Can the "brick wall" model present the same results in different coordinate representations?

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By using the 't Hooft's "brick wall" model and the Pauli-Villars regularization scheme we calculate the statistical-mechanical entropies arising from the quantum scalar field in different coordinate settings, such as the Painlevé and Lemaitre coordinates. At first glance, it seems that the entropies would be different from that in the standard Schwarzschild coordinate since the metrics in both the Painlevé and Lemaitre coordinates do not possess the singularity at the event horizon as that in the Schwarzschild-like coordinate. However, after an exact calculation we find that, up to the subleading correction, the statistical-mechanical entropies in these coordinates are equivalent to that in the Schwarzschild-like coordinate. The result is not only valid for black holes and de Sitter spaces, but also for the case that the quantum field exerts back reaction on the gravitational field provided that the back reaction does not alter the symmetry of the spacetime.

I. INTRODUCTION

In quantum field theory, we can use a timelike Killing vector to define particle states. Therefore, in static spacetimes we know that it is possible to define positive frequency modes by using the timelike Killing vector. However, in these spacetimes there could arise more than one timelike Killing vector which make the vacuum states inequivalent. This means that the concept of particles is not generally covariant in curve spacetime.

Bekenstein and Hawking [1,2] found that, by comparing black hole physics with thermodynamics and from the discovery of black hole evaporation, black hole entropy is proportional to the area of the event horizon. The discovery is one of the most profound in modern physics. However, the issue of the exact statistical origin of the black hole entropy has remained a challenging one. Recently, much effort has been concentrated on the problem [3–30]. The "brick wall" model (BWM) proposed by 't Hooft [11] is an extensively used way to calculate the entropy in a variety of black holes, black branes, de Sitter spaces, and anti–de Sitter spaces [11–30]. In this model the Bekenstein-Hawking entropy of the black hole is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the event horizon.

The concept of particles in quantum field theory is not generally covariant and depends on the coordinate representations. This leads to an interest question: can we get the same results for statistical-mechanical entropy of black holes in different coordinate representations, such as the Painlevé and Lemaitre coordinates, by employing the BWM by making use of the wave modes in this model? At first sight, we might anticipate that the results are different since the wave modes obtained by using semiclassical techniques are the exact modes of the quantum system in the asymptotic region. Thus, if the asymptotic structures of the spacetime are different for any two coordinates, then the semiclassical wave

modes associated with different coordinates will be different. The aim of this paper is to study this question carefully by applying the BWM to two different coordinate representations of the general standard static black hole and studying the statistical-mechanical entropy. The two coordinate representations which we use are the stationary Painlevé coordinate and the time dependent Lemaitre coordinate. In both Painlevé and Lemaitre coordinates, the metrics have no coordinate singularity which are different from the standard Schwarzschild-like coordinate. However, they both acquire singularity at the event horizon in the action function. Therefore, there could be particle production in these coordinates and hence we can use the knowledge of the wave modes of the quantum field in these coordinate settings to calculate the statistical-mechanical entropies.

In order to compare the statistical-mechanical entropies obtained in this paper with the result for the standard Schwarzschild-like coordinate, we first introduce the expression of the entropy for the Schwarzschild-like coordinate in the following. In the BWM, in order to eliminate divergence which appears due to the infinite growth of the density of states close to the horizon, 't Hooft introduces a brick wall cutoff: a fixed boundary Σ_h near the event horizon within the quantum field does not propagate and the Dirichlet boundary condition was imposed on the boundary, i.e., wave function $\phi = 0$ for $r = r(\Sigma_h)$. However, Demers, Lafrance, and Myers [31] found, in the Pauli-Villars regulated theory, that 't Hooft's brick wall can be removed by introducing five regulator fields: ϕ_1 and ϕ_2, which are two anticommuting fields with mass $m_1 = m_2 = \sqrt{\mu^2 + m^2}$ (where μ represents the UV cutoff); ϕ_3 and ϕ_4, which are two commuting fields with mass $m_3 = m_4 = \sqrt{3\mu^2 + m^2}$; and ϕ_5, which is an anti commuting field with mass $m_5 = \sqrt{4\mu^2 + m^2}$. Together with the original scalar field $\phi = \phi_0$ with mass $m = m_0$ these fields satisfy the two constraints $\Sigma_{i=0}^5 \Delta_i = 0$ and $\Sigma_{i=0}^5 \Delta_i m_i^2 = 0$, where $\Delta_i = +1$ for the commuting fields, and $\Delta_i = -1$ for the anticommuting fields. By using the BWM and Pauli-Villars regulators, Demers, Lafrance and Myers [31], and Solodukhin [32] found that the statistical-mechanical entropy

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arising from the minimally coupled quantum scalar field in a general nonextreme static black hole

$$ds^2 = -g(r)dt_s^2 + \frac{1}{g(r)}dr^2 + R^2(r)(d\theta^2 + \sin^2\theta d\varphi^2)$$

(1.1)

(where $g(r)$ is an arbitrary function of r. The event horizon is determined by $g(r)=0$. And $[dg(r)/dr]|_{r_+} \neq 0$ for the nonextreme black holes) can be expressed as

$$S = \frac{A_\Sigma}{48\pi} \sum_{i=0}^{5} \Delta_i m_i^2 \ln m_i^2$$
$$- \frac{A_\Sigma}{288\pi}\left[\mathcal{R} - \frac{1}{5}\left(\frac{\partial^2 g(r)}{\partial^2 r} - \frac{1}{R^2(r)}\frac{\partial g(r)}{\partial r}\frac{\partial R^2(r)}{\partial r}\right)\right]_{r_+}$$
$$\times \sum_{i=0}^{5} \Delta_i \ln m_i^2,$$

(1.2)

where $A_\Sigma = \int d\theta d\varphi [\sqrt{g_{\theta\theta}g_{\varphi\varphi}}]_{r_+}$ is the area of the event horizon, $\mathcal{R}$ is a scalar curvature of the spacetime. The statistical-mechanical entropy (1.2) obtained by this approach consists of two parts: the first part, after taking renormalization of the gravitational constant as $1/G_R = (1/G_B) + (1/12\pi)\Sigma_{i=0}^{5}\Delta_i m_i^2 \ln m_i^2$, gives Bekenstein-Hawking entropy, and the second part can be considered as a quantum correction to the entropy of the black hole due to the quantum scalar field.

The paper is organized as follows. In Sec. II the Painlevé spacetime is introduced and the statistical-mechanical entropy arises from the quantum scalar field in the Painlevé coordinate that is studied. In Sec. III the statistical-mechanical entropy due to the quantum scalar field in the Lemaitre coordinate is investigated. The summary and discussions are presented in Sec. IV.

II. STATISTICAL-MECHANICAL ENTROPY IN THE PAINLEVÉ COORDINATE

We now investigate statistical-mechanical entropy that arises from the quantum scalar field in the Painlevé coordinate system. The time coordinate transformation from the standard Schwarzschild-like coordinate (1.1) to the Painlevé coordinate is

$$t = t_s + \int \frac{\sqrt{1-g(r)}}{g(r)}dr.$$

(2.1)

The radial and angular coordinates remain unchanged. With this choice, the line element (1.1) becomes

$$ds^2 = -g(r)dt^2 + 2\sqrt{1-g(r)}dtdr + dr^2 + R^2(r)(d\theta^2 + \sin^2\theta d\varphi^2),$$

(2.2)

which is the Painlevé coordinate representation. The coordinate has distinguishing features: (a) The spacetime is stationary but not static; (b) the constant-time surfaces is flat if

$R^2(r) = r^2$; and (c) there is now no singularity at $g(r)=0$. That is to say, the coordinate complies with perspective of a free-falling observer, who is expected to experience nothing out of the ordinary upon passing through the event horizon. However, the event horizon manifests itself as a singularity in the expression for the semiclassical action. It is easily to prove that the inverse Hawking temperature

$$\beta_H = 2\pi \frac{1+\sqrt{1-g(r)}}{\frac{dg(r)}{dr}}\Bigg|_{r_+} = 4\pi \Bigg/ \frac{dg(r)}{dr}\Bigg|_{r_+},$$

(2.3)

is recovered in the Painlevé coordinate by using the complex path technique [34,35].

We now try to find an expression of the statistical-mechanical entropy due to the quantum scalar field in thermal equilibrium at temperature $1/\beta$ in the Painlevé coordinate by suing the BWM. Using the WKB approximation with

$$\phi = \exp[-iEt + iW(r,\theta,\varphi)],$$

(2.4)

and substituting the metric (2.2) into the Klein-Gordon equation of the scalar field ϕ with mass m and nonminimal $\xi\mathcal{R}\phi$ ($\mathcal{R}$ is the scalar curvature of the spacetime) coupling

$$\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}g^{\mu\nu}\partial_\nu\phi) - (m^2 + \xi\mathcal{R}) = 0,$$

(2.5)

we find

$$p_r^\pm = \frac{1}{g(r)}\left[\sqrt{1-g(r)}E \pm \sqrt{g(r)}\sqrt{\frac{E^2}{g(r)} - \left(\frac{p_\theta^2}{R^2(r)} + \frac{p_\varphi^2}{R^2(r)\sin^2\theta} + M^2(r)\right)}\right],$$

(2.6)

where $p_r \equiv \partial_r W(r,\theta,\varphi)$, $p_\theta \equiv \partial_\theta W(r,\theta,\varphi)$, and $p_\varphi \equiv \partial_\varphi W(r,\theta,\varphi)$ are the momentum of the particles moving in r, θ, and φ, respectively. The sign ambiguity of the square root is related to the "out-going" (p_r^+) or "in-going" (p_r^-) particle, respectively. If the scalar curvature $\mathcal{R}$ takes a non-zero value at the horizon then this region can be approximated by some sort of constant curvature space. However, the exact result for such a black hole showed that the mass parameter in the solution enters only in the combination $(m^2 - \mathcal{R}/6)$ [32,33], and then $M^2(r) = m^2 - (\frac{1}{6} - \xi)\mathcal{R}$ in the Eq. (2.6). In this paper our discussion is restricted to study minimally coupled ($\xi=0$) scalar fields since the main aim of this paper is to see whether the brick wall model can present the same result in different coordinates.

The partition function is given by

$$z = \sum_{n_q} \exp[-\beta(E_q)n_q],$$

(2.7)

where q denotes a quantum state of the field with energy E_q. The free energy is

$$F = \frac{1}{\beta} \int dp_\theta \int dp_\varphi \int dn(E, p_\theta, p_\varphi) \ln\{1 - \exp[-\beta E]\}$$

$$= -\int dp_\theta \int dp_\varphi \int \frac{n(E, p_\theta, p_\varphi)}{e^{\beta E} - 1} dE$$

$$= -\int \frac{n(E)}{e^{\beta E} - 1} dE, \qquad (2.8)$$

where $n(E) \equiv \int dp_\theta \int dp_\varphi n(E, p_\theta, p_\varphi)$ presents the total number of the modes with energy less than E. In phase space the total number of modes with E is given by

$$n(E) = \frac{1}{\pi} \int d\theta \int_{r_+ + h}^{L} dr \int dp_\theta dp_\varphi \frac{p_r^+ - p_r^-}{2}$$

$$= \frac{1}{\pi} \int d\theta \int_{r_+ + h}^{L} dr \int dp_\theta dp_\varphi \frac{1}{\sqrt{g(r)}}$$

$$\times \sqrt{\frac{E^2}{g(r)} - \left(\frac{p_\theta^2}{R^2(r)} + \frac{p_\varphi^2}{R^2(r)\sin^2\theta} + M^2(r)\right)}. \qquad (2.9)$$

The integral is taken only over those values for which the square root exists. In Eq. (2.9) we utilize the average of the radial momentum (the minus before the p_r^- is caused by a different direction). In this way, the total number of modes is related to all kinds of particles. We checked that this definition can also be used for all previous corresponding works. Carrying out the integrations of the p_θ, p_φ, and r, we get

$$n(E) = -\frac{1}{2\pi} \int d\theta \left\{ \sqrt{g_{\theta\theta}g_{\varphi\varphi}} \left[\frac{2}{3} \left(\frac{\beta_H E}{4\pi}\right)^3 C(r) \right.\right.$$

$$\left.\left. + M^2(r) \left(\frac{\beta_H E}{4\pi}\right) \right] \ln \frac{E^2}{E_{min}^2} \right\}_{r_+}$$

$$- \frac{1}{3\pi} \frac{\beta_H}{4\pi} \int d\theta \left[\sqrt{g_{\theta\theta}g_{\varphi\varphi}} M^2(r) \left(E - \frac{E^3}{E_{min}^2} \right) \right]_{r_+}, \qquad (2.10)$$

where

$$C(r) = \frac{\partial^2 g(r)}{\partial^2 r} - \frac{1}{R^2(r)} \frac{\partial g(r)}{\partial r} \frac{\partial R^2(r)}{\partial r},$$

$$E_{min}^2 = [M^2(r)g(r)]_{\Sigma_h}. \qquad (2.11)$$

We now use the Pauli-Villars regularization scheme introduced in the preceding section. Since each of the scalar fields makes a contribution to the free energy, the total free energy can be expressed as

$$\beta\bar{F} = \sum_{i=0}^{5} \beta\Delta_i F_i. \qquad (2.12)$$

Substituting Eqs. (2.8) and (2.10) into Eq. (2.12) and then taking the integration over E, we have

$$\bar{F} = -\frac{1}{48\pi} \frac{\beta_H}{\beta^2} \int d\theta d\varphi \{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\}_{r_+}$$

$$\times \sum_{i=0}^{5} \Delta_i M_i^2(r_H) \ln M_i^2(r_H)$$

$$- \frac{1}{2880\pi} \frac{\beta_H^3}{\beta^4} \int d\theta d\varphi \left\{ \sqrt{g_{\theta\theta}g_{\varphi\varphi}} \left[\frac{\partial^2 g(r)}{\partial r^2} \right.\right.$$

$$\left.\left. - \frac{1}{R^2(r)} \frac{\partial g(r)}{\partial r} \frac{\partial R^2(r)}{\partial r} \right] \right\}_{r_+} \sum_{i=0}^{5} \Delta_i \ln M_i^2(r_H). \qquad (2.13)$$

Using the assumption that the scalar curvature $\mathcal{R}$ at the horizon is much smaller than each m_i and inserting free energy into the relation

$$S = \beta^2 \frac{\partial F}{\partial \beta}, \qquad (2.14)$$

we obtain the expression of the statistical-mechanical entropy due to a minimally coupled scalar field in the Painlevé coordinates

$$S = \frac{A_\Sigma}{48\pi} \sum_{i=0}^{5} \Delta_i m_i^2 \ln m_i^2$$

$$- \frac{A_\Sigma}{288\pi} \left[\mathcal{R} - \frac{1}{5} \left(\frac{\partial^2 g(r)}{\partial^2 r} - \frac{1}{R^2(r)} \frac{\partial g(r)}{\partial r} \frac{\partial R^2(r)}{\partial r} \right) \right]_{r_+}$$

$$\times \sum_{i=0}^{5} \Delta_i \ln m_i^2, \qquad (2.15)$$

where $A_\Sigma = \int d\varphi d\theta \{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\}_{r_+} = 4\pi R^2(r_+)$ is the area of the event horizon.

By the equivalence principle and the standard quantum field theory in flat space, to construct a vacuum state for the massless scalar field in the Painlevé spacetime we should leave all the positive frequency modes empty. Kraus [36] pointed out that for the metric (2.2) it is convenient to work along a curve

$$dr + \sqrt{1 - g(r)}dt = 0, \qquad (2.16)$$

then the condition is simply a positive frequency with respect to t near this curve. It is easy to prove that the modes used to calculate the entropy are essentially the same as that in the Schwarzschild-like coordinates. Therefore, it is reasonable that the result (2.15) is exactly equal to entropy (1.2).

III. STATISTICAL-MECHANICAL ENTROPY IN THE LEMAITRE COORDINATE

In this section we study statistical-mechanical entropy due to the quantum scalar field in the Lemaitre coordinates. The coordinates that transform from the Painlevé coordinates (2.2) to the Lemaitre coordinates are given by

$$\tilde{r} = t + \int \frac{dr}{\sqrt{1 - g(r)}},$$

$$U = \tilde{r} - t,$$

$$V = \tilde{r} + t, \qquad (3.1)$$

where t is the Painlevé time. The angular coordinates θ and φ remain the same. The line element (2.2), in the new coordinates, is described by

$$ds^2 = \frac{(f(U) - 1)}{4}(dV^2 + dU^2) + \frac{f(U) + 1}{2}dVdU$$

$$+ y(U)(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (3.2)$$

where

$$f(U) \equiv 1 - g(r),$$

$$y(U) \equiv R^2(r). \qquad (3.3)$$

The line element (3.2) is the Lemaitre coordinate representation of the spacetime (1.1). The metric in the Lemaitre coordinate is no singularity at $g(r) = 0$ just as in the Painlevé coordinates. However, the horizon also manifests itself as a singularity in the expression for the semiclassical action. We can also show that the inverse Hawking temperature,

$$\beta_H = -\pi \frac{(1 + \sqrt{f})^2}{\frac{\partial f}{\partial U}}\bigg|_{U_0} = 4\pi \bigg/ \frac{dg(r)}{dr}_{r_+}, \qquad (3.4)$$

is recovered in the Lemaitre coordinate by employing the complex path technique [34,35]. In Eq. (3.4) U_0 represents the root of the equation $(1 - f) = g = 0$.

We can use the WKB approximation with

$$\phi = \exp[-iEV/2 + iW(U, \theta, \varphi)]. \qquad (3.5)$$

The reason for using the modes with positive frequency with respect to the coordinate V is that another coordinate $U = \tilde{r} - t = \int dr/\sqrt{1 - g(r)}$ is related to the space coordinate r of the original coordinates only.

Substituting Eq. (3.5) and metric (3.2) into the Klein-Gordon equation of the scalar field with mass m, Eq. (2.5), we have

$$p_U^{\pm} = \frac{f}{1 - f}\left[\frac{1 + f}{f}E\right.$$

$$\left. \pm \sqrt{\frac{1 - f}{f}}\sqrt{\frac{E^2}{1 - f} - \left(\frac{p_\theta^2}{y} + \frac{p_\varphi^2}{y\sin^2\theta} + M^2(U)\right)}\right], \qquad (3.6)$$

where $p_U \equiv \partial_U W(U, \theta, \varphi)$, $p_\theta \equiv \partial_\theta W(U, \theta, \varphi)$, and $p_\varphi \equiv \partial_\varphi W(U, \theta, \varphi)$ are the momentum of the particle moving in U, θ, and φ, respectively, and $M^2(U) = m^2 - \frac{1}{6}\mathcal{R}$. Therefore, in phase space we obtain the number of modes

$$n(E) = \frac{1}{\pi}\int d\theta d\varphi \int_{U_0 + \tilde{h}}^{\tilde{L}} dU \int dp_\theta dp_\varphi \frac{p_U^+ - p_U^-}{2}$$

$$= \frac{2}{\pi}\int d\theta d\varphi \int_{U_0 + \tilde{h}}^{\tilde{L}} dU \int dp_\theta dp_\varphi \sqrt{\frac{f}{1 - f}}$$

$$\times \sqrt{\frac{E^2}{1 - f} - \left(\frac{p_\theta^2}{4y} + \frac{p_\varphi^2}{4y\sin^2\theta} + \frac{M^2(U)}{4}\right)}, \qquad (3.7)$$

where we make use of the average of the U-direction momentum (the minus before the p_U^- is caused by a different direction). The integral in the second line is taken only over those values for which the square root exists. Carrying out the integrations of the p_θ, p_φ, and U, we get

$$n(E) = -\frac{1}{2\pi}\int d\theta\left\{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left[\frac{2}{3}\left(\frac{\beta_H E}{4\pi}\right)^3 \tilde{C}(U)\right.\right.$$

$$\left.\left. + M^2(U)\left(\frac{\beta_H E}{4\pi}\right)\right]\ln\frac{E^2}{E_{min}^2}\right\}_{U_0}$$

$$- \frac{1}{3\pi}\frac{\beta_H}{4\pi}\int d\theta\left[\sqrt{g_{\theta\theta}g_{\varphi\varphi}}M^2(U)\left(E - \frac{E^3}{E_{min}^2}\right)\right]_{U_0}, \qquad (3.8)$$

where

$$\tilde{C}(U) = \frac{1}{f}\frac{\partial^2 f}{\partial^2 U} - \frac{1}{2f^2}\left(\frac{\partial f}{\partial U}\right)^2 - \frac{1}{fy}\frac{\partial f}{\partial U}\frac{\partial y}{\partial U},$$

$$E_{min}^2 = [M^2(U_0)(1 - f)]_{\Sigma_h}. \qquad (3.9)$$

We now introduce the Pauli-Villars regularization scheme as before. Substituting Eqs. (2.8) and (3.8) into Eq. (2.12) and then taking the integration over E we have

$$\bar{F} = -\frac{1}{48\pi}\frac{\beta_H}{\beta^2}\int d\theta d\varphi \{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\}_{U_0}$$

$$\times\sum_{i=0}^{5}\triangle_i M_i^2(U_0)\ln M_i^2(U_0)$$

$$-\frac{1}{2880\pi}\frac{\beta_H^3}{\beta^4}\int d\theta d\varphi\left\{\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\left[\frac{1}{f}\frac{\partial^2 f}{\partial U^2}-\frac{1}{2f^2}\left(\frac{\partial f}{\partial U}\right)^2\right.\right.$$

$$\left.\left.-\frac{1}{fy}\frac{\partial f}{\partial U}\frac{\partial y}{\partial U}\right]\right\}_{U_0}\sum_{i=0}^{5}\triangle_i\ln M_i^2(U_0). \tag{3.10}$$

Using the assumption that the scalar curvature $\mathcal{R}$ at the horizon is much smaller than each m_i and inserting free energy into the relation $S=\beta^2(\partial F/\partial\beta)$, we obtain the expression of the statistical-mechanical entropy in the Lemaitre coordinate

$$S=\frac{A_\Sigma}{48\pi}\sum_{i=0}^{5}\Delta_i m_i^2\ln m_i^2$$

$$-\frac{A_\Sigma}{288\pi}\left\{\mathcal{R}-\frac{1}{5}\left[\frac{1}{f}\frac{\partial^2 f}{\partial U^2}-\frac{1}{2f^2}\left(\frac{\partial f}{\partial U}\right)^2-\frac{1}{fy}\frac{\partial f}{\partial U}\frac{\partial y}{\partial U}\right]\right\}_{r_+}$$

$$\times\sum_{i=0}^{5}\Delta_i\ln m_i^2, \tag{3.11}$$

where $A_\Sigma = 4\pi y|_{U_0} = 4\pi R^2(r_+)$ is the horizon area.

By using Eq. (3.1), it is easy to prove

$$\left[\frac{1}{f}\frac{\partial^2 f}{\partial U^2}-\frac{1}{2f^2}\left(\frac{\partial f}{\partial U}\right)^2-\frac{1}{fy}\frac{\partial f}{\partial U}\frac{\partial y}{\partial U}\right]_{U_0}$$

$$=\left[\frac{\partial^2 g(r)}{\partial r^2}-\frac{1}{R^2(r)}\frac{\partial g(r)}{\partial r}\frac{\partial R^2(r)}{\partial r}\right]_{r_+}. \tag{3.12}$$

This shows that the result (3.11) for the Lemaitre coordinate is equal to entropy (2.15) for the Painlevé coordinate, and the entropy (1.2) for the standard Schwarzschild coordinate. It is well known that the wave modes obtained by using semiclassical techniques, in general, are the exact modes of the quantum system in the asymptotic regions. Thus, if the asymptotic structure of the spacetime is the same for the two coordinates, then the semiclassical wave modes associated with these two coordinate systems will be the same. From Eq. (3.1) we know that the differential relationship between the Lemaitre time V and the Painlevé time t can be expressed as

$$dV=dt+d\tilde{r}=2dt+\frac{dr}{\sqrt{1-g(r)}}. \tag{3.13}$$

Now let us also work along the curve $dr+\sqrt{1-g(r)}dt=0$, Eq. (3.13) then becomes

$$dV=dt. \tag{3.14}$$

It is shown that the two definitions of positive frequency—with respect to V in the Lemaitre spacetime and with respect to t in the Painlevé spacetime—do coincide. Therefore, it should not be surprised at the entropies driven from the modes in the Lemaitre and Painlevé coordinates are the same.

IV. SUMMARY AND DISCUSSIONS

We have investigated the statistical-mechanical entropies arising from the quantum scalar field in the Painlevé and Lemaitre coordinates by using the 't Hooft brick wall model and the Pauli-Villars regularization scheme. At first glance, we might have anticipated that the results are different from that of the standard Schwarzschild coordinate due to two reasons: (a) both the Painlevé and Lemaitre spacetimes possess a distinguishing property: the metrics do not possess singularity at event horizon; (b) it is not obvious that the time V in the Lemaitre spacetime tends to the time t in the Painlevé spacetime. Nevertheless, for either the Painlevé or Lemaitre coordinate, the event horizon manifests itself as a singularity in the action function and then there could be particles production. Hence we can use the knowledge of the wave modes of the quantum field to calculate the statistical-mechanical entropies. By comparing our results (2.15) and (3.11), which are worked out exactly, with the well-known result (1.2) we find that, up to a subleading correction, the statistical-mechanical entropies arising from the quantum scalar field in both the Painlevé and Lemaitre coordinates are equivalent to that in the standard Schwarzschild-like coordinate. When we construct a vacuum state for the massless scalar field in the Painlevé spacetime we take the condition $dr+\sqrt{1-g(r)}dt=0$, and then we find that the modes used to calculate the entropies in both the Painlevé and Lemaitre coordinates are essentially the same as that in the Schwarzschild-like coordinates since both V and t tend to the Schwarzschild time t_s as r goes to infinity under this condition. Therefore, it should not be a surprise that the entropies driven from the modes in the Lemaitre, Painlevé, and Schwarzschild coordinates are the same.

We should note that all the results are obtained based alone on the most general metric (1.1) and the conditions $g(r)|_{r_+}=0$ and $dg(r)/dr|_{r_+}\neq 0$ (nonextreme black hole). Therefore, the results are valid not only for the spacetimes that we have known, such as the Schwarzschild, the Reissner-Nordström, the Garfinkle-Horowitz-Strominger dilaton [37], the Gibbons-Maeda dilaton [38], the Garfinkle-Horne dilaton [39] black holes, and the Schwarzschild–de Sitter and the Reissner–Nordström–de Sitter spaces, etc., but also for the case that the quantum field exerts back reaction to the gravitational field provided that the back reaction does not alter the symmetry of the spacetime.

[1] J.D. Bekenstein, Lett. Nuovo Cimento Soc. Ital. Fis. **4**, 737 (1972); Phys. Rev. D **7**, 2333 (1973).

[2] S.W. Hawking, Nature (London) **248**, 30 (1974); Commun. Math. Phys. **43**, 199 (1975).

[3] S.N. Solodukhin, Phys. Rev. D **52**, 7046 (1995); **54**, 3900 (1996).

[4] V.P. Frolov and D.V. Fursaev, Class. Quantum Grav. **15**, 2041 (1998).

[5] Jiliang Jing and Mu-Lin Yan, Phys. Rev. D **62**, 104013 (2000); **63**, 024003 (2001).

[6] A.O. Barvinsky, V.P. Frolov, and A.I. Zelnikov, Phys. Rev. D **51**, 1741 (1995).

[7] F. Belgiorno and S. Liberati, Phys. Rev. D **53**, 3172 (1996).

[8] G. Cognola and P. Lecca, Phys. Rev. D **57**, 1108 (1998).

[9] V.P. Frolov, D.V. Fursaev, and A.I. Zelnikov, Phys. Lett. B **382**, 220 (1996).

[10] G. Cognola, Phys. Rev. D **57**, 6292 (1998).

[11] G.'t. Hooft, Nucl. Phys. **B256**, 727 (1985).

[12] Jiliang Jing, Int. J. Theor. Phys. **37**, 1441 (1998).

[13] Jiliang Jing and Mu-Lin Yan, Phys. Rev. D **60**, 084015 (1999); **61**, 044016 (2000).

[14] A. Ghosh and P. Mitra, Phys. Rev. Lett. **73**, 2521 (1994).

[15] M.H. Lee and J.K. Kim, Phys. Lett. A **212**, 323 (1996); Phys. Rev. D **54**, 3904 (1996).

[16] Jiliang Jing and Mu-Lin Yan, Int. J. Theor. Phys. **39**, 1687 (2000).

[17] Jiliang Jing and Mu-Lin Yan, Phys. Rev. D **63**, 084028 (2001).

[18] Jiliang Jing and Mu-Lin Yan, Phys. Rev. D **64**, 064015 (2001).

[19] J. Ho, W.T. Kim, and Y.J. Park, Class. Quantum Grav. **14**, 2617 (1997); J. Ho and G. Kang, Phys. Lett. B **445**, 27 (1998).

[20] Hua Bai and Mu-Lin Yan, J. High Energy Phys. **07**, 058 (2003).

[21] Kumar S. Gupta and Siddhartha Sen, Mod. Phys. Lett. A **18**, 463 (2003).

[22] S.Q. Wu and M.L. Yan, ''Entropy of Kerr-de Sitter black hole due to arbitrary spin fields,'' gr-qc/0303076.

[23] Kyung-Seok Cha, Bum-Hoon Lee, and Chanyong Park, ''dS/CFT correspondence from the Brick Wall method,'' hep-th/0207194.

[24] Li Xiang, Phys. Lett. B **540**, 9 (2002).

[25] A.J.M. Medved, Class. Quantum Grav. **19**, 405 (2002).

[26] Won Tae Kim, John J. Oh, and Young-Jai Park, Phys. Lett. B **512**, 131 (2001).

[27] S.N. Solodukhin, Phys. Rev. D **51**, 609 (1995).

[28] Elizabeth Winstanley, Phys. Rev. D **63**, 084013 (2001).

[29] Won Tae Kim, Phys. Rev. D **59**, 047503 (1999).

[30] R.B. Mann and S.N. Solodukhin, Phys. Rev. D **54**, 3932 (1996).

[31] J.G. Demers, R. Lafrance, and R.C. Myers, Phys. Rev. D **52**, 2245 (1995).

[32] S.N. Solodukhin, Phys. Rev. D **56**, 4968 (1997).

[33] R.B. Birrell and P.C.W. Devies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[34] S. Shankaranarayanan, K. Srinivasan, and T. Padmanabhan, Mod. Phys. Lett. A **16**, 571 (2001).

[35] S. Shankaranarayanan, T. Padmanabhan, and K. Srinivasan, Class. Quantum Grav. **19**, 2671 (2002).

[36] P. Kraus, ''Some applications of a simple stationary line element for the Schwarzschild geometry,'' gr-qc/9406042.

[37] D. Garfinkle, G.T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).

[38] G.W. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).

[39] J.H. Horne and G.T. Horowitz, Phys. Rev. D **46**, 1340 (1992).