Effective chiral Lagrangian in the chiral limit from the instanton vacuum

Hyun-Ah Choi* and Hyun-Chul Kim†
Department of Physics, Pusan National University, 609-735 Busan, Republic of Korea
(Received 18 August 2003; published 12 March 2004)

We study the effective chiral Lagrangian in the chiral limit from the instanton vacuum. Starting from the nonlocal effective chiral action, we derive the effective chiral Lagrangian, using the derivative expansion to order $O(p^4)$ in the chiral limit. The low energy constants L_1, L_2, and L_3 are determined and compared with various models and the corresponding empirical data. The results are in a good agreement with the data. We also discuss the upper limit of the sigma meson, based on the present results.

DOI: 10.1103/PhysRevD.69.054004 PACS number(s): 12.39.Fe, 12.40.−y, 14.20.Dh

I. INTRODUCTION

Chiral perturbation theory (χPT) was introduced as an effective field theory of QCD in a very low energy regime [1]. Based on the chirally invariant Lagrangian with a set of coefficients, χPT has been a great success in describing very low-energy phenomena of the strong interaction [2–5,55]. While the structure of the Lagrangian is determined by the symmetry pattern in QCD, the coefficients are unknown. These unknown coefficients, known as low-energy constants (LECs), contain microscopic information about the quark-gluon dynamics which would be in principle determined by QCD. However, it requires formidable work to derive them from QCD and thus is impractical. In fact, they are fitted to the experimental data such as $\pi\pi$ scattering and K_{l4} decay [2,6] and use them for describing or predicting other processes. However, when one goes beyond the leading order, the number of the LECs start to increase very rapidly. Hence, it is not feasible to fix all LECs to empirical data.

There has been a great amount of works on the LECs within various chiral models [7–24]. Although dynamical ingredients of each model are different, almost all models are in good agreement with empirical data. Apart from some models [15,21], many models are based on local interactions of quarks and mesons. While the nonlocality of the quark can be neglected in the range of quark momenta, for example, $k \ll 1/\bar{\rho} \simeq 600$ MeV in which $\bar{\rho}$ denotes the average size of the instanton, recent works on the pion wave functions [25,26] and skewed parton distribution [27] show that it is of great importance to consider the momentum-dependent quark mass in order to produce the correct end-point behavior of the quark virtuality. Similarly, a very recent study on the effective weak chiral Lagrangian to order $O(p^2)$ from the instanton vacuum [28,29] asserts that the nonlocality of the quark plays an essential role in improving previous results [30] concerning the $\Delta T = 1/2$ rule in the LECs. Furthermore, an appreciable merit of using the momentum-dependent quark mass as a regulator was already pointed out by Ball and Ripka [31]. The momentum-dependent quark mass provides a consistent regularization of the effective action in which its real and imaginary parts are treated on the same footing and thus pertinent observables such as anomalous decays $\pi^0 \to 2\gamma$ are safely recovered even if $M(k)$ acts as a regulator.

In the present work, we shall investigate the effective chiral Lagrangian from the instanton vacuum (see a recent review [32]). We first consider the chiral limit as well as the absence of the external fields. In order to take into account the effect of SU(3)-symmetry breaking, one has to modify the effective chiral action originally obtained by Diakonov and Petrov [33]. Moreover, the vector and axial-vector currents are not conserved in the presence of the nonlocal interaction. Thus, we first shall concentrate on the LECs in the chiral limit.

The outline of the present paper is as follows: In Sec. II we briefly explain the instanton-induced chiral quark model, emphasizing in particular the momentum dependence of the constituent quark mass and explain how to perform the derivative expansion in the presence of the momentum-dependent constituent quark mass. In Sec. III, we show how to derive the $O(p^4)$ effective chiral Lagrangian, using the derivative expansion. In Sec. IV, we discuss the results. In Sec. V we draw conclusion and make summary.

II. CHIRAL QUARK MODEL FROM THE INSTANTON VACUUM

The instanton vacuum elucidates one of the most important low-energy properties of QCD, i.e., the mechanism of spontaneous breaking of chiral symmetry [34–36]. The Banks-Casher relation [37] tells us that the spectral density $\nu(\lambda)$ of the Dirac operator at zero modes is proportional to the chiral condensate known as an order parameter of spontaneous breaking of chiral symmetry:

$$\langle \bar{\psi}\psi \rangle = -\frac{\pi\nu(0)}{V^{(4)}}. \qquad (1)$$

The picture of the instanton vacuum provides a good realization of spontaneous breaking of chiral symmetry. A finite density of instantons and anti-instantons produces the nonvanishing value of $\nu(0)$, which triggers the mechanism of chiral symmetry breaking. The Euclidean quark propagator in the instanton vacuum acquires the following form with a

*Email address: hachoi@pusan.ac.kr
†Email address:hchkim@pusan.ac.kr

momentum-dependent quark mass generated dynamically, identified with the coupling strength between quarks and Goldstone bosons:

$$S_F(k)=\frac{\not k+iM(k)}{k^2+M^2(k)}, \tag{2}$$

with

$$M(k)=\mathrm{const}\times\sqrt{\frac{N\pi^2\bar{\rho}^2}{VN_c}}F^2(k\bar{\rho})=M_0F^2(k\bar{\rho}). \tag{3}$$

The ratio N/V denotes the instanton density at equilibrium and the $\bar{\rho}$ is the average size of the instanton. The form factor function $F(k\bar{\rho})$ is related to the Fourier transform of the would-be zero fermion mode of individual instantons. The instanton density N/V is expressed as a gap equation:

$$\frac{N}{V}=4N_c\int\frac{d^4k}{(4\pi)^4}\frac{M^2(k)}{k^2+M^2(k)}=1\ \ \mathrm{fm}^{-4}. \tag{4}$$

Taking the average instanton size $\bar{\rho}=1/3$ fm, one obtains $M_0\simeq 350$ MeV.

The instanton vacuum induces effective $2N_f$-fermion interactions [34–36]. For example, it has a type of the Nambu–Jona-Lasinio model for $N_f=2$ while for $N_f=3$ it exhibits the 't Hooft determinant [38]. Goldstone bosons appear as collective excitations by quark loops generating a dynamic quark mass. Eventually it is found that at low energies QCD is reduced to an interacting quark-Goldstone boson theory given by the following Euclidean partition function [36]

$$\begin{aligned}\mathcal{Z}=&\int\mathcal{D}\psi\mathcal{D}\psi^\dagger\mathcal{D}\pi^a\exp\int d^4x\Bigg[\psi_f^{\dagger\alpha}(x)i\partial\!\!\!/\psi_f^\alpha(x)\\&+i\int\frac{d^4kd^4l}{(2\pi)^8}e^{i(k-l)\cdot x}\sqrt{M(k)M(l)}\\&\times\psi_f^{\dagger\alpha}(k)(U^{\gamma_5})_{fg}\psi_g^\alpha(l)\Bigg],\end{aligned} \tag{5}$$

where U^{γ_5} stands for the pseudo-Goldstone boson:

$$\begin{aligned}U^{\gamma_5}(x)&=U(x)\frac{1+\gamma_5}{2}+U^\dagger(x)\frac{1-\gamma_5}{2}\\&=\exp(i\pi^a(x)\lambda^a\gamma_5/f_\pi).\end{aligned} \tag{6}$$

The α is the color index, $\alpha=1,\ldots,N_c$ and f and g are flavor indices. $M(k)$ is the constituent quark mass being now momentum-dependent, which is expressed by Eq. (3). Its momentum dependence will play a main role in the present work. If we choose $F(k\bar{\rho})$ to be constant and add a regularization (e.g., Pauli-Villars or proper-time), the partition function becomes just that of the usual χQM. The original expression for the $F(k\bar{\rho})$ [34], which is obtained from the Fourier transformation of the would-be zero fermion mode of individual instantons with the sharp instanton distribution assumed, is as follows:

$$F(k\bar{\rho})=2z\left(I_0(z)K_1(z)-I_1(z)K_0(z)-\frac{1}{z}I_1(z)K_1(z)\right). \tag{7}$$

Here I_0, I_1, K_0, and K_1 denote the modified Bessel functions, z is defined as $z=k\bar{\rho}/2$. When k goes to infinity, the form factor $F(k\bar{\rho})$ has the following asymptotic behavior:

$$F(k\bar{\rho})\to\frac{6}{(k\bar{\rho})^3}. \tag{8}$$

Actually, there are other ways of understanding the nonlocal effective interaction without relying on the instanton vacuum [39–42]. In those cases, the momentum-dependent quark mass can be interpreted as a nonlocal regularization in Euclidean space. Hence, various types of the $M(k)$ as a regulator with the regularization parameter $\Lambda\sim 1/\bar{\rho}$ has been used by different authors. For example, the dipole-type $M(k)$ is used in the study of the pion wave function [26], while the Gaussian is employed in Ref. [43].

Therefore, we will not confine ourselves to the expression given in Eq. (7) but rather try three different types of the $M(k)$:

$$M(k)=\begin{cases}\text{Eqs. (3),(7)},\\ M_0\left(\dfrac{4\Lambda^2}{4\Lambda^2+k^2}\right)^4,\\ M_0\exp\left(-\dfrac{k^2}{\Lambda^2}\right),\end{cases} \tag{9}$$

where the cutoff parameter Λ is taken as the inverse of $\bar{\rho}$. The $M(k)$ is normalized to M_0 at $k=0$. Originally, M_0 is found to be around 350 MeV. However, we will regard M_0 as a free parameter ranging from 200 MeV to 450 MeV and fit for each M_0 the parameter Λ to the pion decay constant $f_\pi=93$ MeV. Figure 1 shows the momentum dependence of the three different types of $M(k)$ with $M_0=350$ MeV. The dipole type displays the largest tale, while the Gaussian takes the strongly suppressed tail, compared to other ones. As will be shown later, this difference appearing in the tail is basically responsible for the different results in the LECs of the effective weak chiral Lagrangian.

This effective theory of quarks and light Goldstone mesons applies to quark momenta up to the inverse size of the instanton, $\bar{\rho}^{-1}\simeq 600$ MeV, which may act as a scale of the model ($\mu_{\chi\mathrm{QM}}$). A merit to derive the χQM from the instanton vacuum lies in the fact that the scale of the model is naturally determined by $\bar{\rho}^{-1}$. Furthermore, mesons and baryons can be treated on the same footing in the χQM. For example, the model has been very successful in describing the properties of the baryons [44].

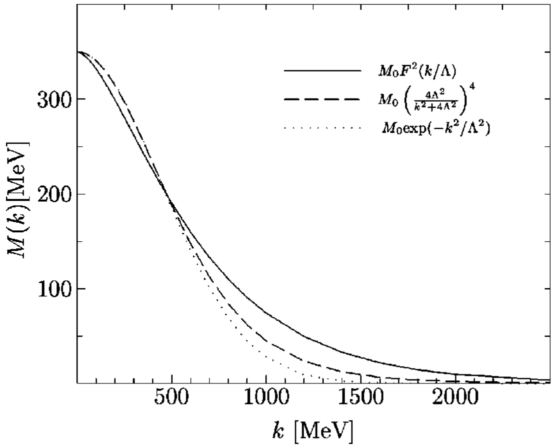

FIG. 1. The dependence of the $M(k)$ on $|k|$. The solid curve draws the Diakonov-Petrov $M(k)$, the dashed one shows the dipole-type parametrization of $M(k)$, and the dotted one corresponds to the Gaussian type of $M(k)$.

III. EFFECTIVE CHIRAL LAGRANGIAN TO ORDER $O(p^4)$

The low-energy effective QCD partition function given in Eq. (5) is the starting point of the present work. Having integrated out the quark fields of Eq. (5), we obtain

$$\mathcal{Z}=\int \mathcal{D}\pi^a \exp(-S_{\rm eff}[\pi^a]), \tag{10}$$

where the $S_{\rm eff}[\pi^a]$ stands for the effective chiral action:

$$S_{\rm eff}[\pi^a]=-N_c \ln\det D(U^{\gamma_5}). \tag{11}$$

Here, the $D(U^{\gamma_5})$ is the Dirac operator defined by

$$D=i\partial\!\!\!/+i\sqrt{M(-i\partial)}U^{\gamma_5}\sqrt{M(-i\partial)}. \tag{12}$$

The Dirac operator is not Hermitian, so that it is useful to divide the effective action into the real and imaginary parts:

$$\mathrm{Re}\, S_{\rm eff}=\frac{1}{2}(S_{\rm eff}+S_{\rm eff}^*)=-\frac{1}{2}N_c \ln\det[D^\dagger D], \tag{13}$$

$$i\,\mathrm{Im}\, S_{\rm eff}=\frac{1}{2}(S_{\rm eff}-S_{\rm eff}^*)=-\frac{1}{2}N_c \ln\det[D/D^\dagger]. \tag{14}$$

It is already known that the imaginary part of the effective chiral action is identical to the Wess-Zumino-Witten action [45,46] with the correct coefficient, which arises from the derivative expansion of the imaginary part to $O(p^5)$ [47–52]. An appreciable merit of using the momentum-dependent quark mass as a regulator was already pointed out by Ball and Ripka [31]. The momentum-dependent quark mass provides a consistent regularization of the effective action given in Eq. (11) in which its real and imaginary parts are treated on the same footing and thus pertinent observables such as anomalous decays $\pi^0\to 2\gamma$ are safely recovered even if $M(k)$ acts as a regulator. Hence, in this work, we will concentrate on the real part of the effective chiral action which will provide us with the effective chiral Lagrangian with the LECs determined. In the present work, we first consider the case of the chiral limit and turn off the external fields. Furthermore, we keep only the leading order in the large N_c.

In order to calculate the real part given in Eq. (13), we subtract the vacuum part and use the derivative expansion. We therefore write

$$\begin{aligned}
\mathrm{Re}\, S_{\rm eff}[\pi^a]-\mathrm{Re}\, S_{\rm eff}[0]&=-\frac{N_c}{2}\mathrm{Tr}\ln\left(\frac{D^\dagger D}{D_0^\dagger D_0}\right)\\
&=-\frac{N_c}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\mathrm{tr}\ln\left(\frac{D^\dagger D}{D_0^\dagger D_0}\right)e^{ikx}\\
&=-\frac{N_c}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\ln\left(\frac{D^\dagger(\partial\to\partial+ik)D(\partial\to\partial+ik)}{D_0^\dagger(\partial\to\partial+ik)D_0(\partial\to\partial+ik)}\right)\cdot 1\\
&=-\frac{N_c}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\ln\left(1-\frac{N}{D_0^\dagger(\partial+ik)D_0(\partial+ik)}\right)\cdot 1\\
&=\frac{N_c}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\ln\left(\frac{1}{D_0^\dagger D_0}N+\frac{1}{2}\frac{1}{D_0^\dagger D_0}N\frac{1}{D_0^\dagger D_0}N\right.\\
&\quad\left.+\frac{1}{3}\frac{1}{D_0^\dagger D_0}N\frac{1}{D_0^\dagger D_0}N\frac{1}{D_0^\dagger D_0}N+\frac{1}{4}\frac{1}{D_0^\dagger D_0}N\frac{1}{D_0^\dagger D_0}N\frac{1}{D_0^\dagger D_0}N\frac{1}{D_0^\dagger D_0}N+\cdots\right)\cdot 1,
\end{aligned} \tag{15}$$

where

$$N = D_0^\dagger(\partial + ik)D_0(\partial + ik) - D^\dagger(\partial + ik)D(\partial + ik). \tag{16}$$

Here we have used a complete set of plane waves for the calculation of the functional trace, summing over all states and taking the trace in x. "tr" then denotes the usual matrix trace over flavor and Dirac spaces. The right-hand side (RHS) of Eq. (15) can now be expanded in powers of the derivatives of the pseudo-Goldstone boson fields, $\partial\!\!\!/ U^{\gamma_5}$, and of $2ik\cdot\partial + \partial^2$. The operators $D^\dagger D$ and $D_0^\dagger D_0$ in Eq. (15) can be expanded as follows:

$$\begin{aligned} D^\dagger(\partial + ik)D(\partial + ik) &= -\partial^2 - 2ik\cdot\partial + k^2 - \sqrt{M(-i\partial + k)}(i\partial\!\!\!/ U^{\gamma_5})\sqrt{M(-i\partial + k)} \\ &\quad + \sqrt{M(-i\partial + k)}U^{-\gamma_5}M(-i\partial + k)U^{\gamma_5}\sqrt{M(-i\partial + k)}, \end{aligned} \tag{17}$$

$$D_0^\dagger(\partial + ik)D_0(\partial + ik) = -\partial^2 - 2ik\cdot\partial + k^2 + M^2(-i\partial + k). \tag{18}$$

Since the dynamic quark mass in Eqs. (17),(18) contains the derivatives, we need to expand it to order $O(\partial^4)$:

$$\begin{aligned} \sqrt{M(-\partial^2 - 2ik\cdot\partial + k^2)} &= \sqrt{M(k^2)}\Bigg(1 - \frac{\tilde{M}'}{2M}\partial^2 - \frac{(\tilde{M}')^2}{8M^2}\partial^4 + \frac{\tilde{M}''}{4M}\partial^4 - i\frac{\tilde{M}'}{M}k_\alpha\partial_\alpha - i\frac{(\tilde{M}')^2}{2M^2}k_\alpha\partial_\alpha\partial^2 + i\frac{\tilde{M}''}{M}k_\alpha\partial_\alpha\partial^2 \\ &\quad + \frac{(\tilde{M}')^2}{2M^2}k_\alpha k_\beta\partial_\alpha\partial_\beta - \frac{\tilde{M}''}{M}k_\alpha k_\beta\partial_\alpha\partial_\beta + \frac{3(\tilde{M}')^3}{4M^3}k_\alpha k_\beta\partial_\alpha\partial_\beta\partial^2 - \frac{3\tilde{M}'\tilde{M}''}{2M^2}k_\alpha k_\beta\partial_\alpha\partial_\beta\partial^2 \\ &\quad + \frac{\tilde{M}'''}{M}k_\alpha k_\beta\partial_\alpha\partial_\beta\partial^2 + i\frac{(\tilde{M}')^3}{2M^3}k_\alpha k_\beta k_\rho\partial_\alpha\partial_\beta\partial_\rho - i\frac{\tilde{M}'\tilde{M}''}{M^2}k_\alpha k_\beta k_\rho\partial_\alpha\partial_\beta\partial_\rho + i\frac{2\tilde{M}'''}{3M}k_\alpha k_\beta k_\rho\partial_\alpha\partial_\beta\partial_\rho \\ &\quad - \frac{5(\tilde{M}')^4}{8M^4}k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma + \frac{3(\tilde{M}')^2\tilde{M}''}{2M^3}k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma - \frac{(\tilde{M}'')^2}{2M^2}k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma \\ &\quad - \frac{2\tilde{M}'\tilde{M}'''}{3M^2}k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma + \frac{\tilde{M}''''}{3M}k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma\Bigg) + O(\partial^5), \end{aligned} \tag{19}$$

$$\begin{aligned} M(-\partial^2 - 2ik\cdot\partial + k^2) &= M(k^2) - \tilde{M}'\partial^2 - 2\tilde{M}''k_\alpha k_\beta\partial_\alpha\partial_\beta + \frac{1}{2}\tilde{M}''\partial^2 + 2\tilde{M}'''k_\alpha k_\beta\partial_\alpha\partial_\beta\partial^2 + \frac{2}{3}\tilde{M}''''k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma \\ &\quad - 2i\tilde{M}'k_\alpha\partial_\alpha + 2i\tilde{M}''k_\alpha\partial_\alpha\partial^2 + \frac{4}{3}i\tilde{M}'''k_\alpha k_\beta k_\rho\partial_\alpha\partial_\beta\partial_\rho + O(\partial^5), \end{aligned} \tag{20}$$

$$\begin{aligned} M^2(-\partial^2 - 2ik\cdot\partial + k^2) &= M^2(k^2) - 2M\tilde{M}'\partial^2 + (\tilde{M}')^2\partial^4 + M\tilde{M}''\partial^4 - 4iM\tilde{M}'k_\alpha\partial_\alpha + 4i(\tilde{M}')^2k_\alpha\partial_\alpha\partial^2 + 4iM\tilde{M}''k_\alpha\partial_\alpha\partial^2 \\ &\quad - 4(\tilde{M}')^2k_\alpha k_\beta\partial_\alpha\partial_\beta - 4M\tilde{M}''k_\alpha k_\beta\partial_\alpha\partial_\beta + 12\tilde{M}'\tilde{M}''k_\alpha k_\beta\partial_\alpha\partial_\beta\partial^2 + 4M\tilde{M}'''k_\alpha k_\beta\partial_\alpha\partial_\beta\partial^2 \\ &\quad + 8i\tilde{M}'\tilde{M}''k_\alpha k_\beta k_\rho\partial_\alpha\partial_\beta\partial_\rho + \frac{8}{3}iM\tilde{M}'''k_\alpha k_\beta k_\rho\partial_\alpha\partial_\beta\partial_\rho + 4(\tilde{M}'')^2k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma \\ &\quad + \frac{16}{3}\tilde{M}'\tilde{M}'''k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma + \frac{4}{3}M\tilde{M}''''k_\alpha k_\beta k_\rho k_\sigma\partial_\alpha\partial_\beta\partial_\rho\partial_\sigma + O(\partial^5), \end{aligned} \tag{21}$$

where

$$M = M(k), \quad \tilde{M}' = \frac{1}{2k}\frac{dM(k)}{dk} = \frac{1}{2k}M'(k),$$

$$\tilde{M}'' = \frac{1}{4k^3}\left(\frac{d^2M(k)}{dk^2}k - \frac{dM(k)}{dk}\right) = \frac{1}{4k^3}(M''(k)k - M'(k)),$$

$$\tilde{M}''' = \frac{1}{8k^5}\left(k^2\frac{d^3M}{dk^3} - 3k\frac{d^2M}{dk^2} + 3\frac{dM}{dk}\right) = \frac{1}{8k^5}(M'''(k)k^2 - 3M''(k)k + 3M'(k)),$$

$$\tilde{M}'''' = \frac{1}{16k^7}\left(k^3\frac{d^4M}{dk^4} - 6k^2\frac{d^3M}{dk^3} + 15k\frac{d^2M}{dk^2} - 15\frac{dM}{dk}\right),$$
$$= \frac{1}{16k^7}(M''''(k)k^3 - 6M'''(k)k^2 + 15M''(k)k - 15M'(k)). \tag{22}$$

Having carried out the necessary arithmetic and grouped terms for each order in the meson momentum, we finally obtain the effective chiral Lagrangian to order $O(p^4)$ with the momentum-dependent quark mass. The effective chiral Lagrangian $\mathcal{L}^{(2)}$ to order $O(p^2)$ is given as follows:

$$\mathcal{L}^{(2)} = \frac{f_\pi^2}{4}\langle \partial_\mu U^\dagger \partial_\mu U\rangle. \tag{23}$$

In Eq. (23) $\langle\,\rangle$ denotes the flavor trace and f_π is the well-known pion decay constant $f_\pi = 93$ MeV expressed by

$$f_\pi^2 = 4N_c\int \frac{d^4k}{(2\pi)^4}\frac{M^2(k) - \frac{1}{2}M(k)M'(k)k + \frac{1}{4}M'^2(k)k^2}{(k^2+M^2(k))^2}. \tag{24}$$

Equation (24) has been already derived (see, for example, Refs. [39,43]). We will use Eq. (24) to fix the cutoff parameter Λ. When we switch off the momentum dependence of the constituent quark mass, we end up with the well-known expression of the χQM for f_π^2:

$$f_\pi^2 = 4N_c^2\int \frac{d^4k}{(2\pi)^4}\frac{M^2}{(k^2+M^2)^2}, \quad M = \text{const}, \tag{25}$$

which is logarithmically divergent.

The $O(p^4)$ effective chiral Lagrangian in the chiral limit is obtained as follows:

$$\mathcal{L}^{(4)} = L_1\langle \partial_\mu U^\dagger \partial_\mu U\rangle^2 + L_2\langle \partial_\mu U^\dagger \partial_\nu U\rangle^2 + L_3\langle \partial_\mu U^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\nu U\rangle, \tag{26}$$

where L_1, L_2, and L_3 denote the LECs for the $O(p^4)$ effective chiral Lagrangian:

$$L_1 = \frac{N_c}{4}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+M^2)^4}\Bigg[M^4 + \frac{1}{6}M^4M'^2 + \frac{1}{24}M^4M'^4 - \frac{1}{6}M^5M'' - \frac{1}{24}M^5M'^2M'' - \frac{1}{2k}M^5M'$$
$$- \frac{1}{24k}M^5M'^3 - \frac{1}{2}kM^3M' + \frac{7}{12}kM^3M'^3 - \frac{1}{6}kM^4M'M'' - \frac{1}{4}k^2M^2M'^4 - \frac{1}{6}k^2M^3M'' + \frac{1}{12}k^2M^3M'^2M''$$
$$- \frac{1}{24}k^3MM'^3 - \frac{1}{6}k^3M^2M'M'' + \frac{1}{8}k^4MM'^2M''\Bigg], \tag{27}$$

$$L_2 = 2L_1, \tag{28}$$

$$L_3 = N_c\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+M^2)^4}\Bigg[-M^4 - \frac{13}{16}M^4M'^2 - \frac{1}{8}M^4M'^4 + \frac{53}{96}M^5M'' + \frac{3}{16}M^5M'^2M'' + \frac{41}{32k}M^5M'$$
$$+ \frac{3}{16k}M^5M'^3 - \frac{19}{32}kM^3M'' - \frac{3}{4}kM^3M'^3 - \frac{1}{8}kM^4M'M'' + \frac{3}{8}k^2M^2M'^4 + \frac{41}{96}k^2M^3M'' + \frac{1}{16}k^3MM'^3$$
$$+ \frac{1}{16}k^3M^2M'M'' - \frac{3}{16}k^4MM'^2M'' - \frac{1}{32}M^6M''^2 - \frac{1}{24}M^6M'M''' + \frac{1}{96}M^7M'''' - \frac{1}{32k^3}M^7M' - \frac{1}{32k^2}M^6M'^2$$
$$+ \frac{1}{32k^2}M^7M'' - \frac{3}{16k}M^6M'M'' + \frac{1}{16k}M^7M''' + \frac{3}{16}kM^5M''' + \frac{23}{32}k^2M^2M'^2 - \frac{1}{16}k^2M^4M''^2 - \frac{1}{12}k^2M^4M'M'''$$
$$+ \frac{1}{32}k^2M^5M'''' + \frac{3}{32}k^3MM' + \frac{3}{16}k^3M^3M''' - \frac{3}{32}k^4MM'' - \frac{1}{32}k^4M^2M''^2 - \frac{1}{24}k^4M^2M'M''' + \frac{1}{32}k^4M^3M''''$$
$$+ \frac{1}{16}k^5MM''' + \frac{1}{96}k^6MM''''\Bigg]. \tag{29}$$

TABLE I. The low energy constants L_1, L_2, L_3.

	M_0(MeV)	Λ(MeV)	$L_1(\times 10^{-3})$	$L_2(\times 10^{-3})$	$L_3(\times 10^{-3})$
Local χQM	350	1905.5	0.79	1.58	−3.17
DP	350	611.7	0.82	1.63	−3.09
Dipole	350	611.2	0.82	1.63	−2.97
Gaussian	350	627.4	0.81	1.62	−2.88
GL			0.9±0.3	1.7±0.7	−4.4±2.5
Bijnens			0.6±0.2	1.2±0.4	−3.6±1.3
Arriola			0.96	1.95	−5.21
VMD			1.1	2.2	−5.5
Holdom (1)			0.97	1.95	−4.20
Holdom (2)			0.90	1.80	−3.90
Bolokhov *et al.*			0.63	1.25	2.50
Alfaro *et al.*			0.45	0.9	−1.8

Equation (28) is the large-N_c relation which was derived from the OZI rule for the meson scattering amplitude [3]. If we turn off the momentum dependence of the constituent quark mass, we reproduce the results of the usual χQM. Equations (27)–(29) are our main results.

IV. RESULTS AND DISCUSSION

The parameters in the present model are the constituent quark mass M_0 at $k^2=0$ and the cutoff parameter Λ in Eq. (9). The cutoff parameter Λ is fixed by reproducing the pion decay constant f_π^2. Having chosen the Λ, we are able to calculate the LECs L_1, L_2, and L_3, numerically. The only free parameter we have is the M_0. In Table I, the results of the L_1, L_2, and L_3 are listed with $M_0=350$ MeV. The results are found to be rather insensitive to the types of $M(k)$. They are compared with those from other models. In Table I, GL denotes the empirical data obtained by Gasser and Leutwyler [2]. The results are in good agreement with Refs. [2,18]. It is interesting to compare the present results with those from Ref. [15], since it emphasizes also the momentum-dependence of the quark mass. Holdom *et al.* [15] used two different values of the quark self-energy $\Sigma(p)$. Holdom (1) represents the quark self-energy $\Sigma(p)_1 = 2M^3/(M^2+p^2)$, while Holdom (2) designates $\Sigma(p)_2 = 4M^3/(3M^2+p^2)$. M denotes the constituent quark mass.

Figures 2, 3 and 4 draw the dependence of the L_1, L_2, and L_3 on the M_0, respectively. While the results with three different $M(k)$ show a similar behavior in smaller M_0, they become rather different as M_0 increases. In particular, the Gaussian type of $F(k)$ drastically suppresses the LECs at higher values of M_0. The reason can be found in the behavior of the $M(k)$. The Gaussian type of $F(k)$ decreases rather strongly as k increases, compared to other two different types of form factors.

Apart from the relation of the large N_c limit, there is an additional relation in the local χQM: $2L_2+L_3=0$. The dual-resonance model has the same relation [23,24]. However, the quantity $2L_2+L_3$ is not equal to zero in the present model. Interestingly, this relation is deeply related to the upper bound of the lightest resonances in $\pi\pi$ scattering. A recent work [53] has shown that the upper bound of the masses of the ρ and σ mesons can be expressed in terms of the LECs

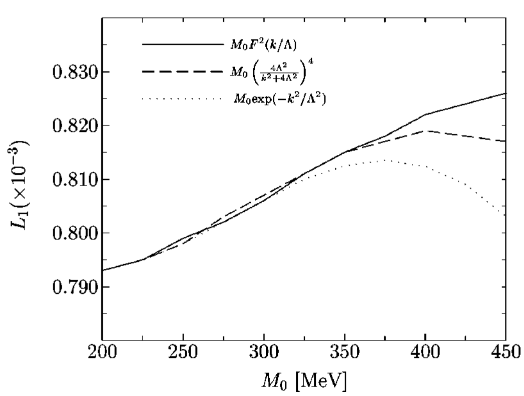

FIG. 2. The dependence of L_1 on M_0. The solid curve stands for the result with the form factor from the instanton vacuum given in Eq. (9), the dashed one draws the result with the dipole type $M(k)$, and the dotted one designates the result with the Gaussian one.

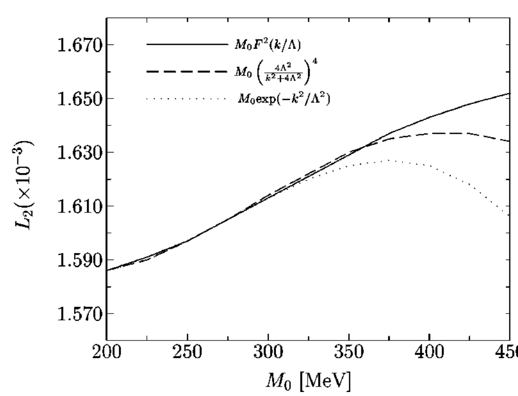

FIG. 3. The dependence of L_2 on M_0. The solid curve stands for the result with the form factor from the instanton vacuum given in Eq. (9), the dashed one draws the result with the dipole type $M(k)$, and the dotted one designates the result with the Gaussian one.

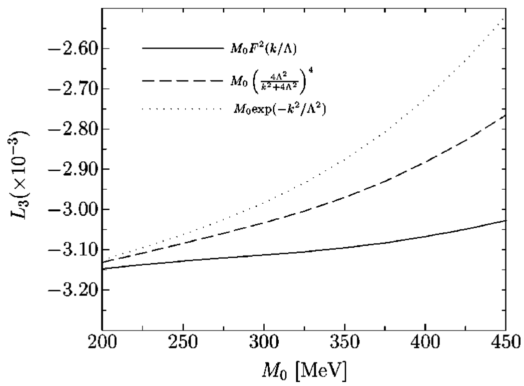

FIG. 4. The dependence of L_3 on M_0. The solid curve stands for the result with the form factor from the instanton vacuum given in Eq. (9), the dashed one draws the result with the dipole type $M(k)$, and the dotted one designates the result with the Gaussian one.

L_2 and L_3. In particular, the following expression for the upper bound of the σ-meson mass was derived:

$$M_\sigma < 665[1+0.44\Delta+0.33\Delta^2+O(\Delta^3)] \text{ MeV}, \quad (30)$$

where

$$\Delta=-\frac{2L_2+L_3}{L_2}. \quad (31)$$

In fact, the ratio Δ is determined by the $\pi\pi$ scattering length as follows [54]:

$$\Delta=-3\frac{a_2^2}{a_2^0}+O(m_\pi^2)\approx-0.2\pm0.6, \quad (32)$$

where a_2^2, a_2^0 denote the D-wave scattering length for $I=0$ and $I=2$, respectively. Though it is hard to judge models based on this empirical value because of the large error, it is still of great interest to see the difference between models. While the present results are similar to those obtained from other models, the ratio Δ in Eq. (30), which is an important quantity to determine the upper limit of the resonances, distinguishes the models. In Fig. 5, the dependence of the ratio Δ on M_0 is drawn. While the result with the form factor in Eq. (9) shows relatively mild dependence on M_0, those with the dipole and Gaussian form factors depend strongly on M_0. It can be easily understood from the dependence of the L_2 and L_3 on M_0 as drawn in Figs. 3 and 4.

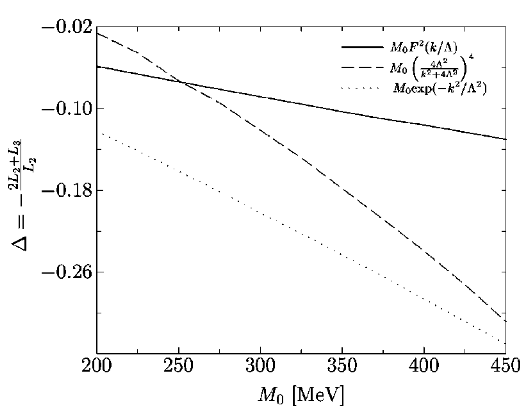

FIG. 5. The dependence of Δ on M_0. The solid curve stands for the result with the form factor from the instanton vacuum given in Eq. (9), the dashed one draws the result with the dipole type $M(k)$, and the dotted one designates the result with the Gaussian one.

TABLE II. Δ and the upper limit of M_σ.

	$2L_2+L_3(\times10^{-3})$	Δ	$\leqslant M_\sigma$(MeV)
local χQM	0	0	665
Type 1	1.67	−0.103	637.2
Type 2	0.29	−0.178	619.9
Type 3	0.387	−0.243	606.9
Arriola	−1.31	0.672	960.7
VMD (Ref. [51])	−1.1	0.5	866.2
Holdom (1)	−0.3	0.154	715.3
Holdom (2)	−0.3	0.167	720
Bolokhov *et al.*	0	0	665
Alfaro *et al.*	0	0	665

In Table II we list the results for Δ and the upper limit of the sigma meson mass. As shown in Table II, we can find a very interesting fact: Except for the present model, all other models presented here give negative values of Δ. As a result, while the present work gives the upper limit of M_σ below 640 MeV, all other models in Table II predict it rather large. In particular, Ref. [19] gives a fairly large value of the upper limit of M_σ: 961 MeV. Though the models of Ref. [15] contain the momentum-dependent quark mass, their values of Δ are quite different from the present one. Thus the values of Δ distinguish the present work from other models.

V. CONCLUSIONS

In the present work, we investigated the $O(p^4)$ effective chiral Lagrangian in the chiral limit, based on the nonlocal chiral quark model derived from the instanton vacuum. Starting from the effective chiral action, we carried out a derivative expansion with respect to the pion momenta in order to get the effective chiral Lagrangian to order $O(p^4)$. The low-energy constants (LECs) which encode QCD dynamics have been obtained. We calculated the LECs, employing three different types of $M(k)$. The LECs are insensitive to the types of the form factors. We found that the results are in a good agreement with the empirical data. Though they are not much different from those of other models, the present results for the ratio Δ turn out to be rather different from them.

A full investigation into the low-energy constants including SU(3) symmetry breaking and external fields is under way.

ACKNOWLEDGMENTS

H-C.K. is grateful to K. Goeke, M. Polyakov, P. Pobylitsa, and M. Musakhanov for valuable discussions and critical comments. The present work is supported by the Korean Research Foundation (KRF-2002-041-C00067).

[1] S. Weinberg, Physica A **96**, 327 (1979).
[2] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).
[3] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).
[4] A. Pich, Rep. Prog. Phys. **58**, 563 (1995).
[5] G. Ecker, Prog. Part. Nucl. Phys. **35**, 1 (1995).
[6] G. Amoros, J. Bijnens, and P. Talavera, Nucl. Phys. **B585**, 293 (2000); **B598**, 665(E) (2001).
[7] D. Diakonov and M.I. Eides, JETP Lett. **38**, 433 (1983) [Pisma Zh. Eksp. Teor. Fiz. Lett. **38**, 358 (1983)].
[8] J. Balog, Phys. Lett. **149B**, 197 (1984).
[9] J.A. Zuk, Z. Phys. C **29**, 303 (1985).
[10] A.A. Andrianov, Phys. Lett. **157B**, 425 (1985).
[11] D. Ebert and H. Reinhardt, Nucl. Phys. **B271**, 188 (1986).
[12] L.H. Chan, Phys. Rev. Lett. **57**, 1199 (1986).
[13] P. Simic, Phys. Rev. D **34**, 1903 (1986).
[14] D. Espriu, E. de Rafael, and J. Taron, Nucl. Phys. **B345**, 22 (1990); **B355**, 278(E) (1991).
[15] B. Holdom, J. Terning, and K. Verbeek, Phys. Lett. B **245**, 612 (1990).
[16] M.V. Polyakov and V.V. Vereshagin, Phys. Rev. D **54**, 1112 (1996).
[17] H.W. Fearing and S. Scherer, Phys. Rev. D **53**, 315 (1996).
[18] J. Bijnens, Phys. Rep. **265**, 369 (1996).
[19] E. Ruiz Arriola, Phys. Lett. B **253**, 430 (1991).
[20] S. Peris, M. Perrottet, and E. de Rafael, J. High Energy Phys. **05**, 011 (1998).
[21] Q. Wang, Y.P. Kuang, M. Xiao, and X.L. Wang, Phys. Rev. D **61**, 054011 (2000).
[22] ALPHA Collaboration, J. Heitger, R. Sommer, and H. Wittig, Nucl. Phys. **B588**, 377 (2000).
[23] A.A. Bolokhov, A.N. Manashov, V.V. Vereshagin, and V. Polyakov, Phys. Rev. D **48**, 3090 (1993).
[24] J. Alfaro, L. Balart, A.A. Andrianov, and D. Espriu, Int. J. Mod. Phys. A **18**, 2501 (2003).
[25] M. Praszalowicz and A. Rostworowski, Phys. Rev. D **64**, 074003 (2001).
[26] V.Y. Petrov, V. Polyakov, R. Ruskov, C. Weiss, and K. Goeke, Phys. Rev. D **59**, 114018 (1999).
[27] V.Y. Petrov, P.V. Pobylitsa, V. Polyakov, I. Bornig, K. Goeke, and C. Weiss, Phys. Rev. D **57**, 4325 (1998).
[28] M. Franz, H-Ch. Kim, and K. Goeke, Nucl. Phys. **A663**, 995 (2000).
[29] M. Franz, H-Ch. Kim, and K. Goeke, Nucl. Phys. **A699**, 541 (2002).
[30] M. Franz, H-Ch. Kim, and K. Goeke, Nucl. Phys. **B562**, 213 (1999).
[31] R.D. Ball and G. Ripka, ‘‘Proceedings of the Conference on Many Body Physics,’’ hep-ph/9312260.
[32] D. Diakonov, Prog. Part. Nucl. Phys. **51**, 173 (2003).
[33] M. Musakhanov, Eur. Phys. J. C **9**, 235 (1999).
[34] D. Diakonov and V. Petrov, Phys. Lett. **147B**, 351 (1984); Nucl. Phys. **B272**, 457 (1986).
[35] D. Diakonov, hep-ph/9602375.
[36] D. Diakonov, hep-ph/9802298.
[37] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[38] G. ’t Hooft, Phys. Rev. D **14**, 3432 (1976). **18**, 2199 (1978).
[39] R.D. Bowler and M.C. Birse, Nucl. Phys. **A582**, 655 (1995).
[40] G. Ripka, *Quarks Bound by Chiral Fields* (Oxford Science Publications, Oxford, 1997) and references therein.
[41] R. Alkofer and L. von Smekal, Phys. Rep. **353**, 281 (2001).
[42] C.D. Roberts and S.M. Schmidt, Prog. Part. Nucl. Phys. **45**, S1 (2000).
[43] B. Golli, W. Broniowski, and G. Ripka, Phys. Lett. B **437**, 24 (1998).
[44] C.V. Christov *et al.*, Prog. Part. Nucl. Phys. **37**, 91 (1996).
[45] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).
[46] E. Witten, Nucl. Phys. **B223**, 433 (1983).
[47] J.R. Aitchison and C. Frazer, Phys. Lett. **146B**, 63 (1984); Phys. Rev. D **31**, 2608 (1985).
[48] Lai-Him Chan, Phys. Rev. Lett. **55**, 21 (1985).
[49] A. Dhar, R. Shankar, and S. Wadia, Phys. Rev. D **31**, 3256 (1985).
[50] D. Diakonov, V.Y. Petrov, and P.V. Pobylitsa, Nucl. Phys. **B306**, 809 (1988).
[51] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, and E. de Rafael, Phys. Lett. B **223**, 425 (1989).
[52] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).
[53] M.V. Polyakov and V.V. Vereshagin, Eur. Phys. J. A **12**, 349 (2001).
[54] M.M. Nagels *et al.*, Nucl. Phys. **B147**, 189 (1979).
[55] For the reviews, see for example, Refs. [4,5] and references therein.