Novel approach to the study of quantum effects in the early Universe

A. Geralico,[1,2,3,*] G. Landolfi,[1,2,†] G. Ruggeri,[1,2,‡] and G. Soliani[1,2,§]
[1]*Dipartimento di Fisica dell'Università di Lecce, I-73100 Lecce, Italy*
[2]*INFN, Sezione di Lecce, I-73100 Lecce, Italy*
[3]*International Center for Relativistic Astrophysics-I.C.R.A., Università di Roma "La Sapienza," I-00185 Roma, Italy*
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We develop a theoretical frame for the study of classical and quantum gravitational waves based on the properties of a nonlinear ordinary differential equation for a function $\sigma(\eta)$ of the conformal time η, called the auxiliary field equation. At the classical level, $\sigma(\eta)$ can be expressed by means of two independent solutions of the ''master equation'' to which the perturbed Einstein equations for the gravitational waves can be reduced. At the quantum level, all the significant physical quantities can be formulated using Bogolubov transformations and the operator quadratic Hamiltonian corresponding to the classical version of a damped parametrically excited oscillator where the varying mass is replaced by the square cosmological scale factor $a^2(\eta)$. A quantum approach to the generation of gravitational waves is proposed on the grounds of the previous η-dependent Hamiltonian. An estimate in terms of $\sigma(\eta)$ and $a(\eta)$ of the destruction of quantum coherence due to the gravitational evolution and an exact expression for the phase of a gravitational wave corresponding to any value of η are also obtained. We conclude by discussing a few applications to quasi–de Sitter and standard de Sitter scenarios.

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I. INTRODUCTION

In a notable paper by Grishchuk and Sidorov [1], it was shown that relic gravitons can be created from the vacuum quantum fluctuations of the gravitational field during cosmological expansion and can be interpreted as squeezed quantum states of the gravitational field, in analogy to what happens for squeezed states in quantum optics [2–4]. A systematic treatment of the particle creation mechanism is contained in [5]. The theory of particle creation is essentially based on the Bogolubov transformations, exploited primary in the formulation of the process of squeezing for the electromagnetic field. An important aspect of the theory of particle creation is the possible loss of coherence in quantum gravitational theories. In a sense, the exploration of decoherence in gravitational theories has been more concerned with the quantum (gravity) effects which can manifest themselves even at scales others than the Planck one. The growing interest in this field relies, among the others, on quite a remarkable mechanism: the amplification of quantum (incidentally vacuum) fluctuations in the metric of a gravitational background (see e.g., [1,6–12]). In the presence of a change of regime under the cosmological evolution, the occupation number of the initial quantum state would get indeed amplified. As long as the change can be considered as adiabatic, the amplification factor approaches one. Nevertheless, in case the change is sudden the amplification mechanism cannot be neglected. In such a case, even the vacuum state transforms into a multiparticle state in the Fock space appropriate to the new regime. The scenario clearly sounds highly attractive under the stronger conviction that, thanks to the great progress we are witnessing in the experimental application of new technologies, the amplification mechanism may provide the possibility to detect quantum effects (e.g., relic gravitons) at scales considerably above the Planck one. Because of the semiclassical approximation underlying these studies, the natural formal arena turns out to be that of coherent states. These states are generated by the displacement operator $D(\alpha)$ (see Sec. II). Squeezed states enter in the matter whenever the quadratic operators $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ are involved. A squeezed state is generated by the action on a coherent state of the so-called squeeze operator defined in Sec. II. As a matter of fact, in a Friedmann-Robertson-Walker (FRW) spacetime the behavior of matter scalar fields as well as of gravitational waves is governed by an equation of the time-dependent type [or time-dependent oscillator (TDO)]. Thus the problem of both particle creation and metric field fluctuation amplification during the cosmological evolution is reduced to that of solving the quantum TDO problem. The latter problem has been the subject of several studies, mainly in connection with quantum optical arguments. In this paper, the main idea is to make use of the machinery built-up in [13] and further developed in [14]. The formalism would enable us to find at every stage information concerning the spectrum of created modes. In the case of a de Sitter gravitational field, the prescription straightforwardly results into full determination of quantities of physical relevance, such as Bogolubov coefficients and the phase of gravitational waves for any value of the conformal time η. All these quantities are determined exactly. This has been possible by solving a nonlinear ordinary differential equation for an auxiliary field $\sigma(\eta)$ which has been expressed in terms of two independent solutions of the η-dependent part of the D'Alembert equation for the gravitational perturbation tensor field. Another interesting result achieved in this paper is an estimate in terms of the auxiliary field $\sigma(\eta)$ and the scale factor $a(\eta)$ of

*Email address: andrea.geralico@le.infn.it
†Email address: giulio.landolfi@le.infn.it
‡Email address: giovanna.ruggeri@le.infn.it
§Email address: giulio.soliani@le.infn.it

the destruction of quantum coherence due to gravitational time-evolution. In our approach, the use of the (nonlinear) auxiliary equation in the linear parametrically excited oscillator equation for $y(\eta)$ [see Eq. (2.12)] reveals therefore to be quite profitable and makes more strict and compact the formal connection between quantum optics and the theory of gravitational waves. The paper is organized as follows. In Sec. II, after a description of some basic properties pertinent to the squeeze operator (2.3), we introduce the nonlinear equation (2.9) for the auxiliary field $\sigma(t)$ in terms of which the position and momentum operators Q and P turn out to be expressed. The role of the matrix element between squeezed states $|\alpha,z\rangle$ of the operator $D(\alpha)S(z)H(t)S^{\dagger}(z)D^{\dagger}(\alpha)$ is investigated. This matrix element results to be evaluated in terms of the auxiliary field $\sigma(t)$. It is worth noticing that in the expression for the matrix element three energy terms appear, one of them, formula (2.25), can be interpreted as the energy related to squeezed states which do not preserve the minimum uncertainty. In the theory of gravitational waves, Eq. (2.25) plays the role of *decoherence* energy of the waves. The Bogolubov transformation is reported whose coefficients are explicitly written in terms of σ. In Sec. III we discuss classical and quantum aspects of the generation of gravitational waves. An exact formula for the phase of a gravitational wave is obtained. A natural approach to the theory of gravitational waves based on the Kanai-Caldirola oscillator is outlined by means of an operator Hamiltonian expressed in terms of the auxiliary field $\sigma(\eta)$. In Sec. IV some applications are displayed. Precisely, we evaluate the decoherence energy in the quasi-de Sitter inflationary model and standard de Sitter spacetime and the role of Bogolubov coefficients in terms of the auxiliary field in the particle creation mechanism is analyzed. Finally, in Sec. V some future perspectives are discussed.

II. PRELIMINARIES ON THE SQUEEZED STATES IN GENERALIZED OSCILLATORS

Let us recall that a squeezed state of a quantum system is defined by [2]

$$|\alpha,z\rangle = D(\alpha)S(z)\,|0\rangle, \tag{2.1}$$

where

$$D(\alpha) = e^{\alpha\hat{a}_0^{\dagger} - \alpha^*\hat{a}_0} \tag{2.2}$$

is the (unitary) displacement (Weyl) operator,

$$S(z) = e^{(1/2)\,z\hat{a}_0^{\dagger\,2} - (1/2)\,z^*\hat{a}_0^2} \tag{2.3}$$

is the (unitary) *squeeze* operator [2,15], $\hat{a}_0$ and $\hat{a}_0^{\dagger}$ stand for $\hat{a}(t)|_{t=t_0}$ and $\hat{a}^{\dagger}(t)|_{t=t_0}$, respectively, where $\hat{a}(t)$ and $\hat{a}^{\dagger}(t)$ denote the annihilation and creation operators of the system. The complex functions $\alpha(t)$ and $z(t)$ are arbitrary, namely

$$\alpha = |\alpha|\;e^{i\varphi}, \qquad z = re^{i\phi}, \tag{2.4}$$

with φ, ϕ arbitrary (real) c numbers. Notice that for $z=0$, Eq. (2.1) reproduces the coherent state $|\alpha,0\rangle = D(\alpha)\;|0\rangle$.

The following relations

$$b \doteq S^{\dagger}\hat{a}_0 S = \hat{a}_0\cosh r + \hat{a}_0^{\dagger}e^{i\phi}\sinh r, \tag{2.5}$$

$$b^{\dagger} \doteq S^{\dagger}\hat{a}_0^{\dagger} S = \hat{a}_0^{\dagger}\cosh r + \hat{a}_0 e^{-i\phi}\sinh r \tag{2.6}$$

hold. This can be readily seen by applying the Baker-Campbell-Hausdorff formula. In other words, the squeeze operator $S(z)$ induces a canonical transformation of the annihilation and creation operators, in the sense that $[b,b^{\dagger}] = \hat{1}$.

A. Transformation of the position and momentum variables under the squeeze operator $S(z)$

From the quantum theory of generalized oscillators, it follows that position and momentum operators Q and P can be expressed by [13,14]

$$Q = \sqrt{\frac{\hbar}{m}}\;\sigma(\hat{a} + \hat{a}^{\dagger}), \qquad P = \sqrt{\hbar m}(\xi\hat{a} + \xi^*\hat{a}^{\dagger}), \tag{2.7}$$

with $\hat{a} = \hat{a}(t)$, $\hat{a}^{\dagger} = \hat{a}^{\dagger}(t)$, and

$$\xi = \frac{-i}{2\sigma} + \left(\dot{\sigma} - \frac{M}{2}\sigma\right), \tag{2.8}$$

where $M = M(t) \equiv \dot{m}/m$, the dot means time derivative, and (the mass) $m = m(t)$ is a given function of time. The function (c number) $\sigma(t)$ satisfies the nonlinear ordinary differential equation [18,19]

$$\ddot{\sigma} + \Omega^2\sigma = \frac{1}{4\sigma^3} \tag{2.9}$$

[Ω is specified below, see Eq. (2.13)], called the auxiliary equation associated with the classical equation of motion

$$\ddot{q} + M\dot{q} + \omega^2(t)q = 0, \tag{2.10}$$

and $\omega(t)$ is the time dependent frequency. Via the transformation

$$q \rightarrow e^{-(1/2)\int_{t_0}^{t} M(t')dt'}y, \tag{2.11}$$

Eq. (2.10) can be cast into the equation

$$\ddot{y} + \Omega^2(t)y = 0, \tag{2.12}$$

where

$$\Omega^2(t) = \frac{1}{4}(4\omega^2 - 2\dot{M} - M^2). \tag{2.13}$$

The quantum theory of the generalized oscillator (2.10) can be described by the Hamiltonian operator [16,13]

$$H(t) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 \tag{2.14}$$

where the canonical variables Q,P are given by Eq. (2.7).

Taking account of Eqs. (2.5) and (2.6), and choosing $\phi=0$ we obtain[1]

$$S^{\dagger}QS=e^{r}Q, \tag{2.15}$$

$$S^{\dagger}PS=2m\sigma\left(\dot{\sigma}-\frac{M}{2}\sigma\right)\sinh rQ+e^{-r}P. \tag{2.16}$$

The physical meaning of the expression $\dot{\sigma}-(M/2)\,\sigma$ will be clarified later. At the present we observe only that whenever the condition

$$\dot{\sigma}-\frac{M}{2}\sigma=0 \tag{2.17}$$

is fulfilled, then the uncertainty product $(\Delta Q)(\Delta P)$ of the variances

$$(\Delta Q)=\sqrt{\langle Q^{2}\rangle-\langle Q\rangle^{2}},\quad (\Delta P)=\sqrt{\langle P^{2}\rangle-\langle P\rangle^{2}}, \tag{2.18}$$

attains its minimum, where the expectation value $\langle\cdots\rangle$ is referred to coherent states [14].

We observe that the operators $S^{\dagger}QS$ and $S^{\dagger}PS$ obey the same commutation relation as Q and P, that is

$$[S^{\dagger}QS,S^{\dagger}PS]=[Q,P]=i\hbar. \tag{2.19}$$

However, in contrast to what happens for the operators b and $b^{\dagger}$ [see Eqs. (2.5), (2.6)], the operators $S^{\dagger}QS$ and $S^{\dagger}PS$ are not Hermitian conjugate. We point out that the property of Hermitian conjugation is enjoyed by the operators $S^{\dagger}QS$ and $S^{\dagger}PS$ in the case in which the condition (2.17) is valid.

A possible physical interpretation of the properties (2.15) and (2.16) is the following. For a quantum system governed by a Hamiltonian preserving the minimum wave packet [i.e., the condition (2.17) holds] Eqs. (2.15) and (2.16) become $S^{\dagger}QS=e^{r}Q$ and $S^{\dagger}PS=e^{-r}P$, respectively. If $|\psi\rangle$ is the state of the system under consideration, then $|\psi'\rangle=S(r)\,|\psi\rangle$ represents the same system squeezed in the space of the position Q by a factor e^{-r} and expanded in the space of the momentum P by the factor e^{r}. In fact, we deduce

$$e^{-r}\langle\psi'|Q|\psi'\rangle=\langle\psi|Q|\psi\rangle,\quad e^{r}\langle\psi'|P|\psi'\rangle=\langle\psi|P|\psi\rangle. \tag{2.20}$$

Now, we shall evaluate a matrix element involving the operator $H(t)$ [see Eq. (2.14)] in the context of squeezing of a quantum system. In doing so, let us consider the following expectation value between squeezed states:

$$\begin{aligned}\langle\alpha,z|D(\alpha)S(z)H(t)S^{\dagger}(z)D^{\dagger}(\alpha)|\alpha,z\rangle &=\langle 0|S^{\dagger}(z)D^{\dagger}(\alpha)D(\alpha)S(z)H(t)S^{\dagger}(z)D^{\dagger}(\alpha)D(\alpha)S(z)|0\rangle\\ &=\langle 0|H(t)|0\rangle\\ &=\frac{1}{2m}\langle 0|P^{2}|0\rangle+\frac{1}{2}m\omega^{2}(t)\langle 0|Q^{2}|0\rangle,\end{aligned} \tag{2.21}$$

where Eqs. (2.1) and (2.14) have been employed.

The expectation values an the right-hand side of Eq. (2.21) can be evaluated from Eq. (2.7). They read

$$\langle 0|P^{2}|0\rangle=\hbar m|\xi|^{2}=\hbar m\left[\frac{1}{4\sigma^{2}}+\left(\frac{M}{2}\sigma-\dot{\sigma}\right)^{2}\right], \tag{2.22}$$

$$\langle 0|Q^{2}|0\rangle=\frac{\hbar}{m}\sigma^{2}, \tag{2.23}$$

where ξ and σ are described by Eqs. (2.8) and (2.9). With the help of Eqs. (2.22) and (2.23), Eq. (2.21) takes the form

$$\begin{aligned}\langle\alpha,z|\ D(\alpha)S(z)H(t)S^{\dagger}(z)D^{\dagger}(\alpha)\ |\alpha,z\rangle &=\langle 0|H(t)|0\rangle\\ &=\left[\frac{\hbar}{8\sigma^{2}}+\frac{\hbar}{2}\left(\frac{M}{2}\sigma-\dot{\sigma}\right)^{2}\right]+\frac{\hbar}{2}\omega^{2}(t)\ \ \sigma^{2}.\end{aligned} \tag{2.24}$$

The term in the square bracket corresponds to the vacuum expectation value of the kinetic energy of the system, while the last term is related to the vacuum expectation value of the potential energy.

The quantity

$$E_{\mathrm{NM}}\doteq\frac{\hbar}{2}\left(\frac{M}{2}\sigma-\dot{\sigma}\right)^{2} \tag{2.25}$$

(NM=nonminimum) can be interpreted as the energy associated with the squeezed states which do not satisfy the criterium of minimum uncertainty $(E_{\mathrm{NM}}\neq 0)$. When the criterium is verified, then $E_{\mathrm{NM}}=0$. In such a case Eq. (2.24) can be written as

$$\begin{aligned}\langle\alpha,z|D(\alpha)S(z)H(t)S^{\dagger}(z)D^{\dagger}(\alpha)\ |\alpha,z\rangle &=\langle 0|\ H(t)\ |0\rangle=\frac{\hbar}{8\sigma^{2}}+\frac{\hbar}{2}\omega^{2}(t)\ \sigma^{2}.\end{aligned} \tag{2.26}$$

Hence in the minimum uncertainty situation the vacuum expectation values of the kinetic energy and the potential energy turn out to be proportional to σ^{2} and $1/\sigma^{2}$, respectively. It is noteworthy that, in general, all the energies appearing in Eq. (2.21) can be expressed in terms of the auxiliary field $\sigma(t)$ obeying the auxiliary Eq. (2.9). This enhances the convenience of our approach to the study of cosmological quantum effects based on the theory of equation (2.9), which we are going to develop in Sec. III.

[1]Stoler [2] saw that, generally, states of the type $|\alpha,z\rangle$ do not describe wave packets relative to the minimum value of the product $(\Delta Q)(\Delta P)$, where Δ means the variance operation [see Eq. (2.18)]. The state $|\alpha,z\rangle$ can describe a wave packet of minimum uncertainty only if z is real (ϕ=0). In the framework of quantum generalized oscillators, this corresponds to the condition (2.17).

We remark that the quantity $(M\sigma/2-\dot{\sigma})$ is connected with the expectation value of the operator $\{Q,P\}=QP+PQ$ between vacuum states, i.e.,

$$\langle 0|QP+PQ|0\rangle = 2\hbar\sigma\left(\frac{M}{2}\sigma-\dot{\sigma}\right). \tag{2.27}$$

Thus the minimum uncertainty requirement $M\sigma/2=\dot{\sigma}$ implies that the expectation value $\langle 0|QP+PQ|0\rangle$ is vanishing.

B. A link between Eq. (2.12) and the auxiliary equation

For later convenience (see Sec. III), we shall report a result establishing a relationship involving the solutions of the (linear) equation of motion and the (nonlinear) auxiliary equation

$$\ddot{\sigma}+\Omega^2(t)\sigma=\frac{\kappa}{\sigma^3}, \tag{2.28}$$

where κ is a constant. If y_1 and y_2 are two independent solutions of Eq. (2.12), then the general solution of the auxiliary equation (2.28) can be written as [17]

$$\sigma=(Ay_1^2+By_2^2+2Cy_1y_2)^{1/2}, \tag{2.29}$$

with A,B,C arbitrary constants such that

$$AB-C^2=\frac{\kappa}{W_0^2}, \tag{2.30}$$

where $W_0=W_0(y_1,y_2)=y_1\dot{y}_2-\dot{y}_1y_2=\text{const}$ is the Wronskian.

It is worth remarking that from the theory of the auxiliary equation (2.28) a phase can be given by the real function $\theta(t)$,

$$\theta(t)=\int_{t_0}^{t}\frac{dt'}{\sigma^2(t')}. \tag{2.31}$$

(See [18,19,20]; for some applications: [13,16].) Here we shall suggest the procedure which can be used to compute the above integral in general cases. To this aim it is convenient to introduce the function

$$\psi(t)=\sqrt{A}\,e^{i\alpha}y_1(t)-\sqrt{B}\,e^{i\beta}\,y_2(t), \tag{2.32}$$

where α,β are real numbers and y_1,y_2 are the two independent solutions apppearing in Eq. (2.29). We have

$$|\psi(t)|^2=Ay_1^2+By_2^2-2\sqrt{AB}\cos\theta_0\,y_1y_2, \tag{2.33}$$

where $\theta_0=\alpha-\beta$. Comparing Eq. (2.33) with Eq. (2.29) we get $C=-\sqrt{AB}\cos\theta_0$, so that the condition (2.30) becomes

$$AB\sin^2\theta_0=\frac{\kappa}{W_0^2}, \tag{2.34}$$

from which $\kappa\neq 0$ whenever $\sin\theta_0\neq 0$. Hence the auxiliary field σ can be expressed by

$$\sigma^2(t)=|\psi(t)|^2. \tag{2.35}$$

To calculate the phase $\theta(t)$ corresponding to the solution (2.29) of Eq. (2.28), we look for a function $F(t)$ defined by

$$F(t)=\frac{1}{2i\sqrt{\kappa}}\ln\frac{\psi(t)}{\psi^*(t)}, \tag{2.36}$$

so that

$$\dot{F}=\frac{1}{2i\sqrt{\kappa}}\,\frac{\dot{\psi}\psi^*-\psi\dot{\psi}^*}{\psi\psi^*}. \tag{2.37}$$

The numerator in Eq. (2.37) can be elaborated to give

$$\dot{\psi}\psi^*-\psi\dot{\psi}^*=2i\sqrt{AB}\sin\theta_0W_0=2i\sqrt{\kappa}, \tag{2.38}$$

where Eq. (2.34) has been exploited. Substitution from Eq. (2.38) in Eq. (2.37) thus yields

$$\dot{F}=\frac{1}{\sigma^2}. \tag{2.39}$$

Then the phase $\theta(t)$ is determined by integrating Eq. (2.31), namely

$$\theta(t)=\int_{t_0}^{t}\frac{dt'}{\sigma^2(t')}=F(t)-F(t_0), \tag{2.40}$$

where $F(t)$ is provided by Eq. (2.36). We shall recall this general result later.

C. The Bogolubov coefficients in terms of σ

By resorting to the operators

$$Q=\sqrt{\frac{\hbar}{2\omega_0m_0}}\;(\hat{a}_0+\hat{a}_0^\dagger),\qquad P=-i\sqrt{\frac{\hbar\omega_0m_0}{2}}(\hat{a}_0-\hat{a}_0^\dagger), \tag{2.41}$$

where $\hat{a}_0=\hat{a}(t_0)$ is the (mode) annihilation operator in the Schrödinger representation, combining Eq. (2.41) and (2.7) we can derive the Bogolubov transformation

$$\hat{a}(t)=\mu(t)\hat{a}_0+\nu(t)\hat{a}_0^\dagger \tag{2.42}$$

whose coefficients are expressed by

$$\mu(t)=\sqrt{\frac{m}{2\omega_0m_0}}\left(-i\xi^*+\frac{\omega_0m_0}{m}\sigma\right),$$

$$\nu(t)=\sqrt{\frac{m}{2\omega_0m_0}}\left(-i\xi^*-\frac{\omega_0m_0}{m}\sigma\right), \tag{2.43}$$

with $m_0 = m(t_0)$, $\omega_0 = \omega(t_0)$, and ξ given by Eq. (2.8). We note that the Bogolubov transformation can be naturally embedded into the relations (2.5) and (2.6), where the operators b, $b^\dagger$ can be identified with $\hat{a}(t)$ and $\hat{a}^\dagger(t)$. Equation (2.43) entails

$$|\mu|^2 - |\nu|^2 = 1. \tag{2.44}$$

On the other hand, the uncertainty product can be formulated as follows [14]:

$$(\Delta Q)(\Delta P) = \frac{\hbar}{2}\sqrt{1 + 4\sigma^2\left(\frac{M}{2}\sigma - \dot{\sigma}\right)^2} = \frac{\hbar}{2}|\mu(t) - \nu(t)||\mu(t) + \nu(t)| \geqslant \frac{\hbar}{2}. \tag{2.45}$$

The uncertainty formula (2.45) is closely related to the concept of coherent states for the generalized oscillators. Such coherent states were constructed by Hartley and Ray in 1982 [21] taking account of the Lewis-Riesenfield theory [16]. These states share all the features of the coherent states of the conventional (time-independent) oscillator except that of the uncertainty formula, in the sense that the product (ΔQ) (ΔP) turns out to be not minimum. A few years later, Pedrosa showed that the coherent states devised by Hartley and Ray are equivalent to squeezed states [22].

III. CLASSICAL VIEW AND QUANTUM THEORY GENERATION OF GRAVITATIONAL WAVES VIA THE KANAI-CALDIROLA OSCILLATOR

In this section we shall develop a model of propagation of gravitational waves based on the application of the auxiliary equation (2.9) for the function $\sigma(\eta)$, in terms of which the Bogolubov coefficients can be built-up. The Bogolubov transformation is a basic concept in the theory of particle creation in external fields. The created particles do exist in squeezed quantum states [1]. According to [1], relic gravitons created form zero-point quantum fluctuations during cosmological evolution should now be in strongly squeezed states. In this context the generation of gravitational waves is of fundamental importance.

The theory of generation of gravitational waves in the inflationary universe scenario is based on the action [11]

$$S = \frac{1}{16\pi G}\int f(R)\ \sqrt{-g}\ d^4x, \tag{3.1}$$

where $f(R)$ is an arbitrary function of the scalar curvature R. The theory defined by the above action is conformally equivalent to a pure Einstein theory with scalar-field matter. In linear theory, the gravitational waves decouple from the matter field, so that the main problem is to fix the background model and to desume the relation between the conformal metric.

Starting from Eq. (3.1), by varying the action with respect to the gravitational perturbation field h^i_j, the equation of motion

$$h'' + 2\frac{\tilde{a}'}{\tilde{a}}\ h' + (2K - \Delta)h = 0 \tag{3.2}$$

is obtained, where prime denotes $d/d\eta$, $\tilde{a}(\eta) = \sqrt{\partial f/\partial R}a(\eta)$, $h = h(\eta,\vec{x})$ is each component of h^i_j, Δ stands for the Laplace-Beltrami operator, and K means the space curvature. By separating in $h(\eta,\vec{x})$ the dependence on η from the dependence on $\vec{x}$, we can write $h \sim h_0(\eta)h_1(\vec{x})$, so that

$$[\Delta + (n^2 - K)]h_1(\vec{x}) = 0, \tag{3.3}$$

and

$$h_0'' + 2\frac{\tilde{a}'}{\tilde{a}}\ h_0' + (n^2 + K)h_0 = 0. \tag{3.4}$$

Equation (3.4) can be applied to describe the evolution of gravitational waves in any state of the evolution of the universe, even when resorting to higher derivative theories of gravity [11]. The elimination of the first derivative in Eq. (3.4) leads to the equation (called master equation in [1])

$$y'' + [(n^2 + K) - V(\eta)]y = 0, \tag{3.5}$$

with

$$V(\eta) = \frac{a''}{a}, \qquad y(\eta) = \frac{a(\eta)}{a(\eta_0)}\ h_0(\eta).$$

We remark that Eq. (3.4) can be regarded as the equation of motion of an oscillator with time-dependent mass m and constant frequency $\overline{\omega}$,

$$q'' + \frac{m'}{m}q' + \overline{\omega}^2 q = 0 \tag{3.6}$$

which is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\overline{\omega}^2}{2}q^2. \tag{3.7}$$

The quantum theory of gravitational waves is therefore equivalent to the quantum theory of the Kanai-Caldirola oscillator [23,24]. The formal analogy is realized upon the identification:

$$m = a^2, \qquad \overline{\omega}^2 = n^2 + K, \qquad q \sim h_0. \tag{3.8}$$

A. Exact solution of the parametrically excited oscillator and its associated auxiliary equation

We are mainly interested in the period under which the universe accelerates, namely in its *inflationary* stage. Inflation is defined to be a period of accelerating expansion. During such a stage, the universe expands adiabatically and the Friedmann equations can be exploited [11]. The prototype of the models of inflationary cosmology is based on the de Sitter spacetime, which is a very interesting case concerned with the constant Hubble rate and the scale factor given by

$$a(\eta)=-\frac{1}{H_0\eta}, \tag{3.9}$$

where $\eta<0$ and H_0 denotes the Hubble constant. In the de Sitter case, Eq. (3.5) takes the form of Eq. (2.12), with $\Omega^2(\eta)=n^2-2/\eta^2$. It admits the general solution

$$y=\sqrt{-n\eta}[\kappa_1 J_{3/2}(-n\eta)+\kappa_2 J_{-3/2}(-n\eta)], \tag{3.10}$$

where $J_{3/2}$, $J_{-3/2}$ are Bessel functions of the first kind and the arbitrary constants κ_1 and κ_2 are determined once the initial conditions are imposed. Then from Eq. (2.29) we infer that Eq. (2.28), which now reads

$$\sigma''+\left(n^2-\frac{2}{\eta^2}\right)\sigma=\frac{\kappa}{\sigma^3}, \tag{3.11}$$

is exactly solved by

$$\sigma=\sqrt{-n\eta}[AJ^2_{3/2}(-n\eta)+BJ^2_{-3/2}(-n\eta)+2CJ_{3/2}(-n\eta)\,J_{-3/2}(-n\eta)]^{1/2}, \tag{3.12}$$

where constants A,B,C satisfy the condition (2.30).

The phase of the auxiliary field $\sigma(t)$

As we have shown in Sec. II B, to calculate the phase θ corresponding to the solution (3.12) of Eq. (3.11) it is convenient to introduce the function

$$\psi(-n\eta)=\sqrt{-n\eta}[\sqrt{A}\;\;e^{i\alpha}\;\;J_{3/2}(-n\eta)-\sqrt{B}e^{i\beta}J_{-3/2}(-n\eta)], \tag{3.13}$$

where α,β are real numbers. Hence the auxiliary field can be expressed via $\sigma^2=|\psi|^2$ provided that

$$AB\sin^2\theta_0=\frac{\kappa}{W_0}, \tag{3.14}$$

where $\theta_0=\alpha-\beta$ and $W_0=W_0[\sqrt{-n\eta}J_{3/2}(-n\eta),\sqrt{-n\eta}J_{3/2}(-n\eta)]$. Therefore it is an easy matter to see that in the case under consideration the phase θ determined by integrating Eq. (2.40) is given by $(z=-n\eta)$

$$\theta(\eta)=-n\int_{z_0}^{z}\frac{dz'}{\sigma^2(z')}=-n[F(z)-F(z_0)], \tag{3.15}$$

where the function $F(z)$ is defined by

$$F(z)=\frac{1}{2i\sqrt{k}}\ln\frac{\psi(z)}{\psi^*(z)} \tag{3.16}$$

and ψ is given by Eq. (3.13).

B. On the phase of gravitational waves

A deep discussion on the phase of gravitational waves is contained in [25], where this topic is dwelt on both at the classical and quantum level.

Here we confine ourselves to tackle the problem classically. The study of the phase of gravitational waves by a quantum point of view will be done elsewhere. For our purpose, first we observe that the Bessel functions $J_{3/2}$ and $J_{-3/2}$ can be explicitly written as follows [26]:

$$J_{3/2}(z)=\sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z}-\cos z\right),$$

$$J_{-3/2}(z)=-\sqrt{\frac{2}{\pi z}}\left(\sin z+\frac{\cos z}{z}\right).$$

Then, the auxiliary field (2.29) can be written as

$$\sigma(z)=\sqrt{\frac{2}{\pi}}\left[A\left(\frac{\sin z}{z}-\cos z\right)^2+B\left(\sin z+\frac{\cos z}{z}\right)^2-2C\left(\frac{\sin z}{z}-\cos z\right)\left(\sin z+\frac{\cos z}{z}\right)\right]^{1/2}, \tag{3.17}$$

where $y_1=\sqrt{z}J_{3/2}$, $y_2=\sqrt{z}J_{-3/2}$, and the condition (2.30) is understood.

Now we shall see that the phase θ_{GW} of the primordial gravitational waves can be obtained by Eq. (3.17) under the choice

$$A=B\neq 0,\quad C=0, \tag{3.18}$$

and assuming asymptotically negative values of the conformal time. In doing so, Eqs. (3.15), (3.17) provide

$$\sigma^2_{in}\sim\frac{2A}{\pi},\qquad \theta_{GW}\equiv\theta_{in}=\frac{\pi}{2A}(z-z_0), \tag{3.19}$$

with

$$A=\frac{\pi\sqrt{\kappa}}{2n}. \tag{3.20}$$

In the case $n^2\gg|V(\eta)|$, the high frequency waves, e.g., the solutions of the equation

$$y''+n^2y=0, \tag{3.21}$$

correspond to the following behavior of the gravitational perturbation field h:

$$h(\eta)=\frac{1}{a}\sin(n\eta+\rho) \tag{3.22}$$

(ρ is an arbitrary phase). In an expanding universe, the amplitude h of the waves decreases adiabatically for all η. The result represented by the calculation of the phase (3.19) of relic gravitational waves suggests one to interpret formula (3.15) as the phase of gravitational waves not only in the case of the primordial cosmological scenario formally corresponding to $\eta \rightarrow -\infty$. Anyway, this subject, which goes beyond the scope of the present paper, deserves further investigation. Here we recall only that the general solution of Eq. (2.12), which holds in the case of the de Sitter cosmological model, can also be written as

$$y = \frac{c}{\sqrt{\kappa}} \sigma(\eta) \cos[\sqrt{\kappa}\theta(\eta) + \delta], \tag{3.23}$$

where θ is given by Eq. (3.15), c is a Noether invariant of Eq. (2.12), and δ is an arbitrary constant. Hence for any conformal time η in the interval $(-\infty,0)$, the amplitude h of the gravitational perturbation field can be expressed by

$$h \sim \frac{1}{a} \sigma(\eta) \sin[\sqrt{\kappa}\theta(\eta) + \text{const}].$$

C. Quantum gravitational waves: Theory in terms of the auxiliary field $\sigma(\eta)$

The approach to the study of gravitational waves we present in this paper is developed starting from the classical Kanai-Caldirola Hamiltonian [23,24], Eq. (3.7). As we have already pointed out, Eq. (3.4) can be regarded, in fact, as the equation of motion of an oscillator with time-dependent mass $m = a^2$ and constant frequency $\overline{\omega} = \sqrt{n^2 + K}$ [see Eq. (3.6)] described by the Hamiltonian (3.7). So, we make the fundamental identification of the whole temporal part h_0 of the metric fluctuation amplitude as the basic ''coordinate'' variable to quantize as such. (Recall that our procedure for the quantization of gravitational waves is based on the identification of $y = a h_0$ as the variable to quantize.) As a consequence, the quantum theory of gravitational waves turns out to be completely equivalent to the quantum theory of the Kanai-Caldirola oscillator which can be described by the quantum version of the Hamiltonian (3.7). On the grounds of what we learned in Sec. II, the above identification suggests a route which can be successfully pursued whenever we are interested in the characterization of physical effects (quantum decoherence, squeezing, particle production, etc.) emerging from the study of inflationary models in the early universe. Section IV will be devoted to a preliminary exploration of the effectiveness of the idea in the context of expanding universe cosmological models.

IV. APPLICATIONS

By taking full advantage of the formalism introduced in Sec. II, we are in the position to study the dynamical system of the cosmological interest which is described by time-dependent oscillators. In doing so, a key point is the characterization of constants A,B,C in Eq. (2.29). It is concerned with the initial (and boundary) conditions. All dynamical aspects under time evolution are enclosed into the function σ, which obeys the second order nonlinear differential equation (2.9). A general condition on A,B,C is provided by Eq. (2.30). It is not enough, however. Specification of the value and the first time derivative of σ at fixed time is thereby needed. Another condition is associated with the requirement that at the initial time η_i the time-dependent annihilation and creation operators, derived from Eq. (2.7), go into the standard Dirac-like form, Eq. (2.41). As for the final condition, it is to be helpful to reveal that in most cases we want the state at initial time to correspond to a vacuum state. This can be achieved easily under the minimization requirement for $E_{NM} = (\hbar/2)((M/2)\sigma - \dot{\sigma})^2$. Indeed, since it provides a measure of the decoherence at the time η, it has to be vanishing when referring to a vacuum state at the initial time $\eta = \eta_i$. Under these circumstances, the whole set of initial conditions for σ is given by

$$\begin{aligned} &AB - C^2 = \frac{\kappa}{W_0^2}, \\ &\sigma(t_0) - (4\omega^2)^{-1/4} = 0, \\ &\dot{\sigma}(t_0) - \frac{M(t_0)}{2}\sigma(t_0) = 0. \end{aligned} \tag{4.1}$$

In a cosmological framework of the FRW type, the above system is translated into

$$\begin{aligned} &AB - C^2 = \frac{\kappa}{W_0^2}, \\ &\sigma(\eta_i) = [4(n^2 + K)]^{-1/4}, \\ &\sigma'(\eta_i) - \frac{a'(\eta_i)}{\sqrt{2n}\,a(\eta_i)} = 0. \end{aligned} \tag{4.2}$$

In order to proceed with concrete analysis, it is very customary to resort to the spatially flat inflationary model based on the de Sitter metric. However, a more general and realistic description of the inflation may be provided by a quasi-de Sitter spacetime (see, e.g., [28]). In this case, the Hubble rate is not exactly constant but, rather, it weakly conformal changes with time according to $\tilde{H}' = -\epsilon a^2 \tilde{H}^2$ [that is, $aa'' = (2-\epsilon) a^4 \tilde{H}^2 = (2-\epsilon) a'^2$] where ϵ is a constant parameter. When ϵ vanishes one gets just the ordinary de Sitter spacetime. For small values of ϵ, a quasi-de Sitter spacetime is associated with the scale factor

$$a(\eta) = \frac{-1}{\tilde{H}(1-\epsilon)\eta} \tag{4.3}$$

($\eta < 0$). In the quasi-de Sitter spatially flat scenario, Eq. (2.12) reads

$$y'' + \left[n^2 - \frac{(2+3\epsilon)}{(1-\epsilon)^2 \eta^2} \right] y = 0 \tag{4.4}$$

and can be solved in terms of Bessel functions. Precisely, one has the two independent solutions

$$y_1=\sqrt{-n\eta}J_\nu(-n\eta),\quad y_2=\sqrt{-n\eta}\;Y_\nu(-n\eta),\tag{4.5}$$

where $\nu=\sqrt{1/4+(2+3\epsilon)/(\epsilon-1)^2}$. The procedure outlined in the previous sections can be applied and we are led to the introduction of the basic function

$$\begin{aligned}\sigma&=(Ay_1^2+By_2^2+2Cy_1y_2)^{1/2}\\&=\sqrt{-n\eta}\{AJ_\nu^2(-n\eta)+BY_\nu^2(-n\eta)\\&\quad+2CJ_\nu(-n\eta)Y_\nu(-n\eta)\}^{1/2},\end{aligned}\tag{4.6}$$

where A,B,C are determined by means of the system (4.2), η_i denoting the conformal time of the beginning of the inflation. Once we are interested in a situation in which the system started very far in the past in a vacuum state, the Bessel function expansions

$$J_\nu(-n\eta)\sim\sqrt{-\frac{2}{\pi n\eta}}\left[\cos\left(-n\eta-\frac{\nu}{2}\pi-\frac{\pi}{4}\right)+O\left(\frac{1}{n\eta}\right)\right],$$

$$Y_\nu(-n\eta)\sim\sqrt{-\frac{2}{\pi n\eta}}\left[\sin\left(-n\eta-\frac{\nu}{2}\pi-\frac{\pi}{4}\right)+O\left(\frac{1}{n\eta}\right)\right],$$

for ν fixed and $n\eta\rightarrow-\infty$ assists us in finding suitable constants A,B,C. By taking arbitrary asymptotically negative initial times, the leading terms of Bessel functions J_ν, Y_ν give rise to the following behavior for the function:

$$\begin{aligned}\sigma(\eta)=\sqrt{\frac{2}{\pi}}\Bigg\{&A+(B-A)\sin^2\left(-n\eta-\frac{\nu}{2}\pi-\frac{\pi}{4}\right)\\&+C\sin\left(-2n\eta-\nu\pi-\frac{\pi}{2}\right)+O\left(\frac{1}{n\eta}\right)\Bigg\}^{1/2}.\end{aligned}$$

Once the limit $\eta_i<<0$ is concerned a natural choice is given by $A=B=\pi/4n$, $C=0$ (recall that we already found this result for the case $\nu=3/2$ associated with the standard de Sitter metric background). So we obtain

$$\begin{aligned}\sigma(\eta)&=\sqrt{-\frac{\pi}{4}\eta}\;\{J_\nu^2(-n\eta)+Y_\nu^2(-n\eta)\}^{1/2}\\&=\sqrt{-\frac{\pi}{4}\eta}\;|H_\nu^1(-n\eta)|.\end{aligned}\tag{4.7}$$

In the light of our previous results, the decoherence energy E_{NM} at the time η of gravitational waves in a quasi-de Sitter model of inflation can be evaluated by inserting (4.7) into formula (2.25). It then results

$$E_{NM}=\frac{\hbar}{2}\left[\sigma'-\frac{a'(\eta)}{a(\eta)}\sigma\right]^2=\frac{\hbar}{2}\left[\sigma'+\frac{\sigma}{(1-\epsilon)\eta}\right]^2,$$

where

$$\begin{aligned}\sigma'+\frac{\sigma}{(1-\epsilon)\eta}=\sqrt{\frac{\pi}{4}}\Bigg\{&\frac{n}{2}\frac{\sqrt{-\eta}}{|H_\nu^1(-n\eta)|}(H_\nu^{1*}H_{\nu+1}^1+\text{c.c.})\\&-\left(\nu+\frac{3-\epsilon}{2(1-\epsilon)}\right)\frac{|H_\nu^1(-n\eta)|}{\sqrt{-\eta}}\Bigg\}.\end{aligned}\tag{4.8}$$

Moreover, since

$$\begin{aligned}\mu(\eta)=\sqrt{\frac{a^2(\eta)}{2na^2(\eta_i)}}\Bigg\{&\left[\frac{1}{2\sigma}+\frac{na^2(\eta_i)}{a^2(\eta)}\sigma(\eta)\right]\\&-i\left[\sigma'-\frac{a'}{a}\sigma\right]\Bigg\},\end{aligned}\tag{4.9}$$

$$\begin{aligned}\nu(\eta)=\sqrt{\frac{a^2(\eta)}{2na^2(\eta_i)}}\Bigg\{&\left[\frac{1}{2\sigma}-\frac{na^2(\eta_i)}{a^2(\eta)}\sigma(\eta)\right]\\&-i\left[\sigma'-\frac{a'}{a}\sigma\right]\Bigg\}\end{aligned}\tag{4.10}$$

at an arbitrary time η the Bogolubov coefficients are given by

$$\begin{aligned}\mu(\eta)=\sqrt{\frac{1}{2n}\left(\frac{\eta_i}{\eta}\right)^{2/(1-\epsilon)}}\Bigg\{&\left[\frac{1}{2\sigma}+\left(\frac{\eta}{\eta_i}\right)^{2/(1-\epsilon)}n\sigma\right]\\&-i\left[\sigma'+\frac{\sigma}{(1-\epsilon)\eta}\right]\Bigg\},\end{aligned}\tag{4.11}$$

$$\begin{aligned}\nu(\eta)=\sqrt{\frac{1}{2n}\left(\frac{\eta_i}{\eta}\right)^{2/(1-\epsilon)}}\Bigg\{&\left[\frac{1}{2\sigma}-\left(\frac{\eta}{\eta_i}\right)^{2/(1-\epsilon)}n\sigma\right]\\&-i\left[\sigma'+\frac{\sigma}{(1-\epsilon)\eta}\right]\Bigg\}\end{aligned}\tag{4.12}$$

with σ and $\sigma'+\sigma/(1-\epsilon)\eta$ furnished by Eq. (4.7) and (4.8), respectively. Finally, the phase θ can be evaluated. Due to Eq. (2.40), we get

$$\theta(\eta)=-i\ln\frac{\psi(\eta)}{\psi^*(\eta)}\Bigg|_{\eta_i}^{\eta},\tag{4.13}$$

where

$$\psi(t)=\sqrt{\frac{\pi}{4n}}\;e^{i\alpha}[y_1+iy_2]=\sqrt{\frac{\pi}{4n}}\;e^{i\alpha}\sqrt{-n\eta}\;H_\nu^1.$$

That is,

$$\theta(\eta)=2\ \theta_\nu^1|_{\eta_i}^{\eta},$$

where θ_ν^1 denotes the phase of the Hankel function H_ν^1.

It is now instructive to focus on a standard de Sitter inflation. In this case $\epsilon=0$, $\nu=\frac{3}{2}$ and

$$\sigma=\frac{1}{\sqrt{2n}}\sqrt{1+\frac{1}{n^2\eta^2}}. \tag{4.14}$$

So, the integration of $1/\sigma^2$ is straight, i.e.,

$$\int_{\eta_i}^{\eta}\frac{d\eta'}{\sigma^2(\eta')}=4n^2\int\frac{n^2\eta^2}{1+n^2\eta^2}d\eta=4n^2\left[\eta-\frac{\tanh^{-1}(n\eta)}{n}\right], \tag{4.15}$$

up to a constant of integration. The above two formulas provide the standard exact (and normalized) solution for the quantum fluctuations of a generic massless scalar field during a de Sitter inflation. Yet, it is interesting to consider a very long inflation by letting the system evolve towards $\eta\to 0$. In this case E_{NM} simply reads

$$E_{NM}(\eta)=\frac{\hbar}{2}\left(\dot{\sigma}-\frac{M}{2}\sigma\right)^2=\frac{\hbar}{4}\frac{n}{1+n^2\eta^2}. \tag{4.16}$$

Interestingly, the decoherence energy at $\eta=0$ is finite. Moreover, in the standard de Sitter phase, from Eq. (4.14) we obtain the Bogolubov coefficients

$$\mu(\eta)=\frac{1}{2}\frac{\sqrt{n^2\eta_i^2}}{\sqrt{1+n^2\eta^2}}\left\{\left[1+\frac{\eta^2}{\eta_i^2}\right]-\frac{i}{n\eta}\right\}, \tag{4.17}$$

$$\nu(\eta)=\frac{1}{2}\frac{\sqrt{n^2\eta_i^2}}{\sqrt{1+n^2\eta^2}}\left\{\left[1-\frac{\eta^2}{\eta_i^2}\right]-\frac{i}{n\eta}\right\}, \tag{4.18}$$

$(n^2\eta_i^2\gg 1)$ which in turn implies that

$$|\nu(\eta)|^2=\frac{1}{4}\left(\frac{\eta_i}{\eta}-\frac{\eta}{\eta_i}\right)^2 \tag{4.19}$$

particles are created out of the vacuum at the time η.

In more refined studies of cosmological effects in the expanding Universe, it turns out to be useful to introduce a cosmological model which allows one to take into account different evolutionary phases of the Universe. Once the model has been specified and Eq. (2.12) solved, one can get an insight into physical effects associated with different cosmological stages. For instance, one can consider a simple cosmological model which includes the inflationary *(i)*, radiation-dominated *(e)*, and matter-dominated *(m)* epochs [29]. The scale factor has the following dependence on the conformal time:

$$a_i(\eta)=-\frac{1}{H_0\eta},\quad \eta_i\leqslant\eta\leqslant\eta_e<0,$$

$$a_e(\eta)=\frac{(\eta-2\eta_e)}{H_0\ \eta_e^2},\quad \eta_e\leqslant\eta\leqslant\eta_m,$$

$$a_m(\eta)=\frac{(\eta+\eta_m-4\eta_e)^2}{4H_0\eta_i^2\ (\eta_2-2\eta_1)},\quad \eta\geqslant\eta_m, \tag{4.20}$$

where H_0 denotes the Hubble constant at the inflationary stage, and η_i represents the beginning of the expansion. In order to determine the η-dependent amplitude h_n for each epoch we have to solve Eq. (2.12) with the corresponding varying frequencies, namely

$$\Omega_i=\left(n^2-\frac{2}{\eta^2}\right)^{1/2},\quad \Omega_e=n,$$

$$\Omega_m=\left[n^2-\frac{2}{(\eta+\eta_m-4\eta_e)^2}\right]^{1/2}. \tag{4.21}$$

Notice that the frequency Ω_e is constant, while Ω_i and Ω_m are varying with the same temporal dependence. The frequency Ω_i characterizes the de Sitter era. The related equation of motion (2.12) has already been solved. Taking care about matching data at η_e and η_m is needed for the knowledge of the complete form of σ. The associated σ's and their derivatives have to join continuosly at η_e and η_m, in fact. This step is needed to obtain all the physical information implied in formulas for the Bogolubov coefficients, the decoherence energy, the gravitational phase, and squeezing. Having in mind our previous discussion, employing the model by considering a quasi-de Sitter phase is straight.

In general, the vacuum expectation value of the number operator and the other quantities of the physical interest vary slowly with time if the expansion rate becomes arbitrarily slow. In case the espansion is stopped one should be able to recover time-independent Dirac operators. However, the circumstance does not mean that Bogolubov coefficients trivialize. This is because loss of coherence previously occurred due to the expansion dynamics. A typical situation may be that $\dot{\sigma}$ goes to zero but σ does not. E_{NM} goes to zero, indicating that when expansion is stopped the time-dependent gravitational pumping stops as well and there is no further decoherence. If expansion stops from time η_a to time η_b, then $\forall\,\eta\in[\eta_a,\eta_b]$ one gets

$$\mu(\eta)=\sqrt{\frac{a^2(\eta_a)}{2na^2(\eta_i)}}\left[\frac{1}{2\sigma(\eta_a)}+n\frac{a^2(\eta_i)}{a^2(\eta_a)}\ \sigma(\eta_a)\right], \tag{4.22}$$

$$\nu(\eta)=\sqrt{\frac{a^2(\eta_a)}{2na^2(\eta_i)}}\left(\frac{1}{2\sigma(\eta_a)}-n\frac{a^2(\eta_i)}{a^2(\eta_a)}\ \sigma(\eta_a)\right), \tag{4.23}$$

which implies

$$|\mu(\eta)|^2=\frac{1}{2}\left[\frac{a^2(\eta_a)}{4na^2(\eta_i)\ \sigma^2(\eta_a)}+n\frac{a^2(\eta_i)}{a^2(\eta_a)}\ \sigma^2(\eta_a)+1\right]. \tag{4.24}$$

To get a more clear insight into the results achieved in this section, a few comments are in order. Specifically, our formula for the decoherence energy associated with the dynamical evolution of the gravitational fluctuation modes on a background of the FRW type is expressed in a very compact form in terms of the auxiliary field $\sigma(\eta)$ and the scale factor $a(\eta)$. On the other hand, the decoherence energy plays an essential role in the relationships for the Bogolubov coefficients. This aspect makes explicit how the energy lost owing to the decoherence effect may be exploited to excite the vacuum state of the model under consideration. Moreover, in our framework this mechanism would be quantified in a compact way by means of the formula

$$|\nu(\eta)|^2=\frac{a^2(\eta)}{2na^2(\eta_i)}\left[\left(\frac{1}{2\sigma}-\frac{na^2(\eta_i)}{a^2(\eta)}\sigma\right)^2+\frac{2}{\hbar}E_{NM}\right], \tag{4.25}$$

where E_{NM} is the decoherence energy ($m=a^2$). With respect to other works, in our paper the role of the decoherence energy is made manifest. Furthermore, we observe that, remarkably, the auxiliary field σ is nothing but the time-dependent amplitude of the mode solutions to Eq. (2.12) for the redshifted gravitational field fluctuations.

V. CONCLUDING REMARKS

The main results achieved in this paper have been presented and widely discussed in the Introduction. Therefore we shall conclude by making some final comments concerning challenging perspectives which should be dwelt upon in future developments. The evolution equation of a mode with comoving wave number n reduces to the harmonic oscillator equation with time-dependent mass and constant frequency. The approximation behind computations leading to the result actually are applicable only to the infrared region. On general grounds, one therefore expects that predictions for observables may depend sensitively on the physics on the length scales smaller than the Planck one. In order to take into account trans-Planckian physics, it has been recently suggested to make use of effective dispersion relations (see, e.g., [27]). The linear dispersion relation is thus replaced by a nonlinear one, $n_{\rm eff}^2=a^2(\eta)F^2(n/a)$, where $F(n/a)$ is an arbitrary function required to behave linearly whenever n/a $(=k)$ is below a certain threshold. A time dependent dispersion relation thus enters in the matter. As a consequence, the underlying dynamical model turns out to be that of the harmonic oscillator with both the mass m $(=a^2)$ and the frequency $\omega(=n_{\rm eff})$ depending on time. In principle, its quantization can still be pursued by resorting to the formalism of Sec. II and it will be studied in details elsewhere. Nevertheless, a comment is in order. In the light of the discussion in [14], one might wonder, in fact, on whether or not in the cosmological framework the minimum uncertainty criterium can be satisfied under time evolution for some physically reasonable function F. It is straightforwardly seen that this is not the case, generally speaking. Indeed, in the cosmological framework the criterium reads as $a^2n_{\rm eff}=$const and implies a purely cubic function F, say $F=\alpha_0 n^3/a^3$. As a consequence, the uncertainty relation can be minimized only approximately. It is worth noting that this happens in the large wave numbers limit of a special case of the generalized Corley-Jacobson dispersion relation introduced in [27] see Eq. (22) in [27].

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