

Large-scale magnetic fields from inflation in dilaton electromagnetism

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The generation of large-scale magnetic fields is studied in dilaton electromagnetism in inflationary cosmology, taking into account the dilaton's evolution throughout inflation and reheating until it is stabilized with possible entropy production. It is shown that large-scale magnetic fields with an observationally interesting strength at the present time could be generated if the conformal invariance of the Maxwell theory is broken through the coupling between the dilaton and electromagnetic fields in such a way that the resultant quantum fluctuations in the magnetic field have a nearly scale-invariant spectrum. If this condition is met, the amplitude of the generated magnetic field could be sufficiently large even in the case that a huge amount of entropy is produced with the dilution factor $\sim 10^{24}$ as the dilaton decays.

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I. INTRODUCTION

It is well known that magnetic fields are present on various scales in the Universe, from planets, stars, galaxies, to clusters of galaxies (for recent detailed reviews see [1–3]). The origin of the cosmic magnetic fields, however, is not well understood yet. Since they have direct influence not only on various astrophysical situations but also on the evolution of the Universe, their origin is one of the most important problems in modern cosmology.

In galaxies of all types, magnetic fields with the field strength $\sim 10^{-6}$ G, ordered on 1–10 kpc scale, have been detected [3,4]. There is some evidence that they exist in galaxies at cosmological distances [5]. Furthermore, in recent years magnetic fields in clusters of galaxies have been observed by means of the Faraday rotation measurements (RMs) of polarized electromagnetic radiation passing through an ionized medium [6]. Unfortunately, however, RMs inform us of only the value of the product of the field strength along the line of sight and the coherence scale, and so we cannot know the strength without assuming the value of the coherence scale and vice versa. In general, the strength and the scale are estimated as 10^{-7} – 10^{-6} G and 10 kpc–1 Mpc, respectively. It is very interesting and mysterious that magnetic fields in clusters of galaxies are as strong as galactic ones and that the coherence scale may be as large as \sim Mpc.

Some elaborated magnetohydrodynamical (MHD) mechanisms have been proposed to amplify very weak seed magnetic fields into the $\sim 10^{-6}$ G fields generally observed in galaxies. These mechanisms, known as the *galactic dynamo* [7], are based on the conversion of the kinetic energy of the turbulent motion of a conductive interstellar medium into magnetic energy. Galactic dynamo, however, is only an amplification mechanism, and so requires initial seed magnetic fields to feed on. Moreover, the effectiveness of the dynamo amplification mechanism in galaxies at high redshifts or clusters of galaxies is not well established.

Scenarios for the origin of seed magnetic fields fall into two broad categories. One is astrophysical processes and the other is cosmological physical processes in the early Universe. The former, by and large, exploits the difference in

mobility between electrons and ions. This difference can lead to electric currents and hence magnetic fields. The latter can also generate magnetic fields. Typically, magnetogenesis requires an out-of-thermal-equilibrium condition and a macroscopic parity violation. These conditions could have been naturally provided by the first-order cosmological electroweak phase transition (EWPT) [8] or quark-hadron phase transition (QCDPT) [9] (see more references in the review [2]). The bubbles of new phase were formed in the old one and strong, though small-scale, turbulent motion is excited in the plasma. In standard model, however, EWPT is the second order, so that such bubbles cannot be formed. Furthermore, it has recently been shown by Durrer and Caprini [10] that causally produced stochastic magnetic fields on large scales, e.g., during EWPT or even later, are much stronger suppressed than usually assumed.

If the scale of cluster magnetic fields is as large as \sim Mpc, it is likely that the origin of such a large-scale magnetic field is in physical processes in the early Universe rather than in astrophysical processes. From the two points, (1) there exists magnetic fields with the field strength $\sim 10^{-6}$ G even in the objects where the effectiveness of the dynamo amplification mechanism is not well established, and (2) there is the possibility that the scale of cluster magnetic fields is as big as \sim Mpc, it is conjectured that large-scale strong magnetic fields are produced in the early Universe and then are trapped in the plasma that collapsed to form galaxies and clusters of galaxies through adiabatic compression, or, in addition, secondary amplification mechanism such as galactic dynamo.

Since the conductivity of the Universe through most of its history is large, the magnetic field B evolves conserving magnetic flux as $B \propto a^{-2}$, where $a(t)$ is the scale factor. On the other hand, the average cosmic energy density $\bar{\rho}$ evolves as $\bar{\rho} \propto a^{-3}$ in the matter dominated epoch. Hence $B \propto \bar{\rho}^{2/3}$. The present ratio of interstellar medium density in galaxies ρ_{gal} to $\bar{\rho}$ and that of intercluster medium density in clusters of galaxies ρ_{cg} are $\rho_{\text{gal}}/\bar{\rho} \approx 10^5$ – 10^6 and $\rho_{\text{cg}}/\bar{\rho} \approx 10^2$ – 10^3 , respectively. Consequently, from these relations, we see that the required strength of the cosmic magnetic field at the structure formation, adiabatically rescaled to the present time, is 10^{-10} – 10^{-9} G in order to explain the observed

fields in galaxies $B_{\text{gal}} \sim 10^{-6}$ G and clusters of galaxies $B_{\text{cg}} \sim 10^{-7}$ G. On the other hand, in general, seed fields with the present strength 10^{-22} – 10^{-16} G is required for the galactic dynamo scenario.

Although first-order cosmological phase transitions in the early Universe generate magnetic fields, the comoving coherence length of the magnetic fields cannot be larger than the Hubble horizon at the phase transition, which is much smaller than Mpc today. Though the coherence length may grow due to MHD effects, this happens at the expense of the magnetic field strength.

The most natural mechanism overcoming the large-coherence-scale problem is *inflation* in the early Universe (for a comprehensive introduction to inflation see Refs. [11,12]). Turner and Widrow [13] (TW) first indicated that large-scale magnetic fields could be generated in the inflationary stage. Inflation naturally produces effects on very large scales, larger than Hubble horizon, starting from microphysical processes operating on a causally connected volume. If electromagnetic quantum fluctuations are amplified during inflation, they could appear today as large-scale static magnetic fields. This idea is based on the assumption that a given mode is excited quantum mechanically while it is sub-horizon sized and then as it crosses outside the horizon “freezes in” as a classical fluctuation. However, there is a serious obstacle on the way of this nice scenario as argued below.

It is well known that quantum fluctuations of massless scalar and tensor fields are very much amplified in the inflationary stage and create considerable density inhomogeneities [14] or relic gravitational waves [15]. This is closely related to the fact that these fields are not conformally invariant even though they are massless. The amplification of the quantum fluctuations can be understood as particle production by an external gravitational field. Since the Friedmann-Robertson-Walker (FRW) metric usually considered is known to be conformally flat, the background gravitational field does not produce particles if the underlying theory is conformally invariant [16]. This is the case for photons since the classical electrodynamics is conformally invariant in the limit of vanishing masses of fermions. Hence electromagnetic waves could not be generated in cosmological background. If the origin of large-scale magnetic fields in clusters of galaxies is electromagnetic quantum fluctuations generated and amplified in the inflationary stage, the conformal invariance must have been broken at that time. Several breaking mechanisms have been proposed, which are mainly classified into the following three types.

(1) A nonminimal coupling of electromagnetic fields to gravity: TW introduced the gravitational couplings $RA_{,\mu}A^{,\mu}$, $R_{\mu\nu}A^{,\mu}A^{,\nu}$, $RF_{\mu\nu}F^{\mu\nu}/m^2$, etc., where R is the curvature scalar, $A_{,\mu}$ the electromagnetic potential, $F_{\mu\nu}$ the electromagnetic field-strength tensor, and m a constant with dimension of mass. The RA^2 terms could generate large-scale magnetic fields with interesting strength, but they also break gauge invariance by giving the photon an effective mass. In contrast, the RF^2 terms are theoretically more plausible, but the strength of the resultant magnetic fields is very weak.

(2) A coupling of a scalar field to electromagnetic fields:

TW first indicated the coupling of an axion field, or that of a charged field which is not conformally coupled to gravity. After that many authors have studied more natural and effective couplings.

Ratra [17] suggested the coupling of the inflation-driving scalar field (inflaton) ϕ in the form $e^{\omega\phi}F_{\mu\nu}F^{\mu\nu}$, and calculated the strength of large-scale magnetic fields in a so-called power-law inflation model induced by the exponential potential of the form $e^{\tilde{\omega}\phi}$, where ω and $\tilde{\omega}$ are constant parameters with dimension $(\text{mass})^{-1}$. As a result, he found that present magnetic fields as large as 10^{-10} – 10^{-9} G could be generated. Recently Giovannini [18] discussed the coupling of a massive scalar field φ_m other than the inflaton in the form $(\varphi_m/M_{\text{Pl}})^{\xi}F_{\mu\nu}F^{\mu\nu}$, where ξ is a constant parameter and M_{Pl} is the Planck mass. According to Giovannini, large-scale magnetic fields with the strength larger than the dynamo requirement could be generated.

Garretson, Field, and Carroll analyzed the amplification of electromagnetic fluctuations by their coupling to a pseudo-Goldstone boson (PGB) φ_g in the form $\varphi_g F_{\mu\nu} \tilde{F}^{\mu\nu}$, where $\tilde{F}^{\mu\nu} \equiv 1/2 \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual tensor of $F_{\mu\nu}$, and found that this coupling leads to exponential growth not for superhorizon modes but only for subhorizon modes. Consequently, large-scale magnetic fields with interesting strength could not be generated [19].

Magnetic fields due to a charged scalar field were considered in a special model in Ref. [20] (for more detailed review see [21]). The authors found that stochastic currents could be generated during inflation due to production of charged scalar particles by the inflaton, and in turn, magnetic fields. Moreover, Davis *et al.* argued that the backreaction of the scalar field gives the gauge field an effective mass thus breaking the conformal invariance [22]. According to Davis *et al.*, magnetic fields with the strength of order 10^{-24} G on a scale of 100 pc could be generated.

(3) The conformal anomaly in the trace of the energy-momentum tensor induced by quantum corrections to Maxwell electrodynamics: It is known that the conformal anomaly, which is related to the triangle diagram connecting two photons to a graviton, breaks the conformal invariance by producing a nonvanishing trace of the energy-momentum tensor. Dolgov [23] pointed out that such an effect may lead to strong electromagnetic fields amplification during inflation. According to Dolgov, however, magnetic fields with interesting strength might not be generated in a realistic case, e.g., the model based on SU(5) gauge symmetry with three-fermion families.

Incidentally, it has been indicated by Bertolami and Mota [24] that the conformal invariance might be broken actually due to the possibility of spontaneous breaking of the Lorentz invariance in the context of string theories, and that generated large-scale magnetic fields could be strong enough to explain the observed fields through adiabatic compression.

In the light of the above various suggestions, it seems at present that the most natural and effective way of breaking the conformal invariance is to introduce the coupling of a scalar field to electromagnetic fields. In particular, Ratra’s suggestion is attractive, claiming present magnetic fields as

large as 10^{-10} – 10^{-9} G could be generated in the model, which would not require any dynamo amplification to account for the observed fields in galaxies and clusters of galaxies.

In the present paper, in addition to the inflaton field we assume the existence of the dilaton field and introduce the coupling of it to electromagnetic fields. Such coupling is reasonable in the light of indications in higher-dimensional theories, e.g., string theories. Then we investigate the evolution of electromagnetic quantum fluctuations generated through the coupling, which breaks the conformal invariance of electrodynamics, and estimate the strength of large-scale magnetic fields at the present time. Particularly, we consider the following two cases. One is the case when dilaton freezes at the end of inflation in the same way as Ratra [17] just for comparison, and the other is the more realistic case that it still evolves after reheating and then decays into radiation with or without entropy production.

Here we emphasize the following point. In Ratra's model, the inflaton and the dilaton are identified and power-law inflation is realized by introducing an exponential potential. There is no reason, however, why we should identify these fields. Furthermore, in the standard inflation models inflation is driven by the potential energy of the inflaton as it slowly rolls over the potential hill. This slow roll over quasi-de Sitter stage is practically necessary to account for the nearly scale-invariant spectrum¹ of the primordial curvature perturbation out of the quantum fluctuations of the inflaton.

The rest of this paper is organized as follows. In Sec. II we describe our model action and derive the equations of motion from it. In Sec. III we investigate the evolution of electromagnetic fields, and then estimate the strength of the large-scale magnetic fields at the present time in Sec. IV, where we assume the dilaton freezes at the end of inflation. On the other hand, in Sec. V, we consider the case where the dilaton still evolves after reheating and then decays into radiation with or without entropy production. Although we consider the evolution of electromagnetic fields in slow-roll exponential inflation models in Secs. II–V, for comparison we discuss it in power-law inflation models in Sec. VI keeping the recent WMAP data in mind. Finally, Sec. VII is devoted to discussion and conclusion.

We use units in which $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by κ^2 so that $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ where $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$ GeV is the Planck mass. Moreover, in terms of electromagnetism we adopt Heaviside-Lorentz units. The suffixes i, 1, R, and 0 represent the quantities at the initial time t_i , the time when a given mode first crosses the horizon during inflation t_1 , the end of inflation (namely, the instantaneous reheating stage) t_R , and the present time t_0 , respectively.

¹The spectral index is estimated as 0.99 ± 0.04 by using the first year Wilkinson Microwave Anisotropy Probe (WMAP) data only [25], where the errors are the 68% confidence interval.

II. MODEL

A. Action

We introduce two scalar fields, the inflaton field ϕ , and the dilaton Φ . Moreover, we introduce the coupling of the dilaton to electromagnetic fields. Our model action is given as follows:

$$S = \int d^4x \sqrt{-g} [\mathcal{L}_{\text{inflaton}} + \mathcal{L}_{\text{dilaton}} + \mathcal{L}_{\text{EM}}], \quad (1)$$

$$\mathcal{L}_{\text{inflaton}} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U[\phi], \quad (2)$$

$$\mathcal{L}_{\text{dilaton}} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V[\Phi], \quad (3)$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} f(\Phi) F_{\mu\nu} F^{\mu\nu}, \quad (4)$$

$$f(\Phi) = \exp(\lambda \kappa \Phi), \quad (5)$$

$$V[\Phi] = \bar{V} \exp(-\tilde{\lambda} \kappa \Phi), \quad (6)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $U[\phi]$ and $V[\Phi]$ are the inflaton and dilaton potentials, \bar{V} is a constant, and f is the coupling between the dilaton and electromagnetic fields with λ and $\tilde{\lambda} (> 0)$ being dimensionless constants. The form of the coupling between the dilaton and electromagnetic fields in Eq. (5) and that of the dilaton potential in Eq. (6) can be motivated by higher-dimensional theories reduced to four dimensions.

We assume the spatially flat Friedmann-Robertson-Walker (FRW) spacetime with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\mathbf{x}^2, \quad (7)$$

where $a(t)$ is the scale factor. In terms of the U(1) gauge field A_μ , the electromagnetic field-strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (8)$$

Before going on, we state the framework we adopt in this paper.

(1) During slow-roll inflation the cosmic energy density is dominated by $U[\phi]$ and the energy density of the dilaton is negligible.

(2) The Universe is reheated immediately after inflation at $t = t_R$. See, e.g., Ref. [26] for an efficient mechanism of reheating.

(3) The conductivity of the Universe σ_c is negligibly small during inflation, because there are few charged particles at that time. After reheating a number of charged particles are produced, so that the conductivity immediately jumps to a large value, $\sigma_c \gg H$ ($t \gg t_R$), here H is the Hubble parameter. This assumption is justified by a microphysical analysis [13].

(4) We consider both the case of the dilaton frozen at $t = t_R$ as in Ratra's model [17] and that it continues to evolve after reheating until it decays with or without entropy production and gets stabilized at a potential minimum.

(5) The value of the coupling f between the dilaton and electromagnetic fields is set to unity when the dilaton gets stabilized so that the standard Maxwell theory is recovered, $f=1$.

B. Equations of motion

From the above action in Eq. (1) the equations of motion for the inflaton, the dilaton, and electromagnetic fields can be derived as follows:

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)+\frac{dU[\phi]}{d\phi}=0, \quad (9)$$

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi)+\frac{dV[\Phi]}{d\Phi}=-\frac{1}{4}\frac{df(\Phi)}{d\Phi}F_{\mu\nu}F^{\mu\nu}, \quad (10)$$

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}f(\Phi)F^{\mu\nu})=0. \quad (11)$$

Since we are interested in the specific case where the background spacetime is inflating, we assume that the spatial derivatives of ϕ and Φ are negligible compared to the other terms (if this is not the case at the beginning of inflation, any spatial inhomogeneities will quickly be inflated away and this assumption will quickly become very accurate). Hence the equations of motion for the background homogeneous scalar fields read

$$\ddot{\phi}+3H\dot{\phi}+\frac{dU[\phi]}{d\phi}=0, \quad (12)$$

$$\ddot{\Phi}+3H\dot{\Phi}+\frac{dV[\Phi]}{d\Phi}=0, \quad (13)$$

together with the background Friedmann equation

$$H^2=\left(\frac{\dot{a}}{a}\right)^2=\frac{\kappa^2}{3}(\rho_\phi+\rho_\Phi), \quad (14)$$

$$\rho_\phi=\frac{1}{2}\dot{\phi}^2+U[\phi], \quad (15)$$

$$\rho_\Phi=\frac{1}{2}\dot{\Phi}^2+V[\Phi], \quad (16)$$

where a dot denotes a time derivative. Here ρ_ϕ and ρ_Φ are the energy density of the inflaton and that of the dilaton. Since we have $\rho_\phi \gg \rho_\Phi$ by assumption, during inflation H reads

$$H^2 \approx \frac{\kappa^2}{3}U[\phi] \equiv H_{\text{inf}}^2, \quad (17)$$

where H_{inf} is the Hubble constant in the inflationary stage.

We consider the evolution of the gauge field in this background. Its equation of motion in the Coulomb gauge, $A_0(t, \mathbf{x})=0$ and $\partial_j A^j(t, \mathbf{x})=0$, becomes

$$\ddot{A}_i(t, \mathbf{x}) + \left(H + \frac{\dot{f}}{f} \right) \dot{A}_i(t, \mathbf{x}) - \frac{1}{a^2} \partial_j \partial_j A_i(t, \mathbf{x}) = 0. \quad (18)$$

III. EVOLUTION OF MAGNETIC FIELDS

In this section, we investigate the evolution of the vector potential and then consider that of the electric and magnetic fields.

A. Evolution of vector potential

To begin with, we shall quantize the vector potential $A_i(t, \mathbf{x})$. It follows from the electromagnetic part of our model Lagrangian in Eq. (4) that the canonical momenta conjugate to the electromagnetic potential $A_\mu(t, \mathbf{x})$ are given by

$$\pi_0=0, \quad \pi_i=f(\Phi)a(t)\dot{A}_i(t, \mathbf{x}). \quad (19)$$

We impose the canonical commutation relation between $A_i(t, \mathbf{x})$ and $\pi_j(t, \mathbf{x})$,

$$[A_i(t, \mathbf{x}), \pi_j(t, \mathbf{y})] = i \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right), \quad (20)$$

where \mathbf{k} is comoving wave number, and k denotes its amplitude $|\mathbf{k}|$. From this relation we obtain the expression for $A_i(t, \mathbf{x})$ as

$$A_i(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} [\hat{b}(\mathbf{k})A_i(t, \mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}^\dagger(\mathbf{k})A_i^*(t, \mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}], \quad (21)$$

where $\hat{b}(\mathbf{k})$ and $\hat{b}^\dagger(\mathbf{k})$ are the annihilation and creation operators which satisfy

$$[\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}'),$$

$$[\hat{b}(\mathbf{k}), \hat{b}(\mathbf{k}')] = [\hat{b}^\dagger(\mathbf{k}), \hat{b}^\dagger(\mathbf{k}')] = 0. \quad (22)$$

From now on we choose the x^1 axis to lie along the spatial momentum direction \mathbf{k} and denote the transverse directions x^I with $I=2,3$. From Eq. (18) we find that the Fourier modes of the vector potential $A_i(t, k)$ satisfy the following equation:

$$\ddot{A}_I(t, k) + \left(H_{\text{inf}} + \frac{\dot{f}}{f} \right) \dot{A}_I(t, k) + \frac{k^2}{a^2} A_I(t, k) = 0, \quad (23)$$

and that the normalization condition for $A_i(t, k)$ reads

$$A_i(t, k) \dot{A}_j^*(t, k) - \dot{A}_j(t, k) A_i^*(t, k) = \frac{i}{fa} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right). \quad (24)$$

For convenience in finding the solutions of Eq. (23), we introduce the following approximate form as the expression of f :

$$f(\Phi) = f[\Phi(t)] = f[\Phi(a(t))] \equiv \bar{f} a^{\beta-1}, \quad (25)$$

where \bar{f} is a constant and β a parameter whose time dependence is weak as will be seen in the next section. Using Eq. (25), we find

$$H_{\text{inf}} + \frac{\dot{f}}{f} = \beta H_{\text{inf}}. \quad (26)$$

It follows from Eq. (26) that Eq. (23) is rewritten to the following form by replacing the independent variable t to η :

$$\frac{d^2 A_I(k, \eta)}{d\eta^2} + \left(\frac{1-\beta}{\eta} \right) \frac{dA_I(k, \eta)}{d\eta} + k^2 A_I(k, \eta) = 0, \quad (27)$$

where $\eta = \int dt/a(t)$ is conformal time. During inflation $\eta = -1/(aH_{\text{inf}})$. If we regard β as a constant, the solution is given by

$$A_I(k, \eta) = C_{I+}(k) (-H_{\text{inf}} \eta)^{\beta/2} H_{\beta/2}^{(1)}(-k\eta) + C_{I-}(k) (-H_{\text{inf}} \eta)^{\beta/2} H_{\beta/2}^{(2)}(-k\eta), \quad (28)$$

where $H_\nu^{(n)}$ is a ν th-order Hankel function of type n ($n = 1, 2$), and $C_{I+}(k)$ and $C_{I-}(k)$ are constants which satisfy

$$|C_{I+}(k)|^2 - |C_{I-}(k)|^2 = \frac{\pi}{4H_{\text{inf}} \bar{f}}. \quad (29)$$

We shall choose

$$C_{I+}(k) = \sqrt{\frac{\pi}{4H_{\text{inf}} \bar{f}}} e^{i(\beta+1)\pi/4}, \quad C_{I-}(k) = 0, \quad (30)$$

so that the vacuum reduces to the one in Minkowski space-time at the short-wavelength limit,

$$\frac{k}{aH_{\text{inf}}} = -k\eta \rightarrow \infty, \quad A_I(k, \eta) \rightarrow \frac{1}{\sqrt{2kf}} e^{-ik\eta}. \quad (31)$$

We therefore obtain

$$A_I(k, a) = \sqrt{\frac{\pi}{4H_{\text{inf}} f(a)}} a^{-1/2} H_{\beta/2}^{(1)}\left(\frac{k}{aH_{\text{inf}}}\right) e^{i(\beta+1)\pi/4}. \quad (32)$$

Being interested in large-scale magnetic fields, we investigate the behavior of this solution in the large-wavelength limit. Expanding the Hankel function in Eq. (32) and taking the first leading order in $k/(aH_{\text{inf}})$, we obtain

$$A_I(k, a) = 2^{\beta/2} \sqrt{\frac{1}{4\pi H_{\text{inf}} f(a)}} \Gamma\left(\frac{\beta}{2}\right) a^{-1/2} \left(\frac{k}{aH_{\text{inf}}}\right)^{-\beta/2} e^{i(\beta-1)\pi/4} \propto k^{-\beta/2} a^0 \quad \text{for } \beta > 0, \quad (33)$$

and

$$A_I(k, a) = 2^{-\beta/2} \sqrt{\frac{1}{4\pi H_{\text{inf}} f(a)}} \Gamma\left(-\frac{\beta}{2}\right) a^{-1/2} \left(\frac{k}{aH_{\text{inf}}}\right)^{\beta/2} e^{i(3-\beta)\pi/4} \propto k^{\beta/2} a^{-\beta} \quad \text{for } \beta < 0. \quad (34)$$

The large-scale expansion bifurcates at $\beta=0$. This is the reason for the two different expressions, Eqs. (33) and (34). From Eqs. (33) and (34), we see that the large-scale vector potential is time independent for $\beta > 0$ and evolves like $a^{-\beta}$ for $\beta < 0$. Furthermore the mean square of the large-scale vector potential in the position space is $\sim k^3 |A_I(k, a)|^2 \propto k^{-|\beta|+3}$ on a comoving scale $r = 2\pi/k$, so the root-mean-square (rms) has a scale-invariant spectrum when $|\beta| = 3$.

B. Parameter β

In the preceding section, for convenience we have introduced the form (25) as the expression of f . In practice, how-

ever, it is a function of Φ as seen in (5). We therefore investigate the expression of parameter β by the comparison of Eq. (5) with Eq. (25).

To begin with, we consider the evolution of the scale factor $a(t)$ and the dilaton $\Phi(t)$ in the inflationary stage. The scale factor $a(t)$ is given by

$$a(t) = a_1 \exp[H_{\text{inf}}(t - t_1)], \quad (35)$$

where a_1 is the scale factor at the time t_1 when a given comoving wavelength $2\pi/k$ of the vector potential first crosses outside the horizon during inflation, $k/(a_1 H_{\text{inf}}) = 1$.

In order to obtain the analytic solution of Eq. (13) we apply slow-roll approximation to the dilaton, that is,

$$\left| \frac{\dot{\Phi}}{H_{\text{inf}}\Phi} \right| \ll 1, \quad (36)$$

and then Eq. (13) is reduced to

$$3H_{\text{inf}}\dot{\Phi} + \frac{dV[\Phi]}{d\Phi} = 0. \quad (37)$$

The solution of this equation is given by

$$\Phi = \frac{1}{\tilde{\lambda}\kappa} \ln \left[\frac{(\tilde{\lambda}\kappa)^2 \bar{V}}{3H_{\text{inf}}} (t - t_{\text{R}}) + \exp(\tilde{\lambda}\kappa\Phi_{\text{R}}) \right], \quad (38)$$

where $\Phi_{\text{R}} (\leq 0)$ is the value of Φ at the end of inflation. In the case the dilaton is frozen at $t = t_{\text{R}}$, we choose $\Phi_{\text{R}} = 0$ so that $f = 1$ at that time.

Next, we investigate the evolution of \dot{f}/f . Using Eqs. (5), (17), and (38), we find

$$\frac{\dot{f}}{f} = \frac{\lambda\tilde{\lambda}\kappa^2 V[\Phi]}{3H_{\text{inf}}} = \lambda\tilde{\lambda}H_{\text{inf}}w, \quad (39)$$

where w is defined as

$$w \equiv \frac{V[\Phi]}{\rho_\phi} \approx \frac{V[\Phi]}{U[\phi]}. \quad (40)$$

Since we have $\rho_\phi \gg \rho_\Phi$ by assumption, $w \ll 1$. Comparing Eq. (26) with Eq. (39), we find

$$\beta \approx 1 + \lambda\tilde{\lambda}w = 1 + X\epsilon, \quad (41)$$

where X and ϵ are defined as

$$X \equiv \frac{\lambda}{\tilde{\lambda}}, \quad (42)$$

$$\epsilon \equiv \tilde{\lambda}^2 w, \quad (43)$$

respectively.

Though $U[\phi]$ is approximately constant in slow-roll exponential inflation, $V[\Phi]$ changes gradually. As a result it follows from Eqs. (6), (38), (40), and (43) that the ratio of w at $t = t_{\text{R}}$ to that at $t = t_i$ is

$$\begin{aligned} \frac{w(t_{\text{R}})}{w(t_i)} &\approx \frac{V[\Phi(t_{\text{R}})]}{V[\Phi(t_i)]} \\ &= 1 - \frac{(\tilde{\lambda}\kappa)^2 V[\Phi_{\text{R}}]}{3H_{\text{inf}}^2} H_{\text{inf}}(t_{\text{R}} - t_i) \\ &= 1 - \epsilon(t_{\text{R}}) H_{\text{inf}}(t_{\text{R}} - t_i), \end{aligned} \quad (44)$$

where $\epsilon(t_{\text{R}}) = \tilde{\lambda}^2 w(t_{\text{R}})$. Since $H_{\text{inf}}(t_{\text{R}} - t_i) = N(t_i \rightarrow t_{\text{R}})$ is about 50, where $N(t_i \rightarrow t_{\text{R}})$ is the number of e -folds during

the period from $t = t_i$ to $t = t_{\text{R}}$, the ratio of $w(t_{\text{R}})$ to $w(t_i)$ is about half in the case $\epsilon(t_{\text{R}}) \approx 1/100$. If $\epsilon \ll 1$, the variation in w in the inflationary stage is small, and then we can regard β as approximately constant.

The slow-roll condition to the dilaton, Eq. (36), is equivalent to the following relation:

$$\left| \frac{\dot{\Phi}}{H_{\text{inf}}\Phi} \right| \ll 1 \Leftrightarrow \epsilon \ll 1. \quad (45)$$

If we assume $\tilde{\lambda} \sim \mathcal{O}(1)$, the relation $\epsilon \ll 1$ is satisfied during inflation because of $w \ll 1$.

C. Evolution of electric and magnetic fields

We consider the evolution of electric and magnetic fields. The proper electric and magnetic fields are given by

$$E_i^{\text{proper}}(t, \mathbf{x}) = a^{-1} E_i(t, \mathbf{x}) = -a^{-1} \dot{A}_i(t, \mathbf{x}), \quad (46)$$

$$B_i^{\text{proper}}(t, \mathbf{x}) = a^{-1} B_i(t, \mathbf{x}) = a^{-2} \epsilon_{ijk} \partial_j A_k(t, \mathbf{x}), \quad (47)$$

where $E_i(t, \mathbf{x})$ and $B_i(t, \mathbf{x})$ are the comoving electric and magnetic fields, and ϵ_{ijk} is the totally antisymmetric tensor ($\epsilon_{123} = 1$).

From Eqs. (32), (46), and (47) we find the Fourier components of the comoving electric and magnetic fields in the inflationary stage,

$$E_I(k, a) = \sqrt{\frac{\pi}{4H_{\text{inf}}f(a)}} \left(\frac{k}{a} \right) a^{-1/2} H_{\beta/2-1}^{(1)} \left(\frac{k}{aH_{\text{inf}}} \right) e^{i(\beta+1)\pi/4}, \quad (48)$$

$$\begin{aligned} B_I(k, a) &= -i(-1)^I \sqrt{\frac{\pi}{4H_{\text{inf}}f(a)}} \left(\frac{k}{a} \right) a^{-1/2} \\ &\quad \times H_{\beta/2}^{(1)} \left(\frac{k}{aH_{\text{inf}}} \right) e^{i(\beta+1)\pi/4}. \end{aligned} \quad (49)$$

We consider the case the dilaton freezes at the end of inflation and after instantaneous reheating the conductivity of the Universe σ_c jumps to a value much larger than the Hubble parameter at that time. The evolutionary equation of the vector potential for an electrically conducting plasma is given by

$$\ddot{A}_i(t, \mathbf{x}) + \left(\frac{\dot{a}}{a} + \sigma_c \right) \dot{A}_i(t, \mathbf{x}) - \frac{1}{a^2} \partial_j \partial_j A_i(t, \mathbf{x}) = 0. \quad (50)$$

The joining conditions at the transition from the inflationary stage (INF) to the radiation-dominated one (RD) at $t = t_{\text{R}}$ are [17]

$$E_i^{(\text{RD})}(t_{\text{R}}, \mathbf{x}) = \exp(-\sigma_c t_{\text{R}}) E_i^{(\text{INF})}(t_{\text{R}}, \mathbf{x}), \quad (51)$$

$$B_i^{(\text{RD})}(t_{\text{R}}, \mathbf{x}) = B_i^{(\text{INF})}(t_{\text{R}}, \mathbf{x}). \quad (52)$$

From these joining conditions, for a large enough conductiv-

ity at the instantaneous reheating stage, we see that the electric fields accelerate charged particles and dissipate. In fact, we solve Eq. (50) in the large-conductivity limit: $\sigma_c \gg H$, so that electric fields vanish and the proper magnetic fields evolve in proportion to $a^{-2}(t)$ in the radiation-dominated

stage and the subsequent matter-dominated stage ($t \geq t_R$) [17].

It follows from Eq. (49) that the Fourier components of the proper magnetic fields in the inflationary stage are given by

$$|B_I^{\text{proper}}(t, k)|^2 \equiv a^{-2} |B_I(k, a)|^2 = a^{-2} \left(\frac{\pi}{4H_{\text{inf}} f(a)} \right) \left(\frac{k}{a} \right)^2 \left(\frac{1}{a} \right) H_{\beta/2}^{(1)} \left(\frac{k}{aH_{\text{inf}}} \right) H_{\beta/2}^{(2)} \left(\frac{k}{aH_{\text{inf}}} \right). \quad (53)$$

Since we are interested in the scales much larger than the Hubble radius in the inflationary stage, we expand Eq. (53) in the large-scale limit. Taking account of the above fact that the proper magnetic fields evolve in proportion to $a^{-2}(t)$ ($t \geq t_R$), we find that the Fourier components of the large-scale proper magnetic fields are expressed as

$$|B_I^{\text{proper}}(t, k)|^2 = \frac{2^{\beta-2}}{\pi} \Gamma^2 \left(\frac{\beta}{2} \right) f^{-1}(a_R) \left(\frac{1}{a_R H_{\text{inf}}} \right) \left(\frac{k}{a_R H_{\text{inf}}} \right)^{-\beta} \left(\frac{k}{a} \right)^2 \left(\frac{1}{a} \right)^2 \quad \text{for } \beta > 0, \quad (54)$$

and

$$|B_I^{\text{proper}}(t, k)|^2 = \frac{2^{-(\beta+2)}}{\pi} \Gamma^2 \left(-\frac{\beta}{2} \right) f^{-1}(a_R) \left(\frac{1}{a_R H_{\text{inf}}} \right) \left(\frac{k}{a_R H_{\text{inf}}} \right)^{\beta} \left(\frac{k}{a} \right)^2 \left(\frac{1}{a} \right)^2 \quad \text{for } \beta < 0, \quad (55)$$

where we have only recorded the first leading term.

Finally, the energy density of the large-scale magnetic fields in Fourier space is given by

$$\rho_B(t, k) = |B_I^{\text{proper}}(t, k)|^2 f(a). \quad (56)$$

Multiplying $\rho_B(t, k)$ by phase-space density, $4\pi k^3/(2\pi)^3$, we obtain the energy density of the large-scale magnetic fields in the position space

$$\rho_B(L, t) = \frac{k^3}{2\pi^2} |B_I^{\text{proper}}(t, k)|^2 f(a), \quad (57)$$

on a comoving scale $L = 2\pi/k$. Note that the expressions in Eqs. (54) and (55) are for one transverse component of the magnetic fields and the total magnetic field contribution is twice as large. Using Eqs. (54), (55), and (57), we find that the energy density of the large-scale magnetic fields at the present time t_0 is given by

$$\rho_B(L, t_0) = \frac{2^{|\beta|-3}}{\pi^3} \Gamma^2 \left(\frac{|\beta|}{2} \right) H_{\text{inf}}^4 \left(\frac{a_R}{a_0} \right)^4 \left(\frac{k}{a_R H_{\text{inf}}} \right)^{-|\beta|+5}. \quad (58)$$

Hence the large-scale magnetic fields have a scale-invariant spectrum when $|\beta|=5$. From now on we shall take the present scale factor $a_0=1$.

D. Consistency

During inflation the energy density of electric and magnetic fields should be smaller than that of the dilaton so that the evolution of the dilaton is governed by its (classical) potential.

We define the ratio of the energy density of the electric and magnetic fields to that of the dilaton as follows:

$$Y(L, t) \equiv \frac{1}{\rho_\Phi} [\rho_B(L, t) + \rho_E(L, t)], \quad (59)$$

$$\rho_E(L, t) = \frac{k^3}{2\pi^2} |E_I^{\text{proper}}(t, k)|^2 f(a), \quad (60)$$

where $\rho_E(L, t)$ is the energy density of the electric fields on comoving scale $L = 2\pi/k$. During inflation Y must be smaller than 1, which we call *the consistency condition*.

Using Eqs. (17), (48), (49), (59), and (60) we find

$$Y \approx \frac{1}{3w} \left(\frac{H_{\text{inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{k}{aH_{\text{inf}}} \right)^5 \left[H_{\beta/2}^{(1)} \left(\frac{k}{aH_{\text{inf}}} \right) H_{\beta/2}^{(2)} \left(\frac{k}{aH_{\text{inf}}} \right) + H_{\beta/2-1}^{(1)} \left(\frac{k}{aH_{\text{inf}}} \right) H_{\beta/2-1}^{(2)} \left(\frac{k}{aH_{\text{inf}}} \right) \right], \quad (61)$$

where the approximate equality follows from the ratio of the energy density of the dilaton to that of the inflaton in the inflationary stage: $\rho_\Phi/\rho_\phi \approx w$, see Eqs. (16) and (40).

Here we note that the upper limit on H_{inf} is determined by the observation of the anisotropy of cosmic microwave background (CMB) radiation. Using the WMAP data on temperature fluctuation [25], which is consistent with the Cosmic-

Background Explorer (COBE) data, we can obtain a constraint on H_{inf} from tensor perturbation [15,27],

$$\frac{H_{\text{inf}}}{M_{\text{Pl}}} \leq 2 \times 10^{-5}. \quad (62)$$

From this relation we find that the upper limit on H_{inf} is 2.4×10^{14} GeV.

When we consider a given scale $2\pi/k$, the value of $k/(aH_{\text{inf}})$ decreases as the Universe evolves. Evaluating Eq. (61) at the horizon crossing $k/(aH_{\text{inf}})=1$, we find $Y < 10^{-10} w^{-1}$, and so the consistency condition is satisfied. In the long wavelength regime, expanding Eq. (61) and taking the first leading order in $k/(aH_{\text{inf}})$, we find

$$Y \approx \frac{1}{3\pi^2 w} \left(\frac{H_{\text{inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{aH_{\text{inf}}}{k} \right)^{\beta-5} \left[2^\beta \Gamma^2 \left(\frac{\beta}{2} \right) + 2^{\beta-2} \Gamma^2 \left(\frac{\beta}{2} - 1 \right) \left(\frac{aH_{\text{inf}}}{k} \right)^{-2} \right] \quad \text{for } \beta > 2, \quad (63)$$

$$Y \approx \frac{1}{3\pi^2 w} \left(\frac{H_{\text{inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{aH_{\text{inf}}}{k} \right)^{\beta-5} \left[2^\beta \Gamma^2 \left(\frac{\beta}{2} \right) + 2^{-(\beta-2)} \Gamma^2 \left(-\frac{\beta}{2} + 1 \right) \left(\frac{aH_{\text{inf}}}{k} \right)^{-2(\beta-1)} \right] \quad \text{for } 0 < \beta < 2, \quad (64)$$

and

$$Y \approx \frac{1}{3\pi^2 w} \left(\frac{H_{\text{inf}}}{M_{\text{Pl}}} \right)^2 \left(\frac{aH_{\text{inf}}}{k} \right)^{-(\beta+5)} \left[2^{-\beta} \Gamma^2 \left(-\frac{\beta}{2} \right) + 2^{-(\beta-2)} \Gamma^2 \left(-\frac{\beta}{2} + 1 \right) \left(\frac{aH_{\text{inf}}}{k} \right)^2 \right] \quad \text{for } \beta < 0, \quad (65)$$

where the first term in the square brackets is the magnetic contribution and the second term the electric one. Y is smaller on larger scales for $-3 < \beta < 5$.

IV. ESTIMATION OF PRESENT LARGE-SCALE MAGNETIC FIELDS

In this section, we estimate the present strength of the large-scale magnetic fields. Since we consider the case where dilaton is frozen at $t=t_{\text{R}}$, we choose $\Phi_{\text{R}}=0$ so that $f=1$ at that time.

A. Estimation of present large-scale magnetic fields

To estimate the present large-scale magnetic fields from Eq. (58) we use the following values [12]:

$$H_0 = 70 h_{70} \text{ km s}^{-1} \text{ Mpc}^{-1} \approx 2.26 h_{70} \times 10^{-18} \text{ s}^{-1}, \quad (66)$$

$$\rho_\phi = \frac{\pi^2}{30} g_* T_{\text{R}}^4 \quad (g_* \approx 200), \quad (67)$$

$$N = 45 + \ln \left(\frac{L}{[\text{Mpc}]} \right) + \ln \left(\frac{[30/(\pi^2 g_*)]^{1/12} \rho_\phi^{1/4}}{10^{38/3} [\text{GeV}]} \right), \quad (68)$$

$$\frac{a_0}{a_{\text{R}}} = \left(\frac{g_*}{3.91} \right)^{1/3} \frac{T_{\text{R}}}{T_{\gamma 0}} \approx \frac{3.7 T_{\text{R}}}{2.35 \times 10^{-13} [\text{GeV}]} \quad (T_{\gamma 0} \approx 2.73 \text{ K}), \quad (69)$$

where H_0 is the present Hubble parameter (throughout this paper we use $h_{70}=1.00$ [28]), g_* is the total number of

degrees of freedom for relativistic particles at the reheating epoch, T_{R} is reheating temperature, N is the number of e -folds between the time t_1 and the end of inflation t_{R} , and $T_{\gamma 0}$ is the present temperature of CMB radiation.

Applying Eqs. (66)–(69) and $k/(a_{\text{R}} H_{\text{inf}}) = \exp(-N)$ to Eq. (58), we find

$$\rho_B(L, t_0) \propto |B(L, t_0)|^2 \propto (\sqrt{H_{\text{inf}} M_{\text{Pl}} L})^{|\beta|-5} \left(T_{\gamma 0} \sqrt{\frac{H_{\text{inf}}}{M_{\text{Pl}}}} \right)^4. \quad (70)$$

From this relation we see that stronger magnetic fields could be generated in the case of a large H_{inf} and a large $|\beta|$. The reason is as follows. From Eqs. (38) and (43), we find $\Phi(t_i) = (\tilde{\lambda} \kappa)^{-1} \ln[1 - \epsilon H_{\text{inf}}(t_{\text{R}} - t_i)]$. Furthermore, from $\dot{f}/f = (\beta - 1)H_{\text{inf}}$, we find that the rate of the change of f is larger in the case of a large $|\beta|$ and a large H_{inf} . Hence the conformal invariance of the Maxwell theory is broken to a larger extent in the case of a large H_{inf} and a large $|\beta|$. Practically, for $\beta > 0$, the initial amplitude of electromagnetic quantum fluctuation becomes larger. On the other hand, for $\beta < 0$, although the initial amplitude smaller, the rate of the amplification is very high. We will consider the detailed values of the present magnetic fields in Sec. IV C.

B. Upper limits on cosmological magnetic fields

Upper limits on cosmological magnetic fields come from the following three sources (see more detailed explanations in Refs. [3,29]).

(1) CMB anisotropy measurements: Homogeneous magnetic fields during the time of decoupling whose scales are larger than the horizon at that time cause the Universe to expand at different rates in different directions. Since aniso-

tropic expansion of this type distorts CMB, measurements of CMB angular power spectrum impose limits on the cosmological magnetic fields. Barrow, Ferreira, and Silk [30] carried out a statistical analysis based on the 4-year COBE data for angular anisotropy and derived the following limit on the primordial magnetic fields that are coherent on scale larger than the present horizon:

$$B_{\text{cosmic}}^{(0)} < 5 \times 10^{-9} \text{ G}. \quad (71)$$

Incidentally, Caprini, Durrer, and Kahniashvili [31] have recently investigated the effect of gravity waves induced by a possible helicity component of a primordial magnetic field on CMB temperature anisotropies and polarization. According to them, the effect could be sufficiently large to be observable if the spectrum of the primordial magnetic field is close to scale invariant and if its helical component is stronger than $\sim 10^{-10}$ G. Moreover, it has also been argued that if tangled magnetic fields of $\approx 3 \times 10^{-9}$ G exist on cosmological scales, their imprint on CMB anisotropy and polarization may be detectable [32].

(2) Big bang nucleosynthesis (BBN): Magnetic fields that existed during the BBN epoch would affect the expansion rate, reaction rates, and electron density. Taking all these effects into account in calculation of the element abundances, and then comparing the results with the observed abundances, one can set limits on the strength of the magnetic fields. The limits on homogeneous magnetic fields on the BBN horizon size $\sim 1.4 \times 10^{-4} h_{70}^{-1}$ Mpc are less than 10^{-6} G in terms of today's values [33].

(3) Rotation measure (RM) observations: RM data for high-redshift sources can be used to constrain the large-scale magnetic fields. For example, Vallée [34] tested for an RM dipole in a sample of 309 galaxies and quasars. The galaxies in this sample extended to $z \approx 3.6$ though most of the objects were at $z \leq 2$. Vallée derived an upper limit of $6 \times 10^{-10} (n_{e0}/10^{-7} \text{ cm}^{-3})^{-1}$ G, where n_{e0} is the present mean density of thermal electrons, on the strength of uniform component of a cosmological magnetic field. Note that the average baryon density is estimated as $n_{b0} = (2.7 \pm 0.1) \times 10^{-7} \text{ cm}^{-3}$ [25] for comparison.

C. Results

We show the results of the present large-scale magnetic fields calculated from Eq. (58). As described in Sec. I, primordial magnetic fields with the present strength 10^{-10} – 10^{-9} G are required to explain the observed fields in galaxies and clusters of galaxies through adiabatic compression. On the other hand, for galactic dynamo scenario, the fields with the present strength 10^{-22} – 10^{-16} G are required.

Here we define $\Theta(L, t_R)$ as the ratio of the energy density of the large-scale electric fields to that of the magnetic counterpart at the end of inflation t_R ,

$$\Theta(L, t_R) \equiv \frac{|E_I^{\text{proper}}(L, t_R)|^2}{|B_I^{\text{proper}}(L, t_R)|^2}. \quad (72)$$

From Eqs. (63)–(65), (68), and (72), we find

$$\Theta(L, t_R) = \frac{\Gamma^2(\beta/2 - 1)}{\Gamma^2(\beta/2)} K^{-2}(L) \quad \text{for } \beta > 2, \quad (73)$$

$$\Theta(L, t_R) = \frac{\Gamma^2(-\beta/2 + 1)}{\Gamma^2(\beta/2)} K^{-2(\beta-1)}(L) \quad \text{for } 0 < \beta < 2, \quad (74)$$

and

$$\Theta(L, t_R) = \frac{\Gamma^2(-\beta/2 + 1)}{\Gamma^2(-\beta/2)} K^2(L) \quad \text{for } \beta < 0, \quad (75)$$

where

$$K(L) \equiv 2e^{45} \left\{ \frac{[30/(\pi^2 g_*)]^{1/12} \rho_\phi^{1/4}}{10^{38/3} [\text{GeV}]} \right\} \left(\frac{L}{[\text{Mpc}]} \right). \quad (76)$$

While the energy density of magnetic fields is dominant for $\beta > 1$, that of the electric fields is dominant for $\beta < 1$ on large scales. Particularly, in the case $\beta < 0$, the energy density of the large-scale electric fields is much larger than that of the magnetic counterpart. When strong enough magnetic field is generated, the accompanying electric field is so strong that the consistency condition is not satisfied. The reason is as follows. As noted in Sec. III A, for $\beta > 0$, the vector potential is time independent in the large-scale limit in the inflationary stage, conversely, for $\beta < 0$, it evolves like $a^{-\beta}$. Hence the electric fields are not generated in the former case, but generated in the latter case. Consequently, stronger magnetic fields could be generated in the case $\beta > 1$.

Figures 1 and 2 depict the curves in the $H_{\text{inf}} - \beta$ parameter space on which the present magnetic fields on 1 Mpc scale with each strength could be generated. The former is for $\beta > 0$ and the latter for $\beta < 0$. In Fig. 1, the shaded area illustrates the excluded region from the observation of the anisotropy of CMB. This region is decided by Eq. (71). From Eqs. (63) and (64) we find that the consistency condition is satisfied in all the area. On the other hand, in Fig. 2, the shaded area corresponds to $Y \geq 1$ and illustrates the excluded region from the consistency condition decided by Eq. (65). The excluded region from CMB is included in this region. From these figures we can find the following results. For $\beta > 0$, the magnetic fields on 1 Mpc scale with the strength larger than 10^{-10} G could be generated in the case $H_{\text{inf}} > 1.2 \times 10^6$ GeV. Furthermore, the magnetic fields strong enough for the galactic dynamo scenario (10^{-22} – 10^{-16} G) could be generated in the wide region of the parameter space. On the other hand, for $\beta < 0$, the maximum strength of the fields are 5×10^{-19} G at most. Parenthetically we note that in all figures we depict 10^{-22} G contours with solid lines to emphasize the critical value for the galactic dynamo scenario. As we can see from Eq. (70), when $\beta > 5$, the larger-scale field is the stronger, so that its strength is constrained from the observation of the anisotropy of CMB, on the other hand, when $\beta < 5$, the stronger is the smaller-scale field, so that its

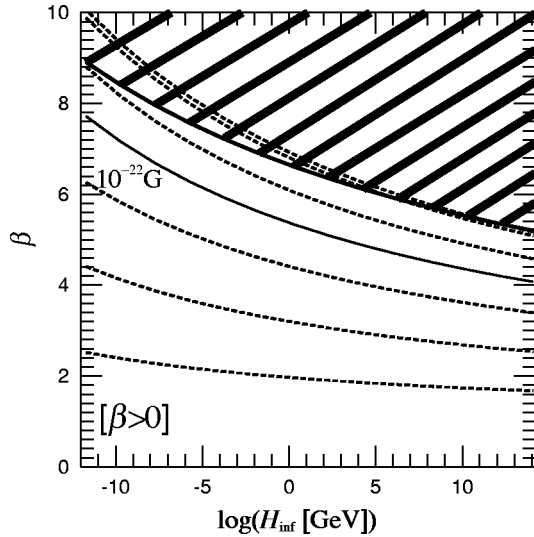


FIG. 1. The curves (dotted lines and a solid line) in the $H_{\text{inf}} - \beta$ parameter space on which the present magnetic fields on 1 Mpc scale with each strength could be generated ($\beta > 0$). $B(t_0) = 10^{-9}$ G, 10^{-10} G, 10^{-16} G, 10^{-22} G (solid line), 10^{-30} G, 10^{-40} G, and 10^{-50} G are shown (top down). The shaded area illustrates the excluded region from the observation of the anisotropy of CMB, Eq. (71).

strength is constrained from the predictions of light element abundances from BBN. The former is more stringent than the latter, and so all results shown in Figs. 1 and 2 satisfy the limit imposed by the latter. Here we show a few characteristic examples. When $\beta = 1$, i.e., in the ordinary Maxwell theory, the present strength of the magnetic field on 1 Mpc

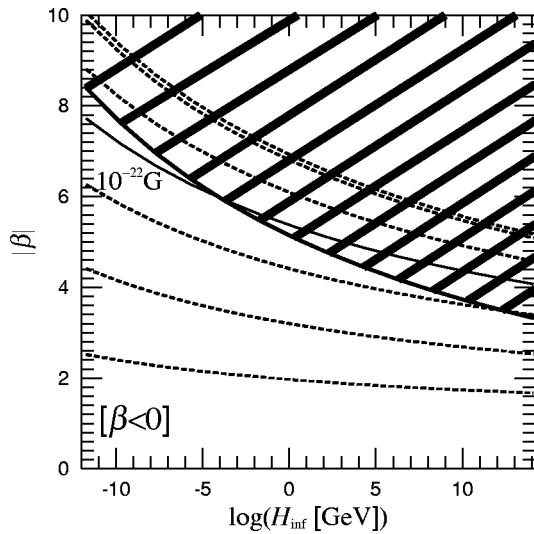


FIG. 2. The curves (dotted lines and a solid line) in the $H_{\text{inf}} - \beta$ parameter space on which the present magnetic fields on 1 Mpc scale with each strength could be generated ($\beta < 0$). $B(t_0) = 10^{-9}$ G, 10^{-10} G, 10^{-16} G, 10^{-22} G (solid line), 10^{-30} G, 10^{-40} G, and 10^{-50} G are shown (top down). The shaded area corresponds to $Y \geq 1$ and illustrates the excluded region from the consistency condition decided by Eq. (65). This area includes the excluded region from the observation of the anisotropy of CMB.

scale is 2.1×10^{-58} G. This value is independent of H_{inf} . Moreover, when $\beta = 5$ (the magnetic field has a scale-invariant spectrum), the maximum value of the field is 1.8×10^{-11} G in the case $H_{\text{inf}} = 2.4 \times 10^{14}$ GeV.

Finally, we note the following point. As discussed in Sec. III B, we have assumed $\epsilon \ll 1$ and regarded β as approximately constant, and analytically investigated the evolution of the electric and magnetic fields. This approximate analysis is proper. In fact we have taken account of the time dependence of β in the case $w(t_R) \ll 1$ and $\tilde{\lambda} \sim \mathcal{O}(1)$: $\beta \sim 1 + \lambda w(t_R)/(1 + w(t_R)H_{\text{inf}}(t - t_R))$, which is derived by using Eqs. (6), (38), (40), and (41), and numerically solved Eq. (27) in the inflationary stage. As a result we have confirmed that the numerical results almost agree with the analytic ones.

V. DILATON DECAY AFTER REHEATING

So far we considered the case the dilaton field freezes at the end of inflation and so the value of the coupling f between the dilaton and electromagnetic fields is set to unity. In practice, however, it is expected that the dilaton continues its evolution along with the exponential potential even after reheating but is finally stabilized when it feels other contributions to its potential from, say, gaugino condensation that generates a potential minimum. As it reaches there, the dilaton starts oscillation with mass m and finally decays into radiation. Then we consider its potential minimum is located at $\Phi = 0$ and $f = 1$ there, and so the field amplitude evolves from $\Phi_R (< 0)$ to nearby zero along with the exponential potential.

During the coherent oscillation the energy density of the dilaton ρ_Φ evolves as $a^{-3}(t)$, so that it decreases more slowly than that of the radiation produced by the inflaton $\rho_r^{(\text{inf})}$. If the Universe is radiation dominated until the dilaton decays, the entropy per comoving volume remains practically constant. If ρ_Φ becomes dominant over $\rho_r^{(\text{inf})}$, significant amount of entropy is produced, which dilutes the energy density of magnetic fields.

A. Dilaton decay

We regard the time $t_{\text{osc}} \approx m^{-1}$ as the epoch when the coherent oscillations commence. When $t > t_R$, the Universe is radiation dominated, and so we shall set $a(t) = a_R(t/t_R)^{1/2}$.

Before the field oscillation regime ($t_R < t < t_{\text{osc}}$), the dilaton evolves with the exponential potential (6). It is known that, if the dilaton evolves with the exponential potential for a sufficiently long time, the dilaton enters a scaling regime as it evolves down its potential [35]. In this scaling regime the friction term from the expansion of the Universe in the equation of motion balances the potential force allowing it to enter this scaling era. If the Universe is radiation dominated in this regime, therefore, the evolution of the dilaton potential is the same as that of the background radiation produced by the inflaton, which is proportional to t^{-2} .

Then we have numerically solved the equation of motion for the dilaton along with the exponential potential, and found that the dilaton evolves near to zero before it enters a

scaling era because the field amplitude at the end of inflation $|\Phi_R|$ is constrained to be relatively small. The reason is as follows. When $t_R < t < t_{\text{osc}}$, the energy density of magnetic fields is enhanced through the coupling with the evolving dilaton. The energy density of magnetic fields on all scales should remain much smaller than that of the dilaton so that (4) should not affect the evolution of the dilaton. As we will show in Eq. (91) in the next section, the energy density of magnetic fields is more enhanced in the case of a large value of $\tilde{\lambda}\kappa|\Phi_R|$. Hence the upper limit on $\tilde{\lambda}\kappa|\Phi_R|$ is determined from this condition. We have therefore numerically solved the equation of motion for the dilaton along with the exponential potential and investigated the region of $\tilde{\lambda}\kappa|\Phi_R|$ in which this condition is satisfied. As a result we have found that the value of $|\Phi_R|$ must be relatively small as $|\Phi_R| \sim 1/\kappa$ in the case $\tilde{\lambda} \sim \mathcal{O}(1)$. From now on we will deal with $\tilde{\lambda}\kappa|\Phi_R|$ as a model parameter, and show the detailed region of this parameter in Figs. 3 and 4 in Sec. V C.

Consequently, even if the energy density of radiation $\rho_r^{(\text{inf})}$ is dominant over that of the dilaton ρ_Φ at the instantaneous reheating stage, that is, $\bar{V} \exp(-\tilde{\lambda}\kappa\Phi_R)/\rho_\Phi \approx w \ll 1$, $\rho_r^{(\text{inf})}$ becomes comparable to ρ_Φ around the field oscillation regime ($t \sim t_{\text{osc}}$). Since ρ_Φ does not become dominant over $\rho_r^{(\text{inf})}$ in $t_R < t < t_{\text{osc}}$, however, it is the field oscillation regime that significant amount of entropy could be produced.

We consider the evolution of ρ_Φ and the energy density of the radiation produced by the dilaton decay $\rho_r^{(\text{dil})}$ in the epoch of the coherent oscillations ($t \geq t_{\text{osc}}$). The equations for them are given as follows [12]:

$$\dot{\rho}_\Phi = - \left(3 \frac{\dot{a}}{a} + \Gamma_\Phi \right) \rho_\Phi, \quad (77)$$

$$\dot{\rho}_r^{(\text{dil})} = -4 \frac{\dot{a}}{a} \rho_r^{(\text{dil})} + \Gamma_\Phi \rho_\Phi, \quad (78)$$

where Γ_Φ is the decay width of Φ . The solutions of these equations are given by

$$\rho_\Phi = \rho_\Phi(t_{\text{osc}}) \left[\frac{a(t)}{a(t_{\text{osc}})} \right]^{-3} \exp[-\Gamma_\Phi(t-t_{\text{osc}})], \quad (79)$$

$$\rho_r^{(\text{dil})} = \Gamma_\Phi \rho_\Phi(t_{\text{osc}}) \left[\frac{a(t)}{a(t_{\text{osc}})} \right]^{-4} \int_{t_{\text{osc}}}^t \left[\frac{a(\tau)}{a(t_{\text{osc}})} \right] \times \exp[-\Gamma_\Phi(\tau-t_{\text{osc}})] d\tau. \quad (80)$$

Here we estimate $\rho_\Phi(t=t_{\text{osc}}) \approx \bar{V}$ because \bar{V} is the value of the exponential potential in the form (6) at $\Phi=0$, and we expect the contribution from stabilization mechanism has the same order of magnitude.

B. Entropy production

If the Universe is radiation-dominated until the dilaton decays $t_\Phi \approx \Gamma_\Phi^{-1}$, we find the following relation:

$$\begin{aligned} \rho_r^{(\text{inf})}(t_\Phi) > \rho_\Phi(t_\Phi) &\Leftrightarrow \rho_\Phi \left[\frac{a(t_\Phi)}{a_R} \right]^{-4} > \rho_\Phi(t_{\text{osc}}) \left[\frac{a(t_\Phi)}{a(t_{\text{osc}})} \right]^{-3} \\ &\Leftrightarrow \frac{\rho_\Phi}{\bar{V}} > \Gamma_\Phi^{-1/2} t_R^{-2} m^{-3/2} \\ &\Leftrightarrow \frac{\rho_\Phi}{\bar{V}} > \left(\frac{M_{\text{Pl}}}{m} \right) \left(\frac{2H_{\text{inf}}}{m} \right)^2, \end{aligned} \quad (81)$$

where the last relation is obtained by using $\Gamma_\Phi \approx m(m/M_{\text{Pl}})^2$ and $t_R \approx 1/(2H_{\text{inf}})$. In this case the entropy per comoving volume remains constant. Hence the necessary condition of entropy production is

$$\frac{\rho_\Phi}{\bar{V}} < \left(\frac{M_{\text{Pl}}}{m} \right) \left(\frac{2H_{\text{inf}}}{m} \right)^2. \quad (82)$$

Here we consider the case $\rho_r^{(\text{inf})}$ becomes equal to ρ_Φ at $t = t_c$ in the epoch of the coherent oscillations ($t_{\text{osc}} < t_c < t_\Phi$),

$$\rho_r^{(\text{inf})}(t_c) = \rho_\Phi(t_c) \Leftrightarrow \rho_\Phi \left[\frac{a(t_c)}{a_R} \right]^{-4} = \rho_\Phi(t_{\text{osc}}) \left[\frac{a(t_c)}{a(t_{\text{osc}})} \right]^{-3}. \quad (83)$$

From this equation we find

$$t_c \approx \left(\frac{\rho_\Phi}{\bar{V}} \right)^2 t_R^4 t_{\text{osc}}^{-3} \approx \left(\frac{\rho_\Phi}{\bar{V}} \right)^2 \left(\frac{1}{2H_{\text{inf}}} \right)^4 m^3, \quad (84)$$

where the first approximate equality follows from $\rho_\Phi(t_{\text{osc}}) \approx \bar{V}$, and the second one from $t_{\text{osc}} \approx m^{-1}$ and $t_R \approx 1/(2H_{\text{inf}})$. When $t \geq t_c$, ρ_Φ is dominant over $\rho_r^{(\text{inf})}$, so that the Universe is matter dominated. Thus we shall set $a(t) = a_c(t/t_c)^{2/3} = a_R t_R^{-1/2} t_c^{-1/6} t^{2/3}$, where the second equality follows from the joining condition of $a(t)$ at $t = t_R$.

We investigate the entropy per comoving volume after the dilaton decay. In general, the entropy per comoving volume is represented as $S = a^3(\rho+p)/T$, where ρ , p , and T are the equilibrium energy density, pressure, and temperature, respectively. It is given by [12]

$$\begin{aligned} S^{4/3} &= S_c^{4/3} + \frac{4}{3} \rho_\Phi(t_c) a_c^4 \left[\frac{2\pi^2 \langle g_* \rangle}{45} \right]^{1/3} \\ &\quad \times \Gamma_\Phi \int_{t_c}^t \left[\frac{a(\tau)}{a_c} \right] \exp[-\Gamma_\Phi(\tau-t_c)] d\tau \end{aligned} \quad (85)$$

$$\begin{aligned} &\approx S_c^{4/3} \left[1 + \Gamma_\Phi t_c^{-2/3} \int_0^\infty (u+t_c)^{2/3} \exp(-\Gamma_\Phi u) du \right] \\ &\approx S_c^{4/3} \left[1 + \Gamma \left(\frac{5}{3} \right) (t_c \Gamma_\Phi)^{-2/3} \right], \end{aligned} \quad (86)$$

where S_c is the entropy per comoving volume at $t = t_c$ and $\langle g_* \rangle$ is the appropriately averaged value of g_* over the decay interval. In the second approximation we have intro-

duced the variable $u \equiv \tau - t_c$ and calculated in the limit $u \rightarrow \infty$. Moreover we have used the relation $\rho_r^{(\text{inf})}(t_c) = \rho_\phi(t_c)$ and the following equation:

$$S = \left[\frac{4}{3} \left(\frac{2\pi^2 g_*}{45} \right)^{1/3} a^4 \rho_r \right]^{3/4}, \quad (87)$$

which is the general relation between entropy per comoving volume S and the energy density of radiation ρ_r . It follows from $\Gamma_\phi \approx m(m/M_{\text{Pl}})^2$, Eqs. (84) and (86) that the ratio of the entropy per comoving volume after decay to that before decay is written as

$$\begin{aligned} \Delta S &\equiv \frac{S}{S_c} \\ &\approx \left\{ 1 + \Gamma \left(\frac{5}{3} \right) \left[\left(\frac{\bar{V}}{\rho_\phi} \right) \left(\frac{2H_{\text{inf}}}{m} \right)^2 \left(\frac{M_{\text{Pl}}}{m} \right) \right]^{4/3} \right\}^{3/4} \\ &\approx \left(\frac{\bar{V}}{\rho_\phi} \right) \left(\frac{2H_{\text{inf}}}{m} \right)^2 \left(\frac{M_{\text{Pl}}}{m} \right), \end{aligned} \quad (88)$$

where the second approximate equality follows from Eq. (82).

If the entropy production occurs, the Universe should expand larger enough to cancel out the effect of the produced entropy. From this point of view, taking account of $\rho_r \propto a^{-4} S^{4/3}$, we find

$$(\Delta S)^{4/3} = \left(\frac{\tilde{a}_0}{a_0} \right)^4, \quad (89)$$

where \tilde{a}_0 is the present scale factor in the case with the entropy production.

Finally, we consider the effect of the dilaton decay on the energy density of the present large-scale magnetic fields. We again assume that after instantaneous reheating the conductivity immediately jumps to a large value and so the amplitude of the vector potential is fixed. It follows from Eqs. (5), (54)–(57), (88), and (89) that the ratio of the energy density of the present large-scale magnetic fields in the case the dilaton still evolves after reheating and then decays into radiation, $\tilde{\rho}_B(L, t_0)$, to that in the case the dilaton freezes at the end of inflation $\rho_B(L, t_0)$ is written as

$$\frac{\tilde{\rho}_B(L, t_0)}{\rho_B(L, t_0)} = f^{-1}(t_R) (\Delta S)^{-4/3} \quad (90)$$

$$\approx \exp(-\tilde{\lambda} \kappa \Phi_R X) \left[\left(\frac{\bar{V}}{\rho_\phi} \right) \left(\frac{2H_{\text{inf}}}{m} \right)^2 \left(\frac{M_{\text{Pl}}}{m} \right) \right]^{-4/3} \quad (91)$$

$$\approx \exp \left[-\tilde{\lambda} \kappa \Phi_R \left(X + \frac{4}{3} \right) \right] \left[w \left(\frac{2H_{\text{inf}}}{m} \right)^2 \left(\frac{M_{\text{Pl}}}{m} \right) \right]^{-4/3}, \quad (92)$$

where the last approximate equality follows from $\bar{V}/\rho_\phi \approx w \exp(\tilde{\lambda} \kappa \Phi_R)$, see Eqs. (6) and (39). In the right-hand side of Eq. (90), the first factor is the contribution of the coupling with the dilaton, through which the energy density of the magnetic fields is enhanced because $\Phi_R < 0$. On the other hand, the second one is that of the produced entropy, which dilutes the energy density of the magnetic fields.

C. Effect of dilaton decay

From Eqs. (58) and (92) we estimate the strength of the present large-scale magnetic fields in the case the dilaton still evolves after reheating and then decays into radiation. Figures 3 and 4 depict the magnetic field strength on 1 Mpc scale at the present time $\tilde{B}(t_0)$. The former is for the case $H_{\text{inf}} = 10^{14}$ GeV and $m = 10^{13}$ GeV, while the latter corresponds to the case $H_{\text{inf}} = 10^{10}$ GeV and $m = 10^9$ GeV. In both these cases entropy is produced (i.e., $\Delta S > 1$), and then the relation (82) is satisfied. The amount of the produced entropy in the former case is smaller than that in the latter. As described in Sec. V A, the region of the parameter $\tilde{\lambda} \kappa |\Phi_R|$ is constrained from the condition that the energy density of magnetic fields on all scales should remain much smaller than that of the dilaton before the field oscillation regime ($t_R < t < t_{\text{osc}}$). This condition is satisfied in the region of $\tilde{\lambda} \kappa |\Phi_R|$ on each line in these figures. Moreover, Fig. 5 depicts the curves in the $H_{\text{inf}} - m$ parameter space on which the present magnetic fields on 1 Mpc scale with each strength could be generated for the case $\beta \approx 5.0$. In Fig. 5, the shaded area illustrates the region with $m > 2H_{\text{inf}}$, where $t_R > t_{\text{osc}}$ and our analysis does not apply. In all the cases in Fig. 5, entropy is also produced. In this figure, we have taken

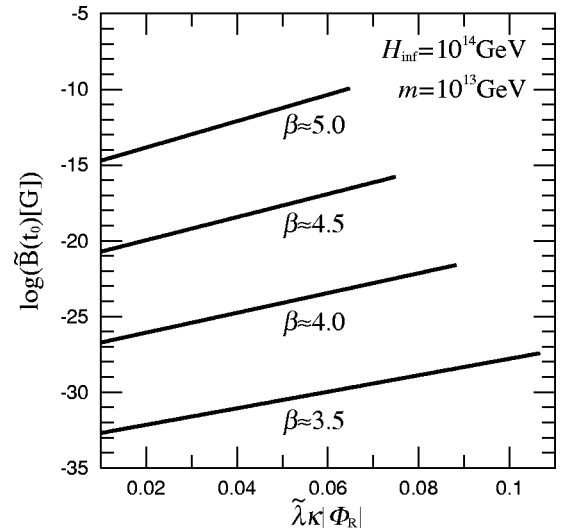


FIG. 3. The magnetic field strength on 1 Mpc scale at the present time $\tilde{B}(t_0)$ in the case with entropy production. The lines are for the case $H_{\text{inf}} = 10^{14}$ GeV and $m = 10^{13}$ GeV. $\beta \approx 1 + \lambda \tilde{\lambda} w \approx 5.0$, $\beta \approx 4.5$, $\beta \approx 4.0$, and $\beta \approx 3.5$ are shown (top down). Here we have taken $w = 0.01$ and $\tilde{\lambda} \sim \mathcal{O}(1)$.

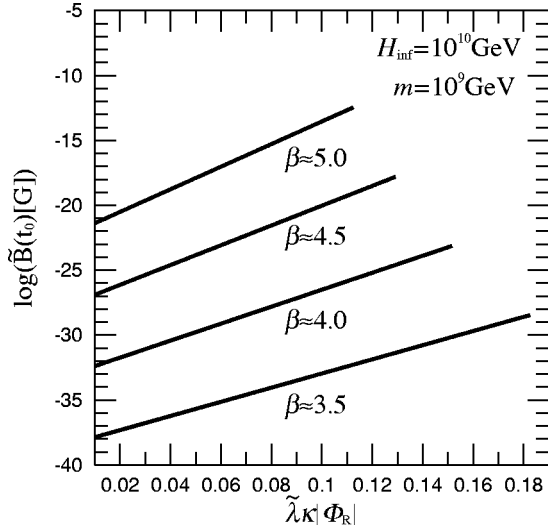


FIG. 4. The magnetic field strength on 1 Mpc scale at the present time $\bar{B}(t_0)$ in the case with entropy production. The lines are for the case $H_{\text{inf}}=10^{10}$ GeV and $m=10^9$ GeV. $\beta \approx 1 + \lambda \tilde{\lambda} w \approx 5.0$, $\beta \approx 4.5$, $\beta \approx 4.0$, and $\beta \approx 3.5$ are shown (top down). Here we have taken $w=0.01$ and $\tilde{\lambda} \sim \mathcal{O}(1)$.

the maximum of $\tilde{\lambda} \kappa |\Phi_R|$ for each case. In Figs. 3–5, we have taken $w=0.01$ and $\tilde{\lambda} \sim \mathcal{O}(1)$.

The energy density of the magnetic fields is more enhanced for larger $\tilde{\lambda} \kappa |\Phi_R|$ and larger X , while it is more diluted in the case more entropy is produced, see Eqs. (90)–(92). In fact, if $\beta \approx 1 + X \epsilon \approx 5.0$, that is, the spectrum of the magnetic fields is close to the scale-invariant one, and produced entropy is relatively small, e.g., in the case $H_{\text{inf}}=10^{14}$ GeV and $m=10^{13}$ GeV, in which $\Delta S=4.6 \times 10^6$, the generated magnetic fields on 1 Mpc scale could be stronger than 10^{-10} G at the present time. Furthermore, in the case $\beta \approx 5.0$, even if the entropy production factor is as large as $\Delta S \sim 10^{24}$, a sufficient magnitude of magnetic fields for the galactic dynamo scenario, $B \geq 10^{-22}$ G, could be generated. More specifically, the amplitude of the generated magnetic field ranges from 10^{-22} G to 10^{-16} G, namely the span favored by the dynamo scenario, if the dilaton mass lies in the range $9.3 \times 10^6 \text{ GeV} \leq m \leq 9.3 \times 10^9 \text{ GeV}$ in the case $H_{\text{inf}}=10^{14}$ GeV and $\beta=5$ with the entropy production factor being $\Delta S=5.8 \times 10^{15} - 5.8 \times 10^{24}$. In the case $H_{\text{inf}}=10^{10}$ GeV and $\beta=5$, the appropriate amplitude of magnetic field results for $1.7 \times 10^4 \text{ GeV} \leq m \leq 1.7 \times 10^7 \text{ GeV}$ in which ΔS ranges $9.2 \times 10^{15} - 9.2 \times 10^{24}$.

Moreover, as shown in Figs. 3 and 4, even if $\beta < 5$, that is, the spectrum of generated magnetic field is blue, sufficiently large seed field for the galactic dynamo scenario could also be generated.

Finally, in the case X is so small (e.g., $X \leq 10$) that β is about unity, the spectrum is too blue for the large-scale magnetic field to be strong enough for the dynamo scenario to work. This result remains unchanged even when $\tilde{\lambda} \kappa |\Phi_R|$ is large and the field amplitude is enhanced with the relation (81) being satisfied, namely, without any entropy production.

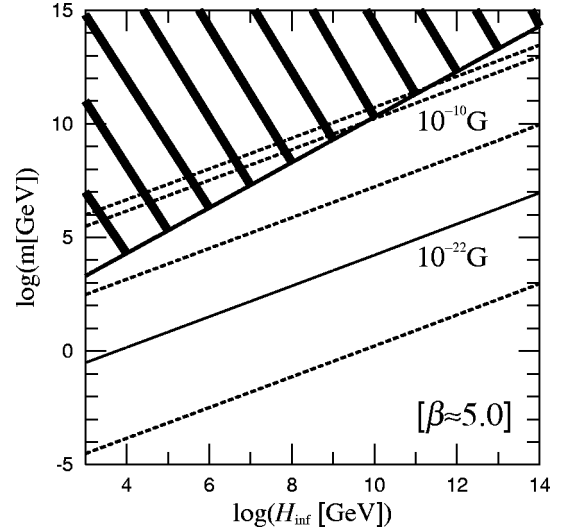


FIG. 5. The curves (dotted lines and a solid line) in the $H_{\text{inf}} - m$ parameter space on which the present magnetic fields on 1 Mpc scale with each strength could be generated for the case $\beta \approx 5.0$. $\bar{B}(t_0)=10^{-9}$ G, 10^{-10} G, 10^{-16} G, 10^{-22} G (solid line), and 10^{-30} G are shown (top down). The shaded area illustrates the region with $m > 2H_{\text{inf}}$, where $t_R > t_{\text{osc}}$ and our analysis does not apply. Here we have taken the maximum of $\tilde{\lambda} \kappa |\Phi_R|$ for each case, and $w=0.01$ and $\tilde{\lambda} \sim \mathcal{O}(1)$.

Thus the essential requirement is that the spectrum should not be too blue.

VI. CASE OF POWER-LAW INFLATION

So far we considered the evolution of electromagnetic fields in slow-roll exponential inflation models in the light of the fact that the recent data of WMAP favors models with small slow-roll parameters. In this section, for comparison with [17], we discuss it in power-law inflation models with the following exponential inflaton potential:

$$U[\phi] = \bar{U} \exp(-\zeta \kappa \phi), \quad (93)$$

where \bar{U} is a constant and ζ is a dimensionless constant. In this case the spectral index of curvature perturbation n_s is given by

$$n_s - 1 = -6\epsilon_U + 2\eta_U = -\zeta^2, \quad (94)$$

$$\epsilon_U \equiv \frac{1}{2\kappa^2} \left(\frac{U'}{U} \right)^2, \quad (95)$$

$$\eta_U \equiv \frac{1}{\kappa^2} \left(\frac{U''}{U} \right). \quad (96)$$

According to the WMAP data [36], $n_s \geq 0.93$, and hence $\zeta \leq 0.26$. In this case the scale factor in the inflationary stage is given by $a(t) \propto t^p$, where $p = 2/\zeta^2 \geq 29$.

If power-law inflation lasts for a sufficiently long time, the dilaton will settle to the scaling solution [35] where $U'' \approx H_{\text{inf}}^2$ with $H_{\text{inf}} = p/t$. Hence the solution of the dilaton in this regime is given by

$$\Phi = \frac{2}{\tilde{\lambda}\kappa} \ln \left(\frac{\sqrt{\tilde{V}\tilde{\lambda}\kappa t}}{p} \right). \quad (97)$$

From Eqs. (5) and (97) we find

$$f(\Phi) = \left(\frac{\sqrt{\tilde{V}\tilde{\lambda}\kappa t}}{p} \right)^{2\lambda/\tilde{\lambda}} \propto a^{2\lambda/(\tilde{\lambda}p)}. \quad (98)$$

Comparing Eq. (25) with Eq. (98), we find

$$\beta = \frac{2\lambda}{\tilde{\lambda}p} + 1. \quad (99)$$

On the other hand, in the inflationary stage the Fourier modes of the vector potential satisfy the following equation:

$$\frac{d^2 A_I(k, \eta)}{d\eta^2} + \frac{2\lambda}{\tilde{\lambda}} \left(\frac{-1}{p-1} \frac{1}{\eta} \right) \frac{dA_I(k, \eta)}{d\eta} + k^2 A_I(k, \eta) = 0, \quad (100)$$

where $\eta = \int dt/a(t)$ is conformal time. Comparing Eq. (27) with Eq. (100), we find

$$\beta = \frac{2\lambda}{\tilde{\lambda}(p-1)} + 1. \quad (101)$$

Here the solution of Eq. (100) is given by the same form as Eq. (28). Since $p \gg 1$ as noted above, we can approximately identify Eq. (99) with Eq. (101). Thus under this approximation we can develop the argument in the same way as in Secs. III–V.

As noted in Secs. IV and V, where we considered the case of exponential inflation, if $\beta \approx 5$, that is, the spectrum of the magnetic fields is close to the scale-invariant one, magnetic fields on 1 Mpc scale with the strength 10^{-10} – 10^{-9} G at the present time could be generated. In the case $w = 0.01$ and $\tilde{\lambda} \sim \mathcal{O}(1)$, if $\beta \approx 5$, $X = \lambda/\tilde{\lambda} \approx 400$. On the other hand, it follows from Eqs. (99) and (101) that, in the case of power-law inflation, if $\beta \approx 5$, $X \approx 2p \approx 58$. Consequently, even in the case of power-law inflation with the maximal breaking of the scale invariance of primordial fluctuations within observational limit [36], X should be much larger than unity in order that the amplitude of the generated magnetic fields could take 10^{-10} – 10^{-9} G on 1 Mpc scale at the present time. That is, the value of n_s obtained by the WMAP data is so close to unity that the power-law inflation model observationally permitted is not practically very much different from slow-roll exponential inflation.

VII. CONCLUSION

In the present paper we have studied the generation of large-scale magnetic fields in inflationary cosmology, breaking the conformal invariance of the electromagnetic field by

introducing a coupling with the dilaton field. First we considered the case the dilaton freezes at the end of inflation automatically as assumed by Ratra [17], to see how the recent detailed observation of the primordial spectrum of density fluctuations in terms of CMB anisotropy [25], which favors slow-rollover inflation, affects Ratra's previous analysis. As a result we have found the resultant magnetic field could be as large as 10^{-10} – 10^{-9} G on 1 Mpc scale at present for $H_{\text{inf}} \gtrsim 10^6$ GeV provided that the model parameters are so chosen that the spectrum of the magnetic field nearly scale invariant or even red.

Next we considered a more realistic case that the dilaton continues its evolution with the exponential potential after inflation until it is stabilized after oscillating around its potential minimum. It has two distinct effects on the final amplitude of the magnetic field. That is, the energy density of the magnetic field is enhanced as the dilaton evolves due to the exponential coupling, while it could produce a huge amount of entropy as it decays at a late time with the gravitational interaction. We have parameterized the evolution of the dilaton in terms of its mass, m , around the potential minimum and its amplitude at the end of inflation, Φ_R , and adopted a view that it starts oscillation at $t \approx m^{-1}$, since the detailed shape of the dilaton potential around the minimum is not known due to the fact that the stabilization mechanism of the dilaton is not fully established yet, although there are a number of proposals. As a result we have found that the magnetic field could be as large as 10^{-10} G even with the entropy increase factor $\Delta S \sim 10^6$ provided that the scale of inflation is maximal and the spectrum is close to scale invariant. Furthermore the seed field for the dynamo mechanism could be accounted for even when ΔS is as large as 10^{24} if model parameters are chosen appropriately to realize nearly scale-invariant spectrum.

Thus the possible dilution due to huge entropy production from decaying dilaton is not the primary obstacle to account for the large-scale magnetic field in terms of quantum fluctuations generated during inflation in this dilaton electromagnetism. The more serious requirement is that the model parameters should be so chosen that the spectrum of generated magnetic field should not be too blue but close to the scale-invariant or the red one, which is realized only if a huge hierarchy exists between λ and $\tilde{\lambda}$, namely, X should be extremely larger than unity. This may make it difficult to motivate this type of model in realistic high energy theories. This feature is independent of whether one considers slow-roll exponential inflation or power-law inflation with an exponential inflaton potential as adopted by Ratra [17], because the latter model is hardly distinguishable from the former under the constraint imposed by WMAP data as far as the evolution of the dilaton is concerned.

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