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# Exact soluble two-dimensional charged wormhole

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We present an exactly soluble charged wormhole model in two dimensions by adding infalling chiral fermions on the static wormhole. The infalling energy due to the infalling charged matter requires the classical backreaction of the geometry, which is solved by taking into account the nontrivial nonchiral exotic energy. Finally, we obtain the exact expression for the size of the throat depending on the total amount of the infalling net energy and discuss the interesting transition from the anti–de Sitter spacetime to the wormhole geometry.

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Two-dimensional soluble gravitational models [1] are much simpler because of less degrees of freedom than the higher-dimensional counterparts in studying black holes and cosmology. In fact, all enormous amount of work related to the intriguing gravitational issues has been accomplished in this lower-dimensional regime [2,4–7]. Recently, wormhole solutions [3] in two-dimensional dilaton gravity [4,5] have been obtained by adding the appropriate exotic matter that gives exact solvability [6]. Therefore, it may be natural to consider the exactly soluble charged wormhole by adding dynamical fermions; of course, the stability of the wormhole seems to be more or less nontrivial.

In this Brief Report, we consider the infalling chiral fermions coupled to the Abelian gauge field in two-dimensional dilaton gravity. The Maxwell kinetic term is slightly modified in order to exactly solve [7]. We consider left-handed chiral fermions coupled to the gauge field, and the Abelian gauge field is suitably gauged away, which is in fact only possible classically without the quantum gauge anomaly. We first derive a static charged wormhole solution in our model without the infalling left-handed fermions and then the exact time-dependent charged wormhole solution is obtained by adding the fermions. The infalling matter spoils the original wormhole structure due to the classical backreaction of the geometry so that the left-right asymmetric exotic matter should be considered to reproduce the exact wormhole geometry in later time.

We now consider Callan-Giddings-Harvey-Strominger (CGHS) dilaton gravity with chiral fermions coupled to the U(1) gauge field:

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2]-\frac{e^{2\phi}}{g_A^2}F^2-\sum_{j=1}^{N}i\overline{\Psi}_j\gamma^\mu(D_\mu-iA_\mu)\Psi_j+\frac{1}{2}(\nabla\mathcal{G})^2\right), \tag{1}$$

where $\phi$ and $\Psi_j=\binom{\psi_j}{0}$ are a dilaton field and the $N$ left-handed chiral fermions, and $\mathcal{G}$ is a ghost scalar field whose energy density is negative. And $g_A$ is the U(1) electromagnetic gauge coupling constant and $D_\mu$ is a covariant derivative. Note that the gauge coupling effectively depends on the dilaton field, which is crucial to obtain the exact solution later. Defining $\Psi=\exp(i\int A_\mu dx^\mu)X$, $\overline{\Psi}=\overline{X}\exp(-i\int A_\mu dx^\mu)$ in terms of new variables $X_j=\binom{\chi_j}{0}$, and then in the conformal gauge of $g_{+-}=-e^{2\rho}/2$, $g_{\pm\pm}=0$, where $x^\pm=(x^0\pm x^1)$, the metric equations of motion are given by

$$e^{-2\phi}(2\partial_\pm\partial_\pm\phi-4\partial_\pm\rho\partial_\pm\phi)=T_{\pm\pm}, \tag{2}$$

$$e^{-2\phi}(2\partial_+\partial_-\phi-4\partial_+\phi\partial_-\phi-\lambda^2e^{2\rho})+\frac{2}{g_A^2}e^{2(\phi-\rho)}F_{+-}^2=0, \tag{3}$$

where the total energy-momentum tensors composed of the fermions and the exotic field are $T_{\pm\pm}=T^f_{\pm\pm}+T^g_{\pm\pm}$, explicitly given as $T^f_{++}=(i/4)\Sigma_{j=1}^N[\chi_j^*\partial_+\chi_j-(\partial_+\chi_j^*)\chi_j]$, $T^f_{--}=0$, and $T^g_{\pm\pm}=-(\partial_\pm\mathcal{G})^2/2$. Then the Maxwell equations are also obtained as

$$\partial_\pm(e^{2(\phi-\rho)}F_{+-})=J_\pm, \tag{4}$$

where the source currents are $J_+=(g_A^2/16)\Sigma_{j=1}^N\chi_j^*\chi_j$ and $J_-=0$. Next, the dilaton, fermions, and the ghost equations of motion are written as

$$-4\partial_+\partial_-\phi+4\partial_+\phi\partial_-\phi+2\partial_+\partial_-\rho+\lambda^2e^{2\rho}-\frac{2}{g_A^2}e^{4\phi-2\rho}F_{+-}^2=0, \tag{5}$$

$$\partial_-\chi_j=0, \tag{6}$$

$$\partial_+\partial_-\mathcal{G}=0, \tag{7}$$

respectively. The fermions are free since we redefined them in terms of new variables classically, which is impossible because of the chiral anomaly in the quantized theory.

By using Eqs. (3) and (5), we get the following simple relation, $\partial_+\partial_-(\rho-\phi)=0$, which is key to the exact solubility of the charged wormhole, and then the residual symmetry

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can be fixed by choosing $\rho=\phi$ corresponding to the Kruskal gauge. Hence, the metric equations of motion can be simply written as

$$-\partial_{\pm}\partial_{\pm}e^{-2\phi}=T^{f}_{\pm\pm}+T^{g}_{\pm\pm}, \tag{8}$$

$$\partial_{+}\partial_{-}e^{-2\phi}+\lambda^{2}-\frac{2}{g_A^2}F_{+-}^{2}=0, \tag{9}$$

and the Maxwell equations of motion (4) become $\partial_{+}F_{+-}=J_{+}$ and $\partial_{-}F_{+-}=0$, and then the general solutions for Eqs. (6) and (7) are $\chi_j=\chi_j(x^+)$ and $\mathcal{G}(x^+,x^-)=\mathcal{G}_{+}(x^+)+\mathcal{G}_{-}(x^-)$. These simple chiral fields are obtained with the help of the special dilaton-gauge coupling in front of the Maxwell kinetic term in the starting action (1).

Before we derive the dynamical wormhole solution, we exhibit a vacuum solution by assuming $X_j=\mathcal{G}=0$, which is given as [7]

$$F_{+-}=\frac{\overline{Q}}{2},\quad e^{-2\phi}=\frac{M}{\lambda}-\tilde{\lambda}^2x^+x^-, \tag{10}$$

where $M$ is a black hole mass and $\tilde{\lambda}$ is defined by $\tilde{\lambda}^2=\lambda^2-2C^2/g_A^2$ and $C$ is a constant corresponding to the background charge. The structure of spacetime is similar to that of the Schwarzschild black holes and simple in contrast to the Reissner-Nordström black hole solutions in four dimensions.

Next, we consider the $X_j=0$ and $\mathcal{G}\neq 0$ case with the special choice of $\mathcal{G}=\sqrt{2}\tilde{\lambda}(x^+-x^-)$, the energy momentum tensor is then negative constant [6],

$$T_{\pm\pm}=T^{g}_{\pm\pm}=-\tilde{\lambda}^2. \tag{11}$$

Integrating Eq. (9), we obtain the metric $e^{-2\phi}=-\tilde{\lambda}^2x^+x^-+a_{+}(x^+)+a_{-}(x^-)$ and the integration functions are determined by the constraints (8) as $a_{\pm}=(1/2)\tilde{\lambda}^2x^{\pm 2}+B_{\pm}x^{\pm}+D_{\pm}$, where $B_{\pm}$ and $D_{\pm}$ are integration constants. Choosing $B_{\pm}=0$ and $D_{+}+D_{-}=D$, the solution is obtained as

$$e^{-2\phi}=D+\frac{1}{2}\tilde{\lambda}^2(x^+-x^-)^2, \tag{12}$$

and the curvature scalar is calculated as $R=4\tilde{\lambda}^2[D-\tilde{\lambda}^2(x^+-x^-)^2/2]/[D+\tilde{\lambda}^2(x^+-x^-)^2/2]$. Note that for $D>0$, it is a traversable charged wormhole solution with $x^+=x^-$, which is a condition of the coincidence for the past and future horizons, whereas there is a naked singularity curve along with $D+\tilde{\lambda}^2(x^+-x^-)^2/2=0$ for $D<0$. Especially, for $D=0$, a constant negative curvature $R=-4\tilde{\lambda}^2$ is given, which is just anti–de Sitter (AdS) spacetime. On the other hand, for $\tilde{\lambda}^2<0$, if $D<0$, the solution is ill-defined as seen from Eq. (12). Until now, we assumed positive $\tilde{\lambda}^2$ for simplicity.

We now study our model for the case of $X_j\neq 0$ and $\mathcal{G}\neq 0$ in order to study the time-dependent wormhole and its maintenance. The infalling fermions carry the positive energy and the electric charge simultaneously, which gives the charged wormhole solution. However, in this dynamical situation in contrast to the static wormhole, the initial wormhole geometry cannot be maintained as time goes on because of the classical back reaction of geometry. Therefore, some additional corrections should be made to maintain the wormhole geometry even at latest time.

We now consider travelling charged matter from our universe to the other by simply assuming $\chi_j(x^+)\neq 0$ with the energy-momentum density expressed by a shock wave as

$$T^{f}_{++}=\alpha\,\delta(x^+-x_1^+),\quad T^{f}_{--}=0, \tag{13}$$

where $\alpha>0$, and the electric charge density as

$$J_{+}=Q\,\delta(x^+-x_1^+),\quad J_{-}=0. \tag{14}$$

Note that we assumed the left-handed chiral fermions since it is nontrivial compared to the left-right symmetric infalling case, and both the charge density and the energy momentum tensor were taken simply as the same shock-wave form. Then from the Maxwell equation of motion (4) and the left-handed current (14),

$$F_{+-}=Q\,\theta(x^+-x_1^+)+\frac{\overline{Q}}{2}, \tag{15}$$

is given where $\overline{Q}/2$ is a background constant charge and $Q$ is added charge through the shock wave at $x^+=x_1^+$.

Next, integrating Eq. (9) with Eq. (15), we obtain the metric solution,

$$e^{-2\phi}=-\tilde{\lambda}^2x^+x^-+\frac{2\beta}{g_A^2}x^-(x^+-x_1^+)\,\theta(x^+-x_1^+)+a_{+}(x^+)+a_{-}(x^-), \tag{16}$$

where $\beta=Q^2+Q\overline{Q}$. The shock wave source (13) indicates irradiation of the wormhole from our universe, which spoils the coincidence of the past and future event horizons. So, in this infalling matter source, we should take into account some corrected exotic matter source instead of the constant exotic background, Eq. (11), so that we assume left-right asymmetric ghost energy-momentum tensors,

$$T^{g}_{++}=-\tilde{\lambda}^2+\frac{2\beta}{g_A^2}x^-\delta(x^+-x_0^+)-\alpha\,\delta(x^+-x_0^+),$$

$$T^{g}_{--}=-\tilde{\lambda}^2-\frac{2\beta}{g_A^2}(x_1^- - x_0^-)\,\delta(x^- - x_0^-), \tag{17}$$

where $x_1>x_0$, which means that the negative energy source is provided in advance before the fermions destroy the original static wormhole structure. From Eqs. (2), (13), and (17), we find the complete metric,

$$e^{-2\phi}=\frac{1}{2}\tilde{\lambda}^2(x^+-x^-)^2-\frac{2\beta}{g_A^2}[x^-(x^+-x_0)\theta(x^+-x_0) -(x_1-x_0)(x^--x_0)\theta(x^--x_0)-x^-(x^+-x_1) \times\theta(x^+-x_1)]+\alpha(x^+-x_0)\theta(x^+-x_0) -\alpha(x^+-x_1)\theta(x^+-x_1)+D, \quad (18)$$

and we assigned $x_0^+=x_0^-=x_0$, $x_1^+=x_1^-=x_1$ to satisfy the wormhole condition in latest time $x^+>x_1^+$ and $x^->x_0^-$, where the metric can be simplified by the desirable wormhole geometry for the limiting cases,

$$e^{-2\phi}=\begin{cases}\frac{1}{2}\tilde{\lambda}^2(x^+-x^-)^2+D, & x^\pm<x_0,\\ \frac{1}{2}\tilde{\lambda}^2(x^+-x^-)^2+\left(\alpha-\frac{2\beta}{g_A^2}x_0\right)(x_1-x_0)+D, & \\ \qquad x^+>x_1 \text{ and } x^->x_0. & \end{cases} \quad (19)$$

Note that the size of the initial throat can be changed by the given parameters, infalling energy density, and electric charge. To see how this infalling matter affects the throat, we calculate the energy defined by [4,5],

$$E=\int dx^+x^+T_{++}+\int dx^-x^-T_{--}, \quad (20)$$

which is explicitly calculated as

$$E=\begin{cases}-\frac{1}{2}\tilde{\lambda}^2(x^{+2}+x^{-2}), & x^\pm<x_0,\\ -\frac{1}{2}\tilde{\lambda}^2(x^{+2}+x^{-2})+\frac{2\beta}{g_A^2}(x^-x_0-x_0x_1+x_0^2) & \\ \qquad +\alpha(x_1-x_0), \quad x^+>x_1 \text{ and } x^->x_0. & \end{cases} \quad (21)$$

Therefore, the change of the infalling energy is simply at $x^-=0$ given by

$$\Delta E=\left(\alpha-\frac{2\beta}{g_A^2}x_0\right)(x_1-x_0), \quad (22)$$

and Eq. (19) is rewritten in terms of $\Delta E$ such as $e^{-2\phi}=(1/2)\tilde{\lambda}^2(x^+-x^-)^2+\Delta E+D$, which implies that the size of the throat is directly influenced by the infalling energy.

We have obtained the exactly soluble charged wormhole solution in the two-dimensional Maxwell-dilaton gravity. The infalling chiral matter affects the static background geometry by adding the energy-momentum and electric charge. The classical backreaction of the geometry has been properly taken into account and the exact self-gravitating solutions are derived. Of course, the corresponding exotic source should be adjusted along with the infalling real matter, which has left-right asymmetric form even if we only consider the left-handed chiral fermions. On the other hand, unfortunately, we did not address the quantum backreaction of the geometry due to the conformal anomaly and the chiral anomaly, which might be more interesting in the lower-dimensional quantum gravity.

As for some candidates for the exotic matter, it seems to be a nontrivial issue in that they must violate the positiveness of the energy. However, it has been shown that the Brans-Dicke (BD) scalar field in the BD theory can play a role of the exotic matter in Ref. [8], where the crucial reason is due to the unconventional negative coupling parameter in BD theory while the conventional Einstein-Hilbert action can be recovered for the positive BD parameter. So, one might wonder whether our dilaton field can be the candidate for the exotic matter or not. It is so plausible because it has a wrong sign in the kinetic term as seen from Eq. (1). We now explicitly investigate the possibility of exotic matter of the dilaton field by using the simplified model aside from its exact solubility. We divide the two-dimensional dilaton gravity action into the two pieces as $S=S_G+S_D$, where $S_G=(1/2\pi)\int d^2x\sqrt{-g}e^{-2\phi}R$,

$$S_D=(1/2\pi)\int d^2x\sqrt{-g}e^{-2\phi}[4(\nabla\phi)^2+4\lambda^2],$$

and assume the static metric to be $ds^2=-e^{2\Phi(r)}dt^2+[1-b(r)/r]^{-1}dr^2$. The undetermined functions $\Phi(r)$ and $b(r)$ are the gravitational redshift and the shape functions, respectively [3]. Let us define orthonormal basis as a proper reference frame $\hat{e}_{\hat{t}}=e^{-\Phi}\hat{e}_t$, $\hat{e}_{\hat{r}}=(1-b/r)^{1/2}\widehat{e_r}$, $g_{\hat{\alpha}\hat{\beta}}=e_{\hat{\alpha}}\cdot e_{\hat{\beta}}=\eta_{\hat{\alpha}\hat{\beta}}=\mathrm{diag}(-1,1)$, in order to obtain the energy-momentum tensors measured by static observers, then they are expressed as

$$\rho=e^{-2\phi}\left[\left(1-\frac{b_0}{r}\right)\left(2\Phi''+3(\Phi')^2+\frac{3b_0\Phi'}{2r(r-b_0)} -2\Phi'\phi'-2\phi''-\frac{b_0\phi'}{r(r-b_0)}\right)-4\lambda^2\right],$$

$$-\tau=e^{-2\phi}\left[\left(1-\frac{b_0}{r}\right)\left(-2\Phi''-(\Phi')^2-\frac{b_0\Phi'}{2r(r-b_0)} -4(\phi')^2+2\Phi'\phi'+2\phi''+\frac{b_0\phi'}{r(r-b_0)}\right)+4\lambda^2\right], \quad (23)$$

where $\rho(r)=T_{\hat{t}\hat{t}}$ is the density of mass-energy and $\tau=-T_{\hat{r}\hat{r}}$ is the tension in the radial direction. Note that we assumed $b=b_0$ for simplicity, which is consistent with the flaring-out condition at the wormhole throat. The exoticity is defined by $\zeta=(\tau-\rho)/|\rho|>0$ [3], which explicitly requires

$$4(\phi')^2-2(\Phi')^2-\frac{b_0}{r(r-b_0)}\Phi'>0 \quad (24)$$

for the wormhole. Especially, choosing $\Phi\sim(r-b_0)^\alpha r^{-\alpha-1}$ where $\alpha\geqslant 2$ for the regularity of the redshift function, $\zeta$ is always positive at the throat, which implies the exoticity is maintained in terms of the dilaton field. Therefore, the dilaton field can be a candidate of the exotic matter near the throat. However, in our model the additional ghost was just added to obtain the exact soluble dynamic wormhole in the conformal gauge.

The other issue is on the integrability of the two-dimensional gravity in connection with the Poincaré gauge gravity [9]. Triviality of the Einstein-Hilbert action in two dimensions has been overcome with introduction of the torsion [10], whereas the dilaton field make the two-dimensional theory nontrivial in our model. The exact black-hole type solution is well obtained in the regime of two-dimensional Poincare gauge gravity. Furthermore, the scalar curvatures in both theories are all constant at the asymptotic limit while at the origin they are different. Superficially, they seem to be conformally equivalent and they deserve to be solved at the action level. So, the conformal transformation implemented by the metric is $g_{\mu\nu} \to e^{-2\phi} g_{\mu\nu}$. Then the gravitational sector in Eq. (1) is transformed into $\int d^2x \sqrt{-g}(e^{-2\phi}R + 4\lambda^2)$, where the dilaton factor in front of the curvature scalar is not removed in two dimensions because $\sqrt{-g}g^{\mu\nu}$ is Weyl invariant, which means that the Einstein-Hilbert action does not appear after some appropriate conformal transformation. Of course, the torsion does not appear under the conformal transformation either [10]. Therefore, the two theories are apparently unrelated although the black hole solutions are almost equivalent.

As a final comment, as seen below Eq. (12), the curvature scalar is constant especially for $D=0$, which is nothing but the AdS space time. So we may consider an interesting transition from the AdS spacetime to the wormhole geometry by adding infalling matter. In fact, the AdS spacetime frequently appears in near-horizon physics of the charged black holes in a certain limit, so the infalling real matter produces the wormhole geometry locally. In our case, simply considering Eq. (19) for $D=0$, the AdS spacetime evolves into the wormhole geometry as far as $\alpha > 2\beta x_0 / g_A^2$. Therefore, it will be interesting to study this kind of transition in more detail.

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