

# $SL(2, \mathbb{C})$ gravity with a complex vierbein and its noncommutative extension

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We show that it is possible to formulate gravity with a complex vierbein based on  $SL(2, \mathbb{C})$  gauge invariance. The proposed action is a four-form where the metric is not introduced but results as a function of the complex vierbein. This formulation is based on the first order formalism. The novel feature here is that integration of the spin-connection gauge field gives rise to kinetic terms for a massless graviton, a massive graviton with the Fierz-Pauli mass term, and a scalar field. The resulting theory is equivalent to bigravity. We then show that by extending the gauge group to  $GL(2, \mathbb{C})$  the formalism can be easily generalized to apply to a noncommutative space with the star product. We give the deformed action and derive the Seiberg-Witten map for the complex vierbein and gauge fields.

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## I. INTRODUCTION

The general theory of relativity can be formulated either as a geometrical theory in terms of a metric tensor over the space-time manifold, or in terms of a vierbein and a spin connection of the local Lorentz algebra [1,2]. Both formulations lead to equivalent results as far as the dynamics of the graviton is concerned. The second approach is more appropriate to couple to spinors [3–5]. There have been attempts to unify gravitation with other interactions, notably the Kaluza-Klein approach of compactifying higher dimensional theories, and the Einstein-Strauss-Schrödinger [6,7] approach of considering a Hermitian metric tensor and interpreting the antisymmetric field as that of the Maxwell field strength. The advantages and disadvantages of the Kaluza-Klein approach are well known, while the uses of a complex space-time metric are less familiar [8,9]. It is now well known that the antisymmetric part of the Hermitian metric cannot be interpreted as the photon field strength but rather as an antisymmetric tensor where the theory is consistent only if the field is massive [10]. Recently, a formulation of complex gravity using the idea of gauging the unitary algebra  $U(2,2)$  was made using a complex vierbein [11]. This was shown to give an action with many desirable properties; the main disadvantage is that the density formed from the complex vierbein is not unique. As one of the motivations for introducing a complex metric is to deform general relativity for a special noncommutative space with a star product, it is necessary to require the full action to be invariant under both the star product and the group transformations. The easiest way to implement this requirement is to construct the action to be a trace of a four-form, ensuring that it is a gauge invariant density. It turned out that in this case it is not easy to obtain a simple action satisfying these properties. By using a constrained gauge group  $U(2,2)$  the construction becomes possible, but only for conformal gravity, not Einstein gravity. Another disadvantage is that it was necessary to use the Seiberg-Witten map [12,13] in order to solve the noncommu-

tative constraints, resulting in complicated expressions.

It is therefore important to have a gauge invariant formulation of deformed complex gravity where the action is written as a four-form. To do this we must first succeed in formulating complex gravity without introducing *a priori* a metric tensor. Taking a close look at the  $SL(2, \mathbb{C})$  formulation of gravity [14], one notes that the following steps are needed. First an  $SL(2, \mathbb{C})$  gauge field is introduced, the spin connection

$$\omega = dx^\mu \omega_\mu = \frac{1}{4} dx^\mu \omega_{\mu ab} \gamma^{ab},$$

where  $\gamma^{ab}$  is the antisymmetrized product of Dirac gamma matrices.<sup>1</sup> The field strength

$$\begin{aligned} R = d\omega + \omega^2 &= \frac{1}{2} dx^\mu \wedge dx^\nu (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu + [\omega_\mu, \omega_\nu]) \\ &= \frac{1}{8} dx^\mu \wedge dx^\nu R_{\mu\nu ab} \gamma^{ab} \end{aligned}$$

transforms covariantly under the  $SL(2, \mathbb{C})$  gauge transformation  $\Omega$ :

$$\begin{aligned} \omega &\rightarrow \Omega d\Omega^{-1} + \Omega \omega \Omega^{-1}, \\ R &\rightarrow \Omega R \Omega^{-1}, \end{aligned}$$

where  $\Omega = \exp(\frac{1}{4} \Lambda_{ab} \gamma^{ab})$  and  $\Lambda_{ab}$  are the infinitesimal gauge parameters. Next, the vierbein  $e$  defined by

$$e = dx^\mu e_\mu = dx^\mu e_\mu^a \gamma_a$$

is introduced, which transforms under  $SL(2, \mathbb{C})$  according to

$$e \rightarrow \Omega e \Omega^{-1}.$$

The  $SL(2, \mathbb{C})$  invariant gravitational action is then given by

<sup>1</sup>We adopt the notation of Ref. [15] for the Dirac gamma matrices. In particular,  $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ ,  $\gamma_a^\dagger = \gamma_a$ ,  $a = 1, \dots, 4$ ,  $\gamma_4 = i\gamma_0$ , and  $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ .

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$$\begin{aligned}
I &= \frac{1}{8} \int_M \text{Tr} \left( (\alpha + \beta \gamma_5) e \wedge e \wedge R + \frac{\delta}{48} \gamma_5 e \wedge e \wedge e \wedge e \right) \\
&= \frac{1}{16} \int_M d^4x \epsilon^{\mu\nu\kappa\lambda} \left[ \epsilon_{abcd} \left( \beta e_\mu^a e_\nu^b R_{\kappa\lambda}{}^{cd} + \frac{\delta}{6} e_\mu^a e_\nu^b e_\kappa^c e_\lambda^d \right) \right. \\
&\quad \left. - 2\alpha e_\mu^a e_\nu^b R_{\kappa\lambda}{}^{ab} \right].
\end{aligned}$$

After the  $\omega_\mu^{ab}$  field is integrated out, this gives

$$I = \frac{1}{4} \int_M d^4x e (R + \delta),$$

which is the Einstein-Hilbert action plus a cosmological constant. Notice that the term with the coefficient  $\alpha$  is of the form  $\epsilon^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda}$  and will vanish on shell by the symmetries of the Riemann tensor. The invariance of the action under  $SL(2, \mathbb{C})$  transformations can be easily verified as  $\Omega$  commutes with  $\gamma_5$ .

In noncommutative geometry where the star product replaces ordinary products, the group  $SL(2, \mathbb{C})$  is first extended to  $GL(2, \mathbb{C})$  so that the star product of two group elements is a group element. The field  $e$  is not preserved under group transformations

$$e \rightarrow \Omega * e * \Omega_*^{-1},$$

where  $\Omega * \Omega_*^{-1} = 1$ . It is easily verified that the field  $e$  will become complex and must be replaced by the field  $L$  defined by

$$L = dx^\mu (e_\mu^a + i \gamma_5 f_\mu^a) \gamma_a,$$

which transforms properly under  $L \rightarrow \Omega * L * \Omega_*^{-1}$ . It is therefore necessary before studying any noncommutative generalization to determine whether the gravitational theory with the field  $e$  replaced by  $L$  is well defined. At first this idea does not seem to be very promising because there will be two vierbeins  $e_\mu^a$  and  $f_\mu^a$  and only one spin connection  $\omega_\mu^{ab}$ . In the Einstein-Hilbert action given above, the field  $\omega_\mu^{ab}$  appears quadratically and can be determined exactly from its equation of motion as a function of the vierbein, and this is equivalent to performing a Gaussian integration. The question we have to address is whether the couplings of  $\omega_\mu^{ab}$  to  $e_\mu^a$  and  $f_\mu^a$  will be in such a way as to ensure the dynamical propagation of both fields. What is needed is to get correct couplings for two symmetric tensors and two antisymmetric tensors that could be formed out of  $e_\mu^a$  and  $f_\mu^a$ . One combination of the antisymmetric tensors could be gauged away by the  $SL(2, \mathbb{C})$  invariance of the action. Moreover, because of the diffeomorphism invariance of the full action, one combination of the symmetric tensors would correspond to the massless graviton. The other symmetric combination would then correspond to a massive graviton coupled to a scalar field (dilatonlike). The remaining antisymmetric field will be massive. In other words, this complex gravity should be equivalent to bigravity [16–19] and yield the interaction of a massless graviton coupled to a massive graviton and to a

scalar field and a massive antisymmetric tensor. It is essential to have  $\omega_\mu^{ab}$  generate the correct kinetic energies for the two tensors. Happily, we shall show that this is indeed the case, and remarkably there exists a coupling of the complex vierbein  $L$  to the curvature tensor that gives precisely the desired form with correct signs. As mentioned before, the metric tensor is not introduced *a priori* but results as a combination of the two fields  $e_\mu^a$  and  $f_\mu^a$ . To deform the action so that ordinary products are replaced with star products it is necessary to extend the group  $SL(2, \mathbb{C})$  to  $GL(2, \mathbb{C})$ . Chiral rotations are present in  $GL(2, \mathbb{C})$  and this further restricts the form of the invariant action. The invariant action taken in the commutative case has to be modified. In this case it will be necessary to impose a torsion-free constraint on the complex vierbein  $L$ . The  $GL(2, \mathbb{C})$  gauge fields have to be determined by solving the torsion-free constraint instead of solving the equations of motion. Again this can only be done perturbatively but it is relatively easy to evaluate the deformed action. It is also possible to derive the Seiberg-Witten map [12,13] between the deformed and undeformed gauge fields and complex vierbein. In contrast to earlier approaches, we shall show that the deformed action could be obtained without the use of this map, and that its form is manageable.

The plan of this paper is as follows. In Sec. II we propose the action for complex gravity in terms of the field  $L$ . In Sec. III we eliminate the field  $\omega_\mu^{ab}$  in terms of  $e_\mu^a$  and  $f_\mu^a$  and show that both tensors obtain the correct kinetic and mass terms. In Sec. IV we extend the complex gravitational action to the noncommutative case where ordinary products are replaced with star products. We also give transformations of the deformed fields, the deformed action as well as the Seiberg-Witten map. Section V contains the conclusion and some comments.

## II. GRAVITY WITH A COMPLEX VIERBEIN

We start by considering the  $SL(2, \mathbb{C})$  gauge connection  $\omega$  and the field  $L$  transforming under  $SL(2, \mathbb{C})$  as  $L \rightarrow \Omega L \Omega^{-1}$ . A generalization of the Einstein-Hilbert action for the complex field

$$L = dx^\mu (e_\mu^a + i f_\mu^a \gamma_5) \gamma_a$$

is given by

$$I_1 = \frac{1}{8} \int_M \text{Tr} [(\alpha + \beta \gamma_5) (L + iL') \wedge (L - iL') \wedge R],$$

where

$$L' = dx^\mu (e_\mu^a - i f_\mu^a \gamma_5) \gamma_a = -CL^T C^{-1}$$

with  $C$  being the charge conjugation matrix with the property  $C \gamma_a^T C^{-1} = -\gamma_a$ . Under  $SL(2, \mathbb{C})$  gauge transformations the field  $L'$  transforms as  $L' \rightarrow \Omega L' \Omega^{-1}$ . Notice that this action is Hermitian because

$$\omega_\mu^\dagger = -\omega_\mu, \quad R_{\kappa\lambda}^\dagger = -R_{\kappa\lambda},$$

$$L_\mu^\dagger = L_\mu, \quad L_\mu^{\prime\dagger} = L_\mu, \quad \gamma_5 L_\mu = -L_\mu \gamma_5.$$

It is possible to construct a different action where the combination  $(L_\mu L'_\nu + L'_\mu L_\nu)$  replaces  $(L_\mu L_\nu + L'_\mu L'_\nu)$ . This would yield the tensor combination  $(e_\mu^a e_\nu^b - f_\mu^a f_\nu^b)$  instead of  $(e_\mu^a e_\nu^b + f_\mu^a f_\nu^b)$ , which is an undesirable result as it gives the wrong sign for the kinetic energy of the massive graviton in the action. There are many possibilities for the cosmological constant and mass terms. We shall choose a combination of terms such that it would be possible to set the cosmological constant to zero, have the linear terms in the fields  $e_\mu^a$  and  $f_\mu^a$  vanish, and get the Fierz-Pauli form [20] for the mass of the spin-2 field. This is given by

$$I_2 = \frac{1}{192} \int_M \text{Tr} \left\{ \alpha_1 \gamma_5 (L \wedge L \wedge L \wedge L + L \wedge L' \wedge L \wedge L') + \frac{i}{8} (L \wedge L' - L' \wedge L) \wedge [\alpha_2 (L + L') \wedge (L + L') - \alpha_3 (L - L') \wedge (L - L')] \right\}.$$

To evaluate this action, we first expand it in terms of the component fields  $e_\mu^a$ ,  $f_\mu^a$ , and  $\omega_\mu^{ab}$  and then simplify the Clifford algebra. The full action  $I = I_1 + I_2$  simplifies to

$$I = \frac{1}{2} \int_M d^4x \epsilon^{\mu\nu\kappa\lambda} \left\{ \epsilon_{abcd} [\beta (e_\mu^a e_\nu^b + f_\mu^a f_\nu^b) + 2\alpha e_\mu^a f_\nu^b] R_{\kappa\lambda}{}^{cd} - 2[\alpha (e_\mu^a e_\nu^b + f_\mu^a f_\nu^b) + 2\beta e_\mu^a f_\nu^b] R_{\kappa\lambda}{}^{ab} + \frac{1}{4!} \epsilon_{abcd} \alpha_1 (e_\mu^a e_\nu^b e_\kappa^c e_\lambda^d + f_\mu^a f_\nu^b f_\kappa^c f_\lambda^d) + \frac{1}{4!} \epsilon_{abcd} (\alpha_2 e_\mu^a e_\nu^b e_\kappa^c f_\lambda^d + \alpha_3 f_\mu^a f_\nu^b f_\kappa^c e_\lambda^d) \right\}.$$

The field  $\omega_\mu^{ab}$  appears quadratically. This means that it can be eliminated from the action by a Gaussian integration. Alternatively, we can solve the  $\omega_\mu^{ab}$  equations of motion and substitute the value of  $\omega_\mu^{ab}$  back into the action. In general this would require inverting the tensor operator

$$\epsilon^{\mu\nu\kappa\lambda} \left\{ \epsilon_{abcd} [\beta (e_\mu^a e_\nu^b + f_\mu^a f_\nu^b) + 2\alpha e_\mu^a f_\nu^b] - 2[\alpha (e_\mu^c e_\nu^d + f_\mu^c f_\nu^d) + 2\beta e_\mu^c f_\nu^d] \right\}.$$

This step could be done perturbatively only as function of  $e_a^\mu$  and  $f_a^\mu$ , the inverses of  $e_\mu^a$  and  $f_\mu^a$ . In fact the analysis is fairly complicated, and in order to determine the dynamical degrees of freedom of the system, it is essential to study the linearized approximation. This is done by expanding  $e_\mu^a$  and  $f_\mu^a$  around a flat background by writing [16]

$$e_\mu^a = c_1 \delta_\mu^a + \bar{e}_\mu^a, \\ f_\mu^a = c_2 \delta_\mu^a + \bar{f}_\mu^a,$$

where  $c_1$  and  $c_2$  are parameters. Keeping only up to the bilinear terms in  $\bar{e}_\mu^a$  and  $\bar{f}_\mu^a$ , we obtain

$$I = 2 \int d^4x \left\{ -[\beta(c_1^2 + c_2^2) + 2\alpha c_1 c_2] (\omega_{dce} \omega_{ced} + \omega_e \omega_e) + \delta_{bcd}^{\nu\kappa\lambda} \partial_\kappa \omega_{\lambda cd} [(\beta c_1 + \alpha c_2) \bar{e}_\nu^b + (\alpha c_1 + \beta c_2) \bar{f}_\nu^b] - [\alpha(c_1^2 + c_2^2) + 2\beta c_1 c_2] \epsilon^{ab\kappa\lambda} \omega_{kae} \omega_{\lambda eb} - 2\epsilon^{a\nu\kappa\lambda} \partial_\kappa \omega_{\lambda ab} [(\alpha c_1 + \beta c_2) \bar{e}_\nu^b + (\beta c_1 + \alpha c_2) \bar{f}_\nu^b] + [\alpha_1(c_1^4 + c_2^4) + \alpha_2 c_1^3 c_2 + \alpha_3 c_1 c_2^3] + (4\alpha_1 c_1^3 + 3\alpha_2 c_1^2 c_2 + \alpha_3 c_2^3) \bar{e} + (4\alpha_1 c_2^3 + 3\alpha_3 c_1 c_2^2 + \alpha_2 c_1^3) \bar{f} + 3\delta_{\mu\nu}^{ab} [(2\alpha_1 c_1^2 + \alpha_2 c_1 c_2) \bar{e}_\mu^a \bar{e}_\nu^b + (2\alpha_1 c_2^2 + \alpha_3 c_1 c_2) \bar{f}_\mu^a \bar{f}_\nu^b] + \dots \right\}.$$

As a first step, we write the  $\omega_\mu^{ab}$  equation of motion, which takes the form

$$\left\{ [\beta(c_1^2 + c_2^2) + 2\alpha c_1 c_2] (\omega_{d\lambda c} - \omega_{c\lambda d} + \delta_{c\lambda} \omega_d - \delta_{d\lambda} \omega_c) + [\alpha(c_1^2 + c_2^2) + 2\beta c_1 c_2] (\epsilon^{ad\kappa\lambda} \omega_{kae} - \epsilon^{ac\kappa\lambda} \omega_{kad}) \right\} = -\delta_{bcd}^{\nu\kappa\lambda} \partial_\kappa [(\beta c_1 + \alpha c_2) \bar{e}_\nu^b + (\alpha c_1 + \beta c_2) \bar{f}_\nu^b] + \{ \epsilon^{c\nu\kappa\lambda} \partial_\kappa [(\alpha c_1 + \beta c_2) \bar{e}_\nu^d + (\beta c_1 + \alpha c_2) \bar{f}_\nu^d] - c \leftrightarrow d \}.$$

This is a difficult equation to solve. To simplify the problem, we first define the tensor

$$X_{mab}^{npq} = \frac{a}{2} (\delta_m^n \delta_{ab}^{pq} + \delta^{np} \delta_{ab}^{qm} - \delta^{nq} \delta_{ab}^{pm}) + \frac{b}{2} (\epsilon_{abnp} \delta_{qm} - \epsilon_{abnq} \delta_{pm}),$$

where

$$a = \beta(c_1^2 + c_2^2) + 2\alpha c_1 c_2, \\ b = \alpha(c_1^2 + c_2^2) + 2\beta c_1 c_2.$$

We then define the tensor

$$Y_{mab} = X_{mab}^{npq} \omega_{npq},$$

so that the  $\omega_{mab}$  equation simplifies to

$$Y_{d\lambda c} - Y_{c\lambda d} = -\partial_\kappa (\delta_{bcd}^{\nu\kappa\lambda} E_{\nu b} - \epsilon_{c\nu\kappa\lambda} F_{\nu d} + \epsilon_{d\nu\kappa\lambda} F_{\nu c}),$$

where

$$E_\nu^b = (\beta c_1 + \alpha c_2) \bar{e}_\nu^b + (\alpha c_1 + \beta c_2) \bar{f}_\nu^b, \\ F_\nu^b = (\alpha c_1 + \beta c_2) \bar{e}_\nu^b + (\beta c_1 + \alpha c_2) \bar{f}_\nu^b.$$

We can easily solve for  $Y_{cd\lambda}$  by a cyclic permutation of the  $Y$  equation to obtain

$$Y_{cd\lambda} = \frac{1}{2} [\partial_c(E_{d\lambda} - E_{\lambda d}) - \partial_d(E_{c\lambda} + E_{\lambda c}) + \partial_\lambda(E_{cd} + E_{dc})] \\ - \delta_{c\lambda}(\partial_b E_{db} - \partial_d E) + \delta_{d\lambda}(\partial_b E_{cb} - \partial_c E) \\ + \epsilon_{d\lambda\kappa\nu} \partial_\kappa F_{\nu c},$$

where  $E = E_{bb}$ . We now define the inverse of the tensor  $X_{mab}^{npq}$  by

$$(X^{-1})_{rst}^{mab} X_{mab}^{npq} = \frac{1}{2} \delta_r^n \delta_{st}^{pq}.$$

To find the inverse we write the most general rank-6 tensor antisymmetric in  $s$  and  $t$  and in  $p$  and  $q$  then determine the coefficients from the above constraint. After a lengthy calculation we obtain

$$(X^{-1})_{rst}^{mab} = \frac{1}{2(a^2 - b^2)} \left[ a \left( \delta_r^n \delta_{st}^{ab} - \frac{1}{2} \delta_{rs} \delta_{mt}^{ab} + \frac{1}{2} \delta_{rt} \delta_{ms}^{ab} \right) \right. \\ \left. - b \left( \epsilon_{stma} \delta_{br} - \epsilon_{stmb} \delta_{ar} + \epsilon_{mabs} \delta_{tr} - \epsilon_{mabt} \delta_{sr} \right. \right. \\ \left. \left. - \frac{1}{2} \epsilon_{rsta} \delta_{mb} + \frac{1}{2} \epsilon_{rstb} \delta_{ma} \right) \right].$$

We can then write

$$\omega_{rst} = (X^{-1})_{rst}^{mab} Y_{mab},$$

and after some algebra one finds

$$\omega_{rst} = \frac{1}{2(a^2 - b^2)} (\partial_r(aE_{st} - bF_{st}) - \partial_s(aE_{rt} - bF_{rt}) \\ + \epsilon_{st\mu\nu} \partial_\mu(aF_{vr} - bE_{vr}) + \delta_{rs} \epsilon_{t\mu\nu} \partial_\mu(aF_{vm} - bE_{vm}) \\ - s \leftrightarrow t).$$

This expression simplifies by noting that

$$aE_{st} - bF_{st} = (\beta^2 - \alpha^2)(c_1^2 - c_2^2)g_{va}, \\ bE_{rt} - aF_{rt} = (\beta^2 - \alpha^2)(c_1^2 - c_2^2)h_{va},$$

where we have defined

$$g_{va} = (c_1 \bar{e}_{va} - c_2 \bar{f}_{va}), \\ h_{va} = (-c_2 \bar{e}_{va} + c_1 \bar{f}_{va}).$$

We finally have

$$\omega_{rst} = \frac{1}{2(c_1^2 - c_2^2)} [\partial_r g_{st} - \partial_s(g_{rt} + g_{tr}) + \epsilon_{st\mu\nu} \partial_\mu h_{vr} \\ + \delta_{rs} \epsilon_{t\mu\nu} \partial_\mu h_{vm} - s \leftrightarrow t], \\ \omega_t = \frac{1}{(c_1^2 - c_2^2)} \left( -\partial_r g_{tr} + \partial_t g + \frac{1}{2} \epsilon_{t\mu\nu} \partial_\mu h_{vr} \right),$$

where  $\omega_t = \omega_{rrt}$ . To avoid degeneracy we shall impose the following constraints on the parameters  $\alpha$ ,  $\beta$ ,  $c_1$ , and  $c_2$ :

$$\alpha \neq \beta, \quad c_1 \neq c_2.$$

Substituting these expressions back into the action, we find that the antisymmetric part of  $g_{ab}$  decouples, while both the symmetric and antisymmetric parts of  $h_{ab}$  couple and acquire kinetic energies. We therefore write

$$g_{\mu\nu} = \frac{1}{2}(s_{\mu\nu} + a_{\mu\nu}), \\ h_{\mu\nu} = (l_{\mu\nu} + B_{\mu\nu}),$$

where  $s_{\mu\nu}, l_{\mu\nu}$  and  $a_{\mu\nu}, B_{\mu\nu}$  are, respectively, the symmetric and antisymmetric parts of  $g_{\mu\nu}$  and  $h_{\mu\nu}$ . Keeping only up to bilinear terms, the action reduces to

$$I = - \frac{4}{(c_1^2 - c_2^2)^2} \int d^4x (\partial_\mu s_{\nu\kappa} \partial_\mu s_{\nu\kappa} - 2 \partial_\mu s_{\mu\kappa} \partial_\nu s_{\nu\kappa} + 2 \partial_\mu s_{\mu\nu} \partial_\nu s - \partial_\mu s \partial_\mu s + \partial_\mu l_{\nu\kappa} \partial_\mu l_{\nu\kappa} - 2 \partial_\mu l_{\mu\kappa} \partial_\nu l_{\nu\kappa} + 2 \partial_\mu l_{\mu\nu} \partial_\nu l \\ - \partial_\mu l \partial_\mu l + \partial_\mu B_{\nu\kappa} \partial_\mu B_{\nu\kappa} - 2 \partial_\mu B_{\mu\kappa} \partial_\nu B_{\nu\kappa}) + \int d^4x [\alpha_1(c_1^4 + c_2^4) + \alpha_2(c_1^3 c_2) + \alpha_3(c_1 c_2^3)] \\ + \frac{1}{(c_1^2 - c_2^2)} \int d^4x \{ 4[\alpha_1(c_1^4 + c_2^4) + \alpha_2 c_1^3 c_2 + \alpha_3 c_1 c_2^3] g + [4\alpha_1 c_1 c_2 (c_1^2 + c_2^2) + \alpha_2 c_1^2 (c_1^2 + 3c_2^2) + \alpha_3 c_2^2 (c_2^2 + 3c_1^2)] h \} \\ + \frac{1}{(c_1^2 - c_2^2)^2} \int d^4x \delta_{\mu\nu}^{\kappa\lambda} (2[\alpha_1(c_1^4 + c_2^4) + \alpha_2(c_1^3 c_2) + \alpha_3(c_1 c_2^3)] g_{\mu\kappa} g_{\nu\lambda} \\ + \{ [4\alpha_1 c_1 c_2 (c_1^2 + c_2^2) + \alpha_2 c_1^2 (c_1^2 + 3c_2^2) + \alpha_3 c_2^2 (c_2^2 + 3c_1^2)] g_{\mu\kappa} h_{\nu\lambda} \} + [4\alpha_1 c_1^2 c_2^2 + (\alpha_2 + \alpha_3) c_1 c_2 (c_1^2 + c_2^2)] h_{\mu\kappa} h_{\nu\lambda}),$$

where  $g = g_{\mu\mu}, h = h_{\mu\mu}$ . By setting the cosmological term and linear terms in  $g$  and  $h$  to zero we get three equations in

the three parameters  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ . Only two of the equations are independent, and they are

$$\alpha_1(c_1^4 + c_2^4) + \alpha_2 c_1^3 c_2 + \alpha_3 c_1 c_2^3 = 0,$$

$$4\alpha_1 c_1^3 + 3\alpha_2 c_1^2 c_2 + \alpha_3 c_2^3 = 0.$$

These can be easily solved to determine  $\alpha_2$  and  $\alpha_3$  in terms of  $\alpha_1$ :

$$\alpha_2 = \frac{k^4 - 3}{2k} \alpha_1,$$

$$\alpha_3 = \frac{1 - 3k^4}{2k^3} \alpha_1,$$

where

$$k = \frac{c_2}{c_1} \neq 1.$$

With this solution one immediately finds that both the mass term  $\delta_{\mu\nu}^{\kappa\lambda} g_{\mu\kappa} g_{\nu\lambda}$  and the mixing term  $\delta_{\mu\nu}^{\kappa\lambda} g_{\mu\kappa} h_{\nu\lambda}$  vanish. There is, however, a mass term for  $h_{\mu\nu}$ :

$$\frac{3\alpha_1(k^4 + 1)}{2k^2} \delta_{\mu\nu}^{\kappa\lambda} h_{\mu\kappa} h_{\nu\lambda},$$

which is of the Fierz-Pauli type [20]. The order of the mass term can be tuned by adjusting the parameters  $\alpha_1$  and  $k$ . Note that both the symmetric field  $l_{\mu\nu}$  and the antisymmetric field  $B_{\mu\nu}$  acquire mass. It is not unexpected that the graviton field remains massless as this is protected by diffeomorphism invariance. However, it is remarkable that through the coupling of the spin connection  $\omega_{\mu ab}$  the correct kinetic energies for both fields  $g_{\mu\nu}$  and  $h_{\mu\nu}$  are generated. The degrees of freedom of this system are well defined. The graviton corresponds to a massless spin-2 field with two dynamical degrees of freedom, while the field  $h_{\mu\nu}$  corresponds to a massive spin-2 field coupled to a dilaton and has six degrees of freedom [17]. The dilaton coupling can be seen only by going to higher order terms as it couples to curvature terms [18,19]. To have a closed form for the fully nonlinear theory, it would be necessary to define an inverse for the tensor  $(e_\mu^a e_\nu^b + f_\mu^a f_\nu^b - e_\nu^a e_\mu^b - f_\nu^a f_\mu^b)$  so as to express the action in terms of this inverse.

Much work remains to be done to fully understand this theory and to determine its full coupling at the nonlinear level, but the above results are very encouraging and strongly indicate that this theory is consistent. It is also very geometrical as it is based on the gauge principle where all terms in the action are four-forms, thus avoiding the use of a density factor to guarantee invariance under general coordinate transformations. It would be very interesting to find some particular solutions to the full field equations such as generalizations of the Schwarzschild or de Sitter solutions.

### III. NONCOMMUTATIVE DEFORMED GRAVITY

The construction of the complex gravity action proposed in the last section suggests that it could be easily generalized

to the noncommutative case where the coordinates of space-time do not commute,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where  $\theta^{\mu\nu}$  are deformation parameters. An immediate step is to extend the  $SL(2, \mathbb{C})$  group to  $GL(2, \mathbb{C})$ . This is necessary because the commutator in a star product involves both ordinary commutators and anticommutators, as can be seen from the relation

$$A * B - B * A = [A, B]_{(*, \text{even})} + \{A, B\}_{(*, \text{odd})},$$

where

$$[A, B]_{(*, \text{even})} = [A, B] + \left(\frac{i}{2}\right)^2 \theta^{\mu\nu} \theta^{\kappa\lambda} [\partial_\mu \partial_\kappa A, \partial_\nu \partial_\lambda B] + O(\theta^4),$$

$$\{A, B\}_{(*, \text{odd})} = \frac{i}{2} \theta^{\mu\nu} \{ \partial_\mu A, \partial_\nu B \} + \left(\frac{i}{2}\right)^3 \theta^{\mu\nu} \theta^{\kappa\lambda} \theta^{\alpha\beta} \{ \partial_\mu \partial_\kappa \partial_\alpha A, \partial_\nu \partial_\lambda \partial_\beta B \} + O(\theta^5).$$

With this modification we first define the  $GL(2, \mathbb{C})$  gauge field  $\tilde{A}_\mu$ ,

$$\tilde{A} = dx^\mu \left( i(\tilde{\alpha}_\mu + \tilde{\beta}_\mu \gamma_5) + \frac{1}{4} \tilde{\omega}_{\mu ab} \gamma^{ab} \right),$$

satisfying the condition  $\tilde{A}_\mu^\dagger = -\tilde{A}_\mu$  and transforming under a gauge transformation according to

$$\tilde{A} \rightarrow \tilde{\Omega} * \tilde{A} * \tilde{\Omega}_*^{-1} + \tilde{\Omega} * d\tilde{\Omega}_*^{-1}$$

where  $\tilde{\Omega} = e^{\tilde{\lambda}}$  with

$$\tilde{\lambda} = i(\tilde{\alpha} + \tilde{\beta} \gamma_5) + \frac{1}{4} \tilde{\lambda}_{ab} \gamma^{ab}.$$

One can easily verify that these transformations close as both the commutators and anticommutators of  $\gamma_{ab}$  with  $\gamma_c$  and  $\gamma_c \gamma_5$  are proportional to  $\gamma_d$  and  $\gamma_d \gamma_5$ . The field strength is

$$\tilde{F} = \frac{1}{2} dx^\mu \wedge dx^\nu \tilde{F}_{\mu\nu},$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + \tilde{A}_\mu * \tilde{A}_\nu - \tilde{A}_\nu * \tilde{A}_\mu,$$

transforming according to

$$\tilde{F}_{\mu\nu} = \tilde{\Omega} * \tilde{F}_{\mu\nu} * \tilde{\Omega}_*^{-1}.$$

The field  $\tilde{L}$  is defined as before,

$$\tilde{L} = dx^\mu (\tilde{\alpha}_\mu^a + i\gamma_5 \tilde{f}_\mu^a) \gamma_a,$$



and transforms according to

$$\tilde{L} \rightarrow \tilde{\Omega} * \tilde{L} * \tilde{\Omega}_*^{-1}.$$

Unlike the commutative case, the field

$$\tilde{L}' = -C\tilde{L}'C^{-1}$$

transforms as

$$\tilde{L}' \rightarrow \tilde{\Omega}' * \tilde{L}' * \tilde{\Omega}'_*^{-1}$$

where  $\tilde{\Omega}' = e^{\tilde{\lambda}'}$  with

$$\tilde{\lambda}' = -i(\tilde{\alpha} + \tilde{\beta}\gamma_5) + \frac{1}{4}\tilde{\lambda}_{ab}\gamma^{ab}.$$

It is therefore not possible to construct a group invariant using both  $\tilde{L}$  and  $\tilde{L}'$  as for  $SL(2, \mathbb{C})$  where  $\lambda$  and  $\lambda'$  coincide. Therefore we are forced to use only the fields  $\tilde{L}$  and  $\tilde{A}$  to construct an action invariant under  $GL(2, \mathbb{C})$ . It can be easily seen from the analysis given in the last section that since the field  $\tilde{L}'$  cannot be used, the coupling of  $\tilde{\omega}_{\mu ab}$  to  $\tilde{e}_\mu^a$  and  $\tilde{f}_\mu^a$  ensures only the propagation of one combination of  $\tilde{e}_\mu^a$  and  $\tilde{f}_\mu^a$ . It is clear that the deformed four-dimensional gravitational action invariant under the noncommutative  $GL(2, \mathbb{C})$  gauge transformations can be written as

$$\begin{aligned} \tilde{I} = & \int_M d^4x \epsilon^{\mu\nu\kappa\lambda} \text{Tr}[(\alpha_1 + \beta_1\gamma_5)(\tilde{L} * \tilde{L} * \tilde{F})] \\ & + \int_M d^4x \epsilon^{\mu\nu\kappa\lambda} \text{Tr}[(\alpha_2 + \beta_2\gamma_5)(\tilde{L} * \tilde{L} * \tilde{L} * \tilde{L})]. \end{aligned}$$

To this it is possible but not necessary to add the torsion-free constraint

$$\tilde{T} = d\tilde{L} + \tilde{A} * \tilde{L} + \tilde{L} * \tilde{A} = 0$$

which can be decomposed in terms of components and then solved.

We first determine the infinitesimal gauge transformations of the gauge fields

$$\delta\tilde{A} = -d\tilde{\lambda} + \tilde{\lambda} * \tilde{A} - \tilde{A} * \tilde{\lambda},$$

where  $\tilde{\Omega} = e^{\tilde{\lambda}}$  and  $\tilde{\lambda} = i(\tilde{\alpha} + \gamma_5\tilde{\beta}) + (1/4)\tilde{\lambda}^{ab}\gamma_{ab}$ . In terms of components, this reads

$$\begin{aligned} \delta\tilde{\alpha}_\mu = & -\partial_\mu\tilde{\alpha} - \theta^{\kappa\lambda} \left( \partial_\kappa\tilde{\alpha}\partial_\lambda\tilde{\alpha}_\mu + \partial_\kappa\tilde{\beta}\partial_\lambda\tilde{b}_\mu + \frac{1}{8}\partial_\kappa\tilde{\lambda}^{ab}\partial_\lambda\tilde{\omega}_{\mu ab} \right) \\ & + O(\theta^3), \end{aligned}$$

$$\begin{aligned} \delta\tilde{b}_\mu = & -\partial_\mu\tilde{\beta} - \theta^{\kappa\lambda} \left( \partial_\kappa\tilde{\beta}\partial_\lambda\tilde{\alpha}_\mu + \partial_\kappa\tilde{\alpha}\partial_\lambda\tilde{b}_\mu \right. \\ & \left. - \frac{1}{16}\epsilon^{abcd}\partial_\kappa\tilde{\lambda}_{ab}\partial_\lambda\tilde{\omega}_{\mu cd} \right) + O(\theta^3), \end{aligned}$$

$$\begin{aligned} \delta\tilde{\omega}_{\mu ab} = & -(\partial_\mu\tilde{\lambda}_{ab} + \tilde{\omega}_{\mu ac}\tilde{\lambda}_{cb} - \tilde{\omega}_{\mu bc}\tilde{\lambda}_{ca}) \\ & - \theta^{\kappa\lambda} \left[ (\partial_\kappa\tilde{\alpha}\partial_\lambda\tilde{\omega}_{\mu ab} + \partial_\kappa\tilde{\lambda}_{ab}\partial_\lambda\tilde{\alpha}_\mu) \right. \\ & \left. + \frac{1}{2}\epsilon_{abcd}(\partial_\kappa\tilde{\beta}\partial_\lambda\tilde{\omega}_{\mu cd} + \partial_\kappa\tilde{\lambda}_{cd}\partial_\lambda\tilde{b}_\mu) \right] \\ & - \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_\alpha\partial_\gamma\tilde{\omega}_{\mu ac}\partial_\beta\partial_\delta\tilde{\lambda}_{cb} - \partial_\alpha\partial_\gamma\tilde{\omega}_{\mu bc}\partial_\beta\partial_\delta\tilde{\lambda}_{ca}) \\ & + O(\theta^3). \end{aligned}$$

Similarly the infinitesimal gauge transformation of the complex vierbein  $\tilde{L}$  is given by

$$\delta\tilde{L} = \tilde{\lambda} * \tilde{L} - \tilde{L} * \tilde{\lambda},$$

which in component form reads

$$\begin{aligned} \delta\tilde{e}_\mu^a = & \tilde{\lambda}^{ac}\tilde{e}_\mu^c - \theta^{\gamma\delta} \left( \partial_\gamma\tilde{\alpha}\partial_\delta\tilde{e}_\mu^a - \frac{1}{8}\epsilon^{abcd}\partial_\gamma\tilde{\lambda}_{bc}\partial_\delta\tilde{f}_\mu^d \right) \\ & - \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_\alpha\partial_\gamma\tilde{\lambda}^{ac}\partial_\beta\partial_\delta\tilde{e}_{\mu c}) + O(\theta^3), \\ \delta\tilde{f}_\mu^a = & \tilde{\lambda}^{ac}\tilde{f}_\mu^c - \theta^{\gamma\delta} \left( \partial_\gamma\alpha\partial_\delta\tilde{f}_\mu^a + \frac{1}{8}\epsilon^{abcd}\partial_\gamma\tilde{\lambda}_{bc}\partial_\delta\tilde{e}_\mu^d \right) \\ & - \frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_\alpha\partial_\gamma\tilde{\lambda}^{ac}\partial_\beta\partial_\delta\tilde{f}_{\mu c}) + O(\theta^3). \end{aligned}$$

The components of the torsion constraints are

$$\tilde{T}_{\mu\nu} = \tilde{T}_{\mu\nu}^a\gamma_a + \tilde{T}_{\mu\nu}^{a5}i\gamma_5\gamma_a = 0,$$

where

$$\begin{aligned} \tilde{T}_{\mu\nu}^a = & \left( \partial_\mu\tilde{e}_\nu^a + \frac{1}{2}\{\tilde{\omega}_{\mu ab}, \tilde{e}_\nu^b\}_* - \frac{i}{4}\epsilon^{abcd}[\tilde{\omega}_{\mu bc}, \tilde{f}_{\nu d}]_* \right. \\ & \left. + i[\tilde{\alpha}_\mu, \tilde{e}_\nu^a]_* - \{\tilde{b}_\mu, \tilde{f}_\nu^a\}_* - \mu \leftrightarrow \nu \right), \\ \tilde{T}_{\mu\nu}^a = & \left( \partial_\mu\tilde{f}_\nu^a + \frac{1}{2}\{\tilde{\omega}_{\mu ab}, \tilde{f}_\nu^b\}_* + \frac{i}{4}\epsilon^{abcd}[\tilde{\omega}_{\mu bc}, \tilde{e}_{\nu d}]_* \right. \\ & \left. + i[\tilde{\alpha}_\mu, \tilde{f}_\nu^a]_* + \{\tilde{b}_\mu, \tilde{e}_\nu^a\}_* - \mu \leftrightarrow \nu \right). \end{aligned}$$

These equations simplify when written in terms of the complex field

$$\tilde{\mathbb{E}}_\mu^a = \tilde{e}_\mu^a + i\tilde{f}_\mu^a$$

as they take the form

$$\begin{aligned} 0 = & \left( \partial_\mu\tilde{\mathbb{E}}_\nu^a + \frac{1}{2}\{\tilde{\omega}_{\mu ab}, \tilde{\mathbb{E}}_\nu^b\}_* - \frac{i}{4}\epsilon^{abcd}[\tilde{\omega}_{\mu bc}, \tilde{\mathbb{E}}_{\nu d}]_* \right. \\ & \left. + i[\tilde{\alpha}_\mu, \tilde{\mathbb{E}}_\nu^a]_* + i\{\tilde{b}_\mu, \tilde{\mathbb{E}}_\nu^a\}_* - \mu \leftrightarrow \nu \right) \end{aligned}$$

as well as the complex conjugate equation.

We now determine the deformed action to second order in  $\theta$ . The gauge field strength is given by

$$\tilde{F}_{\mu\nu} = i(\tilde{a}_{\mu\nu} + \gamma_5 \tilde{b}_{\mu\nu}) + \frac{1}{4} \tilde{R}_{\mu\nu ab} \gamma^{ab}$$

where

$$\begin{aligned} \tilde{a}_{\mu\nu} &= \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + i[\tilde{a}_\mu, \tilde{a}_\nu]_* + i[\tilde{b}_\mu, \tilde{b}_\nu]_* \\ &+ \frac{i}{8} [\tilde{\omega}_\mu^{ab}, \tilde{\omega}_{\nu ab}]_*, \end{aligned}$$

$$\begin{aligned} \tilde{b}_{\mu\nu} &= \partial_\mu \tilde{b}_\nu - \partial_\nu \tilde{b}_\mu + i[\tilde{a}_\mu, \tilde{b}_\nu]_* + i[\tilde{b}_\mu, \tilde{a}_\nu]_* \\ &- \frac{i}{8} \epsilon_{abcd} [\tilde{\omega}_\mu^{ab}, \tilde{\omega}_\nu^{cd}]_*, \end{aligned}$$

$$\begin{aligned} \tilde{R}_{\mu\nu ab} &= \partial_\mu \tilde{\omega}_{\nu ab} + i[\tilde{\omega}_{\mu ab}, \tilde{a}_\nu]_* + \frac{i}{2} \epsilon_{abcd} [\tilde{b}_\mu, \tilde{\omega}_\nu^{cd}]_* \\ &+ \frac{1}{2} \{ \tilde{\omega}_\mu^{ac}, \tilde{\omega}_\nu^b \}_* - \mu \leftrightarrow \nu. \end{aligned}$$

To determine the deformed action we first expand the combination

$$\tilde{L} * \tilde{L} = \frac{1}{2} dx^\mu \wedge dx^\nu [l_{\mu\nu ab} \gamma^{ab} + i(l_{\mu\nu}^{(1)} + \gamma_5 l_{\mu\nu}^{(5)})],$$

where

$$l_{\mu\nu}^{ab} = \{ \tilde{e}_\mu^a, \tilde{e}_\nu^b \}_* + \{ \tilde{f}_\mu^a, \tilde{f}_\nu^b \}_* - \frac{i}{2} \epsilon^{abcd} ([\tilde{e}_{\mu c}, \tilde{f}_{\nu d}]_* - \mu \leftrightarrow \nu),$$

$$l_{\mu\nu}^{(1)} = -i[\tilde{e}_\mu^a, \tilde{e}_{\nu a}]_* - i[\tilde{f}_\mu^a, \tilde{f}_{\nu a}]_*,$$

$$l_{\mu\nu}^{(5)} = -\{ \tilde{e}_\mu^a, \tilde{f}_{\nu a} \}_* + \{ \tilde{e}_\nu^a, \tilde{f}_{\mu a} \}_*.$$

The kinetic part of the action then takes the form

$$\begin{aligned} \tilde{I}_1 &= \int_M d^4x \epsilon^{\mu\nu\kappa\lambda} \left[ -\alpha_1 \left( l_{\mu\nu}^{(1)} * \tilde{a}_{\kappa\lambda} + l_{\mu\nu}^{(5)} * \tilde{b}_{\kappa\lambda} + \frac{1}{2} l_{\mu\nu}^{ab} * \tilde{R}_{\kappa\lambda ab} \right) \right. \\ &\left. - \beta_1 \left( l_{\mu\nu}^{(1)} * \tilde{b}_{\kappa\lambda} + l_{\mu\nu}^{(5)} * \tilde{a}_{\kappa\lambda} + \frac{1}{4} \epsilon_{abcd} l_{\mu\nu}^{ab} * \tilde{R}_{\kappa\lambda}^{cd} \right) \right], \end{aligned}$$

while the cosmological term gives

$$\begin{aligned} \tilde{I}_2 &= \int_M d^4x \epsilon^{\mu\nu\kappa\lambda} \left[ -\alpha_2 (l_{\mu\nu}^{(1)} * l_{\kappa\lambda}^{(1)} + l_{\mu\nu}^{(5)} * l_{\kappa\lambda}^{(5)} + 2l_{\mu\nu}^{ab} * l_{\kappa\lambda ab}) \right. \\ &\left. - \beta_2 (2l_{\mu\nu}^{(1)} * l_{\kappa\lambda}^{(5)} - \epsilon_{abcd} l_{\mu\nu}^{ab} * l_{\kappa\lambda}^{cd}) \right]. \end{aligned}$$

The Seiberg-Witten map [12,13] determining the deformed gauge field in terms of the undeformed one is defined by

$$\tilde{A}(gAg^{-1} + g dg^{-1}) = \tilde{g} * \tilde{A}(A) * \tilde{g}_*^{-1} + \tilde{g} * d\tilde{g}_*^{-1}.$$

Its solution is given by

$$\tilde{A}_\mu = A_\mu - \frac{i}{4} \theta^{\kappa\lambda} \{ A_\kappa, \partial_\lambda A_\mu + F_{\lambda\mu} \} + O(\theta^2),$$

$$\begin{aligned} \tilde{F}_{\mu\nu} &= F_{\mu\nu} + \frac{i}{4} \theta^{\kappa\lambda} (2\{F_{\mu\kappa}, F_{\nu\lambda}\} \\ &- \{A_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu}\}) + O(\theta^2), \end{aligned}$$

$$\tilde{\lambda} = \lambda + \frac{i}{4} \theta^{\alpha\beta} \{ \partial_\alpha \lambda, A_\beta \} + O(\theta^2).$$

The deformed complex vierbein  $\tilde{L}$  is defined by the relation

$$\tilde{L}(gLg^{-1}, gAg^{-1}) + g dg^{-1} = \tilde{g} * \tilde{L}(L, A) * \tilde{g}_*^{-1}.$$

Its solution is given by

$$\tilde{L}_\mu = L_\mu + \frac{i}{2} \theta^{\kappa\lambda} \left\{ \partial_\kappa L_\mu + \frac{1}{2} [A_\kappa, L_\mu], A_\lambda \right\} + O(\theta^2).$$

The component forms of these relations read

$$\begin{aligned} \tilde{a}_\mu &= a_\mu + \frac{1}{2} \theta^{\kappa\lambda} \left[ a_\kappa (2\partial_\lambda a_\mu - \partial_\mu a_\lambda) + b_\kappa (2\partial_\lambda b_\mu - \partial_\mu b_\lambda) \right. \\ &\left. + \frac{1}{8} \omega_\kappa^{ab} (\partial_\lambda \omega_\mu^{ab} + R_{\lambda\mu}{}^{ab}) \right] + O(\theta^2), \end{aligned}$$

$$\begin{aligned} \tilde{b}_\mu &= b_\mu + \frac{1}{2} \theta^{\kappa\lambda} \left[ a_\kappa (2\partial_\lambda b_\mu - \partial_\mu b_\lambda) + b_\kappa (2\partial_\lambda a_\mu - \partial_\mu a_\lambda) \right. \\ &\left. - \frac{1}{16} \epsilon_{abcd} \omega_\kappa^{ab} (\partial_\lambda \omega_\mu^{cd} + R_{\lambda\mu}{}^{cd}) \right] + O(\theta^2), \end{aligned}$$

$$\begin{aligned} \tilde{\omega}_\mu^{ab} &= \omega_\mu^{ab} + \frac{1}{2} \theta^{\kappa\lambda} \left\{ a_\kappa (\partial_\lambda \omega_\mu^{ab} + R_{\lambda\mu}{}^{ab}) \right. \\ &+ \omega_\kappa^{ab} (2\partial_\lambda a_\mu - \partial_\mu a_\lambda) + \frac{1}{2} \epsilon_{abcd} [b_\kappa (\partial_\lambda \omega_\mu^{cd} + R_{\lambda\mu}{}^{cd}) \\ &\left. + \omega_\kappa^{cd} (2\partial_\lambda b_\mu - \partial_\mu b_\lambda)] \right\} + O(\theta^2), \end{aligned}$$

$$\begin{aligned} \tilde{e}_\mu^a &= e_\mu^a - \theta^{\kappa\lambda} \left[ a_\lambda \left( \partial_\kappa e_\mu^a + \frac{1}{2} \omega_\kappa{}^{ae} e_\mu^e \right) \right. \\ &\left. - \frac{1}{4} \epsilon_{abcd} \omega_\lambda{}^{cd} \left( \partial_\kappa f_\mu^b + \frac{1}{2} \omega_\kappa{}^{be} f_\mu^e \right) \right] + O(\theta^2), \end{aligned}$$

$$\begin{aligned} \tilde{f}_\mu^a &= f_\mu^a - \theta^{\kappa\lambda} \left[ a_\lambda \left( \partial_\kappa f_\mu^a + \frac{1}{2} \omega_\kappa{}^{ae} f_\mu^e \right) \right. \\ &\left. + \frac{1}{4} \epsilon_{abcd} \omega_\lambda{}^{cd} \left( \partial_\kappa e_\mu^b + \frac{1}{2} \omega_\kappa{}^{be} e_\mu^e \right) \right] + O(\theta^2). \end{aligned}$$

As an alternative to the deformed action obtained in this section, one can use the Seiberg-Witten (SW) map for the

fields  $\tilde{L}_\mu$  and  $\tilde{A}_\mu$  and then substitute the undeformed solution for  $\omega_{\mu ab}$  in terms of  $e_\mu^a$  and  $f_\mu^a$ . The resulting expressions would be very complicated, which shows that the use of the SW map in obtaining the deformed action is not practical for the gravitational system. These expressions might simplify for specific solutions where  $\omega_{\mu ab}$ ,  $e_\mu^a$ , and  $f_\mu^a$  are given.

#### IV. CONCLUSIONS

The idea that the gravitational field could be complex is not new and was first considered by Einstein and Stauss [6], motivated by the unification of electromagnetism with gravity. The work of Weyl [3] and Cartan [4] on spinors in general relativity and of Utiyama [1] and Kibble [2] relating gravity to a gauge theory of the Lorentz group showed how general relativity could be formulated based on the  $SL(2, \mathbb{C})$  gauge invariance [14]. This symmetry also played a crucial part in determining Ashtekar variables [21,22]. The  $SL(2, \mathbb{C})$  symmetry acts as a gauge symmetry of the spin connection, and in a first order formalism gives the correct kinetic terms for the vierbein. It is also possible to include torsion in the spin connection to accommodate the antisymmetric  $B$  field appearing in string theory and give it a kinetic term. In this paper we have shown that it is possible to go further and complexity the vierbein, keeping the  $SL(2, \mathbb{C})$  symmetry. We have proposed an action with the exceptional property that when the spin connection, which appears quadratically, is eliminated by its equation of motion, then both the real and imaginary parts of the metric propagate. One combination protected by diffeomorphism invariance will produce the massless graviton, while the other will produce a massive graviton coupled to a scalar field. This is identical to the spectrum of bigravity, but has the advantage of using a mini-

mal number of fields. We have worked out only the linearized approximation of the theory and shown that all fields acquire the correct kinetic terms. The computation is not simple, but it is very important to go one step further and determine the higher order interactions. This calculation can only be performed perturbatively because the massless and massive gravitons are linear combinations of the real and imaginary parts of the complex vierbein  $L$ , and these tensor combinations should be inverted. It would be very enlightening to find some special solutions for this theory which are generalizations of the Schwarzschild and de Sitter solutions.

When coordinates do not commute and fields are defined on such noncommutative space, ordinary products must be replaced with star products. Commutators of Lie algebra valued fields using star products, would result in both commutators and anticommutators in terms of the undeformed fields. This makes it necessary to extend the gauge group form  $SL(2, \mathbb{C})$  to  $GL(2, \mathbb{C})$ . Having the proposed action for complex gravity based on the requirement that all terms must be four-forms, the extension carries through without any complications by replacing ordinary products with star products. It is then a straightforward matter to determine the deformed action to second order in the deformation parameter  $\theta^{\mu\nu}$ . We have only touched the surface in this direction, and many questions remain to be addressed, such as the effect of the deformed parameters on quantization of the theory, finding the SW map of some specific solutions, and generalization to nonconstant parameters  $\theta^{\mu\nu}$ . These questions and others will hopefully be addressed in future investigations.

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