Kasner and mixmaster behavior in universes with equation of state $w \geqslant 1$

Joel K. Erickson,[1] Daniel H. Wesley,[1] Paul J. Steinhardt,[1,2,*] and Neil Turok[3]
[1]*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA*
[2]*School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540, USA*
[3]*DAMTP, CMS, Wilberforce Road, Cambridge, CB30WA, United Kingdom*
(Received 30 November 2003; published 24 March 2004)

We consider cosmological models with a scalar field with equation of state $w \geqslant 1$ that contract towards a big crunch singularity, as in recent cyclic and ekpyrotic scenarios. We show that chaotic mixmaster oscillations due to anisotropy and curvature are suppressed, and the contraction is described by a homogeneous and isotropic Friedmann equation if $w > 1$. We generalize the results to theories where the scalar field couples to p forms and show that there exists a finite value of w, depending on the p-forms, such that chaotic oscillations are suppressed. We show that $\mathbb{Z}_2$ orbifold compactification also contributes to suppressing chaotic behavior. In particular, chaos is avoided in contracting heterotic M-theory models if $w > 1$ at the crunch.

DOI: 10.1103/PhysRevD.69.063514 PACS number(s): 98.80.Jk, 04.40.Nr, 04.50.+h, 11.25.Mj

I. INTRODUCTION

In cosmological models with a big-crunch–big-bang transition, a key issue is the behavior as the Universe contracts towards the crunch. From the classic studies of Belinskii, Khalatnikov, and Lifshitz (BKL) [1–3] and others [4–8], it is known that the contraction can either proceed smoothly or chaotically. These studies have focused on models in which the universe contains matter and radiation, or, more generally, an energy component whose equation of state is $w \leqslant 1$ (where $w \equiv p/\rho$ is defined as the ratio of the pressure p to the energy density ρ). If $w < 1$, a contracting homogeneous and isotropic solution is unstable to small perturbations in the anisotropy and spatial curvature. As the overall volume shrinks, the anisotropy causes the universe to expand along one axis and contract along the others, a state that can be approximated by the anisotropic Kasner solution. The spatial curvature causes the axes and rates of contraction to undergo sudden jumps from one Kasner-like solution to another, an effect known as ''mixmaster'' [9,10] behavior. If the curvature is not spatially uniform, then the chaotic behavior in different regions is not synchronized and the universe becomes highly inhomogeneous at the big crunch. Hence, mixmaster behavior could potentially wreak havoc in cosmological models with a big-crunch–big-bang transition, making them inconsistent with the observed large scale homogeneity of the universe.

In this paper we show that the behavior of the universe as it approaches the big crunch is very different if there is an energy component with $w > 1$. The chaotic behavior is suppressed and the universe contracts homogeneously and isotropically as it approaches the singularity. The reason is that the anisotropy and curvature terms in the Einstein equations grow rapidly and become dominant if $w < 1$, but they remain negligible compared to the energy density if $w > 1$. In the latter case, the Einstein equations converge to the Friedmann equations with purely time-dependent terms, a condition sometimes referred to as ''ultralocality.'' The effect can be viewed as a generalization of the ''cosmic no-hair theorem'' invoked in a rapidly inflating universe. Here we demonstrate analogous behavior in a slowly contracting universe with $w > 1$. A related result of Dunsby *et al.* [11–13] shows that models with $0 < w < 1$ but with ρ^2 terms in the stress-energy tensor are also driven towards isotropy.

The cosmic no-hair theorem for a contracting universe containing a perfect fluid with $w \geqslant 1$ is discussed in Sec. II. A common example of a perfect fluid is a scalar field ϕ with a potential $V(\phi)$. In Sec. III we consider the interaction of the scalar ϕ with a p-form field F_{p+1} through an exponential coupling,

$$e^{\lambda\phi} F_{p+1}^2, \tag{1}$$

where λ is a constant [14]. We consider this case because scalar fields with exponential couplings to p-form fields are common in Kaluza-Klein, supergravity and superstring models. For the case $w = 1$, it is known [3,7] that the contraction is not chaotic if λ lies within a bounded interval. Here we show that, for *any* λ and p, there is a critical value $w_{\rm crit}(\lambda, p)$ for which the chaotic behavior is suppressed if $w > w_{\rm crit}(\lambda, p)$.

Our results are of particular importance for the recent ekpyrotic [15] and cyclic [16] cosmological models, which have a big-crunch–big-bang transition with a contraction phase dominated by a scalar field with $w \geqslant 1$ [17]. The evolution of perturbations leading up to and passing through the transition is an important aspect that remains unsettled [18–24], and may depend on the precise physical conditions leading up to the bounce [25,26]. The present work may be relevant since it suggests that the universe can remain homogeneous and isotropic on large scales. Once the evolution becomes ultralocal, the whole universe is following the same homogeneous and isotropic evolution all the way to the big crunch.

In Sec. IV we explore how the time variation of w affects our conclusions, and in particular how w approaching $w_{\rm crit}$ from above may suppress chaotic behavior. In Sec. V we discuss some specific models. In particular, we show how orbifolding can remove p-forms that might induce chaotic

*Electronic address: steinh@princeton.edu

behavior and discuss the special case of heterotic M theory, which, to leading order in the 11-dimensional gravitational coupling κ, is on boundary between chaotic and smooth behavior.

II. A COSMIC NO-HAIR THEOREM FOR CONTRACTING UNIVERSES

The cornerstone of the inflationary paradigm is an argument known as the cosmic no-hair theorem, according to which a universe containing a perfect fluid component with $w<-1/3$ will rapidly approach flatness, homogeneity and isotropy at late times, for a wide range of initial data (namely those for which the space curvature, inhomogeneity and anisotropy are not very large) [27]. In the Friedmann equation, the energy density for a component with equation of state w is proportional to $1/a^x$, where the exponent $x=3(1+w)$. The anisotropy term is proportional to a^{-6} and the spatial curvature term is proportional to a^{-2}. As the universe expands, the contribution with the smallest values of x redshifts away more slowly than components with larger values of x and comes to dominate the Friedmann equation; the components with the smallest value of x overall ultimately dominate. If the energy component with the smallest value of w has $w<-1/3$, then $x<2$ and this component dominates. For a wide range of initial data, convergence to a homogeneous and isotropic expanding universe is assured.

Below, we will present an analogous cosmic no-hair theorem for contracting universes. In a contracting universe, the component with the largest value of x will dominate the Friedmann equation. Starting from an inhomogeneous and anisotropic initial state, we will show that the existence of a perfect fluid with $w>1$ (or $x>6$) will suppress chaotic behavior, and enable a smooth and isotropic contraction to the big crunch. We will find that curvature plays a more complicated role compared to the case of expansion. Hence, we first obtain a cosmic no-hair theorem for the case of zero spatial curvature and then generalize to the case of arbitrary spatial curvature. We intentionally take a pedagogical approach that encompasses known results for $w\leqslant 1$ to make our discussion self-contained. Our analysis assumes the initial inhomogeneity is small; it is possible that the universe evolves towards other attractors for sufficiently large deviations from homogeneity. Our conventions are given in [14].

All of our computations are performed in synchronous gauge,

$$ds^2=-dt^2+h_{ab}(t,\mathbf{x})dx^a dx^b, \tag{2}$$

where we use our freedom to choose a spatial slicing to ensure that the big crunch occurs everywhere at $t=0$ ($\det h_{ab}\to 0$ as $t\to 0$). For a perfect, comoving fluid with equation of state $p=w\rho$, the Einstein equations are [1]

$$\frac{\partial}{\partial t}\kappa_j{}^j+\kappa_j{}^k\kappa_k{}^j=-\left(\frac{1+3w}{2}\right)\rho, \tag{3a}$$

$$\frac{\partial}{\partial x^a}\kappa^j{}_j-\frac{\partial}{\partial x^j}\kappa^j{}_a=0, \tag{3b}$$

$$P_a{}^b+\frac{1}{\sqrt{h}}\frac{\partial}{\partial t}(\sqrt{h}\kappa_a{}^b)=+\left(\frac{1-w}{2}\right)\rho, \tag{3c}$$

where $P_a{}^b$ is the Ricci tensor on spacelike surfaces, and κ_{ab} is defined by

$$\kappa_{ab}=\frac{1}{2}\frac{\partial}{\partial t}h_{ab}, \tag{4a}$$

$$\kappa_a{}^b=\kappa_{aj}h^{jb}. \tag{4b}$$

Near the big crunch, the dynamics of the metric (2) are *ultralocal* [2,7,8,28]. That is, the evolution of adjacent spatial points decouples because spatial gradients increase more slowly than other terms in the equations of motion. Therefore, analyzing the dynamics of this metric near the singularity and at fixed spatial coordinate $\mathbf{x}_0$ is equivalent to analyzing the much simpler system

$$ds^2=-dt^2+\sum_{ij} e^{2\beta_{ij}(t;\mathbf{x}_0)}\sigma^{(i)}(\mathbf{y};\mathbf{x}_0)\sigma^{(j)}(\mathbf{y};\mathbf{x}_0), \tag{5}$$

where the $\sigma^{(i)}$ are $\mathbf{y}$-dependent one-forms that are linearly independent at each point and form a *homogeneous* (but possibly curved) space such as Bianchi type IX [10]. The β_{ij}, which do not depend on $\mathbf{y}$, describe the (generally anisotropic) contraction of this space. Both the $\sigma^{(i)}$ and the β_{ij} depend on the parameter $\mathbf{x}_0$, the spatial point being studied. The dynamics of the inhomogeneous universe at a fixed spatial point can be approximated, near $t=0$, by the dynamics of a homogeneous (but curved and anisotropic) universe. Differences in curvature and anisotropy between different $\mathbf{x}_0$ are encoded in the different $\sigma^{(i)}$ and β_{ij} associated with these points.

In each Kasner-like epoch, we may perform a rotation so that β is diagonal. Furthermore, we may separate out the trace of β and write it as the ''volume scale factor'' $a(t)$, in analogy to the isotropic Friedman-Robertson-Walker universe, to obtain the metric

$$ds^2=-dt^2+a^2(t)\sum_i e^{2\beta_i(t)}(\sigma^{(i)})^2, \tag{6a}$$

$$\beta_1(t)+\beta_2(t)+\beta_3(t)=0, \tag{6b}$$

where the dependence of $a(t)$, the β_i and the $\sigma^{(i)}$ on $\mathbf{x}_0$ has been suppressed. The combination ae^{β_i} can be thought of as the effective scale factor along the ith direction, and the functions β_i then describe the contraction or expansion of each direction relative to the overall volume contraction. We may use our freedom to rescale the σ to ensure that at some time t_0, $a(t_0)=1$, $\beta_i(t_0)=0$ and $\det(\sigma^{(1)},\sigma^{(2)},\sigma^{(3)})=1$. Quantities with a subscript zero (such as ρ_0) refer to their values at this fixed time.

The Einstein equations (3) close with the equation of energy conservation for the fluid,

$$\frac{d\log\rho}{d\log a} = -3(1+w). \tag{7}$$

For constant w, this equation has the familiar solution,

$$\rho(a) = \rho_0 a^{-3(1+w)}. \tag{8}$$

While we could have included several perfect fluids, with different equations of state w_i, the fluid with the largest equation of state will always dominate near the crunch, so it is sufficient to consider only one energy component. We have taken this fluid to be comoving, because small perturbations of a comoving background are suppressed in a $w>0$, contracting universe. In particular, the $T^0{}_i$ terms that would appear on the right-hand side of Eq. (3b) grow only as $t^{-2/(1+w)}$, which is slower than the t^{-2} rate at which the diagonal terms grow [29].

A. The curvature-free case

We first examine the case of Ricci flat spatial 3-surfaces, for which $P_a{}^b = 0$. In this case, we write $\sigma^{(i)} = dx^i$. Then, the Einstein equations (3) reduce to

$$3\left(\frac{\dot a}{a}\right)^2 - \frac{1}{2}(\dot\beta_1^2+\dot\beta_2^2+\dot\beta_3^2) = \rho, \tag{9a}$$

$$\ddot\beta_i + 3\frac{\dot a}{a}\dot\beta_i = 0, \tag{9b}$$

where a dot indicates a derivative with respect to the proper time t. Integration of Eq. (9b) gives

$$\dot\beta_i = c_i a^{-3}, \tag{10}$$

while the constraint (6b) implies,

$$c_1 + c_2 + c_3 = 0. \tag{11}$$

Combining these results, Eq. (9a) becomes a Friedmann equation,

$$3\left(\frac{\dot a}{a}\right)^2 = \rho(a) + \frac{\sigma^2}{a^6} = \frac{\rho_0}{a^{3(1+w)}} + \frac{\sigma^2}{a^6}, \tag{12}$$

where we define

$$\sigma^2 = \frac{1}{2}(c_1^2+c_2^2+c_3^2). \tag{13}$$

An anisotropic universe has $\dot\beta_i \neq 0$, i.e. $c_i \neq 0$. The constant σ^2 parametrizes the anisotropic contribution to the Friedmann equation in Eq. (12). The anisotropy evolves as $1/a^6$ or $x=6$. We define the fractional energy densities Ω_ρ and Ω_σ as

$$\Omega_\rho = \frac{\rho(a)}{\rho(a)+\sigma^2/a^6}, \tag{14a}$$

$$\Omega_\sigma = \frac{\sigma^2/a^6}{\rho(a)+\sigma^2/a^6}. \tag{14b}$$

These quantities represent the contribution of the perfect fluid and anisotropy to the critical density for closure of the universe. Since we are neglecting curvature, $\Omega_\rho + \Omega_\sigma = 1$.

The solution for the β_i as a function of the scale factor a is

$$\beta_i(a) = c_i\sqrt{3}\int_a^1 \frac{da'}{a'}[\rho(a')a'^6+\sigma^2]^{-1/2}. \tag{15}$$

The limits of integration have been chosen to ensure $\beta_i(1) = 0$. For the remainder of the paper, we will assume a universe contracting towards $a\to 0$ as t approaches zero from below. Let us now examine the behavior of these solutions for various w.

1. w<1

When $w<1$, the $\rho(a)$ part of the integral (15) is negligible as $a\to 0$, and so the solution converges to the vacuum ($\rho=0$) Kasner universe during contraction,

$$a(t) = \left(\frac{t}{t_0}\right)^{1/3} \tag{16a}$$

$$\beta_i(t) = \frac{c_j}{\sigma\sqrt{3}}\ln\left(\frac{t}{t_0}\right). \tag{16b}$$

The Kasner universe is parametrized by three *Kasner exponents* p_i,

$$p_i = \frac{1}{3} + \frac{c_j}{\sigma\sqrt{3}}. \tag{17}$$

The scale factors in Eq. (6) are powers of t,

$$a e^{\beta_i} = |t/t_0|^{p_j}, \tag{18}$$

and the relations (11) and (13) become

$$p_1 + p_2 + p_3 = 1 \tag{19a}$$

$$p_1^2 + p_2^2 + p_3^2 = 1, \tag{19b}$$

known as the *Kasner conditions*. These describe the intersection of a plane, the *Kasner plane*, and a unit sphere, the *Kasner sphere*, as illustrated in Fig. 1. We will denote the intersection, which represents the allowed values of the p_i, as the *Kasner circle*. The outermost circle in Fig. 1 corresponds to the limit where $w<1$, as the energy density scales away and only a vacuum, anisotropic universe remains.

There are three degenerate solutions where exactly one of the p_i is one, and the other exponents are zero (the solid black circles in Fig. 1). At all other points on the (dashed) Kasner circle exactly one of the p_i is negative. Thus, although the geometric mean of the three scale factors $a(t)$

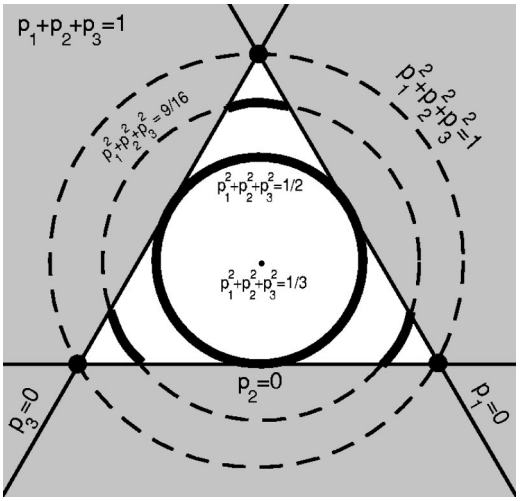

FIG. 1. The Kasner plane $p_1+p_2+p_3=1$ and its intersections (the Kasner circles) with various spheres $p_1^2+p_2^2+p_3^2=1-q^2$ where $q^2=\frac{2}{3}(1-\Omega_\sigma)$; see Eq. (22). The vacuum solution corresponds to $\Omega_\sigma=1$ (the outermost circle). The inner circles are relevant to the case where $w=1$ and $\Omega_\sigma<1$. In the white regions, the Kasner exponents are all positive (corresponding to contraction); in gray regions, one exponent is negative (expanding). If the spatial curvature is non-zero, points along the circles in the white region (thick parts of circles) are stable but points in the gray regions (dashed parts of circles) are unstable, jumping to new values after a short period of contraction. If a model (i.e. a circle) has an open set of stable points (the three innermost circles but not the outermost circle), the contracting phase does not exhibit chaotic mixmaster behavior.

$=|t|$ is contracting, a single scale factor corresponding to the negative Kasner exponent is undergoing expansion to infinity.

For the curvature-free case, the universe becomes increasingly anisotropic near the big crunch if $w<1$. In particular, the isotropic solution, $p_1=p_2=p_3=1/3$, is inconsistent with the Kasner conditions (19a) and (19b).

2. w=1

Inspection of Eq. (8) reveals that, when $w=1$, the matter density and the anisotropy terms in the Friedmann equation (12) scale with the same power of a, so Ω_ρ and Ω_σ remain fixed. The solutions are

$$a(t)=\left(\frac{t}{t_0}\right)^{1/3} \tag{20a}$$

$$\beta_i(t)=\frac{c_j}{\sqrt{3(\sigma^2+\rho_0)}}\ln\left(\frac{t}{t_0}\right). \tag{20b}$$

This solution is very similar to the $\rho=0$ case, and indeed we may define the Kasner exponents,

$$p_i=\frac{1}{3}+\frac{c_j}{\sigma\sqrt{3}}\left(1+\frac{\rho_0}{\sigma^2}\right)^{-1/2}. \tag{21}$$

The Kasner conditions are different. If we define

$$q^2\equiv\frac{2}{3}\frac{\rho_0}{\sigma^2+\rho_0}=\frac{2}{3}(1-\Omega_\sigma) \tag{22}$$

then the Kasner conditions are

$$p_1+p_2+p_3=1, \tag{23a}$$

$$p_1^2+p_2^2+p_3^2=1-q^2=\tfrac{1}{3}+\tfrac{1}{3}\Omega_\sigma. \tag{23b}$$

The first condition is unchanged from Eq. (19b) but the right-hand side of the second condition has been modified. Increasing Ω_σ corresponds to increasing the radius of the Kasner sphere.

The $w=1$ model allows us to explore the behavior of the contracting universe as a function of Ω_σ. The perfectly isotropic case corresponds to $\Omega_\sigma=0$, which is the usual flat Friedmann-Robertson-Walker solution (the innermost circle, in the limit where the circle has shrunk to a point, in Fig. 1). Unlike the vacuum Kasner case, all of the Kasner exponents are positive (i.e., they lie within the white region of Fig. 1) provided that $\Omega_\sigma<1/4$ (within the larger, solid circle inscribed in the triangle). For this range, none of the scale factors is increasing during the contraction, although they are decreasing at different rates. When $\Omega_\sigma>1/4$ (third largest circle), then some points on the Kasner circle have a negative Kasner exponent (dashed part of circle) and other points may have all positive Kasner exponents (solid, thick parts of circle).

Thus, ignoring the curvature, the $w=1$ case with non-zero Ω_σ contracts smoothly but anisotropically to the crunch. In the special case where $\Omega_\sigma=0$, the contraction is isotropic.

3. w>1

For $w>1$, the energy density dominates ($\Omega_\rho\to1$) as $a\to0$, and the metric approaches the approximate form

$$a(t)=\left(\frac{t}{t_0}\right)^{2/3(1+w)}, \tag{24a}$$

$$\beta_i(t)=c_j\frac{2}{\sqrt{3\rho_0}}\frac{1}{w-1}\left[\left(\frac{t}{t_0}\right)^{(w-1)/(w+1)}-1\right], \tag{24b}$$

where we have chosen the constants of integration so $\beta_i=0$ at $t=t_0$. The crucial feature is that the time-varying part of the β_i is proportional to t^α where α is *positive* if $w>1$. This means that the β_i approach a constant and the universe becomes isotropic at the crunch [30].

This simple result is a ''no-hair theorem'' for universes without spatial curvature: When $w>1$, an initially anisotropic universe becomes isotropic ($\Omega_\sigma\to0$) near the big crunch. The $w>1$ case is stable under anisotropic perturbations. For $w<1$, the universe becomes increasingly anisotropic in the sense that $\Omega_\sigma\to1$ as $a\to0$. For $w=1$, Ω_σ remains fixed as $a\to0$. Evolution is smooth (no mixmaster

behavior) in all cases, and is well approximated as a Kasner metric with constant coefficients for sufficiently small a.

B. Curvature and chaos

Complex behavior can arise when there is non-zero spatial curvature in a contracting universe. This may seem surprising at first, since the spatial curvature for a homogeneous and isotropic universe grows as $1/a^2$, which increases more slowly than either the anisotropy or the energy density of a component with $w>-1/3$. However, we have seen above that the contracting phase for $w\leqslant 1$ is anisotropic. We will show below that this can produce rapidly growing curvature perturbations and chaotic behavior. On the other hand, we will see that chaotic behavior is suppressed if $w>1$ and the contraction approaches isotropy as $a\to 0$.

We now allow the $\sigma^{(i)}$ to have an **x** dependence and consider a curved manifold. The spatial Ricci tensor for the metric (6) has the form [1]

$$P_a{}^b=\frac{1}{a^2}\sum_{ijk} S_a{}^b{}_{ijk}(\sigma)e^{2(\beta_i-\beta_j-\beta_k)}. \tag{25}$$

The functions $S_a{}^b{}_{ijk}$ depend only on the σ^i and their space derivatives, and are independent of time.

The expression (25) reveals a crucial connection between the behavior of anisotropy and curvature near the big crunch. In the isotropic limit, $\beta_i=0$ and Eq. (25) reduces to the homogeneous and isotropic $1/a^2$ scaling discussed above. However, the terms in Eq. (25) are essentially ratios of scale factors. Thus, if the anisotropy is growing as $a\to 0$, some terms—involving ratios of expanding and contracting scale factors—will grow, and the corresponding curvature components will scale faster than $1/a^2$. For $w<1$ the anisotropy dominates near the crunch, and, as we will discuss below, this causes the curvature to grow and induce chaos. By contrast, in the $w>1$ model, the anisotropy vanishes at the crunch, and the curvature scales as the usual $1/a^2$, which may be neglected.

1. $w<1$

In this case we begin by assuming that the behavior near the crunch is described by the vacuum Kasner solution, with Kasner conditions (19a) and (19b). Using the Kasner solution, it is readily seen that the Einstein equation (3a) contains a leading order term with time dependence t^{-2}.

The second Einstein equation (3b) is a consistency check for our assumption of ultralocality. For an appropriate choice of the $\sigma^{(i)}$—a basis for one of the Bianchi universes—this equation vanishes identically and the metric (6) solves the Einstein equations.

The third Einstein equation (3c) indicates that the simple Kasner solutions must break down near the big crunch. If we order the Kasner exponents as $p_1\leqslant p_2\leqslant p_3$, then the most divergent term in the third Einstein equation comes from terms in the spatial curvature (25), with leading time dependence,

$$t^{-2(1-2p_1)}. \tag{26}$$

The leading term is more divergent than t^{-2}, since the Kasner conditions (19a) and (19b) imply p_1 is always negative. Therefore, our smoothly contracting solutions are not stable to perturbations in the spatial 3-curvature. A small amount of curvature will grow and come to dominate the dynamics before the big crunch.

The behavior of the universe in this regime has been extensively studied and is known to be chaotic [2–6,8]. The spatial curvature terms cause the Kasner exponents p_i and the principal directions σ^i to become time-dependent during contraction.

More precisely, the exponents and principal directions are nearly constant for stretches of Kasner-like contraction, during which the curvature is negligible. These Kasner-like epochs are punctuated by short intervals when the curvature momentarily dominates. The exponents and principal directions suddenly jump to new values, and then a new stretch of Kasner-like contraction begins during which the curvature terms are again negligible. The universe undergoes an infinite number of such jumps before the big crunch. The chaotic, non-integrable evolution is equivalent to that of a billiard ball [8], which experiences free motion interrupted by collisions with walls. Models with this oscillatory behavior are called *chaotic*.

This presents a problem for cosmological models, as one expects curvature perturbations in any realistic universe will cause the local value of the curvature to vary from point to point. If each spatial point evolves independently and chaotically, the evolution of nearby points diverges very quickly as contraction continues, and the universe rapidly becomes highly inhomogeneous as $a\to 0$. If $w<1$ throughout the contracting phase, it seems unlikely that the observed homogeneous universe could emerge from this state after the bounce to an expanding phase [31].

2. $w=1$

The chaotic behavior is mitigated in the $w=1$ case. Recalling our discussion of the curvature-free scenario, it is clear that there are regions of non-zero measure on the Kasner circle for which all of the p_i are positive. We will refer to these points as *stable*. All choices of p_i when $\Omega_\sigma<1/4$ are stable. If the universe begins at a stable point, the curvature term remains negligible as $a\to 0$ and the contraction is smoothly Kasner-like.

However, when $\Omega_\sigma>1/4$, some choices of the p_i will have one $p_i<0$. If the universe begins at one of these points, the curvature term will grow and become dominant, causing the values of p_i and the principal axes σ_i to change. We refer to these points as *unstable*. A more complete analysis [3] reveals that, after a finite number of jumps, the universe hits a point in the open set of stable p_i. From this point onwards, the universe contracts smoothly and without any further jumps.

We call these models *non-chaotic*, since the universe is guaranteed to arrive at a stable point as $a\to 0$. Non-chaotic models (Kasner circles) may contain both stable and unstable points, but they will always oscillate only a finite number of times before arriving in the set of stable points, after which the behavior is integrable.

3. w>1

For $w>1$, curvature does not affect the contraction. The key is the time dependence of the β_i in Eq. (24), which approach zero as a positive power of t as $t\to 0$. Consequently, the exponential factors e^{β_i} in the metric approach constants. The leading order time-behavior of $P_a{}^b$ is simply that of a homogeneous and isotropic universe,

$$P\sim\frac{1}{a^2}\sim|t|^{-[4/3(1+w)]}. \tag{27}$$

This is always less divergent that t^{-2} for $w>1$. Thus, even in the presence of initial anisotropy and curvature, the solution for $w>1$ converges to the isotropic solution represented by the central point on the Kasner sphere in Fig. 1.

We can generalize our cosmic no-hair theorem (described at the end of Sec. II A) to include models with spatial curvature. The Einstein equations for a contracting universe with anisotropy and inhomogeneous spatial curvature converge to the Friedmann equation for a homogeneous, flat and isotropic universe if it contains energy with $w>1$, and that for a homogeneous, flat but anisotropic universe if $w=1$. The $w<1$ case becomes highly inhomogeneous and the no-hair theorem is inapplicable.

III. COUPLING TO *p*-FORMS AND CHAOS

In Sec. II we assumed that the evolution of the universe was dominated by an energy component with a fixed equation of state evolving independently of other matter in the universe. The component could have been a scalar field or a perfect fluid. We found chaotic behavior for $w<1$ in the presence of curvature but non-chaotic behavior for $w\geqslant 1$.

In this section we want to consider how the behavior for $w\geqslant 1$ can change if the fluid is imperfect or couples to other components. In many theories, including Kaluza-Klein, supergravity and superstring models, the relevant energy consists of a scalar field that is coupled to p-forms. Consequently, we will focus on this important example, as others have in the past [3,5–8].

To determine the effect of the coupling to p-forms on chaotic behavior, our approach is similar to our analysis for spatial curvature, where we assume an initial state in which the spatial curvature is negligible and then check that it remains small. Here we assume that the p-form field strength is initially negligible and ask how its contribution evolves relative to the energy density with equation of state w. Our action is

$$S=\int d^4x\sqrt{-g}\left(\frac{1}{2}R-\frac{1}{2}(\partial\phi)^2-V(\phi)-\frac{1}{2(p+1)!}e^{\lambda\phi}F_{\mu_1\cdots\mu_{p+1}}{}^2\right), \tag{28}$$

where g is the metric, R is the scalar curvature, V is a potential for the scalar field ϕ, p is the rank of the p-form, F is the associated field strength tensor and λ is the coupling constant. The potential $V(\phi)$ is chosen to give a fixed equation of state $w\geqslant 1$ in the absence a p-form coupling:

$$V(\phi)=-V_0e^{-\sqrt{3(1+w)}\phi}, \tag{29}$$

where V_0 is a positive constant. Throughout this paper, we assume without loss of generality that $\phi\to-\infty$ as $a\to 0$.

For a given equation of state w and p-form rank, the behavior of the system as $t\to 0$ depends on the coupling λ. We can extend the terminology introduced earlier to describe the properties for a given λ. We classify the p-form coupling parameter λ as *supercritical* if the p-form terms grow relative to the scalar field energy density. We call these models supercritical, as opposed to chaotic, because if $w>1$ it is not known whether chaos occurs or whether the p-forms merely play a non-negligible role in integrable dynamics. In the special case $w=1$, chaos is known to occur, and we call these models *chaotic* [5–8]. Values of λ for which the contracting solution with negligible p-forms is stable are called *non-chaotic* (some authors use *subcritical*). These two cases are analogous to those introduced in Sec. II. If λ is on the boundary between supercritical and non-chaotic, we call λ *critical*. The behavior of critical models may be novel, and will be discussed at the end of this section.

We are assuming that initially the spatial curvature, the anisotropy and the p-form terms are small, and then we check if these conditions are maintained as the universe contracts. Since we are considering models where $w\geqslant 1$, the model is non-chaotic if the p-forms are negligible. The universe may be approximated initially by the homogeneous isotropic Friedmann-Robertson-Walker form in Eq. (6) with $\beta_i\approx 0$ and $\sigma^{(i)}=dx^i$. If $w>1$ and the p-form terms are negligible, $\Omega_\sigma\to 0$ as the crunch approaches. For $w=1$, Ω_σ remains small but finite. If the isotropic case is unstable, then adding anisotropy cannot restore stability; just as in Sec. II B, the isotropic scale factors are the most stable.

It can be shown that the p-form terms involving the spatial gradients of F grow slower than the leading homogeneous time-derivative terms, another example of the ultralocal behavior discussed previously. Hence, we neglect all spatial derivatives of the field strength.

The components of F with purely spatial indices, $F_{i_1\cdots i_{p+1}}$ are called *magnetic* and the components with one time index, $F^{0i_1\cdots i_p}$, are called *electric*, in analogy with the Maxwell action. We will use the labels E and B to indicate their respective contributions. F has a vanishing exterior derivative $dF=0$. In coordinate notation, neglecting the spatial derivatives of F, this corresponds to

$$\partial_{[0}F_{i_1\cdots i_{p+1}]}=0, \tag{30}$$

where the brackets $[\cdots]$ indicate antisymmetrization. Thus, the magnetic components are constant,

$$F_{i_1\cdots i_{p+1}}=(\text{const}). \tag{31}$$

The equation of motion for F is

$$\nabla_\mu(e^{\lambda\phi}F^{\mu\mu_2\cdots\mu_{p+1}}) = \partial_\mu(e^{\lambda\phi}F^{\mu\mu_2\cdots\mu_{p+1}}) + \Gamma^\mu{}_{\mu\sigma}e^{\lambda\phi}F^{\sigma\mu_2\cdots\mu_{p+1}} = 0. \tag{32}$$

Only one set of Christoffel symbols appears due to the antisymmetry of F. Since $\Gamma^\mu{}_{\mu 0} = \partial/\partial t \log\sqrt{-g}$ and $\Gamma^\mu{}_{\mu i} = 0$, we can integrate to find,

$$F^{0i_1\cdots i_p} = \frac{e^{-\lambda\phi}}{\sqrt{-g}} \times (\text{const}). \tag{33}$$

The p-form part of the stress-energy tensor is

$$T_{\mu\nu} = \frac{e^{\lambda\phi}}{(p+1)!}\left((p+1)F_{\mu\mu_2\cdots\mu_{p+1}}F_\nu{}^{\mu_2\cdots\mu_{p+1}} - \frac{1}{2}g_{\mu\nu}F^2\right). \tag{34}$$

Decomposing (34) into electric and magnetic components, and including factors of the metric, we compute the energy density for the p-forms $\rho_p = -T_0{}^0$, which is,

$$\rho_p = \frac{e^{\lambda\phi}}{(p+1)!}\left(\frac{p+1}{2}F^{0i_1\cdots i_p}F^{0j_1\cdots j_p}g_{i_1j_1}\cdots g_{i_pj_p} + \frac{1}{2}F_{i_1\cdots i_{p+1}}F_{j_1\cdots j_{p+1}}g^{i_1j_1}\cdots g^{i_{p+1}j_{p+1}}\right) \tag{35}$$

$$= \frac{e^{-\lambda\phi}}{a^{2(3-p)}}\alpha_E^2 + \frac{e^{\lambda\phi}}{a^{2(p+1)}}\alpha_B^2, \tag{36}$$

where the positive constants α_E^2 and α_B^2 represent the magnitude of the electric and magnetic energy, respectively. We can now define a new set of fractional energy densities,

$$\Omega_\phi = \rho^{-1}[\dot\phi^2/2 + V(\phi)], \tag{37a}$$

$$\Omega_E = \rho^{-1}\frac{e^{-\lambda\phi}}{a^{2(3-p)}}\alpha_E^2, \tag{37b}$$

$$\Omega_B = \rho^{-1}\frac{e^{\lambda\phi}}{a^{2(p+1)}}\alpha_B^2, \tag{37c}$$

$$\rho = \frac{1}{2}\dot\phi^2 + V(\phi) + \frac{e^{-\lambda\phi}}{a^{2(3-p)}}\alpha_E^2 + \frac{e^{\lambda\phi}}{a^{2(p+1)}}\alpha_B^2, \tag{37d}$$

where Ω_ϕ is the energy density in the scalar field and Ω_E and Ω_M are the energy densities in electric and magnetic modes. We are assuming that the anisotropy is negligible, so $\Omega_\sigma \approx 0$.

The solution for a ϕ-dominated universe with equation of state w is

$$\phi = q\ln|t|, \quad q = \sqrt{\frac{4}{3(1+w)}}, \tag{38}$$

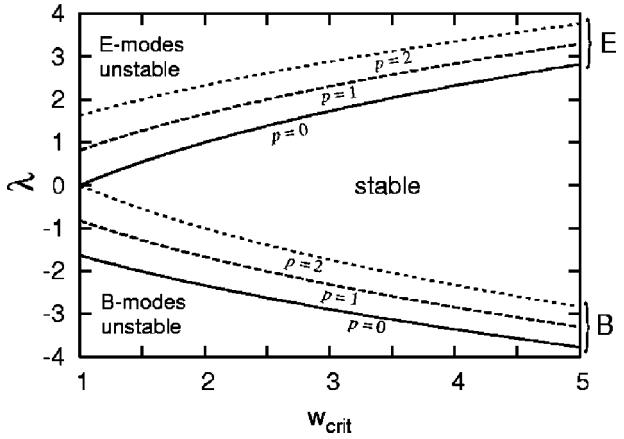

FIG. 2. The four dimensional electric and magnetic couplings λ as a function of the critical equation of state for $p = 0,1,2$. The upper and lower three curves represent the critical electric and magnetic exponents, respectively. A form with given p and λ is stable in a universe with equation of state w if the point (w,λ) lies between the two curves for the given p.

and $a = |t/t_0|^{2/3(1+w)}$. Substituting in Eq. (36), two terms in ρ_p may be written as

$$\rho_p = \alpha_E^2|t|^{p_E} + \alpha_B^2|t|^{p_B}, \tag{39}$$

where p_E and p_B are called the electric and magnetic exponents, respectively. They are

$$p_E = -\frac{4(3-p)}{3(1+w)} - \lambda\sqrt{\frac{4}{3(1+w)}}, \tag{40}$$

$$p_B = -\frac{4(p+1)}{3(1+w)} + \lambda\sqrt{\frac{4}{3(1+w)}}. \tag{41}$$

Note that these expressions are invariant under a duality transformation, which takes $p \to 2-p$, interchanges the electric and magnetic modes, and takes $\phi \to -\phi$.

In the Friedmann equation, the scalar field energy density scales as t^{-2}. Consequently, $\Omega_\phi \to 1$ and $\Omega_{E,B} \to 0$ as the universe contracts if both p_E and p_B are both greater than -2. In this case, the p-form contribution is negligible and λ is non-chaotic. Alternatively, if either p_E or p_B is less than -2, the respective p-form terms become large and alter the dynamics.

For $w = 1$, the non-chaotic values of λ are

$$\begin{aligned} -\sqrt{8/3} < \lambda < 0, &\quad p = 0 \\ -\sqrt{2/3} < \lambda < \sqrt{2/3}, &\quad p = 1 \\ 0 < \lambda < \sqrt{8/3}, &\quad p = 2. \end{aligned} \tag{42}$$

Increasing w causes the interval of non-chaotic couplings to grow, as shown in Fig. 2. In particular, for any p and λ, there exists a critical value $w_{\text{crit}}(\lambda,p)$ such that, for $w > w_{\text{crit}}(\lambda,p)$ the p-form terms remains negligible. For any set

of p-forms and couplings there exists a $\overline{w}_{\rm crit}$, the maximum of 1 (the critical equation of state for curvature) and the $w_{\rm crit}(\lambda,p)$ for each p and λ. Then the contraction is non-chaotic if $w>\overline{w}_{\rm crit}$.

The behavior can be understood in terms of an *effective* equation of state for the action (28), using the conservation equation

$$\dot{\rho}=-3\frac{\dot{a}}{a}(1+w_{\rm eff})\rho, \tag{43}$$

where ρ is given by Eq. (37d). Using Eqs. (37), the equation of motion for ϕ and the Friedmann equation, we find

$$w_{\rm eff}=w_\phi\Omega_\phi+\frac{3-2p}{3}\Omega_E+\frac{2p-1}{3}\Omega_B, \tag{44}$$

where

$$w_\phi=\frac{\dot{\phi}^2/2-V(\phi)}{\dot{\phi}^2/2+V(\phi)} \tag{45}$$

is the equation of state for the decoupled scalar field and $\Omega_\phi+\Omega_E+\Omega_B=1$. The expression (44) is exact, valid for all values of the Ω_i assuming the background is homogeneous, flat and isotropic. For the electric and magnetic contributions, we can introduce $w_E=(3-2p)/3$ and $w_B=(2p-1)/3$, respectively. The $w_{\rm eff}$ is just the Ω-weighted average of w_ϕ, w_E and w_B.

All the λ dependence of $w_{\rm eff}$ is contained in the time evolution of the Ω_i; w_ϕ, w_E and w_B do not depend on λ. Both w_E and w_B are always less than or equal to unity, and at least one is strictly less.

If the p-form coupling λ is non-chaotic, the behavior is simple. The quantities Ω_E and Ω_B rapidly approach zero as Ω_ϕ approaches one, and the universe is dominated by the scalar field, with the equation of state w_ϕ. This is the non-chaotic case, discussed in Sec. III.

Alternatively, if the p-form coupling is supercritical, Ω_E and Ω_B grow. The averaging of the E and B component ensures $w_{\rm eff}<w_\phi$. If $w_\phi=1$ then $w_{\rm eff}<1$. In this case, the anisotropy grows and chaotic oscillations occur. It is not known if this happens in the $w_\phi>1$ case. If in addition, the p-form coupling is critical (so $w_{\rm crit}=1$), it turns out that the model is equivalent to an infinite-dimensional hyperbolic Toda system. There are an infinite number of jumps from one Kasner-like solution to the next, but the system may be formally integrable [7,8]. It is not clear what the physical ramifications of this behavior are.

IV. TIME-VARYING EQUATION OF STATE AND CHAOS

In a realistic cosmological model, the equation of state will not be constant, but will depend on the scale factor and approach some limiting value $w\to\overline{w}$ as $a\to 0$. If $\overline{w}\neq w_{\rm crit}$, none of the above analysis changes substantially. The model is supercritical if $\overline{w}<w_{\rm crit}$ or non-chaotic if $\overline{w}>w_{\rm crit}$. The critical case, $\overline{w}=w_{\rm crit}$, is more subtle, and the time dependence of $\overline{w}$ can be significant. In this section we assume $w_{\rm crit}=1$, as this is the most important case, and analyze what happens when $w_\phi\to 1$ at the crunch. We can expand w_ϕ as

$$w_\phi(a)=1+\gamma(a), \tag{46}$$

where γ is a small function of the scale factor such that $\gamma\to 0$ as $a\to 0$.

If there is no p-form with critical coupling, then using Eqs. (7) and (15) it can be shown that if $\gamma(a)\log a$ approaches a constant as $a\to 0$, the behavior is essentially the same as the $w=1$ case, i.e. non-chaotic. The radius of the Kasner circle in Fig. 1 shrinks if $w\to 1^+$ or expands if $w\to 1^-$. If $\gamma\to 0$ so slowly that $\gamma(a)\log a$ diverges as $a\to 0$, then the anisotropy is eliminated if γ approaches zero from above or the chaos is restored if γ approaches from below.

Alternatively, if the model has a p-form with critical coupling, the Kasner contraction will be stable if the p-form contribution to the equations of motion remain subdominant, or, equivalently, if the ratio of the p-form terms to the other terms vanishes in the $a\to 0$ limit. For magnetic modes with critical coupling $\lambda_{\rm crit}<0$, we find

$$\log\frac{\Omega_M}{\Omega_\phi}=\log\frac{e^{\lambda_{\rm crit}\phi}/a^{2(p+1)}}{\dot{\phi}^2/2+V(\phi)}\sim -C\int_a^{a_0}\frac{da'}{a'}\gamma(a'), \tag{47}$$

where C is a positive constant and by "$\sim$" we mean up to terms finite in the $a\to 0$ limit. The behavior is identical if the electric modes have critical coupling. If $\gamma\to 0$ very slowly, for example,

$$\gamma(a)\sim|\log a|^{-1} \tag{48}$$

so that the integral diverges as $a\to 0$, then the ratio goes to zero and Ω_M becomes negligible in the $a\to 0$ limit. This ensures that the term is small, and never grows to influence the dynamics.

Let us investigate what conditions on the potential will give us a γ of this form. If we combine the Friedmann equation and equation of motion for ϕ, we obtain

$$\frac{d\psi}{d\log a}=3\left(\psi-\frac{V_{,\phi}}{\sqrt{6}V}\right)(\psi-1)(\psi+1), \tag{49}$$

where ϕ denotes a derivative by ϕ and

$$\sqrt{6}\psi=\frac{d\phi}{d\log a}. \tag{50}$$

The equation of state (45) can be expressed in terms of ψ,

$$w_\phi=1+\gamma=2\psi^2-1. \tag{51}$$

We can obtain $w_\phi\to 1^+$ as $a\to 0$ for any negative potential which is bounded (for large negative values of ϕ) by $-Ce^{-\sqrt{6}\phi}$, where C is a positive constant [see Eq. (29)]. The kinetic energy increases more rapidly than the potential energy in these cases, and so w_ϕ approaches unity at the

crunch. In particular, the potential need not be bounded below. In general, any potential which can be expressed in the form

$$V(\phi)=2W'(\phi)^2-3W(\phi)^2 \tag{52}$$

satisfies positive energy [32]. Hertog *et al.* [33] have shown that the potential

$$-V_0 e^{-c\phi}, \tag{53}$$

where V_0 and c are positive constants, can be expressed in this form provided $c<\sqrt{6}$, and so satisfies positive energy. For $c\geqslant\sqrt{6}$, solutions exist with total ADM energy that is unbounded below.

For the potential (53), $V_{,\phi}/V=c$. In the case $c<\sqrt{6}$, we find

$$\gamma\propto a^y, \tag{54}$$

where y is a positive constant. Consequently, $\gamma\log a\to 0$ as $a\to 0$ and the p-form with critical coupling is not suppressed. However, when $c=\sqrt{6}$, the solution to the equations of motion show that $\gamma\log a$ approaches a constant, so the p-form can be suppressed when positive energy is violated.

The potential

$$V(\phi)=-V_0 e^{-\sqrt{6}\phi}|\phi|^n, \tag{55}$$

(or more generally, an exponential times any finite order polynomial) satisfies positive energy [i.e., can be expressed in the form (52)] for $n\leqslant -1$. Solving the equation of motion (49) for large ϕ, we find that for $n\leqslant -1$ the p-form with critical coupling is not suppressed. Surprisingly, for $n>-1$ the ratio (47) goes to zero, and the solution is stable. For the broad class of potentials (53) and (55), the parameters for which they satisfy positive energy turn out to be exactly those which do not suppress the p-form. It is an open question whether any potential can be constructed which will suppress the $w_{\rm crit}=1$ p-form and satisfy positive energy.

V. EXTRA DIMENSIONS, ORBIFOLDS AND CHAOS

In models in which gravity is fundamentally higher dimensional, the detailed global structure of the extra dimensions can suppress or enhance chaos in the four dimensional theory. We consider two simple compactifications of five-dimensional gravity, on S^1 and $S^1/\mathbb{Z}_2$. In the first, the chaotic nature of pure five-dimensional gravity descends to the four-dimensional theory. In the second, the chaotic behavior is suppressed. Models of quantum gravity also generally include additional matter fields in the extra dimensions. As an example, we discuss the compactification of heterotic M theory to four dimensions and find that its behavior during gravitational contraction is on the borderline between smooth and chaotic.

Consider a five-dimensional, flat universe without matter fields. We know from the study of general Kasner universes [4] that it will exhibit chaotic behavior. Now compactify one dimension on S^1. We know that the four-dimensional effective theory describes Einstein gravity coupled to a free scalar field. The scalar field describes the volume of the S^1—it is a simple example of a moduli field. As all of our preceding arguments regarding gravitational contraction are local in nature, we expect that the resulting system should be chaotic as well. However, a free scalar field has equation of state $w=1$. According to our analysis in Sec. II, one might think that the behavior should be non-chaotic. What has happened to the chaos?

The resolution lies in the fact that we have neglected many of the degrees of freedom of the higher-dimensional theory. A general five-dimensional metric G_{MN} can be written

$$G=\begin{pmatrix} h_{\mu\nu}+e^{2q\phi}A_\mu A_\nu & e^{2q\phi}A_\nu \\ e^{2q\phi}A_\mu & e^{2q\phi}\end{pmatrix}. \tag{56}$$

If we neglect the dependence of the metric on the fifth dimension, we may integrate out this dimension and perform a conformal transformation to canonically normalize the four-dimensional Ricci scalar. The coefficient q is then chosen to canonically normalize the scalar field kinetic energy in the resulting action

$$S=\int d^4x\sqrt{-g}\left[\tfrac{1}{2}R-\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{4}e^{\sqrt{6}\phi}F^2\right], \tag{57}$$

which describes a vector field coupled to a free scalar and to gravity. The coupling $\lambda=\sqrt{6}$ is outside the stable range for a 1-form in four dimensions. Therefore, the four-dimensional theory is chaotic, as we would have guessed, but we have to include the interactions with p-forms to see that this is so.

Next, instead of compactifying the fifth dimension on S^1, let us compactify on the orbifold $S^1/\mathbb{Z}_2$. If the coordinate x^5 on S^1 runs from $-\pi$ to π, this orbifold can be realized as S^1 together with the reflection $x^5\to -x^5$. This takes $G_{\mu 5}\to -G_{\mu 5}$, or equivalently $A_\mu\to -A_\mu$. Thus, the Kaluza-Klein zero-mode vector field A_μ is absent in the effective action (57). The absence of this vector field in the effective action thus implies that the four-dimensional theory is no longer chaotic.

While orbifolding suppresses some gauge fields and p-forms that would cause chaotic behavior, in some models there are additional p-forms in the bulk. These p-forms, after dimensional reduction, may themselves lead to chaotic behavior. An illustrative example is heterotic M theory, which includes a three-form field.

The low-energy four-dimensional effective action has been evaluated perturbatively by Lukas *et al.* [34]. To zeroth order in the 11-dimensional gravitational coupling κ, it is

$$S^{(0)}=\frac{\pi\rho V}{\kappa^2}\int dx^4\sqrt{-g}[R-(\partial a)^2-(\partial c)^2-e^{-\sqrt{8/3}c}(\partial\chi)^2 - e^{-\sqrt{8}a}(\partial\sigma)^2], \tag{58}$$

where we have rescaled the fields in Lukas' action so the kinetic energies are canonically normalized. The scalar field c is the radion, which governs the brane separation. The Calabi-Yau volume modulus a and scalar field σ (which

comes from the 11-dimensional 3-form) do not couple to c, and so can be ignored. However, the 3-form modulus χ couples to c and the exponent is critical $\lambda = -\sqrt{8/3}$. Hence, the theory does not lead to stable Kasner contraction. Including the first order ($\kappa^{2/3}$) correction to the action does not change the result.

As this theory is critical, it is quite conceivable that higher order corrections will lead to a different behavior during cosmological contraction. There are a number of kinds of corrections to Eq. (58) that could push the theory away from criticality and render it either chaotic or non-chaotic; however, it is not yet known which behavior occurs.

VI. CONCLUSIONS

The new results presented in this paper build on over three decades of preceding research on the behavior of cosmological models contracting to a big crunch. The classic work focused on cases where the equation of state of the dominant energy component is $w \leqslant 1$ and w is constant. The essential results in this case are as follows.

For a perfect fluid with $w < 1$, the contraction is smooth and anisotropic in the absence of curvature and chaotic mixmaster if there is non-zero curvature.

For a perfect fluid with $w = 1$, the contraction is smooth and *anisotropic* in the absence of curvature. With curvature, the contraction is anisotropic also, although, depending on the initial anisotropy, the contraction may undergo a finite number of jumps from one Kasner-like behavior to another.

For a free scalar field coupled to p-forms with coupling $e^{\lambda\phi}$, the contraction is *chaotic mixmaster* if the coupling λ is outside a finite interval of non-chaotic λ. The *mixmaster case is non-integrable* and the *critical case may be integrable*.

In this paper we have extended this work to include cases where $w > 1$, a situation that arises naturally in some recent models with a big-crunch–big-bang transition, such as the cyclic and ekpyrotic models. We have added the following results.

For perfect fluid with $w > 1$, the contraction is smooth and converges to *isotropic* at the crunch. The Einstein equations converge to ultralocal, homogeneous and isotropic Friedmann equations.

For a scalar field coupled to p-forms, there exists a $w_{\rm crit}$ such that the contraction is smooth and *isotropic* for $w > w_{\rm crit}$.

If w is time varying and approaches one from above sufficiently slowly the contraction is *smooth* and *non-chaotic*, even in the presence of a p-form with critical equation of state $w_{\rm crit} = 1$.

In models with an extra dimension, compactification generically produces a scalar field and p-forms. $\mathbb{Z}_2$ orbifolding forces some p-forms to zero and, thereby, suppresses their contributions to chaos.

In this paper we have studied how chaotic mixmaster behavior may be suppressed in models involving a big-crunch–big-bang transition. In particular, the ekpyrotic and cyclic models already include some of the required ingredients including a scalar field with $w > 1$ and $\mathbb{Z}_2$ orbifolding. We did not present a complete non-chaotic string-motivated model, but we hope the considerations reported here will be helpful in that regard.

ACKNOWLEDGMENTS

We thank G. Horowitz and T. Damour for helpful discussions. This work was supported in part by NSERC of Canada (J.K.E), by the NSF Graduate Research Program (D.H.W.), by U.S. Department of Energy Grant DE-FG02-91ER40671 and by PPARC (N.T.). P.J.S. was supported by the Wm. Keck Foundation and the Monell Foundation.

[1] E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[2] V.A. Belinsky, I.M. Khalatnikov, and E.M. Lifshitz, Adv. Phys. **19**, 525 (1970).
[3] V.A. Belinsky and I.M. Khalatnikov, Sov. Phys. JETP **36**, 591 (1973).
[4] J. Demaret, M. Henneaux, and P. Spindel, Phys. Lett. **164B**, 27 (1985).
[5] T. Damour and M. Henneaux, Phys. Rev. Lett. **85**, 920 (2000).
[6] T. Damour and M. Henneaux, Phys. Rev. Lett. **86**, 4749 (2001).
[7] T. Damour, M. Henneaux, A.D. Rendall, and M. Weaver, Ann. Henri Poincare **3**, 1049 (2002).
[8] T. Damour, M. Henneaux, and H. Nicolai, Class. Quantum Grav. **20**, R145 (2003).
[9] Charles W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).
[10] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973), pp. 800–816 and references therein.
[11] P. Dunsby, N. Goheer, M. Bruni, and A. Coley, hep-th/0312174.
[12] A.A. Coley, Phys. Rev. D **66**, 023512 (2002).
[13] A. Coley, Class. Quantum Grav. **19**, L45 (2002).
[14] Our conventions are $8\pi G = c = 1$; greek letters are for spacetime indices ($\mu, \nu, \ldots = 0,1,\ldots,d$); latin letters are solely for spatial indices ($i,j,\ldots = 1,2\ldots,d$); and $(-+\cdots+)$ is the sign convention for the metric. For simplicity, three spatial dimensions are assumed unless otherwise specified.
[15] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001).
[16] P.J. Steinhardt and N. Turok, Science **296**, 1436 (2002); Phys. Rev. D **65**, 126003 (2002).
[17] J. Khoury, P.J. Steinhardt, and N. Turok, Phys. Rev. Lett. **92**, 031302 (2004).
[18] R. Kallosh, L. Kofman, and A.D. Linde, Phys. Rev. D **64**, 123523 (2001).
[19] R. Kallosh, L. Kofman, A.D. Linde, and A.A. Tseytlin, Phys. Rev. D **64**, 123524 (2001).

[20] R. Brandenberger and F. Finelli, J. High Energy Phys. **11**, 056 (2001).
[21] D.H. Lyth, Phys. Lett. B **524**, 1 (2002).
[22] D.H. Lyth, Phys. Lett. B **526**, 173 (2002).
[23] J. Martin, P. Peter, N. Pinto Neto, and D.J. Schwarz, Phys. Rev. D **65**, 123513 (2002).
[24] J. Martin, P. Peter, N. Pinto-Neto, and D.J. Schwarz, Phys. Rev. D **67**, 028301 (2003).
[25] A.J. Tolley, N. Turok, and P.J. Steinhardt, Phys. Rev. D (to be published), hep-th/0306109.
[26] T.J. Battefield, S.P. Patil, and R. Brandeberger, hep-th/0401010.
[27] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Reading, MA, 1990) pp. 303–309.
[28] L. Andersson and A.D. Rendall, Commun. Math. Phys. **218**, 479 (2001).
[29] In fact, in a homogeneous universe, $T^0{}_i$ scales as a^{-3}. In this case, a the non-comoving component of a scalar field or perfect fluid never grows to dominate in a cosmology with $w>0$. For a universe with a scalar field and negative exponential potential, such as Eq. (29), it is possible to show that inhomogeneous perturbations to the scaling background may be neglected near the crunch in a contracting universe.
[30] Conversely, when $w<1$, α is negative and the β_i grow rapidly. Thus even if the energy density ρ is dominant initially, the anisotropy grows and eventually dominates near the crunch, a result consistent with our earlier analysis.
[31] J.P. Peebles, Comments Astrophys. Space Phys. **4**, 53 (1972).
[32] W. Boucher, Nucl. Phys. **B242**, 282 (1984).
[33] T. Hertog, G.T. Horowitz, and K. Maeda, J. High Energy Phys. **05**, 060 (2003).
[34] A. Lukas, B.A. Ovrut, and D. Waldram, Nucl. Phys. **B532**, 43 (1998).