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Hidden symmetries of the $AdS_5 \times S^5$ superstring

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Attempts to solve Yang-Mills theory must eventually face the problem of analyzing the theory at intermediate values of the coupling constant. In this regime neither perturbation theory nor the gravity dual are adequate, and one must consider the full string theory in the appropriate background. We suggest that in some nontrivial cases the world sheet theory may be exactly solvable. For the Green-Schwarz superstring on $AdS_5 \times S^5$ we find an infinite set of nonlocal classically conserved charges of the type that exist in integrable field theories.

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I. INTRODUCTION

The discovery of AdS/conformal field theory (CFT) duality [1] was a major step towards the long standing goal [2] of recasting large-N QCD as a string theory. The original duality was for highly supersymmetric conformally invariant gauge theories, but these can be deformed in various ways to produce string duals to confining gauge theories with less or no supersymmetry. Thus far the duality provides solutions only for gauge theories where the 't Hooft coupling is strong at all scales, because it is only for these that the world-sheet of the dual string is weakly coupled. However, it implies an existence proof (in the physicists' sense of the term) for a string dual to QCD, by continuous deformation to weak coupling in the UV and so to a strongly coupled world-sheet theory. Thus the program of solving large-N QCD is reduced to two steps: (1) identify the strongly coupled world-sheet field theory of the QCD string; (2) solve it. The hope is that the reduction from a strongly coupled field theory in $3+1$ dimensions to one in $1+1$ dimensions will allow the various special techniques of $(1+1)$-dimensional field theory to be brought to bear.

In this paper we will look ahead to the second step, and report at least one modest positive result. Our focus here is the conformally invariant $\mathcal{N}=4$ theory, where the world-sheet theory is known due to the high symmetry of the problem, but where existing methods of calculation have limited range. That is, at large 't Hooft coupling we can calculate using perturbation theory on the string world-sheet. At small 't Hooft coupling we can calculate in the weakly coupled gauge theory.[1] However, for 't Hooft coupling of order one there are no quantitative methods, since in either dual form the coupling is of order one. Understanding this theory should be strictly easier than solving the QCD string, as in the latter case as well neither the gauge nor the string description is weakly coupled, and there is much less symmetry.

For string theories without Ramond-Ramond (RR) backgrounds, there are familiar strong-coupling methods based primarily on holomorphic currents. The Wess-Zumino-Witten model is the archetype of this [4]: the spacetime symmetry is elevated to an affine Lie algebra. However, with RR backgrounds the bosonic Wess-Zumino term is absent and there is no affine Lie algebra. Also, the RR backgrounds make it impossible to use the standard Ramond-Neveu-Schwarz conformal field theory, within which holomorphic currents can be shown to exist under broad conditions.

However, the Green-Schwarz superstring on $AdS_5 \times S^5$ is similar to field theories for which other forms of higher symmetry algebra are known to exist. That is, the field space can be regarded as a coset [5–7],

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}. \tag{1.1}$$

For certain coset theories there are two kinds of infinite symmetry algebras. One is based on nonlocal currents [8–10] which give rise to charges satisfying a Yangian algebra [11];[2]

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[1]It is an interesting exercise to ask what this implies about the strongly coupled world-sheet in this limit. This will be discussed in a separate paper [3].

[2]Transformations satisfying a familiar affine Lie algebra can also be constructed from the nonlocal currents [12]. These are invariances of the classical equations of motion but not of the Poisson brackets (they are not symplectically generated), and so unfortunately they probably have no quantum analogs.

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for reviews see Refs. [13,14]. The other is based on local right- or left-moving currents [15] which satisfy a *W*-algebra; for a nice recent discussion see Ref. [16].

Of course the Green-Schwarz superstring on $AdS_5\times S^5$ is not precisely a coset sigma model, because of the fermionic Wess-Zumino term and the κ symmetry. However, we will show that it possesses an infinite symmetry algebra of the nonlocal form. These nonlocal charges are conserved in any κ gauge. Our considerations are purely classical, but in related models these charges have been argued to survive quantization [17], with modified algebras.

Our results extend immediately to the superstring on $AdS_3\times S^3\times T^4$ with RR flux, for which the action has the same structure [18]. Of course the theory with NS-NS flux is *S*-dual to this, but in studying the world-sheet theory we are implicitly expanding in $g_{\rm string}$ so this duality is not visible. In fact the NS-NS background is a case where the string side is exactly solvable, and so an example of what we might hope to do for QCD. Some of the methods used in this case, such as those of Ref. [19], may be applicable to gauge theories, but in some ways this example is rather different. The spacetime CFT is less well understood and has no adjustable coupling constant—thus there is no limit in which it has a classical Lagrangian description.

In Sec. II we briefly review the construction of the nonlocal charges for bosonic nonlinear sigma models. This is based on the identification of a one-parameter family of flat connections, constructed from the symmetry currents and their duals. In Sec. III we extend this to the type IIB Green-Schwarz superstring on $AdS_5\times S^5$, and show that a one-parameter family of flat connections exists. In Sec. IV we search for local chiral charges. We find no charges of higher spin, but we find that in conformal gauge the world-sheet CFT separates (at the classical level) into two factors, one associated with AdS_5 and one associated with S^5. In Sec. V we discuss further directions.

Beyond the application to QCD, the possibility of finding an exact solution on one side of the gauge/gravity duality is an exciting prospect. Thus far these higher symmetries and related methods have appeared in gauge/gravity in certain special contexts. On the string side, Maldacena and Maoz [20] have pointed out that one can engineer nonlinear plane wave solutions to produce an integrable world-sheet theory (see also Ref. [21]); Mandal, Suryanarayana, and Wadia [22] have noted that the bosonic part of the $AdS^5\times S_5$ string theory is classically integrable; and, Bershadsky, Zhukov, and Vaintrob [23] have discussed the *W* symmetry of the pure supergroup sigma model. On the gauge side, Minahan and Zarembo [24] have shown that the calculation of one-loop anomalous dimensions for general scalar operators can be recast in terms of an integrable field theory (see further work [25]). Belitsky, Gorsky, and Korchemsky [26] have related the computation of anomalous dimensions of certain higher spin operators to the $SL(2,\mathbb{R})$ spin chain. Also, Lipatov [27] has argued that in certain QCD processes the summation of Feynman graphs leads to an integrable model.

II. REVIEW: NONLOCAL CHARGES IN BOSONIC MODELS

A. Principal chiral models

Consider first the nonlinear sigma model where the field $g(x)$ takes values in the group G, and the Lagrangian is $L \propto \mathrm{Tr}(\partial_i g^{-1}\partial^i g)$. The global symmetry is $G\times G$, left and right multiplication. We will focus on the conserved current corresponding to left multiplication,

$$j_i=-(\partial_i g)g^{-1}. \tag{2.1}$$

Note that the current takes values in the Lie algebra $\mathcal{G}$. Writing the current as a one-form, so that $d*j=0$, one sees that

$$dj+j\wedge j=0. \tag{2.2}$$

Thus the current can be regarded as a flat gauge connection in $\mathcal{G}$. Moreover, by taking general linear combinations

$$a=\alpha j+\beta * j, \tag{2.3}$$

one finds that

$$da+a\wedge a=(\alpha^2-\alpha-\beta^2)j\wedge j. \tag{2.4}$$

We have used the identities $**=+1$, and $*k\wedge l+k\wedge *l=0$ for general one-forms k,l. Thus there are two one-parameter families of flat connections,

$$a^{\lambda\pm}:\quad \alpha=\frac{1}{2}(1\pm\cosh\lambda),$$

$$\beta=\frac{1}{2}\sinh\lambda,\quad -\infty<\lambda<\infty. \tag{2.5}$$

Equivalently, this can be written as

$$\alpha=\frac{y^2}{2y-1},\quad \beta=\frac{y^2-y}{2y-1},\quad -\infty<y<\infty. \tag{2.6}$$

Given any flat connection, the equation

$$dU=-aU \tag{2.7}$$

is integrable: action on both sides with d gives $0=0$. On a simply connected space, given an initial value $U(x_0,x_0)=1$, this defines a group element $U(x,x_0)$. This is just the Wilson line, defining parallel transport with the connection a,

$$U(x,x_0)=P\exp\left(-\int_C a\right), \tag{2.8}$$

where C is any contour running from x_0 to x, and P denotes path ordering of the Lie algebra generators. The flatness of the connection implies that this is invariant under the continuous deformations of C.

This immediately allows the construction of an infinite number of conserved charges [8,9], by taking the unbounded spatial Wilson line at fixed time,

$$Q^{\lambda\pm}(t)=U^{\lambda\pm}(\infty,t;-\infty,t). \tag{2.9}$$

This takes values in G. The contour C here is the $t=$const spatial slice, so conservation of $Q^{\lambda\pm}$ is simply the statement that this is invariant under continuous shift of the contour forward in time. Of course this moves the end points, which is not in general an invariance of the Wilson line, so an appropriate falloff of the fields at infinity is assumed. Explicitly,

$$\begin{aligned}\partial_t U(y,t;z,t) &= -\int_y^z dx\, U(y,t;x,t)\dot{a}_1(x,t)U(x,t;z,t)\\ &= -\int_y^z dx\, U(y,t;x,t)[a_0'-a_0a_1\\ &\quad +a_1a_0]_{(x,t)}U(x,t;z,t)\\ &= a_0(y,t)U(y,t;z,t)-U(y,t;z,t)a_0(z,t),\end{aligned} \tag{2.10}$$

so we need $a_0(\pm\infty,t)$ to go to zero in order for the charges to be conserved. For closed string theory, where the spatial direction is periodic, one takes the trace to form the Wilson loop; for the supercoset case of the next section one would take the supertrace.

These charges can also be presented in other forms, for example by Taylor expanding in λ,

$$Q^{\lambda-}=1+\sum_{n=1}^{\infty}\lambda^n Q_n. \tag{2.11}$$

For $a^{\lambda-}$, which vanishes at $\lambda=0$, we have

$$a^{\lambda-}=\frac{1}{2}\lambda * j-\frac{1}{4}\lambda^2 j+O(\lambda^3). \tag{2.12}$$

Then

$$2Q_1=\int_{-\infty}^{\infty}dx\, j_0(x) \tag{2.13}$$

is just the global left-multiplication charge. Note that we could similarly have started the whole construction with the right-multiplication current, and gotten a second set of charges. The next charge is bilocal,

$$\begin{aligned}Q_2 &= -\frac{1}{4}\int_{-\infty}^{\infty}dx\, j_1(x)\\ &\quad +\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{x}dx'\, j_0(x)j_0(x')\end{aligned} \tag{2.14}$$

and so on. Through its Poisson brackets, Q_2 generates all of the higher Q_n, so for many purposes one can focus on this simplest nonlocal (more precisely, multilocal) charge.

B. Coset models

Now let us consider G/H coset models, where we identify $g(x)\cong g(x)h(x)$. Left multiplication by G is still a global symmetry. To construct the action, define

$$J=g^{-1}jg=-g^{-1}\partial g. \tag{2.15}$$

This is invariant under left multiplication. Further separate J according to the decomposition of the Lie algebra, $\mathcal{G}=\mathcal{H}\oplus\mathcal{K}$,

$$J=H+K. \tag{2.16}$$

Then H transforms as a connection under $\mathcal{H}$-gauge transformations, whereas K transforms covariantly. It follows that

$$k=gKg^{-1} \tag{2.17}$$

is $\mathcal{H}$-gauge invariant. The Lagrangian is then $L\propto \mathrm{Tr}(k_ik^i)=\mathrm{Tr}(K_iK^i)$.

We will use capital letters X to denote currents that are conjugated by right multiplication, generally corresponding to some decomposition under representations of $\mathcal{H}$. Then $x=gXg^{-1}$ is conjugated by left multiplication. We will focus on the $\mathcal{H}$-gauge invariants, which are the x other than h. Notice however that the x do not have simple decompositions under the Lie algebra; to use such decompositions we must refer back to the X. Note also that

$$dx=g(dX)g^{-1}-j\wedge x-x\wedge j. \tag{2.18}$$

The construction of the flat connections can be extended provided the coset is a symmetric space [28,14]. That is, in addition to $[\mathcal{H},\mathcal{H}]\subseteq\mathcal{H}$ and $[\mathcal{H},\mathcal{K}]\subseteq\mathcal{K}$ which follow from the subgroup structure, we must have $[\mathcal{K},\mathcal{K}]\subseteq\mathcal{H}$ as well. To see this, note that $dJ=J\wedge J$, and decompose both sides under $G=\mathcal{H}\oplus\mathcal{K}$:

$$\begin{aligned}dH &= H\wedge H+K\wedge K,\\ dK &= H\wedge K+K\wedge H.\end{aligned} \tag{2.19}$$

If the coset were not a symmetric space, then $K\wedge K$ would be a sum of two pieces, one of which is in $\mathcal{H}$ and the other in $\mathcal{K}$.[3] Transforming to the x forms, we have

$$\begin{aligned}dh &= -k\wedge h-h\wedge k,\\ dk &= -2k\wedge k.\end{aligned} \tag{2.20}$$

The gauge invariant k is also the Noether current for the global symmetry, $d*k=0$. The current $2k$ is both flat and conserved, and so can be used to construct two families of flat connections precisely as above.

The construction of the flat connections and nonlocal charges can be extended to certain sigma models with fermions, including world-sheet supersymmetric sigma models [10,14]. The strategy is slightly different: one separates the global symmetry current into its bosonic and fermion parts, which are not separately conserved, and takes a general linear combination of these currents and their duals. Under appropriate conditions there exist infinite families of flat con-

[3]When $\mathcal{K}$ is a subalgebra, $K\wedge K$ contributes only to dK, and it is again possible to construct flat connections.

nections. We will not review this here, but the construction in the next section has a similar structure.

III. THE GREEN-SCHWARZ SUPERSTRING ON $\mathrm{AdS}_5\times\mathrm{S}^5$

In order to make the construction as transparent as possible, we will in the present subsection use a condensed notation which is parallel to that of the earlier discussion. The Green-Schwarz superstring in $\mathrm{AdS}_5\times\mathrm{S}^5$ can be regarded as a nonlinear sigma model where the field takes values in the coset superspace [5–7]

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}. \tag{3.1}$$

The bosonic part of this space,

$$\frac{SO(4,2)}{SO(4,1)}\times\frac{SO(6)}{SO(5)}=\mathrm{AdS}_5\times\mathrm{S}^5, \tag{3.2}$$

is a symmetric space and so the above construction gives an infinite symmetry algebra. This has recently been remarked in Ref. [22], in the context of finding extended string solutions in $\mathrm{AdS}_5\times\mathrm{S}^5$. The full space (3.1) is not a symmetric space, as the denominator group is too small. The theory also differs from the simple nonlinear sigma model by the presence of a Wess-Zumino term, and by the κ gauge symmetry. Thus the construction of the flat connection is slightly more involved.

The Lie algebra of $PSU(2,2|4)$ can be decomposed [5–7]

$$\mathcal{G}=\mathcal{H}+\mathcal{P}+\mathcal{Q}_1+\mathcal{Q}_2, \tag{3.3}$$

where $\mathcal{H}$ is the denominator algebra, $\mathcal{P}$ contains the remaining bosonic generators, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two copies of the $(\mathbf{4},\mathbf{4})$ representation of $\mathcal{H}$. The algebra respects a $\mathbb{Z}_4$ grading, under which the charges are

$$\mathcal{H}:0,\quad \mathcal{Q}_1:1,\quad \mathcal{P}:2,\quad \mathcal{Q}_2:3. \tag{3.4}$$

Form the current

$$J=-g^{-1}\partial g=H+P+Q_1+Q_2. \tag{3.5}$$

The grading of the Lie algebra implies that the curl $dJ=J\wedge J$ decomposes as

$$\begin{aligned}
dH&=H\wedge H+P\wedge P+Q_1\wedge Q_2+Q_2\wedge Q_1,\\
dP&=H\wedge P+P\wedge H+Q_1\wedge Q_1+Q_2\wedge Q_2,\\
dQ_1&=H\wedge Q_1+Q_1\wedge H+P\wedge Q_2+Q_2\wedge P,\\
dQ_1&=H\wedge Q_2+Q_2\wedge H+P\wedge Q_1+Q_1\wedge P.
\end{aligned} \tag{3.6}$$

It is useful to define also $Q=Q_1+Q_2$ and $Q'=Q_1-Q_2$, in terms of which

$$\begin{aligned}
dH&=H\wedge H+P\wedge P+\frac{1}{2}(Q\wedge Q-Q'\wedge Q'),\\
dP&=H\wedge P+P\wedge H+\frac{1}{2}(Q\wedge Q+Q'\wedge Q'),\\
dQ&=H\wedge Q+Q\wedge H+P\wedge Q+Q\wedge P,\\
dQ'&=H\wedge Q'+Q'\wedge H-P\wedge Q'-Q'\wedge P.
\end{aligned} \tag{3.7}$$

The curls of the lower-case forms are then

$$\begin{aligned}
dh&=-h\wedge h+p\wedge p-h\wedge p-p\wedge h-h\wedge q\\
&\quad -q\wedge h+\frac{1}{2}(q\wedge q-q'\wedge q'),\\
dp&=-2p\wedge p-p\wedge q-q\wedge p+\frac{1}{2}(q\wedge q+q'\wedge q'),\\
dq&=-2q\wedge q,\\
dq'&=-2p\wedge q'-2q'\wedge p-q\wedge q'-q'\wedge q.
\end{aligned} \tag{3.8}$$

In the notation of Ref. [5],

$$\begin{aligned}
H&=\frac{1}{2}(L^{ab}J_{ab}+L^{a'b'}J_{a'b'}),\\
P&=L^aP_a+L^{a'}P_a,\\
Q_I&=L^{\alpha\alpha' I}Q_{\alpha\alpha' I}.
\end{aligned} \tag{3.9}$$

Translating Eqs. 3.18, 3.19 and 3.20 of that paper, the equations of motion are[4]

$$\begin{aligned}
d{*}p&=p\wedge{*}q+{*}q\wedge p+\frac{1}{2}(q\wedge q'+q'\wedge q),\\
0&=p\wedge({*}q-q')+({*}q-q')\wedge p,\\
0&=p\wedge(q-{*}q')+(q-{*}q')\wedge p.
\end{aligned} \tag{3.10}$$

Notice that

$$d{*}\left(p+\frac{1}{2}{*}q'\right)=0. \tag{3.11}$$

[4]These equations, as well as Eqs. (3.8) can be obtained from Ref. [7] with the identifications

$$\begin{aligned}
p_M{}^N&=Z_M{}^aJ_{\langle ab\rangle}Z^{bN}+Z_M{}^{\bar a}J_{\langle\overline{ab}\rangle}Z^{\bar bN}\\
q_M{}^N&=Z_M{}^{\bar a}J_{a\bar b}Z^{bN}+Z_M{}^aJ_{a\bar b}Z^{\bar bN}\\
q'_M{}^N&=Z_M{}^bJ_{a\bar b}Z^{\bar aN}+Z_M{}^{\bar a}J_{b\bar a}Z^{bN}
\end{aligned}$$

where M and N are $PSL(4|4)$ indices, a and $\bar a$ are $Sp(4)$ indices and $\langle\rangle$ denotes the antisymmetric and traceless part.

The conserved current $p+\frac{1}{2}*q'$ is the Noether current of the global left multiplication symmetry. The conservation equation (3.11) is actually the complete equation of motion. By converting to an equation for $d*(P+\frac{1}{2}*Q')$ and decomposing the Lie algebra one obtains all of Eqs. (3.10).

We now construct candidate connections. We do not have equations for $d*q$ and $d*q'$. This is because the local κ symmetry is not yet fixed, so the equations of motion do not determine the full time evolution. In specific gauges one obtains equations for $d*q$ and $d*q'$, but it turns out that we can construct the connection without them. Thus define

$$a=\alpha p+\beta *p+\gamma q+\delta q'. \tag{3.12}$$

Then, noting again the identities $**=+1$ and $*k\wedge l+k\wedge *l=0$, one finds

$$\begin{aligned} da+a\wedge a=&c_1 p\wedge p+c_2(p\wedge q+q\wedge p)\\ &+c_3(p\wedge q'+q'\wedge p)+c_4 q\wedge q+c_5 q'\wedge q'\\ &+c_6(q\wedge q'+q'\wedge q), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} c_1&=-2\alpha+\alpha^2-\beta^2,\\ c_2&=-\alpha+\alpha\gamma-\beta\delta,\\ c_3&=\beta-2\delta+\alpha\delta-\beta\gamma,\\ c_4&=\frac{1}{2}\alpha-2\gamma+\gamma^2,\\ c_5&=\frac{1}{2}\alpha+\delta^2,\\ c_6&=\frac{1}{2}\beta-\delta+\gamma\delta. \end{aligned} \tag{3.14}$$

The vanishing of the c_i gives six equations for four unknowns, but there is a large degree of redundancy, and remarkably there are again two one-parameter families of flat connections. One can use the vanishing of c_5 and c_6 to solve for α and β, and the remaining c_i then all vanish if $\delta^2=\gamma^2-2\gamma$. Thus the connection a is flat for

$$\begin{aligned} \alpha&=-2\sinh^2\lambda,\\ \beta&=\mp 2\sinh\lambda\cosh\lambda, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \gamma&=1\pm\cosh\lambda,\\ \delta&=\sinh\lambda. \end{aligned} \tag{3.16}$$

We do not have any deep understanding of why these flat connections exist. We have found κ gauges in which there are two fewer equations, but this still requires one redundancy. There is some connection between the κ invariance and the nonlocal charges: if we rescale the Wess-Zumino term by σ then both the κ invariance and the higher symmetries are broken (except for $\sigma=-1$, which is just the world-sheet parity transform).

Notice that if we ignore fermions, the bosonic terms in p and $*p$ reproduce the currents for the bosonic coset (though only with the lower sign for the latter). This shows that the currents for different λ are independent—they are not related to one another by κ transformations. Incidentally, the charges cannot be strictly κ invariant, because the generator of κ transformations does not even commute with the global symmetries. It is plausible to conjecture that all the commutators vanish weakly—that is, they are themselves κ transformations.

IV. LOCAL CURRENTS

Consider again the bosonic examples, the group manifold or the coset, where in each case there is a current (either j or $2k$) which is both flat and conserved. In world-sheet light-cone components,

$$\partial_- j_+=-\frac{1}{2}[j_-,j_+]. \tag{4.1}$$

It follows immediately that [15,16]

$$\partial_-\mathrm{Tr}(j_+^n)=0. \tag{4.2}$$

Thus, even though the $G\times G$ currents themselves are not chiral, there are higher spin chiral currents. The traces are not independent, so the number of higher spin currents is finite. Similarly there is a set of left-moving currents.

In the bosonic models, the classical scale invariance is broken by quantum effects. Similarly these chiral currents are anomalous, but under appropriate conditions there will be a conserved though nonchiral higher spin current [15]. In the conformally invariant supergroup models it has been argued that the chiral currents are nonanomalous [23].

In the $AdS_5\times S^5$ case there is no current that is flat and conserved, but the chiral currents can be constructed under weaker conditions. It suffices that

$$\partial_- j_+=\sum_I [a^I,b^I], \quad [j_+,a^I]=0, \tag{4.3}$$

so that

$$\begin{aligned} \partial_-\mathrm{Str}(j_+^n)&=n\sum_I \mathrm{Str}(j_+^{n-1}[a^I,b^I])\\ &=n\sum_I \mathrm{Str}([j_+^{n-1},a^I],b^I)=0. \end{aligned} \tag{4.4}$$

Let us try to proceed without fixing the κ gauge. Since we do not have equations for $d*q$ and $d*q'$, the only candidate for j_+ is p_+. Noting that $(*\omega)_\pm=\pm\omega_\pm$ for any one-form, we find that

$$\partial_- p_+=[p_+,p_-+q_-]-\frac{1}{4}[q_{1-},q_{1+}], \tag{4.5}$$

while the fermionic equations of motion become

$$[p_+,q_{1-}]=[p_-,q_{2+}]=0. \tag{4.6}$$

Together these imply that $\partial_- p_+$ is of the form (4.3), with $a^1=p_+$ and $a^2=q_{1-}$. Unfortunately these currents are actually trivial. Since $\mathrm{Str}(p_+^n)=\mathrm{Str}(P_+^n)$, we only need traces of the broken bosonic generators. By $SO(4,1)\times SO(5)$ invariance, these can only involve products of $P_+^a P_+^a$ and $P_+^{a'} P_+^{a'}$, where a is an $SO(4,1)$ vector index and a' is an $SO(5)$ vector index. In fact one finds that

$$\mathrm{Str}(P_+^{2k})\propto(-P_+^a P_+^a)^k-(P_+^{a'} P_+^{a'})^k. \tag{4.7}$$

The first minus sign is from the Minkowski signature of AdS_5, and the second is from the supertrace. For $k=1$ this is the world-sheet T_{++}, which vanishes by the metric equation of motion, while for all $k>1$ it is a multiple of T_{++}. Thus these would-be chiral currents vanish. Notice however that if we go to conformal gauge, where the vanishing of T_{++} is not imposed as an equation of motion, then the fact that the current (4.7) is chiral for all k implies that $\partial_-(P_+^a P_+^a)=\partial_-(P_+^{a'} P_+^{a'})=0$ separately.

Of course, it could well be that there are currents that are chiral only in certain gauges. In the flat-spacetime case, for example, only after fixing the gauge are the world-sheet fields ∂X and θ chiral. A natural gauge for us is

$$q_{1-}=q_{2+}=0. \tag{4.8}$$

It should be possible to reach such a gauge, at least at the level of the classical solutions: the equations of motion (4.6) already imply half of this, and there is enough gauge freedom to impose the rest. In the flat-spacetime theory, the analogous gauge makes the world-sheet fermions free. Unfortunately, while the field and Maurer-Cartan equations simplify substantially in this gauge, there are no additional currents having the property (4.3).

V. DISCUSSION

The obvious next question is the use of these charges. The classic application is in theories with a mass gap, with the spatial coordinate unbounded. The relevant observable is then the *S*-matrix. The higher spin local conservation laws [15] imply the absence of particle production and the factorization of the *n*-particle *S*-matrix in terms of two-particle *S*-matrices, allowing the full *S*-matrix to be deduced (Ref. [29] and references therein). These same constraints can be derived from the nonlocal charges [17]. The argument appears to be less straightforward, but in at least some circumstances the local charges can be regarded as limits of the nonlocal charges [13], so the latter are sufficient.

These nonlocal charges also form the starting point for the classical and quantum inverse scattering methods, which are related in turn to the Bethe ansatz (Ref. [30] lists some reviews). We note that the existence of these charges is a nontrivial fact: they do not exist in all cases (e.g. the nonsymmetric-space cosets), and in such cases when there is a mass gap there is presumably particle production in scattering. It is less clear how conformal theories with the charges are distinguished.

We are interested in conformally invariant theories, on a bounded space (the open or closed string). In this case the simplest and most obvious observable is the partition function. This encodes the set of operator dimensions (in spacetime conformal theories such as $\mathcal{N}=4$ Yang-Mills) or the meson and glueball masses (in confining theories). In fact, this has been found for certain conformally invariant models based on supergroups [31]. This was done not by direct use of nonlocal charges, but by related methods involving integrable lattice models. Thus, our result on the existence of these charges in the $AdS_5\times S^5$ theory should be taken as motivation to apply the full set of methods of integrable field theory to this system.

For the nonlocal charges, the next step is to determine their classical Poisson bracket algebra, and to extend this to the quantum theory. This requires us to deal with the κ gauge invariance (which we were largely able to sidestep), perhaps along the lines of Ref. [32]. The conformally invariant supergroup and supermanifold models [33,23,34] provide interesting warmup problems, without the complication of κ symmetry. It is also interesting to ask what is dual on the gauge side to the symmetries that we have found. Finally, the extension to less symmetric and more QCD-like theories is challenging.

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[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[2] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[3] J. Polchinski, ''The AdS string worldsheet at weak and strong coupling,'' various seminars, paper in preparation.
[4] E. Witten, Commun. Math. Phys. **92**, 455 (1984).
[5] R.R. Metsaev and A.A. Tseytlin, Nucl. Phys. **B533**, 109 (1998).
[6] R. Kallosh, J. Rahmfeld, and A. Rajaraman, J. High Energy Phys. **09**, 002 (1998).
[7] R. Roiban and W. Siegel, J. High Energy Phys. **11**, 024 (2000).
[8] M. Luscher and K. Pohlmeyer, Nucl. Phys. **B137**, 46 (1978).
[9] E. Brezin, C. Itzykson, J. Zinn-Justin, and J.B. Zuber, Phys. Lett. **82B**, 442 (1979).
[10] T.L. Curtright and C.K. Zachos, Phys. Rev. D **21**, 411 (1980).
[11] N.J. MacKay, Phys. Lett. B **281**, 90 (1992); **308**, 444(E) (1993).

[12] L. Dolan, Phys. Rev. Lett. **47**, 1371 (1981); J.H. Schwarz, Nucl. Phys. **B447**, 137 (1995).
[13] D. Bernard, Int. J. Mod. Phys. B **7**, 3517 (1993).
[14] E. Abdalla, M. C. B. Abdalla, and K. D. Rothe, *Two Dimensional Quantum Field Theory* (World Scientific, Singapore, 2001).
[15] A.M. Polyakov, Phys. Lett. **72B**, 224 (1977); E. R. Nissimov and S. J. Pacheva, "Infinite set of conservation laws of the quantum chiral field in two-dimensional space-time," LOMI-E-1-1978; Y.Y. Goldschmidt and E. Witten, Phys. Lett. **91B**, 392 (1980).
[16] J.M. Evans, M. Hassan, N.J. MacKay, and A.J. Mountain, Nucl. Phys. **B561**, 385 (1999).
[17] M. Luscher, Nucl. Phys. **B135**, 1 (1978).
[18] I. Pesando, J. High Energy Phys. **02**, 007 (1999); J. Rahmfeld and A. Rajaraman, Phys. Rev. D **60**, 064014 (1999).
[19] J. Maldacena and H. Ooguri, J. Math. Phys. **42**, 2929 (2001).
[20] J. Maldacena and L. Maoz, J. High Energy Phys. **12**, 046 (2002).
[21] C. Gomez, J. High Energy Phys. **08**, 039 (2003); I. Bakas and J. Sonnenschein, *ibid.* **12**, 049 (2002).
[22] G. Mandal, N.V. Suryanarayana, and S.R. Wadia, Phys. Lett. B **543**, 81 (2002).
[23] M. Bershadsky, S. Zhukov, and A. Vaintrob, Nucl. Phys. **B559**, 205 (1999).
[24] J.A. Minahan and K. Zarembo, J. High Energy Phys. **03**, 013 (2003).
[25] N. Beisert, C. Kristjansen, and M. Staudacher, Nucl. Phys. **B664**, 131 (2003); N. Beisert, J.A. Minahan, M. Staudacher, and K. Zarembo, J. High Energy Phys. **03**, 010 (2003).
[26] A.V. Belitsky, A.S. Gorsky, and G.P. Korchemsky, Nucl. Phys. **B667**, 3 (2003).
[27] L.N. Lipatov, JETP Lett. **59**, 596 (1994) [Pis'ma Zh. Éksp. Teor. Fiz. **59**, 571 (1994)].
[28] H. Eichenherr and M. Forger, Commun. Math. Phys. **82**, 227 (1981).
[29] A.B. Zamolodchikov and A.B. Zamolodchikov, Nucl. Phys. **B133**, 525 (1978) [JETP Lett. **26**, 457 (1977)].
[30] H.B. Thacker, Rev. Mod. Phys. **53**, 253 (1981); L. D. Faddeev, in *Recent Advances in Field Theory and Statistical Mechanics: Proceedings of the 1982 Les Houches Summer School in Theoretical Physics*, edited J. B. Zuber and R. Stora (North-Holland, Amsterdam, 1984); V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).
[31] N. Read and H. Saleur, Nucl. Phys. **B613**, 409 (2001).
[32] N. Berkovits, J. High Energy Phys. **04**, 018 (2000).
[33] S. Sethi, Nucl. Phys. **B430**, 31 (1994).
[34] N. Berkovits, C. Vafa, and E. Witten, J. High Energy Phys. **03**, 018 (1999).