Two-dimensional quantum-corrected black hole in a finite size cavity

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We consider the gravitation-dilaton theory (not necessarily exactly solvable), whose potentials represent a generic linear combination of an exponential and linear functions of the dilaton. A black hole, arising in such theories, is supposed to be enclosed in a cavity, where it attains thermal equilibrium, whereas outside the cavity the field is in the Boulware state. We calculate quantum corrections to the Hawking temperature T_H, with the contribution from the boundary taken into account. Vacuum polarization outside the shell tends to cool the system. We find that, for the shell to be in thermal equilibrium, it cannot be placed too close to the horizon. The quantum corrections to the mass due to vacuum polarization vanish in spite of nonzero quantum stresses. We discuss also the canonical boundary conditions and show that accounting for the finiteness of the system plays a crucial role in some theories (e.g., Callan-Giddings-Harvey-Strominger), where it enables us to define the stable canonical ensemble, whereas consideration in an infinite space would predict instability.

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I. INTRODUCTION

Two-dimensional (2D) dilaton gravity serves as an excellent tool for studying (at least, on the semiclassical level) quantum effects in gravitation and constructing the prototype of (yet unfinished) 4D quantum gravity. In the first place, it concerns black hole physics, where string-inspired models [1] became very popular during the last decade and enabled us to trace in the simplified context many effects, typical of 4D black hole physics, such as black hole evaporation, thermodynamic features, etc. (for reviews see, e.g., recent papers [2,3]). On the other hand, studies in 2D black hole physics revealed the fact that, in some aspects, such theories look rather unusual and open new interesting possibilities, absent in general relativity and deserving treatment on their own. For example, the Hawking temperature T_H in the classical Callan-Giddings-Harvey-Strominger (CGHS) model and some of its semiclassical generalizations is a constant, not depending on the horizon radius. This makes the question of black hole thermodynamics, which is one of the most important black hole features, quite nontrivial. First, as pure classical thermodynamics is poor for such systems, quantum back reaction and the corresponding quantum corrections to the Hawking temperature become crucial for the calculation of the heat capacity. Second, even with quantum back reaction taken into account, some models that include a wide family of exactly solvable ones still exhibit no quantum corrections to T_H. To obtain substantial thermodynamics, one should take into account that for self-gravitating systems a finite size can be crucial in carefully constructing the canonical ensemble [4]. We will see below that the competition between these two factors, small quantum corrections and large (but finite) spatial size, may lead to well-defined thermal properties even in situations when the pure classical approach gives no sensible answer.

The issue of the finiteness of the system has one more aspect. Consider a black hole enclosed inside a reflecting shell (that, in the 1+1 case, represents a point) in thermal equilibrium with its Hawking radiation. If the shell is perfect, all Hawking radiation is concentrated inside and no radiation comes outside. However, the entire object "black hole +radiation+shell" curves spacetime and serves as a source of gravitation field outside. In turn, this leads to the appearance of quantum stresses even in an otherwise empty space. In other words, the field state is supposed to be the Hartle-Hawking inside the shell and the Boulware one outside. Usually these states are opposed in the 4D world, where the first state is attributed to a black hole, while the second one corresponds to a relativistic star. However, the boundary effects may lead to their overlap and, thus, the Boulware state becomes relevant for black hole thermodynamics, so this effect deserves attention on its own.

The quantum corrections to T_H were calculated in [5] for the black hole in the Hartle-Hawking state for the particular case of the CGHS model, but the contribution of vacuum polarization was neglected there (that looks quite reasonable, if a boundary is situated sufficiently far from the horizon). Recently, these corrections were considered in [6] (in the quite different approach) for a slowly evaporating black hole in the Unruh state. The results differ by the sign that seem to affect the sign of the heat capacity in an infinite space. This prompts us to consider the issue of stability carefully, with proper account for the finite size of the system. We will see that the stable canonical ensemble can be defined even in the cases when consideration in an infinite space would give the negative heat capacity.

The paper is organized as follows. In Sec. II we list basic equations, governing the gravitational-dilatonic system with minimal fields, and, by considering quantum back reaction as perturbation, derive the quantum corrections to the Hawking temperature in an infinite space. This generalizes our previous result which was obtained for the particular case of the CGSH model. In Sec. III we consider a black hole, enclosed inside a perfect reflecting shell, outside of which the quantum field in the Boulwars state, heated to some temperature. For exactly solvable models we find the modified Hawking temperature exactly, for a generic model we find the main

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quantum corrections. In Sec. IV we analyze the canonical ensemble and its stability with account for both quantum back reaction and finiteness of the system. We also discuss briefly the case of the microcanonical ensemble. In Sec. V we summarize the main results.

II. QUANTUM CORRECTIONS TO HAWKING TEMPERATURE IN AN INFINITE SPACE

Let us consider the system governed by the action

$$I=I_{gd}+I_{PL}, \tag{1}$$

where the gravitation-dilaton part

$$I_{gd}=\frac{1}{2\pi}\int_M d^2x\sqrt{-g}[F(\phi)R+V(\phi)(\nabla\phi)^2+U(\phi)], \tag{2}$$

I_{PL} is the Polyakov-Liouville action incorporating effects of Hawking radiation of minimal fields and its back reaction on spacetime for a multiplet of N conformal scalar fields (we omit boundary terms in the action). As is known, it can be written down in the form

$$I_{PL}=-\frac{\kappa}{2\pi}\int_M d^2x\sqrt{-g}\left[\frac{(\nabla\psi)^2}{2}+\psi R\right], \tag{3}$$

$\kappa=\frac{\hbar N}{24}$. The function ψ obeys the equation

$$\Box\psi=R. \tag{4}$$

Varying the action with respect to the metric, we get

$$\delta I=\frac{1}{4\pi}\int d^2x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}=0. \tag{5}$$

For static spacetimes (with which we are dealing with in this paper) in the Schwarzschild gauge

$$ds^2=-fdt^2+f^{-1}dx^2, \tag{6}$$

field equations take the following explicit form [see, for example, Eqs. (23) and (24) of [7]]:

$$G_0^0=2f\frac{\partial^2\tilde{F}}{\partial x^2}+\frac{\partial f}{\partial x}\frac{\partial\tilde{F}}{\partial x}-U-\tilde{V}f\left(\frac{\partial\phi}{\partial x}\right)^2=0, \tag{7}$$

$$G_1^1=\frac{\partial f}{\partial x}\frac{\partial\tilde{F}}{\partial x}-U+\tilde{V}f\left(\frac{\partial\phi}{\partial x}\right)^2=0. \tag{8}$$

Here

$$\tilde{F}=F-\kappa\psi,\tilde{V}=V-\frac{\kappa}{2}\left(\frac{d\psi}{d\phi}\right)^2. \tag{9}$$

In what follows we will use notations $U=4\lambda^2u$ for the potential and $z=\lambda x$ for a coordinate. It is also convenient to take the sum and difference of Eqs. (7) and (8) that gives us

$$\left[\frac{\partial^2\tilde{F}}{\partial\phi^2}-\tilde{V}\right]\left(\frac{\partial\phi}{\partial z}\right)^2+\frac{\partial\tilde{F}}{\partial\phi}\frac{\partial^2\phi}{\partial z^2}=0, \tag{10}$$

$$4u=\left(f\frac{\partial\tilde{F}}{\partial\phi}\frac{\partial\phi}{\partial z}\right)_{,\phi}\frac{\partial\phi}{\partial z}. \tag{11}$$

In the conformal frame

$$ds^2=f(-dt^2+d\sigma^2) \tag{12}$$

we have ($y\equiv\lambda\sigma$)

$$\left[\frac{\partial^2\tilde{F}}{\partial\phi^2}-\tilde{V}-f^{-1}\frac{\partial f}{\partial\phi}\right]\left(\frac{\partial\phi}{\partial y}\right)^2+\frac{\partial\tilde{F}}{\partial\phi}\frac{\partial^2\phi}{\partial x^2}=0 \tag{13}$$

and

$$4u=f^{-1}\frac{\partial^2\tilde{F}}{\partial y^2}. \tag{14}$$

In what follows we will dwell upon the string-inspired models of the form

$$F=\exp(-2\phi)+b\kappa\phi,\quad V=4\exp(-2\phi)+c\kappa,$$
$$u=\exp(-2\phi). \tag{15}$$

Then the solutions of Eqs. (4), (10), and (11) [or (13), (14)], regular on the horizon of a black hole, in the main approximation with respect to κ look like

$$\psi=-2\phi+O(\kappa), \tag{16}$$

$$z=-\phi+\frac{\kappa}{4}e^{2\phi}\left(1-\frac{c}{2}\right), \tag{17}$$

$$f=1-a\exp(2\phi)+\kappa\exp(2\phi)\left\{\frac{q}{2}[1-a\exp(2\phi)]+\left(1-\frac{c}{2}\right)\phi\right\}. \tag{18}$$

$$a=\exp(-2\phi_+)+\kappa\phi_+\left(1-\frac{c}{2}\right). \tag{19}$$

$$T_H=T_0\left[1+\frac{\kappa}{2}\exp(2\phi_+)q\right],\quad T_0\equiv\frac{\lambda}{2\pi}. \tag{20}$$

where $q=b+c/2+1$.

It is seen from Eq. (20) that, classically, the Hawking temperature is a constant and all dependence on the horizon position arises only via quantum corrections. For the CGHS model $c=0=b$, $q=1$ and we return to the result that can be obtained by the limiting transition $\phi_B\to-\infty$ from Eq. (18) of Ref. [5]. If

$$b=2(d-1),\quad c=2(1-2d), \tag{21}$$

that is equivalent to

$$q=b+\frac{c}{2}+1=0, \tag{22}$$

the model becomes exactly solvable [8] and reduces, in particular cases, to the RST ($c=0$, $b=-1$) [9] or BPP ($c=2$, $b=-2$) [10] ones. In this case quantum corrections to the Hawking temperature vanish in accordance with observations made in [11] and [12].

III. MATCHING HARTLE-HAWKING AND BOULWARE STATES AND ROLE OF BOUNDARY

In the 1+1 world, a ''shell'' represents a point. If it is present, field equations modify to

$$G_{\mu}^{\nu}=S_{\mu}^{\nu}, \tag{23}$$

where S_{μ}^{ν} is a dimensionless stress-energy of the shell, containing only delta-like terms. We assume that quantities y, f, ϕ, $\tilde{F}$ are continuous across the shell, while first derivatives $\partial\tilde{F}/\partial y$ may experience jumps. Then it follows from explicit expressions (7), (8) or (10), (11) that G_1^1 is bounded across the shell, so $S_1^1=0$, whereas G_0^0 may contain delta-like singularities:

$$S_0^0=-m\,\delta^{(1)}(y-y_B),\quad m\equiv-2\left[\left(\frac{\partial\tilde{F}}{\partial y}\right)_+-\left(\frac{\partial\tilde{F}}{\partial y}\right)_-\right], \tag{24}$$

where the parameter m can be regarded as the mass of the shell, y_B is its position, ''+'' or ''−'' means ''y_B+0'' and ''y_B-0,'' respectively. The delta function $\delta^{(1)}$ is normalized according to

$$\int dy\,\sqrt{g}\,\delta^{(1)}(y-y_B)=1, \tag{25}$$

where the index ''B'' refers to the boundary.

A. Exactly solvable case

First, we consider the exactly solvable case, when the coefficients obey the relationship (21). As is shown in [12,13], the metric function, describing a black hole in an infinite space, is equal in this case to

$$f=\exp(2\phi+2y). \tag{26}$$

The coefficient in the form of f is chosen in Eq. (26) in such a way that $f\to1$ at right infinity, where the spacetime is flat.

In so doing,

$$\tilde{F}=\exp(2y)+\tilde{F}_+, \tag{27}$$

the index ''+'' refers to the horizon, which is situated at $y=-\infty$, so

$$f=\exp(2\phi)(\tilde{F}-\tilde{F}_+). \tag{28}$$

Let now a 0-dimensional point-like perfect shell between the horizon and right infinity be situated at some y_B. To the left of the shell, the field is in the Hartle-Hawking state, while to the right to the shell it is in the Boulware state. Consider the solution in both regions separately and, afterwards, sew them at $y=y_B$.

First, consider the region $y<y_B$. We can exploit the already obtained solution (26) but with the reservation that there is a freedom in the choice of the conformal coordinate that preserves the conformal gauge (12). The coordinate y can be rescaled as $y\to Ay$, where A is a constant. Apart from this, there is also a freedom in translations $y\to y+\text{const}$. For the solution in an infinite space it was inessential since, due to the condition $f(\infty)=1$, it had to be reduced to Eq. (26). However, now there is no right infinity in the left region and such parameters should be kept arbitrary, their values will be fixed from matching the solutions in two regions (see below). Therefore now we should write

$$\tilde{F}=\tilde{F}^{(0)}=\alpha\exp(2Ay)+\tilde{F}_+, \tag{29}$$

$$f_l=\frac{\alpha A^2}{u}\exp(2Ay)=\frac{A^2}{u}(\tilde{F}-\tilde{F}_+), \tag{30}$$

where α and A are constants. The Hawking temperature for the metric (12) $T_H=(1/4\pi)\lim_{y\to-\infty}f^{-1}(df/dy)$, so

$$T_H=\frac{\lambda}{2\pi}A. \tag{31}$$

Consider now the region to the right of the shell, $y>y_B$. Now we should take into account that the function is determined from Eq. (4) up to the solution of the homogeneous equation that is proportional to y. Again, we may exploit the solution in an infinite spacetime, obtained in [13]:

$$\psi=\psi_0+\frac{\gamma}{\lambda}y. \tag{32}$$

$$\tilde{F}=\tilde{F}^{(0)}-\kappa\frac{\gamma}{\lambda}y, \tag{33}$$

where $\tilde{F}^{(0)}=F-\kappa\psi_0$, ψ_0 is bounded on the horizon (for the exactly solvable under discussion $\psi_0=-2\phi$),

$$\tilde{F}^{(0)}=\exp(2y)-By+E, \tag{34}$$

where E is a constant,

$$B=\kappa\left(1-\frac{T^2}{T_0^2}\right). \tag{35}$$

$$f_r=\frac{e^{2y}}{u}. \tag{36}$$

Here T is the temperature of the thermal gas at the right infinity. As is shown in [13], the constant γ is connected with T according to

$$\gamma = 2\lambda\left(\frac{T}{T_0}-1\right). \tag{37}$$

The attempt of applying the above formulas to the region near the black hole horizon ($y\to-\infty$) shows that the quantity ψ diverges there and so does the Polyakov-Liouville-stresses [14]. However, this problem does not arise now since the region, in which these formulas are valid, is restricted by the condition $y>y_B$ and does not include the horizon.

On the boundary $f_l(\phi_B)=f_r(\phi_B)$, whence

$$\alpha = A^{-2}\exp[2y_B(1-A)]. \tag{38}$$

$$E=(\tilde{F}_B^{(0)}-\tilde{F}_+)(1-A^2)+By_B+\tilde{F}_+ . \tag{39}$$

Calculating the difference $[(\partial\tilde{F}/\partial y)_+-(\partial\tilde{F}/\partial y)_-]$ and remembering Eq. (24), we obtain

$$2\exp(2y_B)\left[1-\frac{1}{A}\right]-\tilde{B}+-\frac{m}{2}=0, \tag{40}$$

$$\tilde{B}=B+\frac{\gamma}{\lambda}\kappa=-\frac{\kappa\gamma^2}{4\lambda^2}=-\kappa\left(1-\frac{T}{T_0}\right)^2. \tag{41}$$

Substituting $\exp(2y_B)=A^2(\tilde{F}_B-\tilde{F}_+)$, we obtain the equation

$$A^2-A-\frac{\tilde{B}-m/2}{2(\tilde{F}_B-\tilde{F}_+)}=0, \tag{42}$$

$$A=\frac{1}{2}\left(1+\sqrt{1-\frac{2|\tilde{B}|+m}{(\tilde{F}_B-\tilde{F}_+)}}\right),\quad \tilde{B}=-|\tilde{B}|. \tag{43}$$

We choose the root of the quadratic equation for which $A=1$, when $\kappa=0=m$.

It follows from Eq. (43) that $A_{\min}=\frac{1}{2}$, when $m=\tilde{F}_B-\tilde{F}_+ +2\tilde{B}$. If $m>0$, $\frac{1}{2}<A<1$. Quantum effects for the temperature are compensated by the shell mass if $m=2\tilde{B}<0$. In the limit $y_B\to\infty$, when $\tilde{F}_B\to\infty$, quantum corrections tend to zero: $\Delta T_H/T\simeq-1/2(|\tilde{B}|+m/2)/(\tilde{F}_B-\tilde{F}_+)$. It is seen from Eq. (43) that both the quantum effects and the shell with a positive mass tend to cool a system.

From the expression (43) it follows the restriction on the position of the shell that cannot be placed too close to the horizon, if we want to maintain thermal equilibrium inside the shell and the Boulware state, heated to the temperature T, outside:

$$\tilde{F}_B-\tilde{F}_+-2|\tilde{B}|-m>0. \tag{44}$$

As the Riemann curvature $R=-\lambda^2/f d^2\ln f/dy^2$, it follows from Eqs. (26) and (24) that for the exactly solvable models the delta-like part of the curvature is equal to $R_s =[m\lambda^2/\tilde{F}'(\phi_B)]\delta^{(1)}(y-y_B)$. Therefore for a massless shell the geometry is smooth across the shell.

B. Generic case: Perturbative approach

Now let the system be of the type (15) with generic coefficients, not necessarily obeying the condition of exact solvability (22). Then the explicit formulas can be obtained perturbatively in κ, matching the solutions to the left and to the right from the shell, following the same line, as in the previous case. In so doing, we retain only terms of the zero and first order in κ. Omitting details of calculations, which are rather straightforward, we list only basic formulas. To the right from the shell the relationship between the dilaton and spatial coordinate reads

$$\frac{d\phi}{dz}=-1-\kappa\frac{\exp(2\phi)}{2}\left(1-\frac{c}{2}-\frac{D}{1-k\exp(2\phi)}\right), \tag{45}$$

the metric function has the form

$$f=1-k\exp(2\phi)+\kappa\exp(2\phi)\left\{\frac{q}{2}[1-k\exp(2\phi)] +\left(1-\frac{c}{2}-D\right)\phi+\frac{D}{2}\ln(1-k\exp(2\phi)\right\}, \tag{46}$$

$$D=-\frac{\gamma}{\lambda}\left(1+\frac{\gamma}{4\lambda}\right)=1-\frac{T^2}{T_0^2}. \tag{47}$$

In the exactly solvable case $q=0$,

$$f=1-k\exp(2\phi)+\kappa\left[(b+2-D)\phi\exp(2\phi) +\frac{D}{2}\exp(2\phi)\ln 1-k\exp(2\phi)\right], \tag{48}$$

that can be also obtained directly from Eq. (34). If $D=0$, Eq. (18) is reproduced.

To the left from the shell $f=A^2\tilde{f}$, where $\tilde{f}$ is given by Eq. (18).

Matching the solution in two regions, we obtain from Eq. (24)

$$A=\frac{1}{2}[1+\sqrt{1-(m+2\kappa|\tilde{D}|)/Q}],$$

$$Q=[\exp(-2\phi_B)-\exp(-2\phi_+)]+\frac{\kappa}{2}\left(1-\frac{c}{2}\right)(\phi_B-\phi_+),$$

$$|\tilde{D}|=\left(1-\frac{T}{T_0}\right)^2. \tag{49}$$

For the massless shell, with the same accuracy (with terms κ^2 and higher discarded)

$$A=1-\frac{\kappa|\tilde{D}|}{2Q}, \tag{50}$$

$$T_H = T_0(1+\kappa\varepsilon), \varepsilon = \frac{q}{2}\exp(2\phi_+) - \frac{|\tilde{D}|}{2Q}. \quad (51)$$

For the CGSH model $b=0=c$, $q=1$, for $T=0$ ($|\tilde{D}|=1$) in the limit $|\phi_B| \gg |\phi_+|$, neglecting the term $\exp(-2\phi_+)$ in the denominator, we obtain that $\varepsilon = (q/2)\exp(2\phi_+) - \frac{1}{2}\exp(-2\phi_B)$ that coincides with Eq. (18) of [5].[1] If $q>0$ and the shell is placed at ϕ_B such that $\exp(-2\phi_B) = \exp(-2\phi_+)(1+|\tilde{D}|/q)$, the boundary and ordinary quantum corrections mutually cancel.

IV. ENERGY, ADM MASS, AND CHOICE OF BACKGROUND

From the physical viewpoint, the perfect shell considered in the previous section realizes microcanonical boundary conditions that fixed the energy (cf. [15]). Meanwhile, another physically relevant type of condition demands fixing the temperature rather than the energy, thus defining the canonical ensemble. This case is also discussed below. In so doing, the correct definition of thermal quantities, such as the energy, heat capacity, etc., can be obtained with the help of the Euclidean action formalism, with account for the finiteness of the system that, in particular, needs specifying the set of boundary data. Generalizing expressions for classical gravitation-dilaton systems [16], one can write down the energy of the quantum-corrected one as [14]

$$E_{gd} = -\frac{1}{\pi}\left(\frac{d\tilde{F}}{dl}\right)_B = -\frac{\lambda}{\pi}\left(\frac{d\tilde{F}}{d\phi}\frac{\sqrt{f}}{z'}\right)_B \equiv -2T_0\left(\frac{d\tilde{F}}{d\phi}\sqrt{f}\frac{\partial\phi}{\partial z}\right)_B. \quad (52)$$

For exactly solvable models [see Eq. (2.7) of Ref. [12]]

$$\frac{dz}{d\phi} = \tilde{F}'\frac{\exp(2\phi)}{2}. \quad (53)$$

Here the common factor in the right-hand side of Eq. (53) is chosen, for the models (15), to give $z = -\phi + \text{const}$ (linear dilaton vacuum) at the right infinity, where spacetime is flat. Thus for exactly solvable models we have

$$E_{gd} = -4T_0\exp(-2\phi_B)\sqrt{f_B}. \quad (54)$$

In general, the energy E is measured with respect to some background whose contribution E_0 is to be subtracted from E_{gd}, so $E = E_{gd} - E_0$. In [11] two reference points were considered: the classical hot flat spacetime (which is obtained by putting $\kappa=0$ in the action) and the black hole configuration with the singular horizon. We adopt another reference configuration: as a background, we choose the "quasi-flat" spacetime which is close to the classical one but differs from it due to the presence of the terms with κ in Eq. (15). Let me remind the reader that the parameter κ enters independently both the Polyakov-Liouville action and the definition of the action coefficients (15). In the second case it was motivated by the demand to construct exactly solvable models but now we relaxed that condition. We discard the first contribution but retain the second one. To avoid confusion, one may replace κ in Eq. (15) by another small parameter τ, effecting the functional form of these coefficients and put $\tau=\kappa$ after calculations. Physically, this means that our reference state is pure classical in the sense that not any quantum back reaction is present, but the functional form of the gravitation-dilaton action is the same as for the quantum-corrected configuration.

To find E_0, we have to solve field equations (10), (11) for these potentials without the contribution of ψ, so tilted quantities should be replaced by the usual ones. Then, with terms of the order κ^2 and higher neglected, we find

$$E_0 = -2T_0\left(\sqrt{f}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial z}\right)_{\text{quasiflat}}, \quad (55)$$

$$f = 1 - \kappa\frac{c}{2}\phi\exp(2\phi) + \frac{\kappa}{2}\exp(2\phi)(q-1). \quad (56)$$

$$\frac{\partial z}{\partial\phi} = -1 - \kappa\frac{c}{4}\exp(2\phi). \quad (57)$$

Asymptotically, for large $|\phi|$, $\phi<0$, we obtain

$$E_0 = -4T_0\exp(-2\phi) + T_0\kappa c\phi + T_0\kappa\frac{(c+2b)}{2}. \quad (58)$$

Now the quantity $E_g - E_0$ can be identified with the ADM mass and we have, after asymptotic expansion of E,

$$E_g - E_0 = M_{BH} + M_{th} + M_0, \quad (59)$$

$$M_{BH} = 2T_0\left[\exp(-2\phi_+) - \kappa\frac{c}{2}\phi_+\right],$$

$$M_{th} = 2T_0\kappa(\phi_+ - \phi), M_0 = \kappa T_0. \quad (60)$$

Here the term M_{BH} does not depend on ϕ and should be identified with the mass of a black hole itself. The quantity M_{th} represents the contribution of thermal gas at the temperature T_0. Remarkably, the coefficients b and c, that characterize the model, are absorbed by these general definitions. One can say that not only for the RST model [11] and even

[1] It was stated in Ref. [5] that the shell should be inevitably massive to maintain equilibrium, whereas in the present paper we mention a massless shell ($m=0$) while comparing the results. There is no contradiction here since these statements refer to different quantities. It follows from Eq. (24) that the mass is a linear functional of the quantity $\tilde{F} = F - \kappa\psi$ and, correspondingly, can be spited in two parts—m_F, connected with F (the gravitational-dilatonic one) and m_ψ, connected with ψ (the Polyakov contribution). It is just $m_F \neq 0$ which was implied in [5], while the total sum $m_F + m_\psi = 0$ (massless shell).

not only for a more general exactly solvable model, but in the general case for the family (15), quantum corrections to the universal form (59) vanish.

In a similar way, the total entropy $S_{tot}=S_{BH}+S_{th}$, where the entropy of the black hole itself

$$S_{BH}=2F(\phi_+)=2[\exp(-2\phi_+)+b\kappa\phi_+] = \frac{M_{BH}}{T_0}+2\kappa(q-1)\phi_+, \tag{61}$$

while the entropy of the thermal gas [11,12,16,17]

$$S_{th}(T_0,\phi_+-\phi_B)=4\kappa(\phi_+-\phi_B). \tag{62}$$

The expression (59) is valid in an infinite space. Now let a wall be placed at $\phi=\phi_B$. Take $\phi=\phi_0$ to the right from the boundary and consider the region between the boundary and infinity. Taking into account Eqs. (45)–(47) and (37) and expanding the expression for the energy for large negative ϕ_0, where spacetime approaches its Minkowski limit, we obtain after simple calculations:

$$E_g=4\kappa(T-T_0)-4T_0\exp(-2\phi)+2T_0k + T_0\kappa[q+2D+(c-2+2D)\phi], \tag{63}$$

$$M_{tot}=M_{BH}+M_{th}(T_0,\phi_+-\phi_B)+M_{th}(T,\phi_B-\phi_0)+M_0, \tag{64}$$

$$M_{th}(T,\phi_1-\phi_2)=\frac{\pi}{6\lambda}T^2(\phi_1-\phi_2)=2\kappa\frac{T^2}{T_0}(\phi_1-\phi_2). \tag{65}$$

The formula (64) generalizes Eq. (59) in a natural way: it includes the contribution of the thermal gas with two different temperatures—T_0 between the horizon and the wall and T between the wall and the point of observation. More surprisingly, vacuum polarization in the Boulware state with $T=0$ (when quantum stresses do not vanish) does not give corrections at all, thus the only contribution of the state outside comes due to thermal excitations, if the Boulware state is heated to some temperature T. This fact can be attributed to the change of the effective coupling between the curvature and dilaton: the quantity $\tilde{F}$ changes to $\tilde{F}+\kappa(\gamma/\lambda)y$ in such a way that the first term in Eq. (63) cancels the vacuum contribution.

In Sec. III we obtained that if we want the shell to maintain thermal equilibrium inside and the Boulware state outside, it cannot be placed too closely to the horizon. Now, the general formulas for the energy obtained above enable us to give a rather simple physical interpretation to the corresponding restriction on the position of the shell. Let the quantum state be in the Boulware state ($T=0$, $|\tilde{B}|=\kappa$, $D=1$) outside. To elucidate the role of different terms containing the parameter κ, we consider the case when the restriction under discussion is obtained exactly (44). Taking into account the explicit expression for the action coefficients (15), the conditions of solvability (22), restoring explicitly the factor $\lambda/\pi=2T_0$ so that the mass of the shell $M_{\rm shell}=2T_0m$, and using the expression for the total mass of black hole plus thermal radiation M_{tot} (64), we obtain

$$M'_{tot}>M_{tot}+M'_{\rm shell}. \tag{66}$$

Here $M'_{\rm shell}=M_{\rm shell}+3M_0$, and $M'_{tot}\equiv 2T_0[\exp(-2\phi_B)-\kappa(c/2)\phi_B]$ represents the mass of a black hole which would form, if thermal radiation completely collapsed, producing a new black hole with the horizon at $\phi'_+=\phi_B$. This horizon would coincide with the radius of the corresponding 1+1 "relativistic star," the quantum field outside being in the Boulware state.

V. CANONICAL ENSEMBLE AND HEAT CAPACITY

In general, for self-gravitating systems the conditions of stability can be different for different types of thermal ensembles. As far as our gravitation-dilaton system is concerned, the stability of the microcanonical ensemble (without account for boundary corrections) follows immediately from Eq. (12) of Ref. [18] (case $a=0$ in their notations). The case of the canonical ensemble is much more subtle since it demands a simultaneous careful account for the finiteness of the system and quantum back reaction. Let us discuss this issue in more detail.

The canonical ensemble is defined by the value of temperature and, possibly, some other parameters, which for self-gravitating systems are fixed on the boundary [4]. According to the Tolman relation,

$$T_B=\frac{T_H}{\sqrt{f_B}}, \tag{67}$$

where T_B is the local temperature on the boundary. For the system under discussion the value of the dilaton ϕ_B is also fixed. The region, external with respect to the boundary, is now discarded and replaced by a heat bath, so there is no sense in speaking about boundary corrections to the Hawking temperature. Nevertheless, the finiteness of the system reveals itself, as we will see below, in the dependence of the horizon radius ϕ_+ and thermodynamic characteristics on the boundary data.

First, we discuss briefly the exactly solvable case. Then $T_H=T_0=\text{const}$ [12]. Then it follows from Eq. (28) that, for given T_B and ϕ_B, there is also one root $\tilde{F}_+$. If the function $\tilde{F}(\phi)$ is monotonic (for example, this happens to the BPP model), there is only one branch and one value ϕ_+. In general, this function can have minima and maxima. For example, in the RST model there are two branches of solutions: the upper branch $\phi_s<\phi<\infty$ and the lower branch $-\infty<\phi<\phi_s$, where ϕ_s corresponds to the singularity [11].

The heat capacity can be found from Eqs. (54), (67) (now the term E_0 does not contribute and can be omitted):

$$C=\frac{dE}{dT_B}=4\exp(-2\phi_B)f_B=4\exp(-2\phi_B)\frac{T_0^2}{T_B^2}>0. \tag{68}$$

Of main interest is the region between the horizon and Minkowski spacetime at infinity, with $-\infty<\phi<\phi_+$. Then it follows from explicit expressions (15), (28) that in this region always $0\leqslant f<1$. Therefore, if $T_B>T_0$, there is one stable root. If $T_B<T_0$, there are no roots at all. Thus if the equilibrium is possible, the system is always locally stable.

Consider now the case with generic coefficients b, c within the perturbative approach with respect to κ. Now, in contrast to the exactly solvable case, quantum corrections to the Hawking temperature do not vanish and an interesting overlap between quantum and boundary effects appears. Remembering Eq. (20) and differentiating the relevant quantities, we obtain

$$\frac{\partial E}{\partial\phi_+}=-2\frac{T_0}{\sqrt{f}}\frac{\partial f}{\partial\phi_+}\exp(-2\phi_B)\left[1-\kappa\frac{q}{2}\exp(2\phi_B)\right]. \tag{69}$$

$$\frac{\partial T}{\partial\phi_+}=-\frac{T_0}{2f\sqrt{f}}\left[\frac{\partial f}{\partial\phi_+}-2f\kappa q\exp(2\phi_+)\right]. \tag{70}$$

$$C=4\exp(-2\phi_B)\frac{f'\left[1-\kappa\frac{q}{2}\exp(2\phi_B)\right]}{f'-2f\kappa q\exp(2\phi_+)}, \tag{71}$$

where $f'=df/d\phi_+$. Let $\phi_B\to-\infty$, $\kappa\to0$, then in the main approximation $\partial f/\partial\phi_+=2\exp(2\phi-2\phi_+)$.

Writing $T/T_0\equiv1+\alpha$ with small, but nonzero α, we obtain from Eq. (67) the equation

$$r^2-2rr_0+s=0,\quad r\equiv\exp(-2\phi_+)>0,$$

$$r_0\equiv\exp(-2\phi_B)\alpha,\quad s\equiv\kappa q\exp(-2\phi_B), \tag{72}$$

where parameters r_0 and s, constructed as the products of small and big quantities, are in general finite. Writing the solution of the quadratic equation as $r_\pm=r_0\pm\sqrt{r_0^2-s}$, we find

$$C=4\exp(-2\phi_B)\frac{r^2}{r^2-s}=\pm2\exp(-2\phi_B)\frac{r_\pm}{\sqrt{r_0^2-s}}, \tag{73}$$

where we took into account that $\kappa\phi_B\exp(2\phi_B)\ll1$. Thus only the root r_+ can correspond to the stable equilibrium. If $r_0^2<s$, there are no black hole solutions in thermal equilibrium at all, so the ground state lies in the same topological sector as the flat spacetime. Let $r_0^2>s$ and discuss now particular cases.

(1) $r_0>0$, $s>0$. Then $r_+>0$, $r_->0$. There are two roots: r_+ is stable, r_- is unstable.

(2) $r_0<0$, $s>0$. $r_+<0$, $r_-<0$. There are no positive roots at all.

(3) $r_0>0$, $s<0$. $r_+>0$, $r_-<0$. One stable root r_+.

(4) $r_0<0$, $s<0$. $r_+>0$, $r_-<0$. One stable root r_+.

Thus in cases (1), (3), and (4) we have the locally stable black hole solution. Pure classical consideration ($\kappa=s=0$) would give only one stable root $r_{cl}=2r_0$, and $C=4\exp(-2\phi_B)>0$, $C\to\infty$ for $\phi_B\to-\infty$. In case (1) the root $r_+<r_{cl}$, whereas in case (3) $r_+>r_{cl}$. According to Eq. (17), the spatial coordinate z grows, when ϕ diminishes. Therefore in case (1) quantum corrections decrease the horizon radius, whereas in case (3) it slightly increases, as compared to the classical case. Case (4) is pure quantum and does not exist in the classical domain. Indeed, for κ so that $s\ll r_0^2$, the root

$$r_+\simeq\frac{|s|}{2|r_0|}=\frac{\kappa}{2}\left|\frac{q}{\alpha}\right|$$

is proportional to the quantum parameter κ.

The Euclidean action $I=\beta E-S_{tot}$, in the main approximation ($\kappa\to0$, $\phi_B\to-\infty$) $I=-2\kappa(\phi_+-\phi_B)<0$. Thus, if a black hole solution exists, it is a favorable phase and is stable not only locally, but also globally. For $q=s=0$, $r_+=2r_0$ and we return to an exactly solvable model.

It is instructive to discuss case (1) in more detail to reveal the role of the finiteness of a system in the issue of stability. Let us suppose, for a moment, that we proceed in an infinite space from the very beginning and, substituting $f=1$ at infinity in Eq. (67), identify $T=T_H$. The formula (20) can be rewritten, in the main approximation with respect to κ, as

$$T_H=T_0\left(1+\kappa\frac{T_0\delta}{M_{BH}}\right),\delta=q. \tag{74}$$

Then, direct differentiation gives us

$$C=\left(\frac{dT_H}{dM_{BH}}\right)^{-1}=-\frac{M_{BH}^2}{\kappa qT_0^2}=-\frac{4\exp(-4\phi_B)}{\kappa q}. \tag{75}$$

It would seem that the sign of the coefficient δ is crucial in that it determines the sign of the heat capacity and stability or instability of the canonical ensemble. In particular, direct application of Eq. (75) or Eq. (18) of [5] to the CGHS model (for which $q>0$) leads to the conclusion about instability [6].

However, such consideration does not exhaust all possible solutions for case (1). Formally, the quantity (75) can be obtained from the first equality in Eq. (73), if the term $r^2=\exp(-4\phi_+)$ is finite, whereas $s=\kappa q\exp(-2\phi_B)$ grows for $\phi_B\to-\infty$. Then r^2 can be neglected in the denominator. Meanwhile, the point is that in case (1) there exist *two* different roots. When a size of a system is large ($-\phi_B\gg1$), in this limit $r_0^2\gg s$. Correspondingly, $r_+\simeq2r_0$, $r_-\simeq s/2r_0=\kappa q/2\alpha$. Thus the root r_- does not depend on ϕ_B in the limit of large $|\phi_B|$, when the boundary is placed in the nearly flat region, whereas the root r_+ itself grows, as it follows from Eq. (72). Therefore the inequality $r^2\ll s$ is valid only for r_-, but not for r_+. As a result, the prediction of the negative heat capacity on the basis of Eq. (75) refers to the root r_- only which is indeed unstable. However, for the root r_+ the horizon radius $\phi_+=\phi_B+\mathrm{const}$ approaches infinity in the same manner as ϕ_B does. Therefore, Eq. (75), the derivation of which tacitly assumes that $|\phi_+|$ is finite, while $|\phi_B|\gg 1$, does not work now. One is forced to use Eq. (73), not discarding the r^2 term in the denominator, whence it is seen that the root is indeed *stable*. In the limit under discussion the heat capacity $C=4\exp(-2\phi_B)$ looks very much like in the case of exactly solvable models (68) in spite of the fact that now $q\neq 0$. In the limit $\phi_B\to-\infty$ the heat capacity diverges, but for any large but finite ϕ_B it is finite and positive.

To a great extent, the situation resembles the one for Schwarzschild black holes in the canonical ensemble [4]. Naive application of the formula for the Hawking temperature $T=(8\pi M)^{-1}$ would give the heat capacity $C=dM/dT=-8\pi M^2<0$ with the conclusion about instability. However, thorough treatment showed that, for a given physical temperature T on the boundary, there exists two different positions of the horizon as roots r_+ and r_- of Eq. (67). The light root $r_-<r_+$ has the horizon radius $2M=(4\pi T)^{-1}$, when the radius of the boundary $r_B\to\infty$, and for it the calculation of the heat capacity in an infinite space is justified with the conclusion about instability of the solution. But the heavy root r_+ itself tends to r_B, when $r_B\to\infty$ and one cannot apply to it formulas in an infinite space, ignoring the boundary. Careful treatment shows that for this root the ensemble is stable [4]. In our 2D system also it is the "heavy" root which is stable, whereas the "light" one is unstable. In both cases (for our system and for Schwarzschild black holes) one loses the heavy solution, which is most important physically, if the presence of the boundary is ignored. On the other hand, the difference between these two situations lies in that this effect manifests itself in [4] on the pure classical level, while for our case it is relevant only if quantum back reaction is taken into account.

VI. SUMMARY

We considered a generic string-inspired gravitation-dilaton model that is characterized by numeric parameters, for particular values of which a model becomes exactly solvable. Quantum corrections to the Hawking temperature of a black hole in an infinite space are found. We analyzed also how the presence of a finite size cavity affects thermal properties of a black hole. Two types of different boundary conditions are considered—microcanonical and canonical ones. In the first case there is a perfectly reflecting shell that fixes the energy inside. We found how vacuum polarization outside the shell affects thermodynamics of a black hole inside the shell and calculated the corrections to the Hawking temperature due to the shell. As a by-product, it turned out that the shell cannot be placed as near to the horizon as one likes. In the second case the outer space is removed and is replaced by the heat bath. It is shown that the canonical ensemble is well-defined and stable in the wide region of parameters. Accounting for the finiteness of the system is important to the extent that in some cases it alters the conclusion about instability (typical of consideration in an infinite space) and gives stable solutions. In so doing, quantum back reaction is also important. In particular, the type of solution is found which exists only due to such a back reaction.

The indirect dependence of black hole thermodynamics on vacuum polarization outside a shell should also be relevant for 4D black holes [19]. In this respect 2D dilaton gravity revealed itself one more time as a clear and simplified tool for understanding overlap between quantum theory and gravitation that occurs in our real world.

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