R_{\parallel}^2 b_{\perp}^2)$. The averaged speed of sound reads

$$\bar{c}_n^2(z, t) = \frac{g}{C_n b_{\perp}^2} \int_{r < r_m} d^2 r \rho_0 \phi_n^2 \quad (10)$$

and the (space and time dependent) effective mass term is, for a given radial mode, obtained to be

$$M_n^2(z, t) = \frac{g}{C_n b_{\perp}^2} \int_{r < r_m} d^2 r \rho_0 [\partial_r \phi_n]^2. \quad (11)$$

The phonon branch of the excitations corresponds to the gapless $n=0$ solution of Eq. (7). In this case the radial wave function ϕ_0 does not depend on the radial variable r [25], and the mass term vanishes, $M_0=0$. We then obtain the following expressions,

$$C_0(z) = \pi R_{\perp}^2 \left(1 - \frac{z^2}{b^2 R_{\parallel}^2} \right), \quad (12)$$

and for the z dependent speed of sound ($\bar{c} \equiv \bar{c}_{n=0}$):

$$\bar{c}^2(z, t) = \frac{c_0^2}{b_{\perp}^2 b} \left(1 - \frac{z^2}{b^2 R_{\parallel}^2} \right). \quad (13)$$

We will see below that we need these expressions in the limit $z \rightarrow 0$ only, because only in this limit we obtain the exact mapping of the phonon field to a quantum field propagating in a 1+1D curved space-time.

III. THE 1+1D DE SITTER METRIC IN THE CONDENSATE CENTER

We identify the action (8) with the action of a minimally coupled scalar field in 1+1D, according to Eq. (1). Remarkably, such an identification is possible only close to the center $z=0$. The contravariant 1+1D metric may generally be written as [4]

$$g^{\mu\nu} = \frac{1}{A_c c^2} \begin{pmatrix} -1 & -v_z \\ -v_z & c^2 - v_z^2 \end{pmatrix}, \quad (14)$$

where A_c is some arbitrary (space and time dependent) factor and $c = \bar{c}(z=0)$. Inverting this expression to get the covariant metric, we obtain

$$g_{\mu\nu} = A_c \begin{pmatrix} -(c^2 - v_z^2) & -v_z \\ -v_z & 1 \end{pmatrix}. \quad (15)$$

The term $\sqrt{-g} g^{\mu\nu}$ contained in the action (1) gives the familiar conformal invariance in a 1+1D space-time, i.e., the conformal factor A_c drops out from the action and thus does not influence the classical equations of motion. We therefore leave out A_c in the formulas to follow, but it needs to be borne in mind that the metric elements are defined always up to the factor A_c .

The actions (1) and (8) can be made consistent if we renormalize the phase field according to $\Phi = Z\bar{\Phi}$ and require that

$$\frac{b_{\perp}^2 C_0(0)}{g} Z^2 = \frac{1}{c} \quad (16)$$

holds. The factor Z does not influence the equation of motion, but does influence the response of a detector (see Sec. IV A below). In other words, it renormalizes the coupling of our ‘‘relativistic field’’ Φ to the laboratory frame detector. More explicitly, Eq. (16) leads to

$$b_{\perp} \sqrt{b} = 8 \sqrt{\frac{\pi}{2}} \frac{1}{Z^2} \sqrt{\rho_m a_s^3} \left(\frac{\omega_{\perp}}{\mu} \right)^2 \equiv B = \text{const.} \quad (17)$$

According to the above relation, we have to impose that the expansion of the cigar in the perpendicular direction proceeds like the square root of the expansion in the axial direction. The constant quantity B can be fixed externally (by the experimentalist), choosing the expansion of the cloud appropriately by adjusting the time dependence of the trapping frequencies $\omega_{\parallel}(t)$ and $\omega_{\perp}(t)$, according to the scaling equations [21,26]

$$\begin{aligned}\ddot{b} + \omega_{\parallel}^2(t) b &= \frac{\omega_{\parallel}^2}{b_{\perp}^2 b^2} = \frac{\omega_{\parallel}^2}{B^2 b^3}, \\ \ddot{b}_{\perp} + \omega_{\perp}^2(t) b_{\perp} &= \frac{\omega_{\perp}^2}{b_{\perp}^3 b} = \frac{\omega_{\perp}^2}{B^3 b^{5/2}}.\end{aligned}\quad (18)$$

Since both C_0 and \bar{c} depend on z , an effective space-time metric for the axial phonons can be obtained only close to the center of the cigar-shaped condensate cloud. This is related to our averaging over the physical perpendicular direction, and does not arise if the excitations are considered in the full D -dimensional situation, where this identification is possible globally. Also note that the action (8) does not contain a curvature scalar contribution of the form $\propto \chi_n^2 \sqrt{-g} R[\mathbf{g}_{\mu\nu}(x)]$, i.e., that it only possesses trivial conformal invariance [27].

We now impose, in addition, the requirement that the metric is identical to that of a 1+1D universe, with a metric of the form of the de Sitter metric in 3+1D [12,28]. We first apply the transformation $c_0 d\tilde{t} = c(t) dt$ to the line element defined by (15), connecting the laboratory time t to the time variable \tilde{t} . Defining $v_z/c = \sqrt{\Lambda} z = (B\dot{b}/c_0)z$ (note that the dot on b and other quantities always refers to ordinary laboratory time), this results, up to the conformal factor A_c , in the line element

$$ds^2 = -c_0^2(1 - \Lambda z^2)d\tilde{t}^2 - 2c_0 z \sqrt{\Lambda} d\tilde{t} dz + dz^2. \quad (19)$$

We then apply a second transformation $c_0 d\tau = c_0 d\tilde{t} + z \sqrt{\Lambda} dz / (1 - \Lambda z^2)$, with a constant Λ . We are thus led to the 1+1D de Sitter metric in the form [12]

$$ds^2 = -c_0^2(1 - \Lambda z^2)d\tau^2 + (1 - \Lambda z^2)^{-1} dz^2. \quad (20)$$

The transformation between t and the de Sitter time τ (on a constant z detector, such that $d\tilde{t} = dt$), is given by

$$\frac{t}{t_0} = \exp[B\dot{b}\tau], \quad (21)$$

where the unit of laboratory time $t_0 \sim \omega_{\parallel}^{-1}$ is set by the initial conditions for the scaling variables b and b_{\perp} . The temperature associated with the effective metric (20) is the Gibbons-Hawking temperature [12]

$$T_{\text{dS}} = \frac{c_0}{2\pi} \sqrt{\Lambda} = \frac{B}{2\pi} \dot{b}. \quad (22)$$

The ‘‘surface gravity’’ on the horizon has the value $a_{\text{H}} = c_0^2 \sqrt{\Lambda} = c_0 B \dot{b}$, and the stationary horizon(s) are located at the constant values of the z coordinate

$$z = \pm z_{\text{H}} = \pm R_{\parallel} \sqrt{\frac{\omega_{\parallel}^2}{2\mu\Lambda}}. \quad (23)$$

Combining (22) and (23), we see that $z_{\text{H}}/R_{\parallel}$ is small if $\omega_{\parallel}/T_{\text{dS}} \ll 4\pi$. Therefore, the de Sitter temperature needs to be at least of the order of ω_{\parallel} for the horizon location(s) to be well inside the cloud. The latter condition then justifies neglecting the z dependence in C_0 and \bar{c} in Eq. (16). Though there is no metric ‘‘behind’’ the horizon, i.e., at large z , this should not affect the low-energy behavior of the quantum vacuum ‘‘outside’’ the horizon.

IV. DETERMINING THE PARTICLE CONTENT OF THE QUANTUM FIELD

The particle content of a quantum field state depends on the observer [1,2,11]. To detect the Gibbons-Hawking effect in de Sitter space, one has to set up a detector that measures frequencies in units of the inverse de Sitter time τ , rather than in units of the inverse laboratory time t . We will show that the de Sitter time interval $d\tau = dt/bB = dt/\sqrt{b}b_{\perp}$ can be effectively measured by an atomic quantum dot (AQD) [29,30]. The measured quanta can then, and only then, be accurately interpreted to be particles coming from a Gibbons-Hawking type process with a constant de Sitter temperature (22). We then use the tunability for other time intervals feasible with our detector scheme, and contrast this de Sitter result with what the detector ‘‘sees’’ if tuned to laboratory and ‘‘adiabatic’’ time intervals.

The AQD can be made in a gas of atoms possessing two hyperfine ground states α and β . The atoms in state α represent the superfluid cigar, and are used to model the expanding universe. The AQD itself is formed by trapping atoms in state β in a tightly confining optical potential V_{opt} . The interaction of atoms in the two internal levels is described by a set of coupling parameters $g_{cd} = 4\pi a_{cd}$ ($c, d = \{\alpha, \beta\}$), where a_{cd} are the s -wave scattering lengths characterizing short-range intra- and interspecies collisions; $g_{\alpha\alpha} \equiv g$, $a_{\alpha\alpha} \equiv a_s$. The on-site repulsion between the atoms β in the dot is $U \sim g_{\beta\beta}/l^3$, where l is the characteristic size of the ground state wave function of atoms β localized in V_{opt} . In the following, we consider the collisional blockade limit of large $U > 0$, where only one atom of type β can be trapped in the dot. This assumes that U is larger than all other relevant frequency scales in the dynamics of both the AQD and the expanding superfluid. As a result, the collective coordinate of the AQD is modeled by a pseudo-spin-1/2, with the spin-up/spin-down state corresponding to occupation by a single atom/no atom in state β .

We first describe the AQD response to the condensate fluctuations in the Lagrangian formalism, most familiar in a field theoretical context. The detector Lagrangian takes the form

$$\begin{aligned} L_{\text{AQD}} = & i \left(\frac{d}{dt} \eta^* \right) \eta - [-\Delta + g_{\alpha\beta}(\rho_0(0,t) + \delta\rho)] \eta^* \eta \\ & - \Omega \sqrt{\rho_0(0,t)l^3} \left(\exp \left[-i \int_0^t g \rho_0(0,t') dt' + i \delta\phi \right] \eta^* \right. \\ & \left. + \exp \left[i \int_0^t g \rho_0(0,t') dt' - i \delta\phi \right] \eta \right). \end{aligned} \quad (24)$$

Here, Δ is the detuning of the laser light from resonance, $\rho_0(z=0,t)$ is the central mean-field part of the bath density, and l is the size of the AQD ground state wave function. The detector variable η is an anticommuting Grassmann variable representing the effective spin degree of freedom of the AQD. The second and third lines represent the coupling of the AQD to the surrounding superfluid, where $\delta\phi$ and $\delta\rho$ are the fluctuating parts of the condensate phase and density at

$z=0$, respectively. The laser intensity and the effective transition matrix element combine into the Rabi frequency Ω ; below we will make use of the fact that Ω can easily experimentally be changed as a function of laboratory time t , by changing the laser intensity with t .

To simplify (24), we use the canonical transformation

$$\eta \rightarrow \bar{\eta} \exp \left[-i \int_0^t g \rho_0(0, t') dt' + i \delta \phi \right]. \quad (25)$$

The above transformation amounts to absorbing the superfluid's chemical potential and the fluctuating phase $\delta \phi$ into the wave function of the AQD, and does not change the occupation numbers of the two AQD states. The transformation (25) gives the detector Lagrangian the form

$$L_{\text{AQD}} = i \left(\frac{d}{dt} \bar{\eta}^* \right) \bar{\eta} - \Omega \sqrt{\rho_0(0, t) l^3} (\bar{\eta} + \bar{\eta}^*) - \left[-\Delta + (g_{\alpha\beta} - g) \rho_0(0, t) + g_{\alpha\beta} \delta \rho + \frac{d}{dt} \delta \phi \right] \bar{\eta}^* \bar{\eta}. \quad (26)$$

The laser coupling (second term in the first line) scales as $b^{-1/2} b_{\perp}^{-1}$, and hence like the de Sitter time interval in units of the laboratory time interval, $d\tau/dt$. We suggest that the detector be operated at the time dependent detuning $\Delta(t) = (g_{\alpha\beta} - g) \rho_0(0, t) = (g_{\alpha\beta} - g) \rho_m / (b^2 B^2 t^2)$, which then leads to a vanishing of the first two terms in the square brackets of (26).

We now reintroduce the wave function of the AQD stemming from a Hamiltonian formulation, $\psi = \psi_{\beta} | \beta \rangle + \psi_{\alpha} | \alpha \rangle$. An ‘‘effective Rabi frequency’’ may be defined to be $\omega_0 = 2\Omega \sqrt{\rho_m l^3}$; at the detuning compensated point, we then obtain a simple set of coupled equations for the AQD amplitudes

$$i \frac{d\psi_{\beta}}{d\tau} = \frac{\omega_0}{2} \psi_{\alpha} + \delta V \psi_{\beta}, \quad i \frac{d\psi_{\alpha}}{d\tau} = \frac{\omega_0}{2} \psi_{\beta}, \quad (27)$$

where τ is the de Sitter time.

We have thus shown that the detector equations (27) are natural evolution equations in de Sitter time τ , if the Rabi frequency Ω is chosen to be a constant, independent of laboratory time t . We will see in Sec. IV B that, adjusting Ω in a certain time dependent manner, within the same detector scheme, we can reproduce time intervals associated to various other effective space-times.

The coupling of the AQD to fluctuations in the superfluid is described by the potential

$$\delta V(\tau) = (g_{\alpha\beta} - g) B b(\tau) \delta \rho(\tau). \quad (28)$$

Neglecting the fluctuations in the superfluid, the level separation implied by (27) is ω_0 , and the eigenfunctions of the dressed two level system are $|\pm\rangle = (|\alpha\rangle \pm |\beta\rangle) / \sqrt{2}$. The quantity ω_0 therefore plays the role of a frequency standard of the detector. By adjusting the value of the laser intensity,

one can change ω_0 , and therefore probe the response of the detector for various phonon frequencies. Note that if $g_{\alpha\beta}$ is very close to g , to obtain the correct perturbation potential, higher order terms in the density fluctuations have to be taken into account in the Rabi term of (26).

To describe the detector response, we first have to solve the equations of motion (8) for the phase fluctuations, and then evaluate the conjugate density fluctuations. The equation of motion $\delta S_0 / \delta \chi_0 = 0$ is, for time independent B , given by

$$B^2 b^2 \frac{d}{dt} (b^2 \dot{\chi}_0) - \frac{1}{C_0(z_b)} \partial_{z_b} [\bar{c}^2(z_b) C_0(z_b) \partial_{z_b} \chi_0] = 0, \quad (29)$$

where $z_b = z/b$ is the scaling coordinate. Apart from the factor $C_0(z_b)$, stemming from averaging over the perpendicular direction, this equation corresponds to the hydrodynamic equation of phase fluctuations in inhomogeneous superfluids [31]. At $t \rightarrow -\infty$, the condensate is in equilibrium and the quantum vacuum phase fluctuations close to the center of the condensate can be written in the following form:

$$\hat{\chi}_0 = \sqrt{\frac{g}{4C_0(0)R_{\parallel}\epsilon_{0,k}}} \hat{a}_k \exp[-i\epsilon_{0,k}t + ikz] + \text{H.c.}, \quad (30)$$

where $\hat{a}_k, \hat{a}_k^{\dagger}$ are the annihilation and creation operators of a phonon. The initial quantum state of phonons is the ground state of the superfluid and is annihilated by the operators \hat{a}_k . With these initial conditions, the solution of (29) is

$$\hat{\chi}_0 = \sqrt{\frac{g}{4C_0(0)R_{\parallel}\epsilon_{0,k}}} \hat{a}_k \exp \left[-i \int_0^t \frac{dt' \epsilon_{0,k}}{B b^2} + ikz_b \right] + \text{H.c.} \quad (31)$$

The solution for the canonically conjugate density fluctuations, consequently, is

$$\delta \hat{\rho} = i \sqrt{\frac{\epsilon_{0,k}}{4C_0(0)R_{\parallel}g}} \frac{\partial}{\partial t} \left(\hat{a}_k \left\{ \exp \left[-i \int_0^t \frac{dt' \epsilon_{0,k}}{B b^2} + ikz_b \right] \right\} \right) + \text{H.c.} \quad (32)$$

Equations (31) and (32) completely characterize the $n=0$ evolution of the condensate fluctuations. Observe that the evolution proceeds without frequency mixing in the adiabatic time interval defined by $d\tau_a = dt/Bb^2$ (the ‘‘scaling time’’ interval dt/B^2b^2 defined in Ref. [26] is proportional to this adiabatic time interval). Therefore, in the ‘‘adiabatic basis,’’ no frequency mixing occurs and thus no quasiparticle excitations are created. This hints at a hidden (low-energy) symmetry, in analogy to the (exact) 2+1D Lorentz group $SO(2,1)$ for an isotropically expanding BEC disk, discussed in Ref. [23].

A. Detection in de Sitter time

The coupling operator $\delta \hat{V}$ causes transitions between the dressed detector states $|+\rangle$ and $|-\rangle$ and thus can be used to effectively measure the quantum state of the phonons. We consider the detector response to fluctuations of $\hat{\Psi}$, by going beyond mean field and using a perturbation theory in $\delta \hat{V}$.

There are two physically different situations. The detector is either at $t=0$ in its ground state, $(|\alpha\rangle + |\beta\rangle)/\sqrt{2}$, or in its excited state, $(|\alpha\rangle - |\beta\rangle)/\sqrt{2}$. We define P_+ and P_- to be the probabilities that at late times t the detector is excited respectively deexcited. Using second order perturbation theory in $\delta\hat{V}$, we find that the transition probabilities for the detector may be written

$$P_{\pm} = \sum_k \frac{g \epsilon_{0,k}}{4R_{\parallel}C_0(0)} \left(\frac{g_{\alpha\beta}}{g} - 1 \right)^2 B^2 |T_{\pm}|^2, \quad (33)$$

where the absolute square of the transition matrix element is given by

$$|T_{\pm}|^2 = \left| \int_0^{\infty} \frac{d\tau}{b(\tau)} \exp \left[\pm i \epsilon_{0,k} \int_0^{\tau} \frac{d\tau'}{b(\tau')} + i \omega_0 \tau \right] \right|^2. \quad (34)$$

Calculating the integrals, we obtain

$$P_{\pm} = J \left(\frac{g_{\alpha\beta}}{g} - 1 \right)^2 B^2 \frac{g \pi}{2B\dot{b}R_{\parallel}C_0(0)} \times \begin{cases} n_B \\ 1 + n_B \end{cases}, \quad (35)$$

where the (formally divergent) sum

$$J = \sum_k \frac{\omega_0}{\epsilon_{0,k}}, \quad (36)$$

and the factors

$$n_B = \frac{1}{\exp[\omega_0/T_{\text{ds}}] - 1} \quad (37)$$

are Bose distribution functions at the de Sitter temperature (22). We conclude that an expansion of the condensate in z direction, with a constant rate faster than the harmonic trap oscillation frequency in that direction, gives an effective space-time characterized by the de Sitter temperature T_{ds} .

We now show that J is proportional to the total *de Sitter* time of observation, so that the probability per unit time is a finite quantity [32]. At late times, the detector measures phonon quanta coming, relative to its space-time perspective, from close to the horizon, at a distance $\delta z = z_{\text{H}} - z \ll z_{\text{H}} = \Lambda^{-1/2}$. The trajectory of such a phonon in the coordinates of the de Sitter metric (20), at late times τ , is given by (cf. Fig. 2)

$$\ln \left[\frac{z_{\text{H}}}{\delta z} \right] = 2\sqrt{\Lambda} c_0 \tau. \quad (38)$$

This implies that the central AQD detector measures quanta that originated at the horizon with large shifted frequency

$$\epsilon_{0,k} = \frac{\omega_0}{\sqrt{2\Lambda}^{1/4} \delta z^{1/2}} = \frac{\omega_0}{\sqrt{2}} \exp[c_0 \tau \sqrt{\Lambda}]. \quad (39)$$

Making use of the above equation, we rewrite the summation over k in (36) as an integral over detector time:

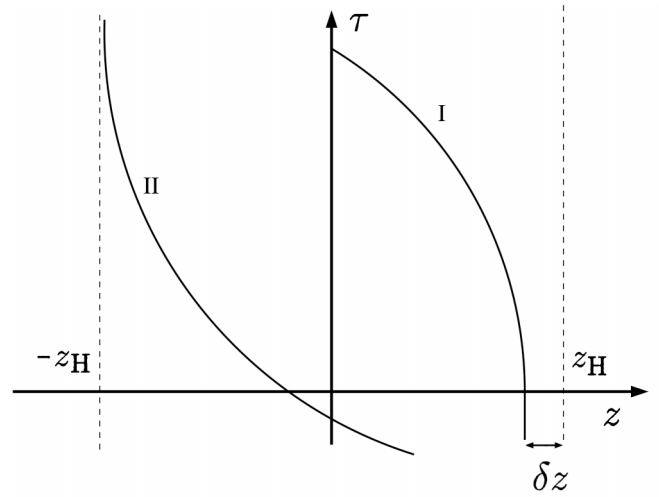


FIG. 2. Typical trajectories of phonons ($ds^2=0$) in the de Sitter metric (20). The path taken by phonon I, which is at early times propagating near the horizon, is described in the text. The path taken by phonon II, which approaches the horizon surface $-z_{\text{H}}$ at late times τ , does not lead to an excitation of the de Sitter detector placed at $z=0$.

$$\begin{aligned} J &= \sum_k \frac{\omega_0}{\epsilon_{0,k}} = \frac{R_{\parallel} \omega_0}{\pi c_0} \int \frac{d\epsilon_{0,k}}{\epsilon_{0,k}} = \frac{R_{\parallel} \omega_0}{\pi c_0} \sqrt{\Lambda} c_0 \int d\tau \\ &= \frac{R_{\parallel} \omega_0}{\pi c_0} B \dot{b} \tau. \end{aligned} \quad (40)$$

Therefore, the probabilities per unit detector time (de Sitter time) read, where upper/lower entries refer to P_+/P_- , respectively:

$$\frac{dP_{\pm}}{d\tau} = \left(\frac{g_{\alpha\beta}}{g} - 1 \right)^2 B^2 \frac{g \omega_0}{2C_0(0)c_0} \times \begin{cases} n_B \\ 1 + n_B \end{cases}. \quad (41)$$

They are finite quantities in the limit that $\tau \rightarrow \infty$. In laboratory time, the transition probabilities evolve according to

$$P_{\pm}(t) = P_0 \frac{\omega_0}{T_{\text{ds}}} \ln \left[\frac{t}{t_0} \right] \times \begin{cases} n_B \\ 1 + n_B \end{cases}, \quad (42)$$

where, from relation (16), $P_0 = Z^2 [(g_{\alpha\beta}/g - 1)B]^2/2$. We see that the detector response is, as it should be, proportional to Z^2 , the square of the renormalization factor of the phase fluctuation field.

The absorption and emission coefficients $dP_{\pm}/d\tau$ satisfy Einsteinian relations. Therefore, the detector approaches thermal equilibrium at a temperature T_{ds} on a time scale proportional to $Z^{-2} \omega_0^{-1}$. Our de Sitter AQD detector thus measures a stationary thermal spectrum, even though its condensed matter background, with laboratory time t , is in a highly nonstationary motional state. Since $Z^2 \propto \sqrt{\rho_m} a_s^3 (\omega_{\perp}/\mu)^2$, not-too-dilute condensates with $\omega_{\perp} \sim \mu$ (i.e., close to the quasi-1D regime [33]) are most suitable for observing the Gibbons-Hawking effect.

The verification of the fact that a thermal detector state has been established proceeds by the fact that the two hyper-

fine states α and β are spectroscopically different states of the same atom, easily detectable by modern quantum optical technology. When the optical potential is switched on, the atoms are in the empty α state originally, which is an equal-weight superposition of $|+\rangle = (|\alpha\rangle + |\beta\rangle)/\sqrt{2}$ and $|-\rangle = (|\alpha\rangle - |\beta\rangle)/\sqrt{2}$. The thermalization due to the Gibbons-Hawking effect takes place in the dressed state basis consisting of the two *detector states*, i.e. of the states $|+\rangle$ and $|-\rangle$, on a time scale given by the quantities P_{\pm} in Eq. (42). For the laboratory observer, the Gibbons-Hawking thermal state will thus appear to cause damping of the Rabi oscillations on the thermalization time scale, i.e., friction on the coherent oscillating motion between the two detector states occurs, due to the thermal phonon bath perceived by the detector. The occupation of the detector states can be measured directly using atomic interferometry: A $\pi/2$ pulse brings one of them into the filled (β) and the other into the empty (α) state. To increase the signal to noise ratio, one could conceive of manufacturing a small array of AQDs in a sufficiently large cigar-shaped host superfluid, and monitor the total population of β atoms in this array.

B. Detection in laboratory time

We contrast the above calculation with the response the AQD detector would see if tuned to laboratory time. This can be realized if we allow $\Omega \propto t$, such that $\Omega \sqrt{\rho_0(0,t)} = \Omega \sqrt{\rho_m}/(B\dot{b}t) = \text{const}$, in the Rabi term on the right-hand side of (26), to be time independent. The detector has, therefore, dt as its natural time interval in this setting. The Painlevé-Gullstrand metric (15) in pure laboratory frame variables, assuming $B^2\dot{b}^2 \gg \omega_{\parallel}^2/b^2$ as in the derivation of the de Sitter metric (20), reads

$$ds^2 = -\frac{c_0^2}{B^2\dot{b}^2 t^2} (1 - \Lambda z^2) dt^2 - \frac{2z}{t} dz dt + dz^2. \quad (43)$$

The metric (43) is asymptotically, for large t , becoming that of Galilei invariant ordinary 1D laboratory space, i.e. it is just measuring length along the z direction, because the speed of sound in the ever more dilute gas decreases like $1/t$ and the “phonon ether” becomes increasingly less stiff.

The transition probabilities for absorption respectively emission are now given by $\tilde{P}_{\pm} = [g \epsilon_{0,k}/4R_{\parallel} |C_0(0)| (g_{\alpha\beta}/g - 1)^2 |\tilde{T}_{\pm}|^2]$, where the matrix elements are, cf. Eq. (34),

$$|\tilde{T}_{\pm}|^2 = \left| \int_{-\infty}^{\infty} \frac{dt}{Bb^2} \exp \left[\pm i \epsilon_{0,k} \int_{-\infty}^{\infty} \frac{dt'}{Bb^2} + i \omega_0 t \right] \right|^2. \quad (44)$$

Substituting the adiabatic time interval $d\tau_a = dt/(B\dot{b}t^2)$ leads for large t to

$$\tau_a = \tau_{0s} - 1/(B\dot{b}t), \quad (45)$$

where $\tau_{0a} = \int_{-\infty}^{+\infty} dt/(B\dot{b}t^2)$. The transformation to adiabatic time maps $t \in [-\infty, +\infty]$ onto $\tau_a \in [-\infty, \tau_{0a}]$ and, by further substituting $y = \epsilon_{0,k}(\tau_a - \tau_{0a})$, we have

$$|\tilde{T}_{\pm}|^2 = \frac{1}{\epsilon_{0k}^2} \left| \int_0^{\infty} dy \exp \left[i \left(y \mp \frac{\omega_0 \epsilon_{0k}}{B\dot{b}^2} \frac{1}{y} \right) \right] \right|^2. \quad (46)$$

The integral is a linear combination of Bessel functions. To test its convergence properties, we are specifically interested in the large ϵ_{0k} limit. Performing a stationary phase approximation for large $A = \pm \omega_0 \epsilon_{0k}/B\dot{b}^2$, we have for positive A (absorption) that the integral above becomes $J(A) = (\pi\sqrt{A})^{1/2} \exp[-2\sqrt{A}]$ and for negative A (emission) $J(A) = (\pi\sqrt{|A|})^{1/2}$. The final result then is

$$\begin{aligned} \tilde{P}_{\pm} &= \left(\frac{g_{\alpha\beta}}{g} - 1 \right)^2 \sqrt{2\pi} \sqrt{\rho_m a_s^3} \left(\frac{\omega_{\perp}}{\mu} \right)^2 \\ &\times \int_0^{E_{\text{Pl}}} d\epsilon_{0k} \sqrt{\frac{\omega_0}{\epsilon_{0k} B \dot{b}^2}} \times \left\{ \frac{\exp[-4\sqrt{\omega_0 \epsilon_{0k}/B\dot{b}^2}]}{1} \right. \\ &= \left(\frac{g_{\alpha\beta}}{g} - 1 \right)^2 \sqrt{2\pi \rho_m a_s^3} \left(\frac{\omega_{\perp}}{\mu} \right)^2 \times \left\{ \begin{array}{l} \approx \frac{1}{2} \\ \sqrt{\frac{4E_{\text{Pl}}\omega_0}{B\dot{b}^2}} \end{array} \right. \end{aligned} \quad (47)$$

where $E_{\text{Pl}} \sim \mu$ is the ultraviolet cutoff in the integral for the emission probability \tilde{P}_{-} , the “Planck” scale of the superfluid. Because of the convergence of the absorption integral for \tilde{P}_{+} , the total number of particles detected remains finite, and there are no particles detected by the effective laboratory frame detector at late times. This is in contrast to the de Sitter detector, which according to (41) still detects particles, in a stationary thermal state.

There is a detector setting that corresponds to a detector at rest in the Minkowski vacuum. This setting is represented by the adiabatic basis, with time interval defined by $d\tau_a = dt/B\dot{b}^2$, realizable with the AQD by setting the Rabi frequency $\Omega \propto 1/t$. Then, no particles whatsoever are detected, i.e., no frequency mixing of the positive and negative frequency parts of (32) does take place. The associated space-time interval

$$ds^2 = b^2 [-c_0^2 d\tau_a^2 + dz_b^2] \quad (48)$$

is simply that of (conformally) flat Minkowski space in the spatial scaling coordinate z_b and adiabatic time coordinate τ_a .

V. SUMMARY AND CONCLUSIONS

We summarize the effective space-times considered in this article, and the associated time intervals in Table I. The major observation of the present investigation is that the physical nature of the effective space-time considered in the condensed matter system reflects itself directly in the quasi-particle (phonon) content measured by a detector that has a natural time interval equal to the time interval of this particular effective space-time. We have demonstrated that the no-

TABLE I. Various time intervals effectively measuring laboratory, de Sitter, and adiabatic time in a 1+1D BEC, respectively, where $b=b(t)$, and B is independent of laboratory time. The third entry specifies if the AQD detector, tuned to the given time interval, detects phonons. In the laboratory frame, the detector has a small, nonthermal response for a finite amount of (initial) laboratory time [cf. Eq. (47)].

Effective space	Time interval	Phonons detected
Laboratory	dt	Yes (nonthermal)
de Sitter	$d\tau=dt/Bb$	Yes (thermal)
Adiabatic	$d\tau_a=dt/Bb^2$	No

tion of observer dependence can be made experimentally manifest by an *atomic quantum dot* placed at the center of a linearly expanding cigar-shaped Bose-Einstein condensate, which has a *tunable* effective time interval. Thus a possibility opens up to confirm experimentally, in an effective curved space-time setting, that, indeed, “a particle detector will react to states which have positive frequency with respect to the detector’s proper time, not with respect to any universal time” [1].

The Gibbons-Hawking effect in the BEC is intrinsically quantum: The signal contains a “dimensionless Planck constant,” i.e. the gaseous (loop expansion) parameter $\sqrt{\rho_m a_s^3}$. This implies that a reasonable signal-to-noise ratio can be achieved only by using initially dense clouds with strong interparticle interactions. On the other hand, the phononic quasiparticles of the superfluid can be regarded as noninteracting only in a first approximation in $\sqrt{\rho_m a_s^3} \ll 1$. The ef-

fects of self-interaction between the phonons, induced by larger values of the gaseous parameter, can lead to decoherence and the relaxation of the phonon subsystem. The same line of reasoning applies to the evolution of quantum fields in the expanding universe. The interactions between quasi-particle excitations and their connection to decoherence processes in cosmological models of quantum field propagation and particle production are, therefore, important topics for future work.

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