2+1 gravity and doubly special relativity

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It is shown that gravity in 2+1 dimensions coupled to point particles provides a nontrivial example of doubly special relativity (DSR). This result is obtained by interpretation of previous results in the field and by exhibiting an explicit transformation between the phase space algebra for one particle in 2+1 gravity found by Matschull and Welling and the corresponding DSR algebra. The identification of 2+1 gravity as a DSR system answers a number of questions concerning the latter, and resolves the ambiguity of the basis of the algebra of observables. Based on this observation a heuristic argument is made that the algebra of symmetries of ultra high energy particle kinematics in 3+1 dimensions is described by some DSR theory.

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I. INTRODUCTION

Recently a proposal has been much discussed concerning how quantum theories of gravity may be tested experimentally. The *doubly* or *deformed special relativity* proposal (DSR)[1] is that quantum gravity effects may lead in the limit of weak fields to modifications in the kinematics of elementary particles characterized by [5–9]

(1) Preservation of the relativity of inertial frames.

(2) Nonlinear modifications of the action of Lorentz boosts on energy-momentum vectors, preserving a preferred energy scale, which is naturally taken to be the Planck energy, E_p. In some cases E_p is a maximum mass and/or momentum that a single elementary particle can attain.

(3) Nonlinear modifications of the energy-momentum relations, because the function of E and $\vec{p}$ that is preserved under the exact action of the Lorentz group is no longer quadratic. This could result in Planck scale effects such as an energy-dependent speed of light and modifications of thresholds for scattering, that may be observable in present and near future experiments.

(4) Modifications in the commutators of coordinates and momentum and/or non-commutativity of space-time coordinates.

Theories with these characteristics are invariant under modifications of the Poincaré algebra, called generically κ-Poincaré algebras, where κ is a dimensional parameter that measures the deformations, usually taken to be proportional to the Planck mass.

In a recent paper [10], it was argued that quantum gravity in 2+1 dimensions [11,12] with vanishing cosmological constant must be invariant under some version of a κ-Poincaré symmetry. The argument there depends only on the assumption that quantum gravity in 2+1 dimensions with the cosmological constant $\Lambda=0$ must be derivable from the $\Lambda\to 0$ limit of 2+1 quantum gravity with nonzero cosmological constant. The argument is simple and algebraic, the point is that the symmetry which characterizes quantum gravity in 2+1 dimensions with $\Lambda>0$ is actually quantum deformed de Sitter to $SO_q(3,1)$, with the quantum deformation parameter q given by [14,15,17]

$$z=\ln(q)\approx l_{Planck}\sqrt{\Lambda}. \tag{1}$$

The limit $\Lambda\to 0$ then affects both the scaling of the translation generators as the de Sitter group is contracted to the Poincaré group, and the limit of $q\to 1$. It is easy to see that because the ratio $\kappa=\hbar\sqrt{\Lambda}/z=G_{2+1}^{-1}$, where G^{2+1} is Newton's constant in 2+1 dimensions, is held fixed, the limit gives the κ-deformed symmetry group in 2+1 dimensions. The conclusion is that the symmetry algebra of (2+1)-dimensional quantum gravity with $\Lambda=0$ is not Poincaré, it is a κ-deformed Poincaré algebra. This means that the theory must be a DSR theory.

Quantum gravity in 2+1 dimensions has been the subject of much study in both the classical and quantum domain,

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[1]Aspects of DSR theories have been proposed or studied more than once in the past, only to be forgotten and then rediscovered again. Early formulations were by Snyder [1] and Fock [2]. During the 1990s the mathematical side of the subject was developed under the name of κ-Poincaré symmetry [3,4]. The recent interest is due to the proposal that the effects of such theories may be both testable and derivable from some versions of quantum gravity, see for example [5–9].

beginning with the work of Deser, Jackiw and 't Hooft [12–23]. If that theory is a DSR theory than the features just listed above must be present, and this could not have been easily missed by investigators.

Indeed, *all of the listed features have been seen in the literature on* $2+1$ *gravity.* In the next section we review some of the long standing results in $2+1$ gravity and show how they may be understood using the language of DSR theories. To clinch the relation, in section III we exhibit an explicit mapping between the phase space of quantum gravity in $2+1$ dimensions coupled to a single point particle, studied in [21], and the algebra of symmetry generators of a DSR theory.

The observation that $2+1$ gravity provides examples of DSR theories can help the study of both sides of the relation. The language of DSR theories and their foundations in terms of general principles can unify and explain some results in the literature of $(2+1)$-dimensional gravity that, when first discovered, seemed strange and unintuitive. We can now see that some of the features of $2+1$ gravity are neither strange nor necessarily unique to $2+1$ dimensions, because they follow only from the general requirement that the transformations between different inertial frames preserve an energy scale.

Furthermore, what one has in the $2+1$ gravity models, such as those with gravity coupled to N point particles, is a class of nontrivial DSR theories that are completely explicit and solvable, both classically and quantum mechanically. The existence of these examples answers a number of questions and challenges that have been raised concerning DSR theories. Some authors have argued [24] that DSR theories are just ordinary special relativistic theories rewritten in terms of some nonlinear combinations of energy and momentum, while, conversely, others have argued that they must be trivial because interactions cannot be consistently included. Both criticisms are shown wrong by the existence of an explicit and solvable class of DSR theories, with interactions, given by quantum gravity in $2+1$ dimensions coupled to point particles and fields.

Furthermore, we see that in $2+1$ dimensions the apparent problem of the freedom to choose the basis of the symmetry algebra of a DSR theory is resolved by the fact that the choice of the coupling of matter to the gravitational field picks out the physical energy and momentum. We see in Sec. III below that for the case of minimal coupling of gravity to a single point particle the basis picked out is the classical basis.

Finally, one can ask whether the fact that $2+1$ gravity is a DSR theory has any implications for real physics in $3+1$ dimensions. In the final section of the paper we present a heuristic argument that it may.

II. SIGNS OF DSR IN 2+1 GRAVITY

In this section we point out where effects characteristic of DSR have been discovered already in the literature on $(2+1)$-dimensional gravity. We consider only the case $\Lambda=0$.

It is important first to note that Newton's constant in $2+1$ dimensions, denoted here by G, has dimensions of inverse mass (with only $c=1$ and no $\hbar$ involved).[2] Thus, if the asymptotic symmetry group knows about gravity, it will have to preserve the scale G^{-1}. Of course, in theories with sufficiently short range interactions the asymptotic symmetry group does not depend on the coupling constants. But in $2+1$ gravity the presence of matter causes the geometry of spacetime to become conical and this deforms the asymptotic conditions in a way that depends on G. Further, since $\hbar$ is not involved in the definition of the mass scale, G^{-1}, the deformation affects also the algebra of the classical phase space. This is the main reason why $2+1$ gravity is a DSR theory.

In $2+1$ gravity coupled to point particles, the Hamiltonian, H, whose value is equal to the ADM mass, and hence is measured by a surface term, is bounded from both above and below [21,23],

$$0\leqslant H\leqslant\frac{1}{4G}. \tag{2}$$

This can be understood in the following way. In $2+1$ dimensions the spacetime is flat, except where matter is present. A particle, or in fact any compactly supported distribution of matter, is surrounded by an asymptotic region, which is locally flat, and whose geometry is thus characterized by a deficit angle α. A standard result is that [12,16–18,21,23],

$$\alpha=8\pi GH. \tag{3}$$

But a deficit angle α must be less than or equal to 2π. Hence there is an upper limit on the mass of any system, as measured by the Hamiltonian. The upper limit holds for all systems, regardless of how many particles there are and what their relative positions or motions are.

This upper mass limit must be preserved by the asymptotic symmetry group. Hence the asymptotic symmetry group cannot be the ordinary Poincaré group, if it include boosts it must be a DSR theory with a maximum energy.[3]

It has further been shown that the spatial components of momentum of a particle in $2+1$ gravity are unbounded [21]. This, together, with a bounded energy, implies a modified energy-momentum relation.

The phase space of a single point particle in $2+1$ gravity was constructed by Matschull and Welling in [21] and it was found that a classical solution is labeled by a three dimensional position $\mathcal{Y}_\mu$, $\mu=0,1,2$ and momentum p_μ. They find explicitly that the energy momentum relations and the action of the Poincaré symmetry are deformed, in a way that pre-

[2] G is identified with inverse of the κ deformation parameter of κ-Poincaré algebra.

[3] Note that in the approach of Matschull and Louko [22,23] which anchor the reference frame to the conical infinity, the asymptotic symmetry group is the two dimensional group of isometry of a conical space time and it does not contain boosts. However the results concerning the phase space structure of the relative motion of particles, like noncommutativity of positions and curved and unbounded space of momenta still hold in this approach.

serves a fixed energy scale. They indeed make explicit reference to the work of Snyder [1], which was an early proposal for DSR.

Furthermore, Matschull and Welling find that the spacetime coordinates $\mathcal{Y}_\mu$ of a particle are noncommutative under the classical Poisson brackets,

$$[\mathcal{Y}_\mu,\mathcal{Y}_\nu]=-2G\epsilon_{\mu\nu\rho}\mathcal{Y}_\rho. \tag{4}$$

This property was found in [23] to extend to systems of N particles.

Matschull and Welling also find that the components of the energy-momentum vector for a point particle in 2+1 gravity live on a curved manifold, which is (2+1)-dimensional anti–de Sitter spacetime. This was shown in [29,31] to be a feature of DSR theories.[4]

Ashtekar and Varadarajan [16] found a relationship between two definitions of energy relevant for 2+1 gravity, which is reminiscent of nonlinear redefinitions of the energy used in changing bases between different realizations of DSR theories. The case they studied has to do with 3+1 gravity, with two Killing fields, one rotational and one axial. One first dimensionally reduces to 2+1 dimensions, in which case the dynamics of GR in 3+1 is expressed as a scalar field coupled to (2+1)-dimensional GR. The ADM Hamiltonian H still exists and still is bounded from above as in Eq. (2). But in the presence of the additional, rotational Killing field, the theory can be represented by a scalar field evolving in a flat reference Minkowski spacetime, with the ordinary Hamiltonian

$$H_{flat}=\frac{1}{2}\int_0^\infty drr[\dot{\phi}^2+(\partial_r\phi)^2]. \tag{5}$$

H_{flat} is of course unbounded above. They find the relationship between them is

$$H=\frac{1}{4G}(1-e^{-4GH_{flat}}). \tag{6}$$

This exact relation is in fact present in the literature on DSR [29]. It holds in the a presentation of the κ-Poincaré algebra known as the "bicrossproduct" basis. In that case $H_{flat}=E$ is, as in the present case, the zeroth component of an energy momentum vector and H is the "physical rest mass, m_0 defined by

$$\frac{1}{m_0}=\lim_{p\to 0}\frac{1}{p}\left.\frac{dE}{dp}\right|_{E=p_0}. \tag{7}$$

It is intriguing that this is the inertial mass, while, for the solutions with rotational symmetry, the ADM energy is the active gravitational mass. Since they are both expressed in terms of the zeroth component of the energy-momentum vector by the same equation, they coincide on the subset of solutions on which they are both defined, which are the rotationally invariant solutions. This appears to be a direct demonstration of the equality of inertial and gravitational mass, within this context.

Indeed, this observation suggests that the Ashtekar-Varadarajan form of the ADM mass is more general than their calculation shows. Indeed it is not hard to see that this is the case. Let us study the free scalar field in (2+1)-dimensional Minkowski spacetime, with no condition of rotational symmetry. This system is *not* a dimensional reduction of general relativity, only a subspace of solutions, those with rotational symmetry, are related to general relativity. But it still may serve as a useful example of a DSR theory. Of course the theory has full Poincaré invariance, with momentum generators P_i and boost generators K_i satisfying the usual Poincaré algebra. But Eq. (6) implies that they form with H a DSR algebra

$$\{K_i,H\}=(1-4GH)P_i$$

$$\{K_i,P_j\}=-\frac{1}{4G}\delta_{ij}\ln[1-4GH] \tag{8}$$

with the other commutators undeformed. The physical energy momentum relations are deformed to

$$P_i^2+m^2=\frac{1}{16G^2}[\ln(1-4GH)]^2. \tag{9}$$

Recent calculations [26] indicate that quantum deformations of symmetries play a role in gravitational scattering of particles in 2+1 dimensions.

All of these pieces of evidence show that 2+1 gravity coupled to matter can be understood as a DSR system.

Of course, the (2+1)-dimensional model system is not completely analogous to real physics in 3+1 dimensions. But this result answers cleanly several queries and criticisms that have been levied against the DSR proposal.

First, some authors have suggested that DSR theories are physically indistinguishable from ordinary special relativity [24]. They argue that in some cases, one can arrive at a DSR system from a nonlinear mapping of energy-momentum space to itself. These results show that argument fails, for there is no doubt that the model system of point particles in 2+1 gravity is physically distinguishable from the model system of free particles in flat (2+1)-dimensional spacetime. This is here a clean result, with no quantization ambiguities, because the deformation parameter $\kappa=1/4G$ is entirely classical and the modification is of the structure of the classical phase space. The two phase spaces are not isomorphic, when gravity is turned on, the phase space is curved and the mass as a maximum, but when $G=0$ the phase space is flat and the mass has no bound.

This is clear also for the multiparticle system, where there are nontrivial interactions, depending on G, which make the system measurably distinct from the free particle case with $G=0$.

[4]Although in Refs. [29,31] the momentum space for a class of DSR theories was shown to be de Sitter spacetime. We discuss below the difference between positively and negatively curved momentum spaces.

The multiparticle system in 2+1 gravity also serves as an example of a counterintuitive property of some DSR models in 3+1 gravity. This is that the upper mass limit $M_{upper} = 1/4G$ is independent of the number of particles in the system. This of course cannot be the case in the real world, so it is good to know that there are implementations of DSR in 3+1 dimensions that do not have an upper mass limit for systems of many particles, or where the upper mass limit grows with the number of particles or the mass of the total system, in such a way as to not violate experience [25].

However, it is also good to know that there is a model system, which is sensible physically, in which this nonintuitive feature is completely realized. Moreover it suggests the start of a physical answer to one of the puzzling questions about DSR models. This is that the addition of energy and momentum in DSR theories is nonlinear. This can be understood as a consequence of the nonlinear action of the Lorentz group, for example it follows from the fact that the energy-momentum space has nonzero curvature. It appears to remain even in realizations of DSR that remove the mass limit for composite systems.

Some physicists have criticized the DSR proposal by pointing out that the nonlinear corrections to addition of energy-momentum vectors for a system of two particles can be interpreted by saying that there is a binding energy between pairs of particles that does not depend on the distance between them, but depends only on the individual energies and momenta.

This may be counter-intuitive, but it is precisely the what happens in 2+1 dimensions. Because spacetime is locally flat, each particle contributes a deficit angle to the overall geometry that affects all the other particles' motions, no matter how far away. The result is that there is a binding energy that is independent of distance.

This suggests a speculative remark: might there be even in 3+1 dimensions a small component of the binding energy of pairs of particles, of order $l_p M_1 M_2$, which is independent of distance? Might this be interpreted as a kind of quantum gravity effect?

In the last section we make some speculative remarks concerning the question of whether these results have any bearing on real physics in 3+1 dimensions.

III. PHASE SPACE OF DSR IN 2+1 DIMENSIONS

In this section we will compare the phase space of (2+1)-dimensional DSR with that of (2+1)-dimensional gravity with one particle. Let us start with the former.

A. Phase spaces of DSR

As in 3+1 dimensions, the starting point to find the phase space of DSR theory in the (2+1)-dimensional case is the (2+1)-dimensional κ-Poincaré algebra [4], the quantum algebra whose generators are momenta[5] $p_\mu = (p_0, p_i)$ and Lorentz algebra generators $J_\mu = (M, N_i)$ boosts. Taking the co-algebra of the κ-Poincaré quantum algebra and using the so-called "Heisenberg-double construction" [27–29] it is possible to derive the position variables, conjugate to momenta, x_μ, as well as the brackets between them and the κ-Poincaré algebra generators.

This (quantum) algebraic construction has a geometrical counterpart, described in [30,31]. Here the manifold on which momenta live is de Sitter space (in the case at hand the 3 dimensional one). The positions and the Lorentz transformations are symmetries acting on the space of momenta. Thus they form the three dimensional de Sitter algebra $SO(3,1)$. It is convenient to define the de Sitter space of momenta as a three dimensional surface

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 = \kappa^2 \tag{10}$$

in the four dimensional Minkowski space with coordinates $(\eta_0, \ldots \eta_3)$. The physical momenta p_μ are then the coordinates on the surface (10). This means that we can think of $\eta_A = \eta_A(p_\mu)$ as of the given functions of momenta, for which Eq. (10) is identically satisfied. In the DSR terminology, the choice of a particular coordinate system on de Sitter space corresponds to a choice of the so called DSR basis (see [29,31]). It turns out that in order to relate DSR to the 2+1 gravity one has to choose the so called classical basis, characterized by $\eta_\mu = p_\mu$. This choice will be implicit below, however we find it more convenient to write down the formulas below in terms of the variables η_A.

The algebra of symmetries of the de Sitter space of momenta (10) can be most easily read off by writing down the action of these symmetries on the four-dimensional Minkowski space with coordinates η_A and then pulling them down to the surface (10). Let us note however that while it is easy to identify the Lorentz generators $J_\mu = (M, N_i)$ as the elements of the $SO(2,1)$ subalgebra of the $SO(3,1)$, it is a matter of convenience which linearly independent combination of generators is to be identified with positions (i.e. the generators of translation in momentum space). Technically speaking we are free to choose the decomposition of $SO(3,1)$ into the sum of $SO(2,1)$ and its remainder.

In the case of the DSR phase space, the action of the symmetries is given by

$$[M, \eta_i] = \epsilon_{ij}\eta_j, \quad [N_i, \eta_j] = \delta_{ij}\eta_0, \quad [N_i, \eta_0] = \eta_i, \tag{11}$$

$$[J_\mu, \eta_3] = 0, \tag{12}$$

with J_μ satisfying the algebra

$$[M, N_i] = \epsilon_{ij}N_j, \quad [N_i, N_j] = -\epsilon_{ij}M \tag{13}$$

$$[x_0, \eta_3] = \frac{1}{\kappa}\eta_0, \quad [x_0, \eta_0] = \frac{1}{\kappa}\eta_3, \quad [x_0, \eta_i] = 0, \tag{14}$$

$$[x_i, \eta_3] = [x_i, \eta_0] = \frac{1}{\kappa}\eta_i, \quad [x_i, \eta_j] = \frac{1}{\kappa}\delta_{ij}(\eta_0 - \eta_3). \tag{15}$$

[5]The Greek indices run from 1 to 3, the Latin ones from 1 to 2, while the capital ones from 0 to 3.

Note that it follows from these equations that

$$[x_0,x_i]=-\frac{1}{\kappa}x_i, \quad [x_i,x_j]=0. \tag{16}$$

It is worth mentioning also that such a decomposition is possible in any dimension. In particular in the 3+1 case the bracket (16) describes the so-called κ-Minkowski type of non-commutativity.

One can repeat this geometric construction in the case when the momenta manifold is the anti–de Sitter space

$$-\eta_0^2+\eta_1^2+\eta_2^2-\eta_3^2=\kappa^2. \tag{17}$$

Now the symmetry algebra is $SO(2,2)$, having again the three dimensional Lorentz algebra $SO(2,1)$ described by Eqs. (11), (12) as its subalgebra. The algebra of positions, which we denote y_μ (i.e. translations of momenta) changes only slightly and now reads

$$[y_0,\eta_3]=-\frac{1}{\kappa}\eta_0, \quad [y_0,\eta_0]=\frac{1}{\kappa}\eta_3, \quad [y_0,\eta_i]=0, \tag{18}$$

$$[y_i,\eta_3]=-\frac{1}{\kappa}\eta_i, \quad [y_i,\eta_0]=\frac{1}{\kappa}\eta_i,$$

$$[y_i,\eta_j]=\frac{1}{\kappa}\delta_{ij}(\eta_0-\eta_3), \tag{19}$$

From Eqs. (18), (19) it follows that

$$[y_0,y_i]=-\frac{1}{\kappa}y_i+\frac{1}{\kappa^2}N_i, \quad [y_i,y_j]=-\frac{2}{\kappa^2}\epsilon_{ij}M. \tag{20}$$

We see that the bracket (20) does not describe the κ-Minkowski type of noncommutativity. Since the noncommutativity type is related to the co-algebra structure of the quantum Poincaré algebra, this result indicates that along with the κ-Poincaré algebra there exists another quantum Poincaré algebra with the same algebra, but different co-algebra, which we expect to be related to the former by a twist.[6]

B. Phase space of 2+1 gravity

The phase space algebra of one particle in (2+1)-dimensional gravity is the algebra of asymptotic charges. This algebra has been carefully analyzed by Matschull and Welling in [21]. They find that the physical momentum manifold is anti–de Sitter space and that $\eta_\mu=p_\mu$, as stated above. This means that 2+1 gravity seems to pick the classical basis of DSR as the one having physical relevance. Further, Matschull and Welling employ a particular decomposition of the $SO(3,1)$ algebra, in which the positions $\mathcal{Y}_\mu$ act on momenta as right multiplication and have the following brackets with η_A:

$$[\mathcal{Y}_0,\eta_3]=-\frac{1}{\kappa}\eta_0, \quad [\mathcal{Y}_0,\eta_0]=\frac{1}{\kappa}\eta_3,$$

$$[\mathcal{Y}_0,\eta_i]=-\frac{1}{\kappa}\epsilon_{ij}\eta_j, \tag{21}$$

$$[\mathcal{Y}_i,\eta_3]=-\frac{1}{\kappa}\eta_i, \quad [\mathcal{Y}_i,\eta_0]=\frac{1}{\kappa}\epsilon_{ij}\eta_j,$$

$$[\mathcal{Y}_i,\eta_j]=\frac{1}{\kappa}(\epsilon_{ij}\eta_0-\delta_{ij}\eta_3). \tag{22}$$

Comparing the expressions (18), (19) with Eqs. (21), (22) we easily find that these decompositions are related by

$$\mathcal{Y}_0=y_0-\frac{1}{\kappa}M, \quad \mathcal{Y}_i=y_i-\frac{1}{\kappa}(N_i-\epsilon_{ij}N_j). \tag{23}$$

It can be also easily checked that

$$[\mathcal{Y}_\mu,\mathcal{Y}_\nu]=-\frac{2}{\kappa}\epsilon_{\mu\nu\rho}\mathcal{Y}^\rho. \tag{24}$$

Thus the DSR anti–de Sitter phase space is (up to a trivial reshuffling of the generators) equivalent to the phase space of a single particle in 2+1 gravity.

It is an open problem whether one can get de Sitter space as a manifold of momenta from 2+1 quantum gravity. It would be interesting to see if this is the case. If so, there exist two kinds of phase spaces of a particle in a (2+1)-gravitational field corresponding to two DSR phase space algebras presented above.

IV. IMPLICATIONS FOR PHYSICS IN 3+1 DIMENSIONS

We present here an argument that suggests that the results of this paper, and of those we reference, concerning (2+1)-dimensional quantum gravity coupled to point particles may have implications for real physics in 3+1 dimensions.

The main idea is to construct an experimental situation that forces a dimensional reduction to the (2+1)-dimensional theory. It is interesting that this can be done in quantum theory, using the uncertainty principle as an essential element of the argument.

Let us consider a system of two relativistic interacting elementary particles in 3+1 dimensions, whose masses are less than G^{-1}. In the center of mass frame the motion will be planar. Let us consider the system as described by an inertial observer who travels perpendicular to the plane of the system's motion, which we will call the z direction. From the point of view of that observer, the system is in an eigenstate of total longitudinal momentum, $\hat{P}_z^{total}$, with some eigenvalue P_z. Since the system is in an eigenstate of $\hat{P}_z^{total}$ the wavefunction of the center of mass will be uniform in z.

[6]This expectation is based on the classification of Poisson structures on Poincaré group presented in [32].

Further, since there was initially zero relative momentum between the particles in the z direction it is also true in the observers frame that

$$P_z^{rel}=P_z^1-P_z^2=0. \tag{25}$$

This implies of course $P_z^1=P_z^2=P_z^{total}/2$. Then the above applies as well to each particle, i.e. their wave functions are uniform in the $\hat{z}$ direction as their wave functions have wavelength $2L$ where

$$L=\frac{\hbar}{P_z^{total}}. \tag{26}$$

At the same time, we assume that the uncertainties in the transverse positions are bounded a scale r, such that $r \ll 2L$.

Then the wave functions for the two particles have support on narrow cylinders of radius r which extend uniformly in the z direction.

Finally, we assume that the state of the gravitational field is semiclassical, so that to a good approximation, within $\mathcal{C}$ the semiclassical Einstein equations hold

$$G_{ab}=8\pi G\langle \hat{T}_{ab}\rangle. \tag{27}$$

Note that we do not have to assume that the semiclassical approximation holds for all states. We assume something much weaker, which is that there are subspaces of states in which it holds.

Since the wave functions are uniform in z, and since we are interested in the particle kinematics in flat space we assume that the dynamical degrees of freedom of the gravitational field are switched off, this implies that the gravitational field seen by our observer will have a spacelike Killing field $k^a=(\partial/\partial z)^a$.

Thus, if there are no forces other than the gravitational field, the scattering of the two particles described semiclassically by Eq. (27) must be the same as that of two parallel cosmic strings. This is known to be described by an equivalent (2+1)-dimensional problem in which the gravitational field is dimensionally reduced along the z direction so that the two "cosmic strings" which are the sources of the gravitational field, are replaced by two punctures.

The dimensional reduction is governed by a length d, which is the extent in z that the system extends. We cannot take $d<L$ without violating the uncertainty principle. It is then convenient to take $d=L$. Further, since the system consists of elementary particles, they have no intrinsic extent, so there is no other scale associated with their extent in the z direction. We can then identify $z=0$ and $z=L$ to make an equivalent toroidal system, and then dimensionally reduce along z. The relationship between the four dimensional Newton's constant G^{3+1} and the three-dimensional Newton's constant $G^{2+1}=G$, which played a role so far in this paper is given by

$$G^{2+1}=\frac{G^{3+1}}{2L}=\frac{G^{3+1}P_z^{tot}}{2\hbar}. \tag{28}$$

Thus, in the analogous (2+1)-dimensional system, which is equivalent to the original system as seen from the point of view of the boosted observer, the Newton's constant depends on the longitudinal momenta.

Of course, in general there will be an additional scalar field, corresponding to the dynamical degrees of freedom of the gravitational field. We will for the moment assume that these are unexcited, but exciting them will not affect the analysis so long as the gravitational excitations are invariant also under the killing field and are of compact support.

Now we note that, if there are no other particles or excited degrees of freedom, the energy of the system can to a good approximation be described by the Hamiltonian H of the two dimensional dimensionally reduced system. This is described by a boundary integral, which may be taken over any circle that encloses the two particles. But this is bounded from above, by Eq. (2). This may seem strange, but it is easy to see that it has a natural four-dimensional interpretation.

The bound is given by

$$M<\frac{1}{4G^{2+1}}=\frac{2L}{4G^{3+1}} \tag{29}$$

where M is the value of the ADM Hamiltonian, H. But this just implies that

$$L>2G^{3+1}M=R_{Sch} \tag{30}$$

i.e. this has to be true, otherwise the dynamics of the gravitational field in 3+1 dimensions would have collapsed the system to a black hole. Thus, we see that the total bound from above of the energy in 2+1 dimensions is necessary so that one cannot violate the condition in 3+1 dimensions that a system be larger than its Schwarzschild radius.

Note that we also must have

$$M>P_z^{tot}=\frac{\hbar}{L}. \tag{31}$$

Together with Eq. (30) this implies $L>l_{Planck}$, which is of course necessary if the semiclassical argument we are giving is to hold.

Now, we have put no restriction on any components of momentum or position in the transverse directions. So the system still has symmetries in the transverse directions. Furthermore, the argument extends to any number of particles, so long as their relative momenta are coplanar. Thus, we learn the following.

Let $\mathcal{H}^{QG}$ be the full Hilbert space of the quantum theory of gravity, coupled to some appropriate matter fields, with $\Lambda=0$. Let us consider a subspace of states $\mathcal{H}^{weak}$ which are relevant in the low energy limit in which all energies are small in Planck units. We expect that this will have a symmetry algebra which is related to the Poincaré algebra $\mathcal{P}^{3+1}$ in 3+1 dimensions, by some possible small deformations parameterized by G^{3+1} and $\hbar$. Let us call this low energy symmetry group $\mathcal{P}_G^{3+1}$.

Let us now consider the subspace of $\mathcal{H}^{weak}$ which is described by the system we have just constructed. It contains two particles, and is an eigenstate of $\hat{P}_z^{tot}$ with large P_z^{tot} and vanishing relative longitudinal momenta. Let us call this subspace of Hilbert space $\mathcal{H}_{P_z}$.

The conditions that define this subspace break the generators of the (possibly modified) Poincaré algebra that involve the z direction. But they leave unbroken the symmetry in the (2+1)-dimensional transverse space. Thus, a subgroup of $\mathcal{P}_G^{3+1}$ acts on this space, which we will call $\mathcal{P}_G^{2+1} \subset \mathcal{P}_G^{3+1}$.

We have argued that the physics in $\mathcal{H}_{P_z}$ is to good approximation described by an analogue system in of two particles in 2+1 gravity. However, we know from the results cited in the previous sections that the symmetry algebra acting there is not by the ordinary (2+1)-dimensional Poincaré algebra, but by the κ-Poincaré algebra in 2+1 dimensions, with

$$\kappa^{-1} = \frac{4G^{3+1}P_z^{tot}}{\hbar}. \tag{32}$$

In particular, there is a maximum energy given by

$$M_{max}(P_z^{tot}) = \kappa = \frac{M_{Planck}^2}{4P_z^{total}}. \tag{33}$$

This gives us a last condition,

$$MP_z^{total} < \frac{M_{Planck}^2}{4} \tag{34}$$

which is compatible with the previous conditions. Thus, when all the conditions are satisfied, the deformed symmetry algebra must be identified with $\mathcal{P}_G^{2+1}$.

Now we can note the following. Whatever $\mathcal{P}_G^{3+1}$ is, it must have the following properties:

It depends on G^{3+1} and $\hbar$, so that its action on *each* subspace $\mathcal{H}_{P_z}$, for each choice of P_z, is the κ deformed 2+1 Poincaré algebra, with κ as above.

It does not satisfy the rule that momenta and energy add, on all states in $\mathcal{H}$, since they are not satisfied in these subspaces.

Therefore, whatever $\mathcal{P}_G^{3+1}$ is, it is not the classical Poincaré group.

Thus the theory of particle kinematics at ultra high energies is not special relativity, and the arguments presented above suggest that it might be doubly special relativity.

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