Black hole formation in perfect fluid collapse

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We construct here a special class of perfect fluid collapse models which generalizes the homogeneous dust collapse solution in order to include nonzero pressures and inhomogeneities into evolution. It is shown that a black hole is necessarily generated as the end product of continued gravitational collapse, rather than a naked singularity. We examine the nature of the central singularity forming as a result of endless collapse and it is shown that no nonspacelike trajectories can escape from the central singularity. Our results provide some insights into how the dynamical collapse works and into the possible formulations of the cosmic censorship hypothesis, which is as yet a major unsolved problem in black hole physics.

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Black hole physics has attracted considerable attention in recent years and has witnessed rapid theoretical developments as well as numerous astrophysical applications. It is to be noted, however, that while few exact static or stationary models of black holes such as the Schwarzschild, Reissner-Nordström, and Kerr-Newman spacetimes are well studied, the actual *formation* of black holes within the framework of a *dynamical* gravitational collapse process is not really an arena where much is known.

In a realistic physical scenario, (stellar mass) black holes will be typically born when a massive star exhausts its nuclear fuel and then collapses endlessly under the influence of its own gravitational field. Towards modeling such a physical process, a well-known model that has served as the basic paradigm in black hole physics is that of the Oppenheimer-Snyder spherically symmetric collapse solution [1], where a dust cloud undergoes a continued collapse to form a black hole. Here the collapse initiates from regular initial data, when there is no trapping of light (i.e., light rays from the star can escape to faraway observers). Subsequently, as the collapse advances the process of the formation of an event horizon and closed trapped surfaces takes place, thus leading to the formation of a black hole and the eventual spacetime singularity. The trapped surfaces and the event horizon form here well in advance of the epoch of the formation of the spacetime singularity, which is hence necessarily hidden within the black hole.

Even though this collapsing Friedmann model already tells us a homogeneous dust collapse will always end in a black hole rather than a naked singularity, it should be noted that this scenario in fact has several limitations. For example, in this case the cloud has no pressures included, whereas any physically realistic collapse must include pressures. Another restrictive assumption here is that the density profile is assumed to be strictly homogeneous in space, at all times throughout the evolution of the cloud. For any isolated object such as a star, one may rather like to study a physically realistic density distribution which would be typically higher at the center and decreasing as we move away from the center.

It is thus essential to study and analyze more general collapse situations in order to understand black hole formation in more realistic stellar collapse scenarios. This is also essential in order to make any possible progress towards the *cosmic censorship hypothesis* [2], which broadly states that any physically realistic gravitational collapse must result in the development of a black hole. Such a conjecture has been absolutely fundamental to the theory of black holes and has played a major role in astrophysical applications of black hole physics. This, however, remains a major unresolved open problem in general relativity and black hole physics today.

From such a perspective, we study here a specific class of collapse models where matter obeys the perfect fluid equation of state and construct models where the collapse always necessarily ends in the formation of a black hole. The models we study here are somewhat special in that the mass function is assumed to be separable in the variables which are the physical radius of the cloud and the time coordinate. However, this is a class which generalizes the Oppenheimer-Snyder dust collapse models in two important respects; namely, inhomogeneities of density distribution are included and also nonzero pressures have been incorporated now. As the collapse always ends here in black hole formation as we show, it is hoped that dynamical considerations such as these will provide some useful insights into physically realistic collapse and the actual process of black hole formation. It is not unlikely that it is only such dynamical considerations which would prove essential to resolve the issue of cosmic censorship. The model here may be of interest as it includes pressures which may be important in the later stages of collapse and because the equation of state is that of a perfect fluid, which is physically a well-studied equation of state widely used in various astrophysical considerations.

The fluid content of the cloud is in the form of a perfect fluid with an equation of state of the form $p=k\rho$; i.e., at all epochs the radial and tangential pressures are equal and isotropic, and are proportional to the density function of the cloud. Though the case of a general inhomogeneous *dust* collapse with $k=0$ can be completely solved [3], there are still a number of open questions regarding the end state of a general perfect fluid collapse. Our purpose here is to examine a class of solutions of the Einstein equations for a spherically

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symmetric perfect fluid to understand explicitly how an inhomogeneous density profile should behave in the later stages of collapse and near the singularity, so that the final state of the collapse would always be a black hole necessarily.

The spacetime geometry within the spherically symmetric collapsing cloud can be described by the metric in the comoving coordinates (t,r,θ,ϕ) as given by

$$ds^2=-e^{2\nu(t,r)}dt^2+e^{2\psi(t,r)}dr^2+R^2(t,r)d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the line element on a two-sphere. The energy-momentum tensor for any matter fields of *type I* [4] (this is a broad class which includes most of the physically reasonable matter fields, including dust, perfect fluids, massless scalar fields, and such others) is then given in a diagonal form

$$T_t^t=-\rho(t,r), \quad T_r^r=p_r(t,r), \quad T_\theta^\theta=T_\phi^\phi=p_\theta(t,r). \tag{2}$$

The quantities ρ, p_r, and p_θ are the energy density, radial, and tangential pressures, respectively, of the cloud. We take the matter fields to satisfy the *weak energy condition*; i.e., the energy density measured by any local observer is non-negative. Then for any timelike vector V^i, we must have

$$T_{ik}V^iV^k\geqslant 0, \tag{3}$$

which amounts to

$$\rho\geqslant 0, \quad \rho+p_r\geqslant 0, \quad \rho+p_\theta\geqslant 0. \tag{4}$$

Now for the metric (1) the Einstein equations take the form, in the units $(8\pi G=c=1)$,

$$\rho=\frac{F'}{R^2R'}, \quad p_r=-\frac{\dot{F}}{R^2\dot{R}}, \tag{5}$$

$$\nu'=\frac{2(p_\theta-p_r)}{\rho+p_r}\frac{R'}{R}-\frac{p_r'}{\rho+p_r}, \tag{6}$$

$$-2\dot{R}'+R'\frac{\dot{G}}{G}+\dot{R}\frac{H'}{H}=0, \tag{7}$$

$$G-H=1-\frac{F}{R}, \tag{8}$$

where

$$G(t,r)=e^{-2\psi}(R')^2, \quad H(t,r)=e^{-2\nu}(\dot{R})^2. \tag{9}$$

The arbitrary function $F=F(t,r)$ here has an interpretation of the mass function for the cloud, and it gives the total mass in a shell of comoving radius r on any spacelike slice $t=$const. We have $F\geqslant 0$ from the energy conditions. In order to preserve regularity at the initial epoch, we have $F(t_i,0)=0$; that is, the mass function should vanish at the center of the cloud. Since we are considering collapse, we have $\dot{R}<0$; i.e., the physical radius R of the cloud keeps decreasing and ultimately reaches $R=0$. As seen from Eq. (6), there is a density singularity in the spacetime at $R=0$ and at $R'=0$. However, the later ones are due to shell crossings and these weak singularities can be possibly removed from the spacetime [5], so we shall consider here only the shell-focusing singularity at $R=0$, which is the genuine physical singularity where all matter shells collapse to a zero physical radius.

Now let us incorporate the perfect fluid form of matter, where the radial and tangential pressures are equal, and take the equation of state for the collapsing matter to be

$$p_r(t,r)=p_\theta(t,r)=k\rho(t,r), \tag{10}$$

where $k<1$ is a constant. Then Eqs. (5) and (6) become

$$\rho=\frac{F'}{R^2R'}=-\frac{1}{k}\frac{\dot{F}}{R^2\dot{R}}, \tag{11}$$

$$\nu'=-\frac{k}{k+1}[\ln(\rho)]'. \tag{12}$$

Thus we see that there are five dynamical variables—namely, ρ, ψ, ν, R, and F—and there are five total field equations. Also, using the scaling independence we can write $R(t_i,r)=r$ at the initial epoch $t=t_i$ from where the collapse commences. The time $t=t_s(r)$ corresponds to the formation of the shell-focusing singularity at $R=0$, where all matter shells collapse to a vanishing physical radius.

Now let us assume that the mass function can be explicitly written as a function of the physical radius R of the cloud and t:

$$F(R,t)=R^3M(R)Q(t). \tag{13}$$

That is, we consider the class of mass functions F to be separable in R and t. Apart from that it is general in the sense that M is any C^2 function, whereas the function $Q(t)$ yet to be determined by the field equations is a suitably differentiable function of t for $t<t_{s_0}$, where $t=t_{s_0}$ is the time for the occurrence of the central singularity. One may consider Eq. (13) to be a somewhat strong assumption on the nature of the mass function. However, our basic purpose here is to construct a class of dynamical collapse models, where the two main constraints of the homogeneous dust collapse mentioned above are relaxed; namely, the density need not be homogeneous at the initial epoch and later as well during the collapse evolution, and second we want to allow for nonzero pressures while considering a dynamical collapse situation. As both these purposes are met by the above form of mass function as we have chosen here, it is adequate for our present purposes, as it allows us to construct explicit collapse models which are more general and which necessarily end up in a black hole as we shall see.

Another requirement that is frequently imposed on physical grounds on the initial data from which the collapse evolves is that the physical variables such as the density and pressures be taken to be smooth or analytic functions at the initial surface. We can then write

$$M(R)=\frac{1}{3}+\frac{1}{5}M_2R^2+\cdots. \tag{14}$$

Then from Eq. (11) we get $\rho(r,t)=\rho(R,t)$ with

$$\rho(R,t)=(3M+RM_{,R})Q(t)=A(R)Q(t), \tag{15}$$

where the function $A(R)$ is given by

$$A(R)=1+M_2R^2+\cdots. \tag{16}$$

As seen from above, at the initial epoch $t=t_i$, the density function is given by

$$\rho(r,t_i)=\rho_0(r)=Q(t_i)[1+M_2r^2+\cdots]. \tag{17}$$

Thus we see that the gradients of the density and pressures of the cloud vanish at the center at the initial epoch as required by the smoothness. Also, for the density to diverge at the singularity, we must have

$$\lim_{t\to t_{s_0}} Q(t)\to\infty. \tag{18}$$

It follows that for the given mass function the perfect fluid condition can be written as

$$(k+1)QA+\frac{R}{\dot{R}}M\dot{Q}=0. \tag{19}$$

The solution of the above equation determines the mass function completely.

In order now to construct a class of collapsing solutions, let us consider the case when $\nu=\nu(R)$; i.e., let the metric function ν be a function of the physical radius R only. Again, this restriction is good enough for us as it allows us to construct the collapse models which include inhomogeneity and nonzero pressures and which end up in black holes, generalizing the collapsing Friedmann models. A further useful feature of these choices may be considered to be that it allows to be incorporated a perfect fluid with a reasonable equation of state, rather than any arbitrary forms of pressures (e.g., a purely tangential pressure, while assuming that the radial pressures identically vanish) as is done sometimes when considering gravitational collapse.

One can now integrate Eq. (12) to get

$$\nu(R)=-\frac{k}{k+1}\ln[C_1A(R)]. \tag{20}$$

Here C_1 is a constant of integration. Now putting the value of $H(t,r)$ in Eq. (7) and simplifying we get

$$R'\dot{G}-2\dot{R}\nu'G=0. \tag{21}$$

It is now possible to solve the above equation and the function G has the form

$$G(R)=A(R)^{-2k/(k+1)}. \tag{22}$$

In other words, the above forms of ρ, ν, and G solve the Einstein's equations (6) and (7). Obtaining now the function $Q(t)$ will complete the solution. Putting in these functions in Eq. (8) we get

$$\dot{R}=-C_2A^{-k/(k+1)}\sqrt{A^{-2k/(k+1)}-1+R^2MQ}, \tag{23}$$

where $C_2=C_1^{-k/(k+1)}$ is another constant. The negative sign denotes the collapse condition $\dot{R}<0$. Finally, substituting the values of the functions A and M we get, ignoring the higher-order terms, as we are interested to find a solution close to the singularity,

$$\dot{R}=-C_2R\left[1-\frac{2kR^2}{k+1}\right]\sqrt{\left[\frac{1}{3}+\frac{1}{5}M_2R^2\right]Q-\frac{2kM_2}{k+1}}. \tag{24}$$

We note that in Eq. (24), the velocity $\dot{R}$ changes sign at the value $R=\sqrt{(1+k)/2k}$. This would correspond to a class of dynamic perfect fluid models where there is a bounce at the above value of the physical radius. However, since we are interested in the collapse models only presently, we do not consider this bouncing branch of the solutions. This can be achieved by fixing the boundary conditions suitably. For example, for the extreme value $k=1$, this change corresponds to $R=1$. Now as we have the scaling $R=r$ at the initial epoch, this means that the boundary of the object $r=r_b$ at the initial epoch is to be given by $0<r_b<1$. Then at all later epochs this condition will be of course respected because $\dot{R}<0$, and the physical radius R monotonically decreases with t and will be less than 1 for all shells at all future times. For smaller values of k—that is, $k<1$—we of course have larger values of the boundary of the cloud available, and in the extreme case $k=0$—i.e., the dust collapse models—we can have arbitrarily large r_b without the velocity ever changing sign, or the cloud can be as big as we want, and there will be no bounce at all possible in the dust case.

Now let us solve Eq. (19) close to the spacetime singularity at $R=0$. In this approximation, $A(R)\to 1$ and $M(R)\to\frac{1}{3}$. Using these approximations and Eq. (24) in Eq. (19) we get

$$(k+1)Q-\frac{1}{3C_2\sqrt{\frac{Q}{3}-\frac{2k}{k+1}M_2}}\dot{Q}=0. \tag{25}$$

Considering that $M_2<0$ and solving the above equation with the boundary condition $Q(t_{s_0})\to\infty$ as pointed out earlier, we get

$$Q(t)=-\alpha+\left[\alpha+\frac{2\alpha}{[\exp\{-\sqrt{3}C_2(k+1)\alpha(t_{s_0}-t)\}-1]}\right]^2, \tag{26}$$

where

$$\alpha=\sqrt{\frac{6k|M_2|}{k+1}}. \tag{27}$$

Thus we see that the above $Q(t)$ is a solution to Einstein's equations in the vicinity of the singularity with respect to the given forms of ρ, ν, and G. Now we can also solve for the metric function R, which is the physical radius for the cloud. Using Eq. (24) we get

$$R(r,t)=f(r)e^{-B(t)}, \tag{28}$$

where $f(r)$ is an arbitrary function of r. To avoid any shell crossing singularity we consider f to be an increasing function of r, and since the area radius of the geometrical center of the cloud vanishes, we must have $f(0)=0$. The function $B(t)$ is given as

$$B(t)=C_2\int \sqrt{\frac{Q}{3}-\frac{2k}{k+1}M_2}dt. \tag{29}$$

As noted earlier, the spacetime singularity occurs at $R=0$. We now need to decide if the singularity in the present case is necessarily covered within an event horizon of gravity (which is the case of a black hole formation) or it could be visible to faraway observers in the spacetime. The way to decide this is to examine if there are any future directed families of null geodesics which go out to external observers in future and which in the past terminate at the singularity. If such families do exist, then the singularity is naked, which in principle can communicate with outside world, and in the case otherwise we have a black hole forming as the end state of collapse. We thus need to consider the existence or otherwise of such families of paths from the singularity.

With the form of R as given above in Eq. (28) and as $\dot{R}<0$, the singularity happens at a time $t=t_s$ when the physical radius for all the shells with different values of the comoving coordinate r becomes zero. In other words, there is a *simultaneous* collapse of all shells to a singularity, and as $t\rightarrow t_{s_0}$ all shells labeled by the coordinate r collapse simultaneously to the singularity at $R=0$. This necessarily gives rise to a covered central singularity at $R=0$, $r=0$, as there are no outgoing future-directed nonspacelike geodesics coming out from the same. Because if there were any such outgoing geodesics, given by, say, $t=t(r)$ in the (t,r) plane, which came out from $t=t_s$, $r=0$, then the time coordinate must increase along these paths, which is impossible as there is complete collapse at $t=t_s$ and there is no spacetime beyond that. Hence no values $t>t_s$ are allowed within the spacetime which does not extend beyond the singularity. Thus, the collapse gives rise necessarily to a black hole in the spacetime.

Our main purpose here has been to generalize the homogeneous dust collapse scenario to include nonzero pressures and the inhomogeneities of density and pressures, which is physically more realistic situation, while ensuring that the collapse end state is a black hole only. While the perfect fluid collapse models we considered here allow for inhomogeneities in density and pressure profiles, and do necessarily give rise to black holes as we have shown here, it should be kept in mind, as we have noted above, that the classes of mass functions and the velocity profiles for the collapsing shells as determined by the choice of metric function $\nu(R)$ considered here are rather special. It is an open problem to explore if we could generalize these assumptions further, and if so to what extent, and still continue to get black holes only and not the naked singularities as the final end product of gravitational collapse. The point is it is known, for example, for inhomogeneous dust collapse [6] that as long as the inhomogeneities are within certain limits, the result of collapse is a black hole. However, beyond that criticality of inhomogeneities, the collapse could end in a naked singularity. Investigating further specific, but physically more realistic models may illustrate better such features of gravitationally collapsing configurations.

[1] J. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).
[2] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969); Nature (London) **236**, 377 (1972); S.W. Hawking, Gen. Relativ. Gravit. **10**, 1047 (1979); R. Penrose, in *General Relativity—An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979); W. Israel, Found. Phys. **14**, 1049 (1984).
[3] R.C. Tolman, Proc. Natl. Acad. Sci. U.S.A. **20**, 410 (1934); H. Bondi, Mon. Not. R. Astron. Soc. **107**, 343 (1947); G. Lemaître, Ann. Soc. Sci. Bruxelles, Ser. 1 **53**, 51 (1933).
[4] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).
[5] C.J.S.C. Clarke, *The Analysis of Spacetime Singularities* (Cambridge University Press, Cambridge, England, 1993).
[6] See, e.g., P.S. Joshi and I.H. Dwivedi, Phys. Rev. D **47**, 5357 (1993), and references therein.