QCD strings with spinning quarks

Theodore J. Allen
Department of Physics, Hobart and William Smith Colleges, Geneva, New York 14456, USA

M. G. Olsson
Department of Physics, University of Wisconsin, 1150 University Avenue, Madison, Wisconsin 53706, USA

Jeffrey R. Schmidt
Department of Physics, University of Wisconsin–Parkside, 900 Wood Road, Kenosha, Wisconsin 53141, USA
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We construct a consistent action for a massive spinning quark on the end of a QCD string that leads to a pure Thomas precession of the quark's spin. The string action is modified by the addition of Grassmann degrees of freedom to the string such that the equations of motion for the quark spin follow from boundary conditions, just as do those for the quark's position.

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I. INTRODUCTION

A consistent description of spin within a QCD string theory has been sought for many years. The addition of dynamical spin to the bosonic string led to the development of supersymmetry and superstring theory [1]. Such theories are more realistic as unified theories of elementary particle physics than as phenomenological descriptions of hadronic states.

A more realistic description of hadronic states involves the replacement of the free end of the dual resonance string by the addition of a massive point quark to the end of the string. In 1977, Ida [2] analyzed the motion of a spinless massive quark on the end of a bosonic string. The relativistic flux tube model [3], derived from different assumptions, is mathematically equivalent to a bosonic string with a spinless quark end and produces realistic meson spectra on average, but there is no place for quark spin in this model. In this paper we make a modification of the bosonic string plus bosonic quark model to introduce quark spin.

Our clue to constructing a consistent action comes from the suggestion of Buchmüller [4] that the spin of the quark should undergo pure Thomas precession because the quark sees a purely chromoelectric field in its rest frame. This seems to be supported by experimental data [5,6] and is in agreement with QCD [7,8].

We begin in Sec. II by discussing the treatment of spin in pseudoclassical language. We show how to construct actions for a free fermion as well as a fermion with background scalar and vector potentials. We analyze the case of a scalar potential in detail and show how the Thomas precession manifests itself in this language.

In Sec. III we show in detail that the Fermi-Walker transport of the spin vector, which is the equation of motion of the spin vector for a particle in a scalar potential, leads to Thomas precession of the spin in its rest frame.

In Sec. IV we use the example of a spinless quark coupled to the end of a string to argue for the form of the action for a spinning particle coupled to a modified Polyakov string action. The key idea is to obtain the equations of motion of the spin of the quark from boundary conditions, just as the equations of motion of the quark's position arise from boundary conditions. To this end, we introduce new Grassmann-valued fields on the string worldsheet.

In Sec. V we use the consistency of the equations of motion of the quark and the requirement of Thomas precession to fix the parameters in the string action. The result is that the only modification of a free spinning quark plus free bosonic string action is the replacement of the bosonic string position variable by the string position variable plus a term bilinear in worldsheet fermionic variables.

In Sec. VI we explore the fermionic gauge invariance of our string action. In the phenomenologically interesting case, we find that the worldsheet fermionic variables are pure gauge degrees of freedom.

We find the momentum and angular momentum from Noether's theorem in Sec. VII. These conserved quantities are the usual starting point for the numerical quantization of the relativistic flux tube model. Finally, we conclude in Sec. VIII.

II. SPIN IN PSEUDOCLASSICAL MECHANICS

We choose to work within the framework of pseudoclassical mechanics [9] because the formalism is elegant as well as physically transparent; the transition from pseudoclassical to quantum mechanics is immediate. In this section we construct actions that produce the Dirac equation, both free and in background potentials, as an equation of motion and we show how the Thomas precession in a scalar potential manifests itself in this language. The main disadvantage is that it requires some familiarity with the technical details of Dirac's constrained Hamiltonian mechanics [10,11] as well as classical mechanics with Grassmann variables [9,11].

The easiest way to construct pseudoclassical actions for fermions is to consider the Dirac equation as a phase-space constraint and to construct consistent actions that yield this constraint. The first actions of this type were found by Berezin and Marinov [12], Barducci, Casalbuoni, and Lusanna [13], and Brink, Deser, Zumino, Di Vecchia, and Howe [14]. To represent the spin degrees of freedom of a fermion, a set of five Grassmann coordinates, ξ_μ and ξ_5, are introduced. Upon quantization, the Grassmann coordinates will become generators of a Clifford algebra and can be identified with Dirac's gamma matrices. The kinetic piece of the action for the Grassmann variables

$$S_{\text{kinetic}} = \int d\tau \, \frac{i}{2}(\xi_\mu \dot{\xi}^\mu + \xi_5 \dot{\xi}_5), \tag{2.1}$$

leads to the canonical second-class constraints

$$\chi_\mu = \pi_\mu - \frac{i}{2}\xi_\mu \approx 0, \quad \chi_5 = \pi_5 - \frac{i}{2}\xi_5 \approx 0. \tag{2.2}$$

Here we use Dirac's wavy equal sign notation [10,11] for "weak equality," which reminds us that the equalities cannot be taken before Poisson brackets are calculated. We denote the canonical momenta to ξ^μ and ξ_5, defined to be the derivative of the Lagrangian from the right with respect to the velocities $\dot{\xi}^\mu$ and $\dot{\xi}_5$ respectively, by π_μ and π_5. With this convention, we obtain the following Poisson brackets:

$$\{\xi^\mu, \pi_\nu\} = \{\pi_\nu, \xi^\mu\} = \delta^\mu_\nu, \tag{2.3}$$

$$\{\xi_5, \pi_5\} = \{\pi_5, \xi_5\} = 1, \tag{2.4}$$

with all others being zero. Our conventions for pseudoclassical mechanics are given in Appendix A.

The weak equalities in Eq. (2.2) can be replaced by strong ones if we introduce the Dirac brackets [10]. From the definition in Appendix B and the Poisson brackets above, we find

$$\{\xi_\mu, \xi_\nu\}_D = -i\,\eta_{\mu\nu}, \tag{2.5}$$

$$\{\xi_\mu, \xi_5\}_D = 0, \tag{2.6}$$

$$\{\xi_5, \xi_5\}_D = -i, \tag{2.7}$$

where $\eta_{\mu\nu}$ is the metric. Our convention is $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

The meaning of the Grassmann numbers becomes clear upon quantization. When we make the replacement of $i\hbar$ times Dirac brackets by anticommutators, we find that the quantum operators $\hat{\xi}_\mu$ and $\hat{\xi}_5$ obey a Clifford algebra,

$$\hat{\xi}_\mu \hat{\xi}_\nu + \hat{\xi}_\nu \hat{\xi}_\mu = \hbar\,\eta_{\mu\nu},$$

$$\hat{\xi}_\mu \hat{\xi}_5 + \hat{\xi}_5 \hat{\xi}_\mu = 0, \tag{2.8}$$

$$\hat{\xi}_5 \hat{\xi}_5 = \frac{\hbar}{2}.$$

From these anticommutation relations, we see that the operators $\hat{\xi}_\mu$ and $\hat{\xi}_5$ can be represented as gamma matrices,

$$\hat{\xi}_\mu = \sqrt{\frac{\hbar}{2}}\gamma_5 \gamma_\mu, \tag{2.9}$$

$$\hat{\xi}_5 = \sqrt{\frac{\hbar}{2}}\gamma_5. \tag{2.10}$$

The free Dirac equation is proportional to

$$\hat{\phi}|\psi\rangle = (\hat{p}_\mu \hat{\xi}^\mu + m\hat{\xi}_5)|\psi\rangle = 0. \tag{2.11}$$

Thus, we should introduce the constraint

$$\phi = p_\mu \xi^\mu + m\xi_5 \approx 0 \tag{2.12}$$

into our action. This constraint does not have vanishing Dirac brackets with itself, but yields the Klein-Gordon operator:

$$K \equiv \frac{i}{2}\{p_\mu \xi^\mu + m\xi_5,\ p_\mu \xi^\mu + m\xi_5\}_D$$

$$= \frac{1}{2}(p^2 + m^2) \approx 0. \tag{2.13}$$

In order to be able to impose the constraint $\hat{\phi}$ as in Eq. (2.11), ϕ and any constraints, such as K, arising from it must be first-class, which means the Dirac brackets of any pair of them yield a combination of other first-class constraints. In order for the set of constraints to close under Dirac brackets, this last constraint must have vanishing Dirac brackets with ϕ. This is guaranteed by the (graded) Jacobi identity,

$$\{\phi, K\}_D = \frac{i}{2}\{\phi, \{\phi, \phi\}_D\}_D = 0. \tag{2.14}$$

The dynamics of this system are given by the free action, plus these constraints put in with Lagrange multipliers λ, and e:

$$S = \int d\tau \Big[p_\mu \dot{x}^\mu + \frac{i}{2}(\xi_\mu \dot{\xi}^\mu + \xi_5 \dot{\xi}_5) + i\frac{\lambda}{m}(p_\mu \xi^\mu + m\xi_5) - e\frac{1}{2}(p^2 + m^2) \Big]. \tag{2.15}$$

We may eliminate p from Eq. (2.15), by using its (purely algebraic) equation of motion. Similarly, we may then eliminate e from the intermediate action to find the action given by Berezin and Marinov [12],

$$S = \int d\tau \Big[-m\sqrt{-\dot{x}^2} + \frac{i}{2}(\xi_\mu \dot{\xi}^\mu + \xi_5 \dot{\xi}_5) + i\lambda(\xi_5 + u\cdot\xi) \Big], \tag{2.16}$$

where we have used the usual notation for the four-velocity, $u = \dot{x}/\sqrt{-\dot{x}^2}$.

The Dirac equation in a background scalar field φ and vector field A_μ is obtained from the free equation by minimal substitution for A_μ and the addition of φ to the mass:

$$\phi = (p - A)_\mu \xi^\mu + (m + \varphi)\xi_5 \approx 0. \tag{2.17}$$

We wish to use this as a constraint to construct an action in the same manner. We must again consider that the constraint ϕ have vanishing Dirac brackets with itself. We find

$$K\equiv \frac{i}{2}\{\phi,\phi\}_D=\frac{1}{2}(p-A)^2+\frac{1}{2}(m+\varphi)^2-\frac{i}{2}\xi^\mu\xi^\nu F_{\mu\nu} + i\xi_5\xi^\mu\partial_\mu\varphi\approx 0, \tag{2.18}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. The Jacobi identity again insures that there are no further constraints.

As before, we implement these constraints by use of Lagrange multipliers, a commuting one, e, and an anticommuting one, λ,

$$S=\int d\tau\left[p_\mu\dot{x}^\mu+\frac{i}{2}(\xi_\mu\dot{\xi}^\mu+\xi_5\dot{\xi}_5)+i\frac{\lambda}{m}\phi-eK\right]. \tag{2.19}$$

We note that the action for a spinless particle can be obtained by taking the spin variables to zero: $\xi_\mu\to 0$, $\xi_5\to 0$. The Thomas-Bargmann-Michel-Telegdi equations of motion [15–17] for the spin can be found from an analysis [12,18–21] of the action (2.19) with $\varphi=0$ and $A_\mu\neq 0$.

The action for a particle interacting with a background Yang-Mills field can be constructed by using additional Grassmann variables for the internal degrees of freedom [22].

Because we are interested only in the Thomas precession here, from now on we consider the action with a scalar potential only, so we set $A_\mu=0$. Eliminating first p_μ, and then e, in the action Eq. (2.19) with $A_\mu=0$, we find

$$S=\int d\tau\left[-\left(m+\varphi+\frac{i\xi_5\xi^\mu\partial_\mu\varphi}{m+\varphi}\right)\sqrt{-\dot{x}^2}+\frac{i}{2}(\xi_\mu\dot{\xi}^\mu+\xi_5\dot{\xi}_5) + i\lambda(\xi_5+u\cdot\xi)\right]. \tag{2.20}$$

This action is the same as the one analyzed by Martemyanov and Shchepkin [21].

The equations of motion from the action (2.20) are

$$\dot{p}_\mu=F_\mu, \tag{2.21}$$

$$\dot{\xi}_\mu=\lambda u_\mu+\frac{\xi_5 F_\mu}{m+\varphi}, \tag{2.22}$$

$$\dot{\xi}_5=\lambda-\frac{\xi\cdot F}{m+\varphi}, \tag{2.23}$$

where

$$F_\mu=-\partial_\mu\left(\varphi+\frac{i\xi_5\xi\cdot\partial\varphi}{m+\varphi}\right)\sqrt{-\dot{x}^2}, \tag{2.24}$$

$$p_\mu=\frac{\partial L}{\partial\dot{x}^\mu} =(m+\varphi)u_\mu-\frac{i}{m+\varphi}\xi_5\xi^\nu F_\nu\frac{u_\mu}{\sqrt{-\dot{x}^2}} + i\frac{\lambda}{\sqrt{-\dot{x}^2}}P_{\mu\nu}\xi^\nu, \tag{2.25}$$

and we have used the convenient notation

$$P_{\mu\nu}=\eta_{\mu\nu}+u_\mu u_\nu \tag{2.26}$$

for the projection operator perpendicular to the four-velocity.

In order to clarify the algebra in the rest of this section, we follow Martemyanov and Shchepkin [21] and work to lowest order in the fermionic variables. In this approximation we have

$$\dot{u}_\mu=\frac{P_{\mu\nu}F^\nu}{m+\varphi}, \tag{2.27}$$

$$\dot{\xi}_\mu=\left(\lambda-\frac{\xi_5 u\cdot F}{m+\varphi}\right)u_\mu+\dot{u}_\mu\xi_5, \tag{2.28}$$

$$\dot{\xi}_5=\lambda-\frac{\xi_5 u\cdot F}{m+\varphi}-\dot{u}\cdot\xi. \tag{2.29}$$

The momentum and angular momentum of the system can be found by Noether's theorem. We make an infinitesimal Poincaré transformation of the variables

$$\delta x^\mu=a^\mu+\omega^\mu{}_\nu x^\nu, \qquad \delta\xi^\mu=\omega^\mu{}_\nu\xi^\nu, \tag{2.30}$$

and extract the conserved quantities from

$$\delta S=\Delta\left(\frac{\partial L}{\partial\dot{x}^\mu}\delta x^\mu\right)+\Delta\left(\frac{\partial^R L}{\partial\dot{\xi}^\mu}\delta\xi^\mu\right) =a^\mu\Delta p_\mu+\frac{1}{2}\omega^{\mu\nu}\Delta J_{\nu\mu}, \tag{2.31}$$

where $\partial^R/\partial\dot{\xi}^\mu$ denotes the derivative acting from the right, and Δ denotes the difference in values between final and initial times. In Eq. (2.31) we have also used the equations of motion.

We find that the total angular momentum is a sum of orbital and spin pieces

$$J_{\mu\nu}=L_{\mu\nu}+S_{\mu\nu}=x_{[\mu}p_{\nu]}-i\xi_\mu\xi_\nu. \tag{2.32}$$

The total angular momentum, as well as each piece separately, obeys the Dirac brackets relation

$$\{J_{\mu\nu},J_{\mu'\nu'}\}_D=-\eta_{\mu\nu'}J_{\nu\mu'}-\eta_{\nu\mu'}J_{\mu\nu'}+\eta_{\nu\nu'}J_{\mu\mu'}+\eta_{\mu\mu'}J_{\nu\nu'}. \tag{2.33}$$

The Pauli-Lubanski vector,

$$s_\mu=-\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\nu S^{\alpha\beta}, \tag{2.34}$$

represents the spin of the particle and is purely spatial in the rest frame of the particle;

$$u\cdot s=0. \tag{2.35}$$

We use the convention that $\epsilon_{0123}=+1$. Using the identity

$$\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\delta}=-\delta^\mu_{[\alpha}\delta^\nu_\beta\delta^\rho_{\gamma]}, \tag{2.36}$$

we may revert Eq. (2.34) to find

$$i\xi^\alpha\xi^\beta=-\epsilon^{\alpha\beta\gamma\delta}u_\gamma s_\delta+i(u^\alpha\xi^\beta-u^\beta\xi^\alpha)(u\cdot\xi). \tag{2.37}$$

Using Eq. (2.37), we find the rate of change of s_μ

$$\begin{aligned}\dot{s}_\mu&=\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\dot{u}^\nu\xi^\alpha\xi^\beta+i\epsilon_{\mu\nu\alpha\beta}u^\nu\dot{\xi}^\alpha\xi^\beta,\\ &=u_\mu(\dot{u}\cdot s)+i\epsilon_{\mu\nu\alpha\beta}u^\nu\dot{u}^\alpha(u\cdot\xi)\xi^\beta\\ &\quad+i\epsilon_{\mu\nu\alpha\beta}u^\nu\dot{\xi}^\alpha\xi^\beta.\end{aligned} \tag{2.38}$$

We observe that the equation of motion for ξ^μ must have the form

$$\dot{\xi}^\mu=-\dot{u}^\mu(u\cdot\xi)+u^\mu(\text{anything}), \tag{2.39}$$

in order for the Pauli-Lubanski vector to be Fermi-Walker transported along the worldline of the particle. That is, for s_μ to obey

$$\dot{s}_\mu=u_\mu\dot{u}_\nu s^\nu=(u_\mu\dot{u}_\nu-\dot{u}_\mu u_\nu)s^\nu. \tag{2.40}$$

Equation (2.40) is the condition that there is no torque on the spin. The spin thus undergoes Thomas precession, as we will see in the next section.

III. THOMAS PRECESSION

In this section we demonstrate that a vector that undergoes Fermi-Walker transport in a circular orbit will precess in its rest frame at the Thomas frequency.

The spin vector of a gyroscope moved along a spacetime path $x^\mu(\tau)$ in the absence of net torque undergoes Fermi-Walker transport. We take laboratory time to be the worldline parameter; $\tau=t$. The rate of change of its spin vector then is

$$\frac{ds^\mu}{dt}=\Omega^\mu{}_\nu s^\nu, \tag{3.1}$$

with

$$\Omega^\mu{}_\nu=u^\mu\dot{u}_\nu-\dot{u}^\mu u_\nu, \tag{3.2}$$

where the u^μ is the four velocity tangent to $x^\mu(t)$ and a dot means derivative with respect to t.

We make the 3+1 identifications

$$u^0=\gamma,\qquad \mathbf{u}=\gamma\mathbf{v}, \tag{3.3}$$

and we note that the spin vector in its (noninertial) rest frame is

$$s_0^{\mu'}=\Lambda^{\mu'}{}_\nu s^\nu, \tag{3.4}$$

where $\Lambda^{\mu'}{}_\nu$ is the Lorentz transformation to the rest frame of the particle

$$\Lambda^{\mu'}{}_\nu=\delta^{\mu'}{}_\nu+\begin{pmatrix}\gamma-1 & -\gamma\mathbf{v}\\ -\gamma\mathbf{v} & \frac{(\gamma-1)}{v^2}\mathbf{v}\mathbf{v}\end{pmatrix}. \tag{3.5}$$

The equation of motion satisfied by the rest-frame spin vector is

$$\begin{aligned}\frac{ds_0}{dt}&=\frac{d}{dt}(\Lambda s)=\dot{\Lambda}s+\Lambda\dot{s}\\ &=\dot{\Lambda}\Lambda^{-1}s_0+\Lambda\Omega\Lambda^{-1}s_0.\end{aligned} \tag{3.6}$$

The rotation matrix (3.2) is

$$\Omega^\mu{}_\nu=\gamma^2\begin{pmatrix}0 & \dot{\mathbf{v}}\\ \dot{\mathbf{v}} & \mathbf{v}\dot{\mathbf{v}}-\dot{\mathbf{v}}\mathbf{v}\end{pmatrix}. \tag{3.7}$$

Simplifying the right hand side of Eq. (3.6), we find

$$\frac{ds_0}{dt}=\frac{\gamma-1}{v^2}\begin{pmatrix}0 & 0\\ 0 & \mathbf{v}\dot{\mathbf{v}}-\dot{\mathbf{v}}\mathbf{v}\end{pmatrix}s_0. \tag{3.8}$$

Since the rest frame spin, s_0, has no time component, we have

$$\frac{d\mathbf{s}_0}{dt}=\frac{\gamma-1}{v^2}(\mathbf{v}\dot{\mathbf{v}}-\dot{\mathbf{v}}\mathbf{v})\cdot\mathbf{s}_0=-\frac{\gamma-1}{v^2}(\mathbf{v}\times\dot{\mathbf{v}})\times\mathbf{s}_0. \tag{3.9}$$

The acceleration of a particle in uniform circular motion with angular velocity $\boldsymbol{\omega}$ is

$$\dot{\mathbf{v}}=\boldsymbol{\omega}\times\mathbf{v}. \tag{3.10}$$

In the case of uniform circular motion, Eq. (3.9) becomes

$$\frac{d\mathbf{s}_0}{dt}=-(\gamma-1)\boldsymbol{\omega}\times\mathbf{s}_0=\boldsymbol{\Omega}_T\times\mathbf{s}_0, \tag{3.11}$$

where $\boldsymbol{\Omega}_T$ is the Thomas frequency.

IV. STRING WITH ONE FIXED AND ONE MASSIVE END

A. Spinless quark

A string with one fixed end and a massive quark on the other end is described by an action that is the sum of the free massive point particle action and a free string action, which we take in Polyakov [23] form,

$$S=-\frac{T}{2}\int d\tau\int_0^1 d\sigma\sqrt{-h}h^{ab}\partial_a X^\mu\partial_b X_\mu - m\int d\tau\sqrt{-\dot{x}^2}. \tag{4.1}$$

Here $X^\mu(\sigma,\tau)$ are the coordinates of the string worldsheet parametrized by τ and σ, h_{ab} is the metric on the string worldsheet with $h=\det(h_{ab})$, $x^\mu(\tau)$ are the coordinates of the quark worldline, T is the string tension, and m is the quark mass. We use small latin letters for worldsheet tensor indices.

We require that the string end at $\sigma=0$ is fixed at the origin, $\mathbf{X}(0,\tau)=\mathbf{0}$. To make this an interacting theory, we must impose the condition that the end at $\sigma=1$ ends on the quark:

$$X^\mu(1,\tau)=x^\mu(\tau). \tag{4.2}$$

The variation of the action under variations that preserve the end-point conditions,

$$\delta X^\mu(0,\tau)=0, \tag{4.3}$$

$$\delta X^\mu(1,\tau)=\delta x^\mu(\tau), \tag{4.4}$$

is

$$\begin{aligned}\delta S&=\int d\tau\left(m\frac{\dot{x}^\mu\delta\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\right)-T\int d\sigma\, d\tau\,\sqrt{-h}h^{ab}\partial_a(\delta X^\mu)\partial_b X_\mu,\\ &=\int d\tau\,\delta x^\mu\left[\mathfrak{p}^1_\mu|_{\sigma=1}-\frac{d}{d\tau}\left(\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}\right)\right]\\ &\quad-\int d\sigma\, d\tau\,\delta X^\mu\partial_a\mathfrak{p}^a_\mu,\end{aligned} \tag{4.5}$$

after an integration by parts. Here we have used the notation $\mathfrak{p}^a_\mu$ for the current density of spacetime momentum on the worldsheet,

$$\mathfrak{p}^a_\mu=\delta S/\delta(\partial_a X^\mu)=-T\sqrt{-h}h^{ab}\partial_b X_\mu. \tag{4.6}$$

We see in Eq. (4.5) that the force that moves the quark arises from the boundary condition (4.2). The key idea of our work is to make a parallel construction with fermionic variables in the case of a spinning quark. In our construction, the motion of the quark's spin comes about as a result of introducing new fermionic variables on the string and the boundary conditions imposed upon them.

B. Spinning quark

In this section we make an ansatz for the form of the action. In order to have pure Thomas precession, we need an action for the fermionic variables ξ^μ whose variation has the form

$$\delta S\propto\int d\tau\,\delta\xi_\mu[i\dot{\xi}^\mu+i\dot{u}^\mu(u\cdot\xi)], \tag{4.7}$$

so that we obtain Eq. (2.39), the condition necessary for Thomas precession.

The term $i\delta\xi\cdot\dot{u}(u\cdot\xi)$ looks like $-i\delta\xi\cdot F^\mu\xi_5/m=-i\delta\xi\cdot\mathfrak{p}^{1\mu}\xi_5/m$, if we use the equations of motion $u\cdot\xi=-\xi_5$ and make the identification $m\dot{u}^\mu=F^\mu$.

We can obtain such a boundary variation by introducing worldsheet fermionic variables $\Xi^\mu(\sigma,\tau)$ and $\Xi_5(\sigma,\tau)$ whose boundary conditions are

$$\begin{aligned}\Xi^\mu(1,\tau)&=\xi^\mu(\tau),\\ \Xi_5(1,\tau)&=\xi_5(\tau),\end{aligned} \tag{4.8}$$

and then replacing $\partial_a X^\mu$ in the string action (4.1) by

$$\Pi^\mu_a\equiv\partial_a X^\mu-\alpha\frac{i}{m}\partial_a\Xi^\mu\Xi_5-\beta\frac{i}{m}\Xi^\mu\partial_a\Xi_5. \tag{4.9}$$

We will fix the parameters α and β by requiring consistency of the equations of motion and pure Thomas precession of the spin.

In analogy to the spinless case, we take our action to be the sum of the free Berezin-Marinov [12] action (2.16) for the particle and a Polyakov action modified by the replacement of $\partial_a X^\mu$ by Π^μ_a defined in Eq. (4.9):

$$\begin{aligned}S&=\int d\tau\left[-m\sqrt{-\dot{x}^2}+\frac{i}{2}(\xi_\mu\dot{\xi}^\mu+\xi_5\dot{\xi}_5)+i\lambda(\xi_5+u\cdot\xi)\right]\\ &\quad-\frac{T}{2}\int d\tau\int_0^1 d\sigma\sqrt{-h}h^{ab}\Pi^\mu_a\Pi_{b\mu}.\end{aligned} \tag{4.10}$$

C. Equations of motion

Under variations of the X^μ and x^μ that obey the boundary conditions Eqs. (4.3), we find the variation of the action to be

$$\begin{aligned}\delta S&=\int d\tau\left[m\frac{\dot{x}^\mu\delta\dot{x}_\mu}{\sqrt{-\dot{x}^2}}+i\lambda\left(\frac{\delta\dot{x}^\mu P_{\mu\nu}\xi^\nu}{\sqrt{-\dot{x}^2}}\right)\right]\\ &\quad-T\int d\sigma\, d\tau\sqrt{-h}h^{ab}\partial_a(\delta X^\mu)\Pi_{b\mu},\\ &=\int d\tau\,\delta x^\mu\left[\mathfrak{p}^1_\mu|_{\sigma=1}-\frac{d}{d\tau}\left(\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}+i\lambda\frac{P_{\mu\nu}\xi^\nu}{\sqrt{-\dot{x}^2}}\right)\right]\\ &\quad-\int d\sigma\, d\tau\,\delta X^\mu\partial_a\mathfrak{p}^a_\mu,\end{aligned} \tag{4.11}$$

where we have again used the notation $\mathfrak{p}^a_\mu$ for the current density of spacetime momentum on the worldsheet, which in this case is

$$\mathfrak{p}^a_\mu = \delta S/\delta(\partial_a X^\mu) = -T\sqrt{-h}h^{ab}\Pi_{b\mu}. \tag{4.12}$$

The vanishing of the variation δS leads to equations of motion for the quark and the string,

$$\frac{dp_\mu}{d\tau} = F_\mu = -T\sqrt{-h}h^{1b}\Pi_{b\mu}|_{\sigma=1}, \tag{4.13}$$

$$0 = \partial_a(\sqrt{-h}h^{ab}\Pi_{b\mu}), \tag{4.14}$$

where the quark's momentum is given by

$$p_\mu = m u_\mu + i\lambda \frac{P_{\mu\nu}\xi^\nu}{\sqrt{-\dot{x}^2}}, \tag{4.15}$$

with the usual projector, $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$.

Under variations of the fermionic variables $\Xi^\mu(\sigma,\tau)$, $\Xi_5(\sigma,\tau)$, $\xi^\mu(\tau)$ and $\xi_5(\tau)$, obeying

$$\delta\Xi^\mu(1,\tau) = \delta\xi^\mu(\tau),$$

$$\delta\Xi_5(1,\tau) = \delta\xi_5(\tau), \tag{4.16}$$

that preserve the boundary conditions (4.8), we find the variation of the action to be

$$\begin{aligned}\delta S &= \int d\tau\left[i\lambda\,\delta\xi^\mu u_\mu + i\lambda\,\delta\xi_5 + \frac{i}{2}(\delta\xi^\mu\dot{\xi}_\mu + \xi^\mu\delta\dot{\xi}_\mu + \delta\xi_5\dot{\xi}_5 + \xi_5\delta\dot{\xi}_5)\right] - \frac{i}{m}\int d\sigma\,d\tau[\alpha(\partial_a\delta\Xi^\mu\Xi_5 + \partial_a\Xi^\mu\,\delta\Xi_5)\\ &\quad + \beta(\delta\Xi^\mu\partial_a\Xi_5 + \Xi^\mu\partial_a\delta\Xi_5)]\mathfrak{p}^a_\mu,\\ &= \int d\tau\left[\left(i\lambda u_\mu - i\dot{\xi}_\mu + \alpha\frac{i}{m}\xi_5 F_\mu\right)\delta\xi^\mu + \left(i\lambda - i\dot{\xi}_5 - \beta\frac{i}{m}\xi^\mu F_\mu\right)\delta\xi_5\right] + \frac{i}{m}\int d\tau[(\alpha\,\delta\Xi^\mu\Xi_5 + \beta\Xi^\mu\,\delta\Xi_5)\mathfrak{p}^1_\mu]|_{\sigma=0}\\ &\quad - \frac{i}{m}\int d\sigma\,d\tau\{[\alpha\Xi_5\partial_a\mathfrak{p}^a_\mu + (\alpha-\beta)\partial_a\Xi_5\mathfrak{p}^a_\mu]\delta\Xi^\mu + [\beta\Xi^\mu\partial_a\mathfrak{p}^a_\mu - (\alpha-\beta)\partial_a\Xi^\mu\mathfrak{p}^a_\mu]\delta\Xi_5\}.\end{aligned} \tag{4.17}$$

Using the notation of Eq. (4.13), and the equation of motion (4.14), we find the equations of motion

$$\dot{\xi}^\mu = \lambda u^\mu + \alpha\xi_5\frac{F^\mu}{m}, \tag{4.18}$$

$$\dot{\xi}_5 = \lambda - \beta\xi_\mu\frac{F^\mu}{m}, \tag{4.19}$$

$$0 = (\alpha-\beta)\sqrt{-h}h^{ab}\Pi_{a\mu}\partial_b\Xi_5, \tag{4.20}$$

$$0 = (\alpha-\beta)\sqrt{-h}h^{ab}\Pi_{a\mu}\partial_b\Xi^\mu. \tag{4.21}$$

These last two equations of motion, Eqs. (4.20) and (4.21), would be automatically satisfied if $\alpha = \beta$.

The equation of motion for the metric h_{ab} yields the vanishing of the stress-energy tensor, also known as the Virasoro constraint,

$$T_{ab} = \Pi^\mu_a\Pi_{b\mu} - \frac{1}{2}h_{ab}h^{cd}\Pi^\mu_c\Pi_{d\mu} = 0. \tag{4.22}$$

Variation of the multiplier λ yields the equation of motion,

$$u^\mu\xi_\mu + \xi_5 = 0, \tag{4.23}$$

that becomes the Dirac equation constraint in canonical language

$$p_\mu\xi^\mu + m\xi_5 \approx 0. \tag{4.24}$$

The Klein-Gordon mass-shell condition,

$$\frac{1}{2}(p^2 + m^2) \approx 0, \tag{4.25}$$

arises directly from squaring the momentum (4.15). Equation (4.25) can also be found by taking the Dirac bracket of the constraint (4.24) with itself, as in Eq. (2.13).

We also need boundary conditions on the string fermionic variables at the fixed end, $\Xi^\mu(0,\tau)$ and $\Xi_5(0,\tau)$, in order to make the second integral in Eq. (4.17) vanish. We cannot impose $0 = T\sqrt{-h}h^{1b}\Pi_{b\mu}|_{\sigma=0}$ because that is the force on the fixed end, which cannot vanish. The correct boundary conditions are Dirichlet, of which the simplest are

$$\Xi^\mu(0,\tau) = 0, \tag{4.26}$$

$$\Xi_5(0,\tau) = 0. \tag{4.27}$$

V. DETERMINATION OF α AND β

A. Conservation of the Dirac equation constraint

We begin by looking at the equations of motion in $\lambda = 0$ gauge in order to make the ideas clearer. With $\lambda = 0$, Eq. (4.13) becomes

$$\dot{u}^\mu = \frac{F^\mu}{m}. \tag{5.1}$$

Using this, we simplify Eq. (4.18) and Eq. (4.19) to

$$\dot{\xi}^\mu = \alpha \xi_5 \dot{u}^\mu, \tag{5.2}$$

$$\dot{\xi}_5 = -\beta \xi_\mu \dot{u}^\mu. \tag{5.3}$$

The equation of motion (4.23) that leads to the Dirac equation,

$$u^\mu \xi_\mu + \xi_5 = 0, \tag{5.4}$$

must be constant in time for consistency. We find

$$\begin{aligned}\frac{d}{d\tau}(u^\mu \xi_\mu + \xi_5) &= \dot{u}\cdot\xi + u\cdot\dot{\xi} + \dot{\xi}_5\\ &= \dot{u}\cdot\xi + \alpha\xi_5 u\cdot\dot{u} - \beta\xi\cdot\dot{u}\\ &\equiv (1-\beta)\dot{u}\cdot\xi = 0.\end{aligned} \tag{5.5}$$

Thus, for consistency we must have $\beta = 1$.

In a general gauge with $\lambda \neq 0$, we obtain a similar result:

$$\begin{aligned}\frac{d}{d\tau}(p^\mu \xi_\mu + m\xi_5) &= \alpha\xi_5 \frac{p\cdot F}{m} + (1-\beta)\frac{F\cdot\xi}{m}\\ &= (1-\beta)\frac{F\cdot\xi}{m} = 0,\end{aligned} \tag{5.6}$$

as long as $p\cdot F = 0$, which is required for the consistency of the mass-shell relation (4.25). We take up this issue at the end of this section.

B. Thomas precession

Using the equation of motion (4.23) in Eq. (5.2), we find

$$\dot{\xi}^\mu = -\alpha \dot{u}^\mu (u\cdot\xi) + \lambda u^\mu. \tag{5.7}$$

The analysis of Sec. II showed that it was necessary for Eq. (2.39) to hold in order to have pure Thomas precession. Comparing Eq. (5.7) to Eq. (2.39), we find it necessary that $\alpha = 1$ in order to have pure Thomas precession.

C. Consistent action and boundary conditions

Because $\alpha = \beta = 1$ from the consistency and pure Thomas precession requirements, the string variable Π_a^μ is a total derivative,

$$\Pi_a^\mu = \partial_a \mathcal{X}^\mu, \tag{5.8}$$

with

$$\mathcal{X}^\mu \equiv X^\mu - \frac{i}{m}\Xi^\mu \Xi_5. \tag{5.9}$$

Remarkably, this combination is also the key to simplifying potential interactions of two fermions [24].

The consistent action for a QCD string with a spinning quark on one end that undergoes pure Thomas precession can be written using Eq. (5.9) as

$$\begin{aligned}S = \int d\tau\left[-m\sqrt{-\dot{x}^2} + \frac{i}{2}(\xi_\mu\dot{\xi}^\mu + \xi_5\dot{\xi}_5) + i\lambda(\xi_5 + u\cdot\xi)\right]\\ - \frac{T}{2}\int_0^1 d\sigma \int d\tau \sqrt{-h}h^{ab}\partial_a\mathcal{X}^\mu\partial_b\mathcal{X}_\mu.\end{aligned} \tag{5.10}$$

Because we have $\alpha = \beta$, the equations of motion (4.20) and (4.21) are automatically satisfied and the boundary conditions on Ξ^μ and Ξ_5 at the fixed end can be relaxed slightly,

$$\Xi^\mu(0,\tau)\Xi_5(0,\tau) = 0. \tag{5.11}$$

D. Conservation of the mass-shell constraint

We used the condition $p\cdot F = 0$ in Eq. (5.6). This condition is also necessary for the conservation of the mass-shell relation (4.25),

$$0 = \frac{1}{2}\frac{d}{d\tau}(p^2 + m^2) = p\cdot F. \tag{5.12}$$

We show that Eq. (5.12) follows from the equations of motion of the full action (5.10). To begin, we use the equations of motion (4.13), (4.18), and (4.19) and the expression (4.15) for the quark's momentum to calculate the boundary value

$$\begin{aligned}m\mathcal{X}^\mu|_{\sigma=1} &= m\dot{x}^\mu - i\dot{\xi}^\mu\xi_5 - i\xi^\mu\dot{\xi}_5\\ &= m\dot{x}^\mu - i\left(\lambda u^\mu + \frac{\xi_5}{m}F^\mu\right)\xi_5 - i\xi^\mu\left(\lambda - \frac{\xi\cdot F}{m}\right)\\ &= \sqrt{-\dot{x}^2}p^\mu + \frac{i}{m}\xi^\mu\xi\cdot F.\end{aligned} \tag{5.13}$$

Using the nilpotency of $\xi\cdot F$ and the Virasoro constraint (4.22), we find

$$\begin{aligned}p\cdot F &= \frac{m}{\sqrt{-\dot{x}^2}}\partial_0\mathcal{X}^\mu|_{\sigma=1}F_\mu\\ &= \left[\frac{m\sqrt{-h}h^{1b}}{\sqrt{-\dot{x}^2}}\partial_0\mathcal{X}^\mu\partial_b\mathcal{X}_\mu\right]\Bigg|_{\sigma=1}\\ &= \left[\frac{m\sqrt{-h}h^{1b}}{2\sqrt{-\dot{x}^2}}h_{0b}h^{cd}\partial_c\mathcal{X}^\mu\partial_d\mathcal{X}_\mu\right]\Bigg|_{\sigma=1}\\ &= \delta_0^1\left[m\frac{\sqrt{-h}}{2\sqrt{-\dot{x}^2}}h^{cd}\partial_c\mathcal{X}^\mu\partial_d\mathcal{X}_\mu\right]\Bigg|_{\sigma=1} = 0.\end{aligned} \tag{5.14}$$

We have used $h^{1b}h_{0b}=\delta_0^1=0$ in the last line. If we do the same analysis keeping α and β arbitrary, after a bit of algebra we find

$$p\cdot F=i(1-\beta)\frac{\lambda}{\sqrt{-\dot{x}^2}}(\xi\cdot F), \tag{5.15}$$

again showing the necessity of having $\beta=1$.

VI. FERMIONIC GAUGE INVARIANCE

The string portion of the action (5.10) has two fermionic constraints,

$$\Phi_\mu=\Pi_\mu-\frac{i}{m}P_\mu\Xi_5\approx 0, \tag{6.1}$$

$$\Phi_5=\Pi_5+\frac{i}{m}P_\mu\Xi^\mu\approx 0, \tag{6.2}$$

where Π_μ, Π_5, and P_μ are the momenta conjugate to Ξ^μ, Ξ_5, and X^μ respectively. The fermionic constraints together with the Virasoro constraints (4.22) are all first-class. It is easy to compute the Poisson brackets

$$\{\Phi_\mu,\Phi_\nu\}=\{\Phi_\mu,\Phi_5\}=\{\Phi_5,\Phi_5\}=0. \tag{6.3}$$

Because the stress tensor (4.22) is traceless, there are only two independent Virasoro constraints, which we may take in the form [1]

$$L_\pm=\frac{1}{2}(P\pm T\mathcal{X}')^2. \tag{6.4}$$

After a bit of algebra, we find the Poisson brackets

$$\begin{aligned}
&\{L_\pm(\sigma),L_\pm(\varrho)\}=T[L_\pm(\sigma)+L_\pm(\varrho)]\delta'(\sigma-\varrho),\\
&\{L_\pm(\sigma),L_\mp(\varrho)\}=0,\\
&\{L_\pm(\sigma),\Phi_\mu(\varrho)\}=0,\\
&\{L_\pm(\sigma),\Phi_5(\varrho)\}=0.
\end{aligned} \tag{6.5}$$

By acting in combination on the fields A through Poisson brackets,

$$\delta_H A=\{A,H^\mu\Phi_\mu+H_5\Phi_5\}, \tag{6.6}$$

the constraints (6.1) generate the following fermionic gauge invariance of the action:

$$\begin{aligned}
\delta_H X^\mu&=\frac{i}{m}(H^\mu\Xi_5+\Xi^\mu H_5),\\
\delta_H\Xi^\mu&=H^\mu,\\
\delta_H\Xi_5&=H_5.
\end{aligned} \tag{6.7}$$

Here $H^\mu=H^\mu(\sigma,\tau)$ and $H_5=H_5(\sigma,\tau)$ are Grassmann-valued functions on the string worldsheet. This is not an invariance of the particle action, so the gauge parameters H must vanish at the boundary. Obviously $\mathcal{X}^\mu$ is gauge invariant,

$$\delta_H\mathcal{X}^\mu\equiv 0, \tag{6.8}$$

so the string action (5.10) is invariant as well. Because we have as many first-class constraints as fermionic variables, there are no dynamical fermionic degrees of freedom on the string; except for their values at the boundary, they are pure gauge.

Just as for a free Dirac particle action, the particle piece of the action (4.10) has a local supersymmetry generated by the Dirac constraint

$$\phi=p\cdot\xi+m\xi_5\approx 0. \tag{6.9}$$

The gauge variation of x^μ is

$$\delta_\eta x^\mu=\{x^\mu,\, i\eta\phi\}_D=i\eta\xi^\mu. \tag{6.10}$$

The gauge variations of the other variables are

$$\begin{aligned}
\delta_\eta p_\mu&=0,\\
\delta_\eta\xi^\mu&=-\eta p^\mu,\\
\delta_\eta\xi_5&=-\eta m.
\end{aligned} \tag{6.11}$$

The Lagrange multiplier fields λ and e have gauge variations

$$\begin{aligned}
\delta_\eta\lambda&=-\dot{\eta}m,\\
\delta_\eta e&=-\frac{2i\lambda\eta}{m}.
\end{aligned} \tag{6.12}$$

VII. ENERGY AND ANGULAR MOMENTUM

The numerical quantization of the relativistic flux tube model starts from the conserved quantities of the system. The canonical form of these quantities for the string with a spinning quark is only slightly different from those for the spinless case. In this section we calculate the four-momentum and the angular momentum of our system.

The action, Eq. (5.10), is invariant under infinitesimal translations and Lorentz transformations,

$$\begin{aligned}
\delta x^\mu(\tau)&=a^\mu+\omega^\mu{}_\nu x^\nu(\tau),\\
\delta X^\mu(\sigma,\tau)&=a^\mu+\omega^\mu{}_\nu X^\nu(\sigma,\tau),\\
\delta\xi^\mu(\tau)&=\omega^\mu{}_\nu\xi^\nu(\tau),\\
\delta\Xi^\mu(\sigma,\tau)&=\omega^\mu{}_\nu\Xi^\nu(\sigma,\tau).
\end{aligned} \tag{7.1}$$

Noether's theorem guarantees the existence of conserved total momentum $\mathcal{P}_\mu$ and conserved total angular momentum $\mathcal{J}_{\mu\nu}$, which can be computed from the vanishing change in the action:

$$\delta S = a^{\mu}\Delta\mathcal{P}_{\mu} + \frac{1}{2}\omega^{\mu\nu}\Delta\mathcal{J}_{\nu\mu} = 0, \tag{7.2}$$

assuming the use of the equations of motion. From Eq. (7.2), we find explicitly

$$\mathcal{P}^{\mu} = p^{\mu} + \int_0^1 d\sigma P^{\mu} \tag{7.3}$$

$$\mathcal{J}_{\mu\nu} = x_{[\mu}p_{\nu]} - \xi_{[\mu}\pi_{\nu]} + \int_0^1 d\sigma (X_{[\mu}P_{\nu]} - \Xi_{[\mu}\Pi_{\nu]})$$

$$= x_{[\mu}p_{\nu]} - i\xi_{\mu}\xi_{\nu} + \int_0^1 d\sigma \mathcal{X}_{[\mu}P_{\nu]}, \tag{7.4}$$

where $\mathcal{X}^{\nu} = X^{\nu} - (i/m)\Xi^{\nu}\Xi_5$. In the last line of Eq. (7.4) we have used the constraints (2.2) and (6.1).

VIII. DISCUSSION

Starting from the requirement that an action for a spinning quark on the end of a string should lead to pure Thomas precession of the quark spin, we have shown how to construct a consistent action for a massive spinning quark on the end of a QCD string whose other end is fixed. To do so, we introduced additional fermionic variables on the QCD string itself and required that the equations of motion of the quark's fermionic variables arise from boundary conditions on the string. The two parameters we introduced, α and β, are fixed to unity by the requirements of pure Thomas precession and consistency of the equations of motion, respectively.

Our action has a consistent fermionic symmetry that allows us to gauge away the stringy fermionic degrees of freedom. Other authors [25–27] have argued for the existence of a supersymmetry on the QCD string worldsheet when there are spinning quarks on the boundary. It would be interesting to know if our fermionic symmetry is that supersymmetry. As one piece of evidence, we can make contact with the Wilson super-loop approach [26–28] by making a change of variables in our action, Eq. (5.10). When we make the variable change

$$X^{\mu} \to X^{\mu} + \frac{i}{m}\Xi^{\mu}\Xi_5,$$

$$x^{\mu} \to x^{\mu} + \frac{i}{m}\xi^{\mu}\xi_5, \tag{8.1}$$

in the string and quark actions respectively, we are led to the Polyakov bosonic string action plus a quark action

$$S_q = \int d\tau (-m\sqrt{-\dot{x}^2} - iu^{\mu}\dot{u}^{\nu}\xi_{\mu}\xi_{\nu} + \cdots), \tag{8.2}$$

where we have neglected to write the kinetic terms for the fermions, the Dirac constraint, some higher-order fermionic pieces, and a total time derivative. The second term yields the interaction of the quark's spin with the string worldsheet. Unfortunately, this action is not simple and does not seem to have a reasonable canonical formulation, unlike Eq. (5.10). We also note that the second term of Eq. (8.2) is similar to one added by Martemyanov and Shchepkin [21], though there are additional terms in Eq. (8.2) not present in their action.

We have not considered the case of a quark at each end, however the generalization is immediate. We introduce a set of Grassmann variables for each quark, ξ_i^{μ}, $\xi_{5\,i}$, with $i = 1,2$. In this case, however, the variables for one quark commute with those of the other [24], just as the gamma matrices of two different fermions commute. We also introduce a set of worldsheet fermionic variables for each quark and make the generalization in the string action

$$X^{\mu} \to \mathcal{X}^{\mu} = X^{\mu} - i\sum_{i=1,2}\frac{1}{m_i}\Xi_i^{\mu}\Xi_{5\,i}. \tag{8.3}$$

Though the Ξ variables of each quark are Grassmann-valued, the Ξ variables of one quark should commute with the Ξ variables of the other.

We have not considered the case of a massless quark, which appears to be somewhat problematic in our formalism. On the other hand, there is no problem treating very light, but still massive, quarks in this formalism.

Although we have partially analyzed our action in the general case of $\alpha \neq \beta$, we have not pursued the analysis with $\alpha \neq 1$ because we are most interested in the phenomenologically relevant $\alpha = \beta = 1$ case. In the more general case, the fermionic constraints on the worldsheet are not all first-class and some of the fermionic variables on the worldsheet may become dynamical, though additional terms in the action may be necessary to preserve the first-class nature of the Virasoro constraints.

We hope to present soon a numerical quantization of this action.

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APPENDIX A: PSEUDOCLASSICAL MECHANICAL CONVENTIONS

We take a canonical form for an action to have the velocities to the right of the momenta,

$$S = \int L\, d\tau = \int d\tau [p_i\dot{q}^i + \pi_{\alpha}\dot{\xi}^{\alpha} - H(q,p,\pi,\xi)], \tag{A1}$$

where q^i and p_i are bosonic variables and ξ^{α} and π_{α} are fermionic variables and H is a Grassmann even function. The variation of H under a change of a fermionic variable such as $\delta\xi^{\alpha}$ is

$$\delta H = \frac{\partial^R H}{\partial \xi^{\alpha}}\delta\xi^{\alpha}, \tag{A2}$$

where $\partial^R H/\partial \xi^\alpha$ denotes the derivative from the right. We could equally well have used

$$\delta H = \delta \xi^\alpha \frac{\partial^L H}{\partial \xi^\alpha}, \tag{A3}$$

since it has the same value. Variation of the action leads to the usual canonical equations of motion,

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

$$\dot{\xi}^\alpha = -\frac{\partial^R H}{\partial \pi_\alpha} = \frac{\partial^L H}{\partial \pi_\alpha}$$

$$\dot{\pi}_\alpha = -\frac{\partial^R H}{\partial \xi^\alpha} = \frac{\partial^L H}{\partial \xi^\alpha}. \tag{A4}$$

The last two relations of Eq. (A4) follow because H is an even Grassmann parity function. These relations can be succinctly summarized by the introduction of a Poisson bracket,

$$\dot{z} = \{z, H\}, \tag{A5}$$

where z is any of q^i, p_i, ξ^α, or π_α and the Poisson bracket of any two functions A and B is

$$\{A,B\} = \sum_i \left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} \right) + \sum_\alpha \left(\frac{\partial^R A}{\partial \xi^\alpha}\frac{\partial^L B}{\partial \pi_\alpha} + \frac{\partial^R A}{\partial \pi_\alpha}\frac{\partial^L B}{\partial \xi^\alpha} \right). \tag{A6}$$

APPENDIX B: DIRAC BRACKETS

When a system has second-class constraints and one wishes to set them strongly to zero, consistency requires that the Poisson brackets of the system be modified so that the Poisson bracket of any second-class constraint with any other phase space function is identically zero. This modified Poisson bracket is called a Dirac bracket.

The simplest example is illuminating, though artificial. We imagine a dynamical system in which there are $2N$ phase space variables and two second-class constraints, $q_N \approx 0$ and $p_N \approx 0$. These constraints hold for all time so q_N and p_N are irrelevant variables; no physical quantity should depend upon them. The correct procedure is to ignore these variables and replace the Poisson bracket,

$$\{A,B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} \right), \tag{B1}$$

by the Dirac bracket

$$\{A,B\}_D = \sum_{i=1}^{N-1} \left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} \right). \tag{B2}$$

The Dirac bracket (B2) of q_N or p_N with any other phase space function will obviously vanish.

For a more complicated system with second-class constraints $\chi_i \approx 0$, the Dirac bracket is less obvious. The matrix of Poisson brackets of the second-class constraints

$$\{\chi_i, \chi_j\} = \Delta_{ij}, \tag{B3}$$

has a nonvanishing determinant, and is therefore invertible. We denote the matrix inverse to Δ_{ij} by Δ^{ij}, and define the Dirac bracket of any two functions A and B as

$$\{A,B\}_D \equiv \{A,B\} - \{A,\chi_i\}\Delta^{ij}\{\chi_j, B\}. \tag{B4}$$

The desired property now follows,

$$\begin{aligned}\{A,\chi_k\}_D &= \{A,\chi_k\} - \{A,\chi_i\}\Delta^{ij}\{\chi_j,\chi_k\} \\ &= \{A,\chi_k\} - \{A,\chi_i\}\Delta^{ij}\Delta_{jk} \\ &= \{A,\chi_k\} - \{A,\chi_i\}\delta^i_k \equiv 0.\end{aligned} \tag{B5}$$

We note that some authors use $\{A,B\}^*$ to denote the Dirac bracket.

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