A parametrization for the neutrino mixing matrix

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We propose a flexible and model independent parametrization of the neutrino mixing matrix, which takes advantage of the fact that there are up to three small quantities in neutrino mixing phenomenology: (i) the deviation from maximal mixing of solar neutrinos, (ii) the mixing matrix element U_{e3}, and (iii) the deviation from maximal mixing of atmospheric neutrinos. It is possible to quantify those three observations with a parameter $\lambda \sim 0.2$, which appears at least linearly in all elements of the mixing matrix. The limit $\lambda \to 0$ corresponds to exact bimaximal mixing. Present and future experiments can be used to pin down the power of λ required to usefully describe the observed phenomenology. Observing that the ratio of the two measured mass squared differences is roughly λ^2 allows us to further study the structure of the Majorana mass matrix. We comment on the implications of this parametrization for neutrinoless double beta decay and on the oscillation probabilities in long-baseline experiments.

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I. INTRODUCTION

Neutrino physics has made impressive progress in recent years [1]. In particular, the structure of the neutrino mixing matrix has been identified to a reasonable precision. The final step for the determination of its structure has come from the KamLAND experiment [2], which confirmed the large mixing angle solution for the solar neutrino problem, after a preference for this parameter space was already implied by the data of the various solar neutrino experiments [3]. Very recently, the SNO salt phase data [4] rejected maximal solar mixing by more than 5σ [4–6]. The (almost) maximal mixing of atmospheric neutrinos has been found by the SuperKamiokande experiment [7] and confirmed by the K2K Collaboration [8]. Finally, the presence of a small if not zero angle was implied by reactor experiments [9].

In the present paper we wish to propose a parametrization of the neutrino mixing matrix in terms of a small parameter λ, whose magnitude is interestingly around 0.2, i.e., close to the Wolfenstein parameter used to parametrize the Cabibbo-Kobayashi-Maskawa (CKM) matrix [10]. In any parametrization of the neutrino mixing matrix (for earlier attempts, see Refs. [11–13]), it is convenient to start from a reference matrix and describe deviations from it. Our reference matrix is the one corresponding to exact bimaximal neutrino mixing. The parameter λ describes up to three small deviations from this mixing scheme; namely, the deviation from maximal mixing of solar neutrinos, the deviation from zero U_{e3}, and the deviation from maximal mixing of atmospheric neutrinos. The magnitude of λ is defined by the observed nonmaximality of solar neutrino mixing [4–6] and future precision experiments can be used to pin down the power of λ to usefully describe the other two deviations. In addition, the ratio of the mass squared differences governing solar and atmospheric neutrino oscillations is given by λ^2, so that it is possible to analyze also the structure of the neutrino mass matrix (provided neutrinos are Majorana particles). We also analyze the oscillation probabilities for long baseline experiments and the effective mass as measured in neutrinoless double beta decay within our parametrization.

The paper is organized as follows: In Sec. II we describe the neutrino mixing parameters as implied by current data and outline the idea of our parametrization. Then, in Sec. III we give the form of the mixing matrix for various special cases of the parametrization and analyze in Sec. IV the form of the neutrino mass matrix. In Sec. V we apply our parametrization to the effective mass as measured in neutrinoless double beta decay and to long-baseline oscillation experiments. We conclude in Sec. VI.

II. QUARK VERSUS LEPTON MIXING

The Wolfenstein parametrization [10] of the CKM matrix uses the fact that the quark mixing is very small, i.e., the mixing matrix is approximately the unit matrix with only small corrections to the off-diagonal entries. In terms of mixing angles, a hierarchy of the form $\theta_{12} \sim 0.1 > \theta_{23} \sim 0.01 > \theta_{13} \sim 0.001$ is observed. This has been used by Wolfenstein to introduce an expansion parameter λ describing the mixing between u and s quarks. The observation that $c-b(u-b)$ mixing is roughly one (two) orders of magnitude suppressed then leads to

$$V_{\rm CKM} \simeq \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho+i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{1}$$

Of course, λ corresponds to the Cabibbo angle $\sin\theta_C \simeq 0.22$, whereas the other parameters are about [14] $A \simeq 0.83$, $\rho \simeq 0.23$, and $\eta \simeq 0.36$. The latter parameter describes the CP violation in the quark sector; all such effects have to be proportional to [15]

$$J_{CP}=\mathrm{Im}\{V_{ud}V_{cb}V_{ub}^{*}V_{cd}^{*}\}\simeq -A^2\lambda^6\eta\sim -3\times 10^{-5}. \tag{2}$$

Therefore, *CP* violation in the quark sector is a small effect.

A. Neutrino mixing

The neutrino oscillation data can consistently be described within a 3-neutrino mixing scheme with massive neutrinos, in which the flavor states ν_α ($\alpha=e,\mu,\tau$) are mixed with the mass states $\nu_i (i=1,2,3)$ via $U_{\rm PMNS}$, the unitary Pontecorvo-Maki-Nagakawa-Sakata [16] lepton mixing matrix. It can be parametrized as

$$U_{\rm PMNS}=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13}e^{i\delta} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}e^{i\delta}\end{pmatrix} \mathrm{diag}(1,e^{i\alpha},e^{i\beta}), \tag{3}$$

where δ is a Dirac *CP* violating phase, α and β are possible Majorana *CP* violating phases [17], $c_{ij}=\cos\theta_{ij}$, and $s_{ij}=\sin\theta_{ij}$. We shall not consider the two Majorana phases in this section. The angles θ_{12} and θ_{23} control the oscillations of solar and atmospheric neutrinos, respectively. The angle θ_{13} is mainly limited by reactor $\overline{\nu_e}$ experiments: The Dirac phase δ can be measured in long baseline neutrino oscillation experiments (see, e.g., Ref. [18]).

To obtain information about the PMNS matrix one fits [5,6,19] the results of neutrino experiments to the hypothesis of neutrino oscillations. The relevant formula for the oscillation probabilities is

$$P(\nu_\alpha\to\nu_\beta)=\delta_{\alpha\beta}-2\,\mathrm{Re}\sum_{j>i}U_{\alpha i}U_{\alpha j}^{*}U_{\beta i}^{*}U_{\beta j}\times\left(1-\exp\frac{i\Delta m_{ji}^2 L}{2E}\right), \tag{4}$$

where $\Delta m_{ji}^2=m_j^2-m_i^2$. The 1σ (3σ) ranges of recent analyses read in terms of the parametrization Eq. (3):

$$\begin{aligned}(0.27)0.35\leqslant\tan^2\theta_{12}\leqslant 0.52(0.72)\ (\text{Ref.}[5]),\\ (0.45)0.75\leqslant\tan^2\theta_{23}\leqslant 1.3(2.3)\ (\text{Ref.}[19]),\\ 0(0)\leqslant\sin^2\theta_{13}\leqslant 0.029(0.074)\ (\text{Ref.}[5]).\end{aligned} \tag{5}$$

The best-fit points are given by

$$\tan^2\theta_{12}=0.43\ \ (\text{Ref.}[5]),\quad \tan^2\theta_{23}=1\ \ (\text{Ref.}[19]). \tag{6}$$

For θ_{23} there is an ambiguity corresponding to $\theta_{23}\leftrightarrow\pi/2-\theta_{23}$, i.e., the angle lies on the "light" or "dark side." Matter effects in future long-baseline experiments will resolve this. In terms of the often used parameter $\sin^2 2\theta_{23}$, which is blind to this ambiguity, one has at 1σ (3σ)$\sin^2 2\theta_{23}\geqslant 0.86$ (0.84).

Regarding the mass squared differences, the best-fit values are $(\Delta m_\odot^2)_{\rm BF}=7.2\times 10^{-5}$ eV^2 [5] and $(\Delta m_A^2)_{\rm BF}=2.6\times 10^{-3}$ eV^2 [19]. A recent preliminary analysis of the SuperKamiokande collaboration, taking into account, e.g., improved identification criteria and neutrino fluxes, yields a value of $(\Delta m_A^2)_{\rm BF}=2.0\times 10^{-3}$ eV^2 [20].

There are two possible mass orderings, the normal and the inverse mass ordering:

normal mass ordering:

$$\Delta m_\odot^2=\Delta m_{21}^2\ll\Delta m_{32}^2\simeq\Delta m_{31}^2=\Delta m_A^2,$$

inverse mass ordering:

$$\Delta m_\odot^2=\Delta m_{21}^2\ll -\Delta m_{31}^2\simeq -\Delta m_{32}^2=\Delta m_A^2. \tag{7}$$

Extreme cases are the normal (inverse) hierarchy with $m_3=\Delta m_A^2\gg m_2=\Delta m_\odot^2\gg m_1$ ($m_2=\Delta m_A^2\simeq m_1\gg m_3$) and the quasidegenerate mass scheme for which $m_3^2\simeq m_2^2\simeq m_1^2\gg\Delta m_A^2$. The latter is fulfilled for values of the neutrino masses larger than ~ 0.2 eV.

Ignoring the phases, the "best-fit PMNS matrix" reads

$$U_{\rm PMNS}^{\rm BF}=\begin{pmatrix} 0.84 & 0.55 & 0 \\ -0.39 & 0.59 & 0.71 \\ 0.39 & -0.59 & 0.71\end{pmatrix}. \tag{8}$$

In the pre-SNO salt-phase analysis of Ref. [19] there was given the 3σ range of the PMNS matrix:

$$|U_{\rm PMNS}|=\begin{pmatrix} 0.73-0.88 & 0.47-0.67 & 0-0.23 \\ 0.17-0.57 & 0.37-0.73 & 0.56-0.84 \\ 0.20-0.58 & 0.40-0.75 & 0.54-0.82\end{pmatrix}, \tag{9}$$

where the phase δ was allowed to take arbitrary values.

Next-generation long-baseline experiments will be able to probe Δm_A^2 and $\sin 2\theta_{23}$ to % accuracy [21]. The element U_{e3} can be probed down to the level 10^{-3} in future long-baseline or reactor experiments [22]. Neutrino factories [23] can improve these bounds considerably. The solar neutrino mixing angle $\tan^2\theta_{\rm sol}$ will see its error reduced below 10% by experiments investigating the low energy neutrino fluxes from the sun [24].

Currently no information about leptonic *CP* violation exists. In oscillation experiments one can detect *CP* violating effects [18], which have to be proportional to [25]

$$\begin{aligned} J_{CP} &= \mathrm{Im}\{U_{e1}\ \ U_{\mu 2}\ \ U^*_{\mu 1}U^*_{e2}\} \\ &= \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta \\ &\simeq \frac{\theta_{13}}{4}(1-\text{corrections from }\ \theta_{13}\ \text{ and} \\ &\qquad \text{nonmaximal }\ \theta_{12}\ \text{ and }\ \theta_{23}), \end{aligned} \tag{10}$$

where the value $\theta_{13}/4$ is the limit for small θ_{13}, maximal θ_{12} and θ_{23}.

B. The strategy

As a very useful limit, the bimaximal mixing pattern [26], corresponding to $\theta_{12}=\theta_{23}=\pi/4$ and $\theta_{13}=0$, can be considered. The resulting mixing matrix, ignoring the *CP* violating phases, reads

$$U^{\rm bimax}_{\rm PMNS}=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1/\sqrt{2} \\ \frac{1}{2} & -\frac{1}{2} & 1/\sqrt{2} \end{pmatrix}. \tag{11}$$

The nonzero entries therefore take values of $1/\sqrt{2}$ and $\pm\frac{1}{2}$. This form of $U_{\rm PMNS}$ shall be our reference matrix, whose deviations are to be described by some small parameter λ. In some sense, the matrix (11) is the analogue to the unit matrix in case of quark mixing. Corrections of order λ and higher to the unit matrix lead to the CKM matrix as parametrized in Eq. (1). In the same way, corrections to $U^{\rm bimax}_{\rm PMNS}$ of order λ and higher can lead to the observed neutrino mixing phenomenology with nonmaximal solar neutrino mixing, possible nonmaximal atmospheric neutrino mixing and nonzero U_{e3}. One might state that the unit matrix in the quark sector and the bimaximal mixing matrix in Eq. (11) are the zeroth order form of the relevant mixing matrix. We shall comment on a possible origin of corrections to bimaximal mixing (along the lines of Ref. [27]) in Sec. III G.

Tries to parametrize the PMNS matrix in analogy to the CKM matrix suffer in general from the fact that from the 9 elements of $U_{\rm PMNS}$ only one is small, namely the element $|U_{e3}|\lesssim 0.27$. The other eight entries take typically values around 1/2 or $1/\sqrt{2}$. There are to our knowledge two other approaches to do something similar to the PMNS matrix as has been done so successfully with the CKM matrix. The analysis from Ref. [11] uses $U_{e2}\simeq\sqrt{2}\lambda$, atmospheric neutrino mixing remains maximal, and U_{e3} is proportional to λ^2. The work [12] chooses the expansion parameter $\lambda = U_{\mu 3}\simeq 1/\sqrt{2}$. Solar neutrino mixing is governed by $U_{e2} = A\lambda^2$ and the small quantity U_{e3} has to be introduced at least the eights power of λ. Effects due to *CP* violation in neutrino oscillations are—courtesy of Eq. (10)—proportional to at least λ^{11}. An ansatz for $U_{\rm PMNS}$ corresponding to $\tan^2\theta_{12}=0.5$, maximal atmospheric mixing, and zero U_{e3} has been used in Ref. [13]. To describe deviations from it, it has been multiplied with a Wolfenstein-like matrix.

In this article we wish to propose a purely phenomenological and model independent parametrization of the PMNS matrix by using a small "expansion" parameter λ. For a useful analysis in terms of a small parameter one requires small quantities in the mixing matrix. The basic idea is given by the identification of up to three such small numbers in neutrino mixing phenomenology, namely:

[(i)] the deviation from maximal mixing of solar neutrinos,

[(ii)] the small mixing element U_{e3}, and

[(iii)] the possible deviation from maximal mixing of atmospheric neutrinos.

Those three aspects describe all possible deviations from the bimaximal mixing scheme in Eq. (11). Observation (i) is now solid experimental evidence, after inclusion of the SNO salt phase data [4]; it now holds that $\tan^2\theta_\odot<1$ at more than 5σ [4–6]. Regarding observation (ii), only the mentioned limit of $|U_{e3}|^2\leqslant 0.07$ (at 3σ) exists. Best-fit points of three flavor analyzes of all neutrino data typically yield very small if not vanishing values for this quantity. Finally, atmospheric neutrino mixing must be described by solutions with a best fit corresponding to maximal mixing. This remains true also when the K2K data are included or separately analyzed (e.g., Ref. [19]). Though exactly maximal mixing and zero U_{e3} would hint at some underlying symmetry in the lepton sector, one cannot expect radiative corrections to allow these extreme values to persist down to low energy [28]. Thus, one expects nonextreme values for θ_{13} and θ_{23}. See, e.g., Refs. [27,29] for ways to generate deviations from the bimaximal mixing scheme.

All in all, the three observations (i) to (iii) together with the mixing matrices (8) and (9) lead us to parametrize three elements of the mixing matrix as

$$\begin{aligned} U_{e2} &= \sqrt{\frac{1}{2}}(1-\lambda), \\ U_{e3} &= A\lambda^n, \\ U_{\mu 3} &= \sqrt{\frac{1}{2}}(1-B\lambda^m)e^{i\delta}. \end{aligned} \tag{12}$$

For $\lambda=0$ one would have the bimaximal scheme from Eq. (11).[1] The two Majorana phases are left out for the moment. Unitarity of $U_{\rm PMNS}$ suffices to calculate the remaining elements. The parameters A and B are numbers of order one.

[1]Deviation from maximal solar neutrino mixing has also been analyzed in terms of the parameter $\epsilon=1-2\sin^2\theta_{12}$ [30], which roughly corresponds to λ: $\epsilon\simeq\lambda+\mathcal{O}(\lambda^2)$.

The $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity reflects in a sign ambiguity of B. The power of λ in the expressions for U_{e3} ($U_{\mu 3}$) can be adjusted when more stringent limits (more precision data) are available.

We can take the best-fit value from Eq. (6) to calculate $\lambda \simeq 0.22$, which is remarkably similar to the Wolfenstein parameter or the sine of the Cabibbo angle. The maximal allowed value of $|U_{e3}|^2 = 0.07$ corresponds to $U_{e3} = A\lambda$ with $A \simeq 1.2$. At 1σ (3σ), the range of λ lies in

$$\lambda \simeq (0.08)0.18 - 0.28(0.35). \tag{13}$$

For the best-fit points of the many available analyzes [6], which lie in the range between 0.41 and 0.44 for $\tan^2\theta_{12}$, λ is between 0.24 and 0.22.

If indeed $\lambda \simeq 0.22$, then for $m=1$ it must hold that $B \lesssim 0.91$ in order to fulfill the requirement $\sin^2 2\theta_{23} \gtrsim 0.85$. In the following we shall work with the "best-fit" value of $\lambda = 0.22$. If the limit on $|U_{e3}|^2$ goes below $\sim 10^{-2}$ one should take the power $n=2$ in Eq. (12) in order to keep A of order one. If $|U_{e3}|^2 \lesssim 10^{-4}$, then $n=3$ is advantageous to choose.

Analogously, since typically (see below) $\sin^2 2\theta_{23} \simeq 1 - 4B^2\lambda^{2m}$ one should for values larger than $\sin^2 2\theta_{23} \simeq 0.95$ (0.99) use the power $m=2$ ($m=3$) in Eq. (12). Values of $m=4$ would be required if a precision in $\sin^2 2\theta_{23}$ of order 10^{-4} was present, which seems improbable unless a neutrino factory will be operative. In terms of $\tan^2\theta_{23}$, which in the future will be more appropriate to use, one will find that $\tan^2\theta_{23} \simeq 1 - 4B\lambda^m$. Thus, for $\tan^2\theta_{23} \gtrsim 0.7$ (or $\lesssim 1.3$) one should take $m=2$, while for $\tan^2\theta_{23} \gtrsim 0.9$ (or $\lesssim 1.1$) the value $m=3$ is more useful.

We shall now consider several different cases for the powers of λ in Eq. (12). The considerations from this section indicate that current data and the precision of future experiments on θ_{23} and θ_{13} limit the realistic values of m and n between 1 and 3.

III. THE MIXING MATRIX

A. Case $m=n=1$

In this case we have $U_{e3} = A\lambda$ and $U_{\mu 3} = \sqrt{\frac{1}{2}}(1 - B\lambda)e^{i\delta}$. It corresponds to rather large deviations from the extreme bimaximal values. One "predicts" U_{e3} very close to its current limit and also $\sin^2 2\theta_{23}$ is on the edge of its 3σ range. We identify

$$\begin{aligned}
\tan^2\theta_{12} &\simeq 1 - 4\lambda + 2(5+A^2)\lambda^2 + \mathcal{O}(\lambda^3),\\
\sin^2 2\theta_{23} &\simeq 1 - 4B^2\lambda^2 + \mathcal{O}(\lambda^3),\\
\tan^2\theta_{23} &\simeq 1 - 4B\lambda + 2(A^2+5B^2)\lambda^2 + \mathcal{O}(\lambda^3),\\
\sin^2\theta_{13} &= A^2\lambda^2.
\end{aligned} \tag{14}$$

The fact that $\sin^2 2\theta_{23}$ is blind to the $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity is reflected in the fact that B appears quadratically in the last expression. The form of U_{PMNS} is rather lengthy, and we shall give it therefore only to order λ:

$$U_{\mathrm{PMNS}} \simeq \begin{pmatrix} \sqrt{\frac{1}{2}}(1+\lambda) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda \\ -\frac{1}{2}[1-(1-B-Ae^{i\delta})\lambda] & \frac{1}{2}[1+(1+B-Ae^{i\delta})\lambda] & \sqrt{\frac{1}{2}}(1-B\lambda)e^{i\delta} \\ \frac{1}{2}[1-(1+B+Ae^{i\delta})\lambda] & -\frac{1}{2}[1+(1-B+Ae^{i\delta})\lambda] & \sqrt{\frac{1}{2}}(1+B\lambda)e^{i\delta} \end{pmatrix} + \mathcal{O}(\lambda^2). \tag{15}$$

The precise form to a given order of λ is easily obtained by using the unitarity of the mixing matrix. The corrections quadratic in λ are functions of A and B except for U_{e1}, which receives only corrections depending on A. It is important to note that the corrections from λ are responsible for the deviations from the values $\pm 1/2$ of the entries in the lower left 12 block. Finally, the invariant measure of CP violation in neutrino oscillations is

$$J_{CP} = \frac{A\lambda}{4}[1 - (2 + A^2 + 2B^2)\lambda^2]\sin\delta + \mathcal{O}(\lambda^4). \tag{16}$$

Noting that $A\lambda \simeq \theta_{13}$, the corrections stemming from θ_{13} and nonmaximal $\theta_{12,23}$—as indicated in Eq. (10)—are easily identified. The larger the deviations from maximal $\theta_{12,23}$, i.e., the larger A and B, the smaller becomes J_{CP}. The "prediction" is that CP violating effects are up to $\sim 5\%$. Note, however, that actual experiments searching for leptonic CP violation will not just measure J_{CP} (see Sec. V B).

B. Case $m=1$ and $n=2$

For these values we have $U_{e3} = A\lambda^2$ and $U_{\mu 3} = \sqrt{\frac{1}{2}}\,(1 - B\lambda)e^{i\delta}$. We can identify

$$\begin{aligned}
\tan^2\theta_{12} &\simeq 1 - 4\lambda + 10\lambda^2 + \mathcal{O}(\lambda^3),\\
\sin^2 2\theta_{23} &\simeq 1 - 4B^2\lambda^2 + \mathcal{O}(\lambda^3),\\
\tan^2\theta_{23} &\simeq 1 - 4B\lambda + 10B^2\lambda^2 + \mathcal{O}(\lambda^3),\\
\sin^2\theta_{13} &= A^2\lambda^4.
\end{aligned} \tag{17}$$

The "predictions" are $|U_{e3}|^2 \sim 10^{-3}$ and atmospheric mixing very close to the end of its allowed 3σ range. The mixing matrix $U_{\rm PMNS}$ reads

$$U_{\rm PMNS} \simeq \begin{pmatrix} \sqrt{\frac{1}{2}}(1+\lambda) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda^2 \\ -\frac{1}{2}[1-(1-B)\lambda] & \frac{1}{2}[1+(1+B)\lambda] & \sqrt{\frac{1}{2}}(1-B\lambda)e^{i\delta} \\ \frac{1}{2}[1-(1+B)\lambda] & -\frac{1}{2}[1+(1-B)\lambda] & \sqrt{\frac{1}{2}}(1+B\lambda)e^{i\delta} \end{pmatrix} + \mathcal{O}(\lambda^2). \tag{18}$$

It is obtained by removing the term $Ae^{i\delta}$ from the PMNS matrix in the case of $m=n=1$ as given in Eq. (15). The corrections of order λ^2 for the lower left 2 by 2 submatrix are functions of A and B. They are constant for U_{e1} and only depending on B for $U_{\tau 3}$. Effects of CP violation are proportional to λ^2,

$$J_{CP} \simeq \frac{A\lambda^2}{4}[1-2(1+B^2)\lambda^2]\sin\delta + \mathcal{O}(\lambda^5), \tag{19}$$

and not more than a few %.

C. Case $m=2$ and $n=1$

Now our parameters read $U_{e3}=A\lambda$ and $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^2)e^{i\delta}$. The mixing angles are

$$\begin{aligned} \tan^2\theta_{12} &\simeq 1-4\lambda+2(5+A^2)\lambda^2+\mathcal{O}(\lambda^3), \\ \sin^2 2\theta_{23} &\simeq 1-(A^2-2B)^2\lambda^4+\mathcal{O}(\lambda^5), \\ \tan^2\theta_{23} &\simeq 1+2(A^2-2B)\lambda^2+\mathcal{O}(\lambda^4), \\ \sin^2\theta_{13} &= A^2\lambda^2. \end{aligned} \tag{20}$$

Thus, U_{e3} is close to its current limit and the deviation from $\sin^2 2\theta_{23}=1$ is not more than a few %. The mixing matrix $U_{\rm PMNS}$ is given by

$$U_{\rm PMNS} \simeq \begin{pmatrix} \sqrt{\frac{1}{2}}(1+\lambda) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda \\ -\frac{1}{2}[1-(1-Ae^{i\delta})\lambda] & \frac{1}{2}[1+(1-Ae^{i\delta})\lambda] & \sqrt{\frac{1}{2}}(1-B\lambda^2)e^{i\delta} \\ \frac{1}{2}[1-(1+Ae^{i\delta})\lambda] & -\frac{1}{2}[1+(1+Ae^{i\delta})\lambda] & \sqrt{\frac{1}{2}}(1+B\lambda^2)e^{i\delta} \end{pmatrix} + \mathcal{O}(\lambda^2), \tag{21}$$

which is obtained from Eq. (15) by removing B from the lower left 2 by 2 submatrix and by the presence of λ^2 in $U_{\mu 3}$ and $U_{\tau 3}$. The quadratic corrections are functions of A and B except for U_{e1}, which only depends on A. The rephasing invariant CP violation measure is

$$J_{CP} \simeq \frac{A\lambda}{4}[1-(2+A^2)\lambda^2]\sin\delta + \mathcal{O}(\lambda^4) \tag{22}$$

being rather sizable but not exceeding 5%.

D. Case $m=n=2$

Now it holds $U_{e3}=A\lambda^2$ and $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^2)e^{i\delta}$.

$$
\begin{aligned}
\tan^2\theta_{12} &\simeq 1-4\lambda+10\lambda^2+\mathcal{O}(\lambda^3),\\
\sin^2 2\theta_{23} &\simeq 1-4B^2\lambda^4+\mathcal{O}(\lambda^5),\\
\tan^2\theta_{23} &\simeq 1-4B\lambda^2+\mathcal{O}(\lambda^3),\\
\sin^2\theta_{13} &= A^2\lambda^4.
\end{aligned}
\tag{23}
$$

The deviation from $\sin^2 2\theta_{23}=1$ is not more than a few percent and $|U_{e3}|$ is on the level of 10^{-3}. The mixing matrix $U_{\rm PMNS}$ is given by

$$
U_{\rm PMNS} \simeq \begin{pmatrix}
\sqrt{\frac{1}{2}}(1+\lambda-\lambda^2) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda^2 \\
-\frac{1}{2}[1-\lambda+(B+Ae^{i\delta})\lambda^2] & \frac{1}{2}[1+\lambda-(1-B+Ae^{i\delta})\lambda^2] & \sqrt{\frac{1}{2}}(1-B\lambda^2)e^{i\delta} \\
\frac{1}{2}[1-\lambda-(B+Ae^{i\delta})\lambda^2] & -\frac{1}{2}[1+\lambda-(1+B-Ae^{i\delta})\lambda^2] & \sqrt{\frac{1}{2}}(1+B\lambda^2)e^{i\delta}
\end{pmatrix} + \mathcal{O}(\lambda^3).
\tag{24}
$$

It is seen that for the lower left 2 by 2 submatrix the linear corrections to the "bimaximal" values $\pm 1/2$ are constant and the quadratic ones are functions of A and B. The rephasing invariant CP violation measure is given by

$$
J_{CP} \simeq \frac{A\lambda^2}{4}(1-2\lambda^2)\sin\delta + \mathcal{O}(\lambda^5),
\tag{25}
$$

again on the level of a few percent.

E. The "Wolfenstein case" $m=2$ and $n=3$

How could one not be tempted to put the third power of the expansion parameter in the U_{e3} and the second power in the $U_{\mu 3}$ element. This would resemble the Wolfenstein parametrization Eq. (1). In this case, i.e., $U_{e3}=A\lambda^3$ and $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^2)e^{i\delta}$, we have

$$
\begin{aligned}
\tan^2\theta_{12} &\simeq 1-4\lambda+10\lambda^2+\mathcal{O}(\lambda^3),\\
\sin^2 2\theta_{23} &\simeq 1-4B^2\lambda^4+\mathcal{O}(\lambda^5),\\
\tan^2\theta_{23} &\simeq 1-4B\lambda^2+\mathcal{O}(\lambda^3),\\
\sin^2\theta_{13} &= A^2\lambda^6.
\end{aligned}
\tag{26}
$$

The "prediction" for $\sin^2 2\theta_{23}-1$ is not more than a few percent and $|U_{e3}|^2$ is on the level of 10^{-4}. For the PMNS matrix one has

$$
U_{\rm PMNS} \simeq \begin{pmatrix}
\sqrt{\frac{1}{2}}(1+\lambda-\lambda^2+\lambda^3) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda^3 \\
-\frac{1}{2}[1-\lambda+B\lambda^2-(B-Ae^{i\delta})\lambda^3] & \frac{1}{2}[1+\lambda-(1-B)\lambda^2+(1+B-Ae^{i\delta})\lambda^3] & \sqrt{\frac{1}{2}}(1-B\lambda^2)e^{i\delta} \\
\frac{1}{2}[1-\lambda-B\lambda^2+(B-Ae^{i\delta})\lambda^3] & -\frac{1}{2}[1+\lambda-(1+B)\lambda^2+(1-B+Ae^{i\delta})\lambda^3] & \sqrt{\frac{1}{2}}(1+B\lambda^2)e^{i\delta}
\end{pmatrix} + \mathcal{O}(\lambda^4).
\tag{27}
$$

In contrast to the Wolfenstein parametrization, however, λ appears linearly in e.g. the $U_{\mu 2}$ element. Also, *CP* violation is proportional to the third power of λ,

$$J_{CP} \simeq \frac{A\lambda^3}{4}(1-2\lambda^2)\sin\delta + \mathcal{O}(\lambda^6), \tag{28}$$

not exceeding one percent.

F. The remaining cases

The remaining interesting cases are $m=3$ with $n=2$ and also $m=n=3$. The former case—that is $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^3)e^{i\delta}$ and $U_{e3}=A\lambda^2$—yields

$$\begin{aligned}
\tan^2\theta_{12} &\simeq 1-4\lambda+10\lambda^2+\mathcal{O}(\lambda^3),\\
\sin^2 2\theta_{23} &\simeq 1-4B^2\lambda^6+\mathcal{O}(\lambda^7),\\
\tan^2\theta_{23} &\simeq 1-4B\lambda^3+\mathcal{O}(\lambda^4),\\
\sin^2\theta_{13} &= A^2\lambda^4,
\end{aligned} \tag{29}$$

with $|U_{e3}|^2 \sim 10^{-3}$ and $\sin^2 2\theta_{23} \neq 1$ with a precision of less than 1%. For the PMNS matrix holds:

$$U_{\text{PMNS}} \simeq \begin{pmatrix}
\sqrt{\frac{1}{2}}(1+\lambda-\lambda^2+\lambda^3) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda^2 \\
-\frac{1}{2}[1-\lambda+Ae^{i\delta}\lambda^2+(B+Ae^{i\delta})\lambda^3] & \frac{1}{2}[1+\lambda-(1+Ae^{i\delta})\lambda^2+(1+B+Ae^{i\delta})\lambda^3] & \sqrt{\frac{1}{2}}(1-B\lambda^3)e^{i\delta} \\
\frac{1}{2}[1-\lambda-Ae^{i\delta}\lambda^2-(B+Ae^{i\delta})\lambda^3] & -\frac{1}{2}[1+\lambda-(1-Ae^{i\delta})\lambda^2+(1-B-Ae^{i\delta})\lambda^3] & \sqrt{\frac{1}{2}}(1+B\lambda^3)e^{i\delta}
\end{pmatrix} + \mathcal{O}(\lambda^4). \tag{30}$$

As for the case $m=n=2$ one obtains

$$J_{CP} \simeq \frac{A\lambda^2}{4}(1-2\lambda^2)\sin\delta + \mathcal{O}(\lambda^5), \tag{31}$$

again at most a few percent.

If $m=n=3$, i.e., $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^3)e^{i\delta}$ and $U_{e3}=A\lambda^3$, then it holds for the mixing parameters:

$$\begin{aligned}
\tan^2\theta_{12} &\simeq 1-4\lambda+10\lambda^2+\mathcal{O}(\lambda^3),\\
\sin^2 2\theta_{23} &\simeq 1-4B^2\lambda^6+\mathcal{O}(\lambda^7),\\
\tan^2\theta_{23} &\simeq 1-4B\lambda^3+\mathcal{O}(\lambda^4),\\
\sin^2\theta_{13} &= A^2\lambda^6,
\end{aligned} \tag{32}$$

i.e., except for $\sin^2\theta_{13}$ to the given order identical for the $m=3$ and $n=2$ case above. The parameters A and B appear in the mixing matrix only a third order in λ:

$$U_{\rm PMNS} \simeq \begin{pmatrix} \sqrt{\frac{1}{2}}(1+\lambda-\lambda^2+\lambda^3) & \sqrt{\frac{1}{2}}(1-\lambda) & A\lambda^3 \\ -\frac{1}{2}[1-\lambda+(B+Ae^{i\delta})\lambda^3] & \frac{1}{2}[1+\lambda-\lambda^2+(1+B-Ae^{i\delta})\lambda^3] & \sqrt{\frac{1}{2}}(1-B\lambda^3)e^{i\delta} \\ \frac{1}{2}[1-\lambda-(B+Ae^{i\delta})\lambda^3] & -\frac{1}{2}[1+\lambda-\lambda^2+(1-B+Ae^{i\delta})\lambda^3] & \sqrt{\frac{1}{2}}(1+B\lambda^3)e^{i\delta} \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{33}$$

There are no quadratic terms in the elements $U_{\mu 1}$ and $U_{\tau 1}$. Finally, *CP* violation is governed by

$$J_{CP} \simeq \frac{A\lambda^3}{4}(1-2\lambda^2)\sin\delta + \mathcal{O}(\lambda^6) \tag{34}$$

being below one percent.

The remaining cases $m=3$ and $n=1$ ($m=1$ and $n=3$) are obtained from the cases $m=2$ and $n=1$ ($m=1$ and $n=2$) by setting $B=0$ ($A=0$) in the relevant expressions for the mixing parameters and J_{CP}.

G. Speculations

One can speculate about the origin of the corrections induced by the λ terms. It is possible to imagine, e.g., that the bimaximal mixing scheme from Eq. (11) stems from the diagonalization of the neutrino mass matrix (this is possible, e.g., when a $L_e - L_\mu - L_\tau$ symmetry is present [31]) and any corrections are implied by the unitary matrix U_ℓ that diagonalizes the charged lepton mass matrix [27]. Recall that in a basis in which the charged lepton mass matrix is not diagonal the PMNS matrix is given by $U_\ell^\dagger U$, where U diagonalizes the neutrino mass matrix in that basis. If we define the matrix U_λ, which induces the correction to the bimaximal scheme, we may write $U_{\rm PMNS} \equiv U_\lambda U_{\rm PMNS}^{\rm bimax}$, where $U_{\rm PMNS}^{\rm bimax}$ is given in Eq. (11).[2] Then one can simply solve for U_λ. Taking for definiteness the example $m=3$ and $n=2$, one finds

$$U_\lambda \simeq \begin{pmatrix} 1-\lambda^2/2 & -\lambda/\sqrt{2}+\frac{1+2A}{2\sqrt{2}}\lambda^2 & \lambda/\sqrt{2}+\frac{2A-1}{2\sqrt{2}}\lambda^2 \\ \lambda/\sqrt{2}-\frac{1+2Ae^{i\delta}}{2\sqrt{2}}\lambda^2 & \frac{1+e^{i\delta}}{2}-\frac{\lambda^2}{4} & \frac{-1+e^{i\delta}}{2}+\frac{\lambda^2}{4} \\ -\lambda/\sqrt{2}+\frac{1-2A}{2\sqrt{2}}\lambda^2 & \frac{-1+e^{i\delta}}{2}+\frac{\lambda^2}{4} & \frac{1+e^{i\delta}}{2}-\frac{\lambda^2}{4} \end{pmatrix} + \mathcal{O}(\lambda^3). \tag{35}$$

It is seen that 23 and 32 entries are in general of order one but reduce to order λ^2 for *CP* conservation. Those entries can also be of order λ, however, only for the cases $m=n=1$ and $m=1$ with $n=2$. Thus, if *CP* is conserved and atmospheric neutrino mixing is very close to maximal, the matrix U_λ takes the unit matrix as the dominant form with corrections of order λ. The typical "CKM structure" with very small λ^3 terms is however not necessary.

IV. THE MAJORANA MASS MATRIX

A. Basics

Up to now our analysis assumed only the neutrino oscillation explanation of the experimental data. Now we assume in addition that neutrinos are Majorana particles, which is, e.g., a prediction of the see-saw mechanism [32]. Thus, the neutrino mass matrix m_ν in the basis in which the charged lepton mass matrix is diagonal is given by

$$m_\nu = U_{\rm PMNS} m_\nu^{\rm diag} U_{\rm PMNS}^T. \tag{36}$$

Here $m_\nu^{\rm diag}$ is a diagonal matrix containing the masses $m_{1,2,3}$ of the three massive Majorana neutrinos. An immediate consequence of the Majorana nature of the neutrinos is the presence of two Majorana phases α and β to which neutrino oscillations are insensitive [33]. Information about these phases can be obtained by studying processes in which the total lepton charge L changes by two units, e.g., neutrinoless

[2]This is similar to the strategy in [13], where however a different mixing matrix to start with was used.

double beta decay, $K^+ \to \pi^- + \mu^+ + \mu^+$, etc. Realistically, only neutrinoless double beta decay can expected to be measured [34]. The decay width of this process is sensitive to the *ee* element of m_ν.

An interesting observation is that the ratio of typical best-fit values of the mass squared differences corresponds roughly to the expansion parameter λ:

$$R \equiv \sqrt{\frac{(\Delta m_\odot^2)_{\rm BF}}{(\Delta m_A^2)_{\rm BF}}} \simeq \sqrt{\frac{7.2\times 10^{-5}}{2.0\times 10^{-3}}} \simeq 0.19 \sim \lambda. \quad (37)$$

We took for Δm_A^2 the best-fit point of the preliminary new analysis of the SuperKamiokande collaboration [20]. Using for Δm_A^2 the 90% confidence limit (CL) analysis from [20], which is $(1.3\text{–}3.1)\times 10^{-3}$ eV^2, with the 90% CL range of $\Delta m_\odot^2$ from Ref. [5], we find that R lies between 0.13 and 0.28. This corresponds to a good precision to the 3σ range of λ as given in Eq. (13). In the following we shall assume that $R \simeq \lambda$ and study the resulting structure of the neutrino mass matrix. The results do not change much unless Δm_A^2, $\Delta m_\odot^2$, and $\tan^2\theta_{12}$ are on the very edges of their allowed ranges. Before we perform this analysis, it is useful to study the mass matrix again in the limit of exact bimaximal mixing. In the following, we will neglect the *CP* violating phases, see, e.g., Ref. [35] for an analysis of the structure of m_ν in case of complex entries. Using Eqs. (3) and (36), the mass matrix reads

$$m_\nu = \begin{pmatrix} A & B & -B \\ \cdot & D+\frac{A}{2} & D-\frac{A}{2} \\ \cdot & \cdot & D+\frac{A}{2} \end{pmatrix}, \quad (38)$$

where

$$A = \frac{m_1+m_2}{2}, \quad B = \frac{m_2-m_1}{2\sqrt{2}}, \quad D = \frac{m_3}{2}. \quad (39)$$

As mentioned, there are three extreme cases for the mass hierarchies, the normal hierarchy (NH) with $\sqrt{\Delta m_A^2} = m_3 \gg m_2 \simeq \sqrt{\Delta m_\odot^2} \gg m_1 \simeq 0$, the inverse hierarchy (IH) with $\sqrt{\Delta m_A^2} = m_2 \simeq m_1 \gg m_3 \simeq 0$, and quasidegenerate neutrinos (QD) with $m_0 \equiv m_3 \simeq m_2 \simeq m_1$. Depending on the relative signs of the mass states, several extreme forms of the mass matrix result. In case of NH, one finds for $m_{1,2}=0$:

$$m_\nu = \frac{\sqrt{\Delta m_A^2}}{2} \begin{pmatrix} 0 & 0 & 0 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}, \quad (40)$$

i.e., a mass matrix with a leading $\mu\tau$ block. Regarding IH, the third mass m_3 can safely be neglected. The form of m_ν then depends on the relative sign of the two mass states m_1 and m_2:

$$m_\nu = \sqrt{\Delta m_A^2} \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ \cdot & \frac{1}{2} & -\frac{1}{2} \\ \cdot & \cdot & \frac{1}{2} \end{pmatrix} & \text{same sign} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \cdot & 0 & 0 \\ \cdot & \cdot & 0 \end{pmatrix} & \text{opposite sign.} \end{cases} \quad (41)$$

For the QD spectrum one finds

$$m_\nu = m_0 \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ \cdot & 1 & 0 \\ \cdot & \cdot & 1 \end{pmatrix}, & \mathrm{sgn}(m_1) = \mathrm{sgn}(m_2) = \mathrm{sgn}(m_3) \\ \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \frac{1}{2} \end{pmatrix}, & \mathrm{sgn}(m_1) = \mathrm{sgn}(m_2) = \mathrm{sgn}(m_3) \\ \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \frac{1}{2} \end{pmatrix}, & \mathrm{sgn}(m_1) = \mathrm{sgn}(m_2) = \mathrm{sgn}(m_3) \\ \begin{pmatrix} 1 & 0 & 0 \\ \cdot & 0 & -1 \\ \cdot & \cdot & 0 \end{pmatrix}, & \mathrm{sgn}(m_1) = \mathrm{sgn}(m_2) = \mathrm{sgn}(m_3) \end{cases} \quad (42)$$

We can expect that in our parametrization the parameter λ will appear in the neutrino mass matrix at least linearly in order to correct the extreme values $0, \pm 1/\sqrt{2}, \pm 1$ and $\pm 1/2$.

B. Normal hierarchy

In case of the normal hierarchy we have

$$m_3 = \sqrt{\Delta m_A^2},\ m_2 = \sqrt{\Delta m_\odot^2}\lambda \text{ and } m_1 = D\sqrt{\Delta m_A^2}\lambda^{2+l},\ l \geqslant 0. \quad (43)$$

The expression for m_1 with $D = \mathcal{O}(1)$ expresses our lack of its knowledge. A similar ansatz for the structure of m_ν in case of a normal hierarchical mass scheme has been made in Ref. [11]. For $m=n=l=1$ and all mass states positive the mass matrix looks like

$$m_\nu = \frac{\sqrt{\Delta m_A^2}}{2}\begin{pmatrix} \lambda & [(1+2A)/\sqrt{2}]\lambda & [(2A-1)/\sqrt{2}]\lambda \\ \cdot & 1+\left(\frac{1}{2}-2B\right)\lambda & 1-\lambda/2 \\ \cdot & \cdot & 1+\left(\frac{1}{2}+2B\right)\lambda \end{pmatrix}, \tag{44}$$

neglecting terms of order $\mathcal{O}(\lambda^2)$. The characteristic "leading $\mu\tau$ block" structure of m_ν from Eq. (40) is identified. Corrections at order λ depend on A in the e row of m_ν and on B for the $\mu\mu$ and $\tau\tau$ entries. Higher powers of λ in m_1 will—to order λ^2—lead to the disappearance of D in the formula for m_ν. Setting $U_{e3}=A\lambda^2$ leads to a mass matrix in which to order λ^2 the parameter A does not appear in the $\mu\tau$ submatrix as well as in the ee entry. It is obtained by removing A from the last equation in the indicated entries. The matrix for $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^2)$ but $U_{e3}=A\lambda$ is given by Eq. (44) by removing B from the first row and from the linear terms of the $\mu\mu$ and $\tau\tau$ entries. The "Wolfenstein-like" parametrization with $U_{e3}=A\lambda^3$ and $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^2)$ together with $m_1 = D\sqrt{\Delta m_A^2}\lambda^3$ leads to the particularly simple form

$$m_\nu = \frac{\sqrt{\Delta m_A^2}}{2}\begin{pmatrix} \lambda-2\lambda^2 & \lambda/\sqrt{2} & -\lambda/\sqrt{2} \\ \cdot & 1+\lambda/2+(1-2B)\lambda^2 & 1-\lambda/2-\lambda^2 \\ \cdot & \cdot & 1+\lambda/2+(1+2B)\lambda^2 \end{pmatrix} + \mathcal{O}(\lambda^3). \tag{45}$$

The $\theta_{23}\leftrightarrow \pi/2-\theta_{23}$ ambiguity, which translates into a sign ambiguity of B, is seen to have origin in the size of the $\mu\mu$ and $\tau\tau$ entries, e.g., for all masses positive and $\theta_{23}>\pi/4$ the $\tau\tau$ entry is larger. The structure of the mass matrix does not depend on the exactness of the relation $R=\lambda$ or the relative signs of the mass states. When $\mathrm{sgn}(m_1)=-\mathrm{sgn}(m_2)=-\mathrm{sgn}(m_3)$, then the mass matrix looks as above. For $\mathrm{sgn}(m_1)=-\mathrm{sgn}(m_2)=\mathrm{sgn}(m_3)$ and $\mathrm{sgn}(m_1)=\mathrm{sgn}(m_2)=-\mathrm{sgn}(m_3)$ one has to replace[3] A with $-A$, B with $-B$ and the 1 in the $\mu\tau$ block with -1. If we further choose $U_{\mu 3}=\sqrt{\frac{1}{2}}(1-B\lambda^3)e^{i\delta}$, then we obtain a mass matrix which up to order λ^2 the parameters A, B, and D do not appear at all. It is obtained by setting in the last equation $B=0$.

C. Inverse hierarchy

In this case we have

$$m_2=\sqrt{\Delta m_A^2},\ m_1\simeq\sqrt{\Delta m_A^2}(1-\lambda^2/2),$$

$$m_3 = D\sqrt{\Delta m_A^2}\lambda^{2+l},\quad l\geqslant 0. \tag{46}$$

The dependence on the power of λ in m_3 is almost vanishing. The form of m_ν depends strongly on the signs of the masses m_1 and m_2. For identical relative signs between m_1 and m_2, the $e\mu$ and $e\tau$ entries are suppressed by λ or λ^2, depending on the powers of λ in $U_{\mu 3}$ or U_{e3}. If, e.g., $m=2$ and $n=1$ or $m=n=1$, then the entries are of order λ. For all other cases under consideration, these terms are of order λ^2. The remaining independent entries of m_ν are order one. If m_1 and m_2 have opposite relative signs, then $m_{e\mu}$ and $m_{e\tau}$ are of order one and the remaining entries of m_ν are linear in λ, independent of m and n. One finds for $m=n=1$ that for same signs of m_1 and m_2

$$m_\nu \simeq \sqrt{\Delta m_A^2}\begin{pmatrix} 1 & -(A/\sqrt{2})\lambda & -(A/\sqrt{2})\lambda \\ \cdot & \frac{1}{2}+B\lambda & -\frac{1}{2}+B\lambda \\ \cdot & \cdot & \frac{1}{2}-B\lambda \end{pmatrix} + \mathcal{O}(\lambda^2), \tag{47}$$

while for opposite signs

$$m_\nu \simeq \sqrt{\Delta m_A^2}\begin{pmatrix} 2\lambda & -1/\sqrt{2}-(B/\sqrt{2})\lambda & 1/\sqrt{2}-(B/\sqrt{2})\lambda \\ \cdot & (A-1)\lambda & \lambda \\ \cdot & \cdot & -(A+1)\lambda \end{pmatrix} + \mathcal{O}(\lambda^2). \tag{48}$$

The parameter λ appears at least linearly to correct the extreme "bimaximal" mass matrices from Eq. (41). For same signs, A appears in the e row and B in the $\mu\tau$ sector, whereas for opposite signs it is vice versa. Taking as another example again the "Wolfenstein-like" case $m=2$ and $n=3$ one finds to order λ^2:

[3]The convention here and in the following will be such that the sign of the ee entry is positive.

$$m_\nu \simeq \sqrt{\Delta m_A^2} \times \begin{cases} \begin{pmatrix} 1-\frac{1}{4}\lambda^2 & (1/4\sqrt{2})\lambda^2 & -(1/4\sqrt{2})\lambda^2 \\ \cdot & \frac{1}{2}+\left(B-\frac{1}{8}\right)\lambda^2 & -\frac{1}{2}+\frac{1}{8}\lambda^2 \\ \cdot & \cdot & \frac{1}{2}-\left(B+\frac{1}{8}\right)\lambda^2 \end{pmatrix} & \text{same signs} \\ \begin{pmatrix} 2\lambda-\frac{5}{4}\lambda^2 & -1/\sqrt{2}+[(9-4B)/4\sqrt{2}]\lambda^2 & 1/\sqrt{2}-[(9+4B)/4\sqrt{2}]\lambda^2 \\ \cdot & -\lambda+\frac{3}{8}\lambda^2 & \lambda-\frac{3}{8}\lambda^2 \\ \cdot & \cdot & -\lambda+\frac{3}{8}\lambda^2 \end{pmatrix} & \text{opposite signs.} \end{cases} \tag{49}$$

As usual, B will not appear for higher orders of λ in $U_{\mu 3}$, leading to a mass matrix that is at order λ^2 only a function of λ. It is again obtained by setting $B=0$ in the last equation.

D. Quasidegenerate neutrinos

For quasidegenerate neutrinos, i.e., $m_3^2 \simeq m_2^2 \simeq m_1^2 \gg \Delta m_A^2$, there is another small quantity introduced, namely the ratio of the common mass scale m_0 with Δm_A^2. For simplicity we work with the normal mass ordering. In this case we can express the three mass states as

$$m_3 \equiv m_0, \quad m_2 = a m_0, \quad m_1 = b m_0, \tag{50}$$
$$\text{where} \quad a = 1-\eta, \quad b = 1-\eta(1+\lambda^2), \quad \eta = \frac{\Delta m_A^2}{2m_0^2}.$$

The common mass scale is denoted by m_0. These expressions for the masses are valid to order η. Since the spectrum is quasidegenerate for $m_0 \gtrsim 0.2$ eV, we can estimate $\eta \lesssim 0.04$ eV, therefore $\lambda^2 > \eta$.

First, we take the case that all mass states have the same relative sign. For $m=n=1$ we find:

$$(m_\nu)_{+++} = m_0 \begin{pmatrix} 1-\eta & (A/\sqrt{2})\,\eta\lambda & (A/\sqrt{2})\,\eta\lambda \\ \cdot & 1-\eta/2-B\,\eta\lambda & \eta/2 \\ \cdot & \cdot & 1-\eta/2+B\,\eta\lambda \end{pmatrix} + \mathcal{O}(\eta\lambda^2). \tag{51}$$

Taking the case $m=2$ and $n=3$ we have

$$(m_\nu)_{+++} = m_0 \begin{pmatrix} 1-\eta-\eta\lambda^2/2 & \eta\lambda^2/2\sqrt{2} & -\eta\lambda^2/2\sqrt{2} \\ \cdot & 1-\eta/2-[(1+4B)/4]\,\eta\lambda^2 & \eta/2+\eta\lambda^2/4 \\ \cdot & \cdot & 1-\eta/2+[(4B-1)/4]\,\eta\lambda^2 \end{pmatrix} + \mathcal{O}(\eta\lambda^3). \tag{52}$$

It is seen that for $\eta=0$ the mass matrix is proportional to the unit matrix, irrespective of m and n. The corrections to the zero values of the extreme bimaximal form from Eq. (42) are very small.

If $\mathrm{sgn}(m_1) = -\mathrm{sgn}(m_2) = \mathrm{sgn}(m_3)$, then the dependence on η is not so important. Neglecting η with respect to terms of order 1, we find for $m=n=1$:

$$(m_\nu)_{+-+} = m_0 \begin{pmatrix} 2\lambda & -1/\sqrt{2}+[(A-B)/\sqrt{2}]\lambda & 1/\sqrt{2}+[(A-B)/\sqrt{2}]\lambda \\ \cdot & \frac{1}{2}+(A-B-1)\lambda & \frac{1}{2}+\lambda \\ \cdot & \cdot & \frac{1}{2}-(A+1-B)\lambda \end{pmatrix} + \mathcal{O}(\eta,\lambda^2) \tag{53}$$

and for $m=2,\ n=3$:

$$(m_\nu)_{+-+} = m_0 \begin{pmatrix} 2\lambda - \lambda^2 & -(1-\eta)/\sqrt{2} + [(2-B)/\sqrt{2}]\lambda^2 & (1-\eta)/\sqrt{2} - [(B+2)/\sqrt{2}]\lambda^2 \\ \cdot & \frac{1}{2} - \lambda + \left(\frac{1}{2} - B\right)\lambda^2 & \frac{1}{2} + \lambda - \frac{1}{2}\lambda^2 \\ \cdot & \cdot & \frac{1}{2} - \lambda + \left(\frac{1}{2} + B\right)\lambda^2 \end{pmatrix} + \mathcal{O}(\eta\lambda, \lambda^3). \tag{54}$$

For the case $\text{sgn}(m_1) = -\text{sgn}(m_2) = -\text{sgn}(m_3)$ one finds for $m = n = 1$

$$(m_\nu)_{+--} = m_0 \begin{pmatrix} 2\lambda & -1/\sqrt{2} - [(A+B)/\sqrt{2}]\lambda & 1/\sqrt{2} - [(A+B)/\sqrt{2}]\lambda \\ \cdot & -\frac{1}{2} + (A+B-1)\lambda & -\frac{1}{2} + \lambda \\ \cdot & \cdot & -\frac{1}{2} - (A+B+1)\lambda \end{pmatrix} + \mathcal{O}(\eta, \lambda^2) \tag{55}$$

and for $m = 2$, $n = 3$:

$$(m_\nu)_{+--} = m_0 \begin{pmatrix} 2\lambda - \lambda^2 & -(1-\eta)/\sqrt{2} + [(2-B)/\sqrt{2}]\lambda^2 & (1-\eta)/\sqrt{2} - [(B+2)/\sqrt{2}]\lambda^2 \\ \cdot & -\frac{1}{2} - \lambda + \left(\frac{1}{2} + B\right)\lambda^2 & -\frac{1}{2} + \lambda - \frac{1}{2}\lambda^2 \\ \cdot & \cdot & -\frac{1}{2} - \lambda + \left(\frac{1}{2} - B\right)\lambda^2 \end{pmatrix} + \mathcal{O}(\eta\lambda, \lambda^3). \tag{56}$$

These two last cases look very similar. Finally, the situation for $\text{sgn}(m_1) = \text{sgn}(m_2) = -\text{sgn}(m_3)$ looks simpler: e.g., when $m = n = 1$:

$$(m_\nu)_{++-} = m_0 \begin{pmatrix} 1 - 2A^2\lambda^2 & -\sqrt{2}A\lambda + \sqrt{2}AB\lambda^2 & -\sqrt{2}A\lambda - \sqrt{2}AB\lambda^2 \\ \cdot & 2B\lambda - B^2\lambda^2 & -1 + (A^2 + 2B^2)\lambda^2 \\ \cdot & \cdot & -2B\lambda + (2A^2 + B^2)\lambda^2 \end{pmatrix} + \mathcal{O}(\eta, \lambda^3). \tag{57}$$

The corrections to the entry 1 (0) that is present in the extreme bimaximal form from Eq. (42) are at least quadratic (linear). For $m = 2$ and $n = 3$ the following holds:

$$(m_\nu)_{++-} = m_0 \begin{pmatrix} 1 - \eta - \frac{1}{2}\eta\lambda^2 & (1/2\sqrt{2})\eta\lambda^2 & -(1/2\sqrt{2})\eta\lambda^2 \\ \cdot & -\frac{\eta}{2} + [B(2-\eta) - \eta/4]\lambda^2 & -1 + \eta/2 + \frac{1}{4}\eta\lambda^2 \\ \cdot & \cdot & -\eta/2 - [B(2-\eta) + \eta/4]\lambda^2 \end{pmatrix} + \mathcal{O}(\lambda^3). \tag{58}$$

E. Summary for the mass matrix

Looking at the cases considered in the last subsections, the following summarizing statements can be made:

Roughly, for $|U_{e3}| \sim 0.01$ and $\sin^2 2\theta_{23} \lesssim 0.9$, corrections to the extreme forms of the mass matrices in Eqs. (40)–(42) are linear in λ. When $|U_{e3}| \lesssim 10^{-3}$ and $\sin^2 2\theta_{23} \gtrsim 0.95$, the corrections become quadratic.

For the normal hierarchy, corrections to the exact bimaximal form are at least order λ. To lowest order, the parameter A appears in the e row of m_ν and B in the $\mu\mu$ and $\tau\tau$ elements. The number D, parametrizing the unknown lightest mass state, appears in all entries. There is basically no dependence on the relative signs of the mass states.

For the inverse hierarchy, the dependence on D is highly suppressed. For identical signs of the two heaviest mass states, the correction to the ee entry, whose extreme value in case of bimaximal mixing is (in units of Δm_A^2) 1, is at least order λ^2. The remaining elements receive at least linear corrections. A appears at leading order in the e row and B in the $\mu\tau$ sector. Opposite signs of the two leading mass states lead to linear corrections to the entries and the appearance of A in the $\mu\tau$ sector and B in the e row.

In case of a quasidegenerate spectrum and identical signs of the masses, the corrections to the unit matrix are at least quadratical. The cases $\text{sgn}(m_1) = -\text{sgn}(m_2) = \text{sgn}(m_3)$ and $\text{sgn}(m_1) = -\text{sgn}(m_2) = -\text{sgn}(m_3)$ look very similar. For $\text{sgn}(m_1) = \text{sgn}(m_2) = -\text{sgn}(m_3)$ there are only quadratic corrections to the ee and $\mu\tau$ entries.

The $\mu\tau$ entry is special in our parametrization since the parameters A and B do typically only appear there for rather large deviations from zero U_{e3} and from maximal atmospheric neutrino mixing.

If we would consider the inverse ordering in the QD mass spectrum, one has to change Eq. (50) to $m_2=m_0$, $m_1=am_0$ and $m_3=bm_0$. Only simple sign changes for A and/or B in some elements of m_ν would be the result. The main difference would be for the case $\mathrm{sgn}(m_1)=\mathrm{sgn}(m_2)=\mathrm{sgn}(m_3)$, where the corrections to the $e\mu$ and $e\tau$ elements are now order η and not just $\eta\lambda$ or $\eta\lambda^2$.

V. APPLICATIONS

It is surely useful to study formulas that are obtained by expansions in small parameters or by certain simplifications within our parametrization. We shall perform this analysis now for the effective mass as measurable in neutrinoless double beta decay and the oscillation probabilities for long-baseline neutrino oscillations.

A. Neutrinoless double beta decay

We shall analyze now within our parametrization the form of the ee element of m_ν, which is denoted by $\langle m\rangle$. In a given mass scheme or hierarchy one can considerably simplify the expression for $\langle m\rangle$ [36]. We first note that since

$$\langle m\rangle=\left|\sum m_i U_{ei}^2\right|, \tag{59}$$

the results are independent of the power of λ in $U_{\mu3}$.

1. Normal hierarchy

With the help of Eqs. (43) and (59) we can evaluate the effective mass in case of the normal hierarchy. We find for $U_{e3}=A\lambda$ and $m_3=D\lambda^2$:

$$\langle m\rangle\simeq\frac{\sqrt{\Delta m_A^2}}{2}\left[\lambda-2\lambda^2\left(1-\frac{D}{2}c_{2\alpha}-A^2c_{2(\alpha-\beta)}\right)\right]+\mathcal{O}(\lambda^3), \tag{60}$$

where $c_{2\alpha}=\cos 2\alpha$ and so on. Terms of order λ^3 depend on A,D and the two Majorana phases. Choosing $m_3=D\lambda^3$ or higher powers of λ leads to the disappearance of D in the formula. For $m_3=D\lambda^2$ and higher orders of λ in U_{e3}, i.e., $n\geqslant 2$, the following holds:

$$\langle m\rangle\simeq\frac{\sqrt{\Delta m_A^2}}{2}\left[\lambda-2\lambda^2\left(1-\frac{D}{2}c_{2\alpha}\right)\right.$$
$$\left.+\frac{\lambda^3}{4}[4(1+2Dc_{2\alpha})+D^2(1-c_{4\alpha})]\right]+\mathcal{O}(\lambda^4). \tag{61}$$

The formulas for $m_3=D\lambda^3$ correspond to setting $D^2=0$ in this last equation. Roughly, we can estimate the effective mass in the normal hierarchy as

$$\langle m\rangle\sim\frac{\lambda}{2}\sqrt{\Delta m_A^2}\lesssim 0.005\ \text{eV}. \tag{62}$$

2. Inverse hierarchy

From Eqs. (46) and (59) one sees that in the expression for $\langle m\rangle$ the dependence on β practically vanishes. The result for $U_{e3}=A\lambda$ is

$$\langle m\rangle\simeq\sqrt{\Delta m_A^2}\sqrt{c_\alpha^2+\frac{1}{4}[7-4A^2-(9+4A^2)c_{2\alpha}]\lambda^2}$$
$$+\mathcal{O}(\lambda^3). \tag{63}$$

Higher powers of λ in U_{e3} lead to the disappearance of A in this equation. The maximal and minimal values are obtained when α takes the values 0 and $\pi/2$, respectively. Thus,

$$2\sqrt{\Delta m_A^2}\lambda\lesssim\langle m\rangle\lesssim\sqrt{\Delta m_A^2}\left[1-\left(A^2+\frac{1}{4}\right)\lambda^2\right], \tag{64}$$

up to corrections of $\mathcal{O}(\lambda^3)$. For no extreme values of α, the scale of the effective mass is

$$\langle m\rangle\sim\sqrt{\Delta m_A^2}\gtrsim 0.05\ \text{eV}. \tag{65}$$

Comparing this with the value of $\langle m\rangle$ in the normal hierarchy in Eq. (62), one sees that the expansion parameter λ shows up as the ratio of the typical values of $\langle m\rangle$ in the inverted and normal hierarchy. It is known that extraction of information from a measurement of $0\nu\beta\beta$ suffers from a large uncertainty stemming from the calculation of the nuclear matrix elements. This uncertainty is a number of order one [37]. It is therefore an important question to ask and an even more important one to answer whether future $0\nu\beta\beta$ experiments can distinguish [36,38] between the normal and inverted mass hierarchy. Let us parametrize the nuclear matrix element uncertainty with a factor ξ as done in Ref. [39]. In order to distinguish the normal from the inverted hierarchy it must hold that the maximal value of $\langle m\rangle$ in the normal hierarchy times the uncertainty ξ has to be smaller than the minimal value of $\langle m\rangle$ in the inverted hierarchy. Therefore, choosing $U_{e3}=A\lambda$ and small m_3 we find from Eqs. (60) and (64)

$$\xi\lesssim 4[1+2\lambda(1\pm A^2)]+\mathcal{O}(\lambda^3). \tag{66}$$

Needless to say, A vanishes for smaller values of U_{e3}. If that is the case, then $\xi\lesssim 6$, which is a very realistic number. Thus, with our expansion parameter $\lambda\simeq 0.2$ and $|U_{e3}|^2\lesssim 0.01$ it is easily possible to distinguish between the normal and inverted mass hierarchy.

3. Quasidegenerate neutrinos

The formulas for the mass states are given in Eq. (43). Ignoring η and taking $U_{e3}=A\lambda$ one finds

$$\frac{\langle m\rangle}{m_0}\simeq\sqrt{c_\alpha^2+[2-A^2-(2+A^2)c_{2\alpha}+2A^2c_\alpha c_{\alpha-2\beta}]\lambda^2}+\mathcal{O}(\lambda^3). \tag{67}$$

Interesting cases correspond to CP conservation, which are obtained by setting α,β to $\pi/2,\pi$. They read up to $\mathcal{O}(\eta\lambda^2,\lambda^3)$:

$$\frac{\langle m\rangle}{m_0}\simeq\begin{cases}1-\eta, & \alpha=\beta=0 \quad\leftrightarrow \mathrm{sgn}(m_1)=\mathrm{sgn}(m_2)=\mathrm{sgn}(m_3)\\ 2\lambda, & 2\alpha=\beta=\pi \leftrightarrow \mathrm{sgn}(m_1)=-\mathrm{sgn}(m_2)=\mathrm{sgn}(m_3)\\ 2\lambda, & \alpha=\beta=\pi/2 \leftrightarrow \mathrm{sgn}(m_1)=-\mathrm{sgn}(m_2)=-\mathrm{sgn}(m_3)\\ 1-\eta-2A^2\lambda^2, & \alpha=2\beta=\pi \leftrightarrow \mathrm{sgn}(m_1)=\mathrm{sgn}(m_2)=-\mathrm{sgn}(m_3).\end{cases} \tag{68}$$

As usual, for $U_{e3}=A\lambda^2$ and above, the dependence on A (and thus also on β) drops and appears only to order λ^4. Noting that the minimal value of $\langle m\rangle$ is $2\lambda m_0$, we can investigate if future experiments can distinguish between the normal and quasidegenerate mass hierarchy [39]. In analogy to the discussion leading to Eq. (66), it follows from Eqs. (60) and (68)

$$\xi\lesssim 2\sqrt{\frac{2}{\eta}}[1+2\lambda(1\pm A^2)], \tag{69}$$

which will be easily possible. It is a bit more tricky to distinguish between the quasidegenerate and the inverted hierarchies. The requirement for ξ is

$$\xi\lesssim\sqrt{\frac{2}{\eta}}\lambda\left[1+\left(A^2+\frac{1}{4}\right)\lambda^2\right], \tag{70}$$

which is suppressed roughly by a factor λ with respect to the limit on ξ in order to distinguish the normal and quasidegenerate mass hierarchy. Also in this aspect the parameter λ shows up as a scaling factor.

B. Long baseline oscillation experiments

There is another field of neutrino physics in which expansion in small parameters gives insight in the physics involved and which is therefore useful to study within our parametrization. These are the oscillation probabilities for long-baseline experiments [18]. The determination of some currently unknown neutrino parameters, namely, U_{e3}, the sign of Δm_A^2, and the Dirac-like CP violating phase, are the purpose of such experiments. There are helpful expansions of the relevant oscillation probabilities in vacuum [40]. Here, we do not consider matter effects since they will not change our conclusions. Let us first comment on CP violation. Using Eq. (4), one finds for the difference of the oscillation probabilities:

$$\Delta P\equiv P(\nu_e\to\nu_\mu)-P(\overline{\nu_e}\to\overline{\nu_\mu})$$
$$=8J_{CP}\left(\sin\frac{\Delta m_{31}^2L}{2E}-\sin\frac{\Delta m_{32}^2L}{2E}-\sin\frac{\Delta m_{21}^2L}{2E}\right), \tag{71}$$

which, using $\Delta m_{32}^2=\Delta m_{31}^2-\Delta m_{21}^2$, can easily be shown to vanish for two masses being equal. The invariant J_{CP} was defined in Eq. (10). Since $\Delta m_{21}^2/\Delta m_{31}^2=\pm\lambda^2$, where '+' is for the case of normal ordering and '−' for inverse ordering, we can expand the last equation:

$$\Delta P\simeq\pm 4J_{CP}\lambda^2\frac{\Delta m_A^2L}{2E}\sin^2\frac{\Delta m_A^2L}{4E}+\mathcal{O}(\lambda^3). \tag{72}$$

Thus, the CP violating effects in realistic experiments are suppressed by another two orders of λ in addition to the suppression present in J_{CP}. If n is the power of λ in U_{e3}, then the total suppression is λ^{2+n}.

One can also consider the bare oscillation probability for the "golden channel," which is given by $\nu_e\to\nu_\mu$ oscillations. Using the form of $P(\nu_e\to\nu_\mu)$ as given, e.g., in Ref. [22], one finds for the oscillation probability in case of $m=n=1$:

$$P(\nu_e\to\nu_\mu)\simeq 2A^2\sin^2\Delta_{32}\lambda^2$$
$$-2A\sin^2\Delta_{32}[2AB+\cos(\delta\mp\Delta_{32})]\lambda^3$$
$$+\mathcal{O}(\lambda^4). \tag{73}$$

Here the −' sign is for neutrinos and the + for antineutrinos. We defined $\Delta_{32}=(m_3^2-m_2^2)L/4E$. The first term proportional to λ^2 is the term that probes U_{e3} whereas the second term proportional to λ^3 is the one probing the CP phase δ.

As another example, assume $n=m=2$. Then, the terms probing U_{e3} and CP violation will both be proportional to λ^4:

$$P(\nu_e\to\nu_\mu) \simeq \frac{\sin^2\Delta_{32}}{2}[1+4A^2-4A\cos(\delta\mp\Delta_{32})]\lambda^4+\mathcal{O}(\lambda^6). \tag{74}$$

The parameter B only appears at order λ^6, since there are no terms of order λ^5. For $m=2$ and $n=3$ the following holds:

$$P(\nu_e\to\nu_\mu) \simeq \frac{\sin^2\Delta_{32}}{2}\lambda^4-2A\sin^2\Delta_{32}\cos(\delta\mp\Delta_{32})\lambda^5+\mathcal{O}(\lambda^6). \tag{75}$$

A characteristic combination of the oscillation parameters that appears in the relevant probabilities is $\Delta m_{21}^2/\Delta m_{31}^2\sin 2\theta_{12}$ [40]. Neglecting terms of order λ^6, we find for this parameter in our parametrization that

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\sin 2\theta_{12} \simeq \pm\lambda^2\times\begin{cases}(1-2\lambda^2+2(1+A^2)\lambda^3) & \text{for } n=1\\ (1-2\lambda^2+2\lambda^3) & \text{for } n=2\\ (1-2\lambda^2+2\lambda^3) & \text{for } n=3,\end{cases} \tag{76}$$

where again the $+$ is for normal ordering and the $-$ for inverse ordering. The difference between the cases $n=2$ and $n=3$ appears only at the seventh order in λ. The characteristic parameter is therefore to order λ^4 independent of the precise form of the parametrization.

VI. CONCLUSIONS

The zeroth order approximation for neutrino mixing can be the bimaximal scheme with two maximal and one zero angle in the mixing matrix. It can be used as a reference matrix, whose corrections can be described in a similar manner as the Wolfenstein parametrization describes corrections to the unit matrix. Indeed, at least one of the angles in neutrino mixing is different from the extreme value corresponding to bimaximal mixing, namely the angle describing solar neutrino oscillations. To take this into account, a flexible parametrization of the neutrino mixing matrix was proposed in which the expansion parameter $\lambda\simeq 0.2$ is introduced to quantify this deviation from maximal mixing of solar neutrinos. It can also be used to quantify the possible deviation from zero U_{e3} and maximal mixing of atmospheric neutrinos. The power of λ to usefully describe these two latter aspects can be adjusted to future data. Depending on the power of λ, rather simple forms of the PMNS matrix are obtained, where the deviations from the ‘‘bimaximal’’ values 0, $\pm 1/2$, and $\pm 1/\sqrt{2}$ are implied by λ. If $U_{\mu 3}$ and U_{e3} are close to their maximally allowed values, λ appears at first order in all elements of $U_{\rm PMNS}$. For values of $|U_{e3}|\lesssim 10^{-3}$ and $\sin^2 2\theta_{23}\gtrsim 0.95$, the corrections become quadratic. The invariant measure for leptonic *CP* violation is proportional to λ^n, where n is the power of λ in U_{e3}. One can interpret these corrections to the exact bimaximal mixing scheme in the same way as corrections to the unit matrix lead to the CKM matrix for the quark sector. Observing further that the ratio of the mass squared differences as measured in experiments is roughly λ^2 allows us to study the form of the Majorana neutrino mass matrix m_ν. Also here, the corrections to the extreme forms of m_ν in case of bimaximal mixing and extreme hierarchies are linear or quadratic in λ, depending on the precise values of U_{e3}, $U_{\mu 3}$ or the value of the smallest mass state. The *ee* element of m_ν can be measured in experiments probing neutrinoless double beta decay. Here, λ appears as the scale factor of the typical values of $\langle m\rangle$ in the normal and inverted hierarchy. It also influences the maximal value of the uncertainty in the calculations of the nuclear matrix elements allowed to distinguish the normal, inverted, or quasidegenerate mass hierarchies. We furthermore commented on how our parametrization applies to realistic long-baseline oscillation experiments. Simple forms of the relevant oscillation probabilities are obtained. In particular, due to the small ratio of the two independent mass squared differences, effects of *CP* violation are suppressed by another two orders of λ.

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