Effective action for scalar fields and generalized zeta-function regularization

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Motivated by the study of quantum fields in a Friedmann-Robertson-Walker space-time, the one-loop effective action for a scalar field defined in the ultrastatic manifold $R\times H^3/\Gamma$, H^3/Γ being the finite volume, noncompact, hyperbolic spatial section, is investigated by a generalization of zeta-function regularization. It is shown that additional divergences may appear at the one-loop level. The one-loop renormalizability of the model is discussed and, making use of a generalization of zeta-function regularization, the one-loop renormalization group equations are derived.

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I. INTRODUCTION

Within the so-called one-loop approximation in quantum field theory, the Euclidean one-loop effective action may be expressed in terms of the sum of the classical action and a contribution depending on a functional determinant of an elliptic differential operator, the so called fluctuation operator. The ultraviolet one-loop divergences which are eventually present, have to be regularized by means of a suitable technique (for recent reviews, see Refs. [1–3]). In general, one works on a Euclidean version of the spacetime and deals with a self-adjoint, non-negative, second-order differential operator of the form

$$L=-\Delta+M^2, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator and M^2 a potential term depending on the classical (constant) background solution ϕ_c and in general containing the mass, the nonminimal coupling with the gravitational field and a possible self-interaction term.

Within the one-loop approximation, one usually splits the original field ϕ into two parts: the classical background ϕ_c and a quantum fluctuation Φ. As a result, the theory can be conveniently described by the (Euclidean) one-loop partition function

$$Z[\phi]=e^{-S[\phi_c]}\int D\Phi\, e^{-\int dV\Phi L\Phi}=e^{-W[\phi]}. \tag{1.2}$$

Here $S\equiv S[\phi_c]$ is the classical action, while $W\equiv W[\Phi]$ is the one-loop effective action, which can be related to the determinant of the field operator L by

$$W=-\ln Z=S+\frac{1}{2}\ln\det\frac{L}{\mu^2}, \tag{1.3}$$

μ^2 being a renormalization parameter, which appears for dimensional reasons.

The functional determinant is formally divergent, thus, the last term in the latter equation must be regularized and in order to study the one-loop divergences, it is convenient to make use of a variant of the zeta-function regularization [4–6]. To this end, we select the regularization function [2]

$$\rho(\varepsilon,t)=\frac{t^{\varepsilon}}{\Gamma(1+\varepsilon)}, \tag{1.4}$$

which goes to one as soon as the regularization parameter ε goes to zero. Then we may write

$$W(\varepsilon)=S-\frac{1}{2}\int_0^\infty dt\frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)}\mathrm{Tr}\,e^{-tL/\mu^2}$$
$$=S-\frac{1}{2\varepsilon}\zeta(\varepsilon|L/\mu^2), \tag{1.5}$$

where, as usual, for the elliptic operator L, the zeta function is defined by means of the Mellin-like transform

$$\zeta(s|L)=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\mathrm{Tr}\,e^{-tL},\qquad \zeta(s|L/\mu^2)=\mu^{2s}\zeta(s|L). \tag{1.6}$$

For a second order differential operator in four-dimensions, the integral is convergent for $\mathrm{Re}\,s>2$.

We see that the heat kernel trace $\mathrm{Tr}\,e^{-tL}$ plays a preeminent role in the investigation of the analytical properties of the zeta function. In fact, for a second-order, elliptic nonnegative operator L in a boundaryless smooth manifold, one has the short-t asymptotic expansion

$$\mathrm{Tr}\,e^{-tL}\simeq\sum_{j=0}^{\infty}A_j(L)t^{j-2}, \tag{1.7}$$

where $A_j(L)$ are the well known Seeley-DeWitt coefficients [7,8]. As a consequence, $\zeta(s|L)$ is regular at the origin and one gets the well known result $\zeta(0|L)=A_2(L)$. This quantity is easily computable (see, for example, the recent works [9,10]) and depends only on coupling constants and geometrical invariants.

By expanding Eq. (1.5) in Taylor's series one obtains

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$$W(\varepsilon)=S-\frac{\zeta(0|L/\mu^2)}{2\varepsilon}-\frac{\zeta'(0|L/\mu^2)}{2}+O(\varepsilon) \quad (1.8)$$

and the regularized one-loop effective action W_R can be defined by taking the finite part of $W(\varepsilon)$ in the limit $\varepsilon\to 0$, that is

$$W_R=S-\frac{\zeta'(0|L/\mu^2)}{2}=S-\frac{\zeta'(0|L)}{2}-\ln\mu^2\frac{\zeta(0|L)}{2}. \quad (1.9)$$

This leads to the usual zeta-function regularization prescription [5], i.e.,

$$\ln\det L=-\zeta'(0|L). \quad (1.10)$$

The one loop-divergences are governed by $\zeta(0|L)=A_2(L)$, which does not depend on the arbitrary scale parameter μ. The coefficient $A_2(L)$ also determines the beta functions of the model, namely its one-loop renormalization group equations (RGEs) and its one-loop renormalizability. In fact, the RGEs can be obtained by assuming that all coupling constants appearing in the renormalized effective action W_R are depending on μ and requiring

$$\mu\frac{d}{d\mu}W_R\equiv\mu\frac{d}{d\mu}S-\frac{\mu}{2}\frac{d}{d\mu}\zeta'(0|L/\mu^2)=0. \quad (1.11)$$

In this way, at one-loop level one obtains the renormalization group equations in the form [11,12]

$$\mu\frac{d}{d\mu}S=\zeta(0|L)=A_2(L). \quad (1.12)$$

In this paper, we would like to discuss a more general case corresponding to the presence of logarithmic terms in the heat-trace asymptotics. One may have logarithmic terms in the heat-kernel trace in the case of nonsmooth manifolds, for example when one considers the Laplace operator on higher dimensional cones [13,14], but also in four-dimensional space-times with a three-dimensional, noncompact, hyperbolic spatial section of finite volume [15]. More recently the presence of logarithmic terms in self-interacting scalar field theory defined on manifolds with noncommutative coordinates have also been pointed out [16,17].

The content of the paper is the following. In Sec. II the Heat kernel asymptotics with logarithmic terms are considered and the consequences of their presence discussed in some detail. A generalization of zeta-function regularization is proposed and the generalized one-loop RGEs are derived. In Sec. III an explicit example related to the work of Friedmann, Robertson, and Walker (FRW) by a conformal transformation is investigated in detail, and the generalized RGEs are explicitly written down. The conclusions and two Appendixes end the paper.

II. HEAT KERNEL ASYMPTOTICS WITH LOGARITHMIC TERMS

In this section we will discuss the modification of the formalism due to the presence of logarithmic terms in the heat kernel asymptotics. The starting point are Eqs. (1.3) and (1.5). We have to discuss the meromorphic extension of the zeta function, which depends on the form of the heat-kernel asymptotics.

Let us suppose to deal with a quite general expression of the kind

$$\mathrm{Tr}\, e^{-tL}\simeq\sum_{j=0}^{\infty}B_j t^{j-2}+\sum_{j=0}^{\infty}P_j\ln t\, t^{j-2}, \quad (2.1)$$

where now, with respect to the standard case, new terms containing $\ln t$ are present in the expansion.

In order to obtain the meromorphic continuation of the zeta function, we use the Mellin-like representation, Eq. (1.6). The original integral over t can be split into two integrals, the first from 1 to ∞, which gives analytic contributions to the zeta function and the second from 0 to 1, which gives rise the poles and can be explicitly computed using the small t expansion (2.1). In this way we get

$$\zeta(s|L)=\frac{1}{\Gamma(s)}\left(\sum_{j=0}^{\infty}\frac{B_j(L)}{s+j-2}-\sum_{j=0}^{\infty}\frac{P_j(L)}{(s+j-2)^2}+J(s)\right), \quad (2.2)$$

the function $J(s)$ being analytic.

We see that in contrast with the standard situation, here the zeta function has also double poles and, if P_2 is nonvanishing, it is no longer regular at the origin, but it has a simple pole with residue $-P_2$. Another important consequence for physics is that, due to the presence of logarithmic terms, the heat-kernel coefficients B_n, with respect to scale transformations, transforms in a nonhomogeneous manner. This can be easily seen by replacing the dimensionless parameter t with t/μ^2 in the heat expansion. In this way

$$\begin{aligned}\mathrm{Tr}\, e^{-tL/\mu^2}&\simeq\sum_{j=0}^{\infty}B_j(L/\mu^2)t^{j-2}+\sum_{j=0}^{\infty}P_j(L/\mu^2)\ln t\, t^{j-2}\\&=\sum_{j=0}^{\infty}B_j(L)\left(\frac{t}{\mu^2}\right)^{j-2}\\&\quad+\sum_{j=0}^{\infty}P_j(L)\ln\left(\frac{t}{\mu^2}\right)\left(\frac{t}{\mu^2}\right)^{j-2},\end{aligned} \quad (2.3)$$

from which it follows that

$$\begin{aligned}B_n(L/\mu^2)&=\mu^{4-2n}[B_n(L)-\ln\mu^2 P_n(L)],\\P_n(L/\mu^2)&=\mu^{4-2n}P_n(L).\end{aligned} \quad (2.4)$$

In particular, in contrast with the standard case, the coefficient B_2 is not scale invariant. It is convenient to split the B_n coefficients in two parts, that is $B_n=A_n+Q_n$, where A_n represent the standard coefficients, obtained as integral of the local geometric quantities a_n, namely,

$$A_n(L)=\frac{1}{(4\pi)^2}\int dVa_n(x|L), \quad A_n(L/\mu^2)=\mu^{4-2n}A_n(L), \tag{2.5}$$

while the second part Q_n is strictly connected with the presence of logarithmic terms and transforms according to

$$\begin{aligned} Q_n(L/\mu^2)&=\mu^{4-2n}[Q_n(L)-\ln\mu^2 P_n(L)],\\ Q_2(L/\mu^2)&=Q_2(L)-\ln\mu^2 P_2(L),\\ P_2(L/\mu^2)&=P_2(L). \end{aligned} \tag{2.6}$$

The consequences of the presence of such a pole at the origin on the one-loop effective action can be investigated by using the regularization of Sec. I, namely,

$$\begin{aligned} \ln\det(L/\mu^2)_\varepsilon&=-\int_0^\infty dt\frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)}\operatorname{Tr}e^{-tL/\mu^2}\\ &=-\frac{\zeta(\varepsilon|L/\mu^2)}{\varepsilon}=-\frac{\omega(\varepsilon|L/\mu^2)}{\varepsilon^2}. \end{aligned} \tag{2.7}$$

In the latter equation we have conveniently introduced the new kind of zeta function ω, regular at the origin, by means of the relation

$$\omega(s|L)=s\zeta(s|L), \quad \omega(s|L/\mu^2)=s\mu^{2s}\zeta(s|L)=\mu^{2s}\omega(s|L). \tag{2.8}$$

We may expand ω in Taylor's series around $s=0$, obtaining in this way

$$\begin{aligned} \ln\det(L/\mu^2)_\varepsilon=&-\frac{1}{\varepsilon^2}\omega(0|L/\mu^2)-\frac{1}{\varepsilon}\omega'(0|L/\mu^2)\\ &-\frac{1}{2}\omega''(0|L/\mu^2)+O(\varepsilon). \end{aligned} \tag{2.9}$$

As a consequence, the one-loop divergences are governed by the two coefficients $\omega(0|L/\mu^2)$ and $\omega'(0|L/\mu^2)$, while the nontrivial finite part is given by $\frac{1}{2}\omega''(0|L/\mu^2)$. This suggests a generalization of the zeta-function regularization for a functional determinant associated with an elliptic operator L, namely [3],

$$\ln\det L=-\frac{1}{2}\omega''(0|L). \tag{2.10}$$

Of course, this reduces to the usual zeta-function regularization when $\zeta(s|L)$ is regular at the origin.

The two coefficients governing the one-loop divergences can be computed making use of the meromorphic structure

$$\begin{aligned} \omega(s|L)=&\frac{1}{\Gamma(s)}\sum_{j=0}^{\infty}\frac{sB_j}{s+j-2}\\ &-\frac{1}{\Gamma(s)}\sum_{j=0}^{\infty}\frac{sP_j}{(s+j-2)^2}+\frac{s}{\Gamma(s)}J(s). \end{aligned} \tag{2.11}$$

One has

$$\omega(s|L)=-P_2(L)+[B_2(L)-\gamma P_2(L)]s+O(s^2) \tag{2.12}$$

and so, using Eq. (2.8) or alternatively Eq. (2.4), one has

$$\begin{aligned} \omega(0|L/\mu^2)&=-P_2(L),\\ \omega'(0|L/\mu^2)&=B_2(L)-(\gamma+\ln\mu^2)P_2(L), \end{aligned} \tag{2.13}$$

γ being the Euler-Mascheroni constant.

Recall that the model is one-loop renormalizable, if the dependence of B_2 and P_2 on the background field has the same algebric structure of the classical action and the divergences may be reabsorbed by the redefinition of mass and coupling constants.

With regard to the derivation of one-loop RGEs, they may be obtained again by assuming that the mass and all coupling constants appearing in the classical action are depending on μ and requiring

$$\mu\frac{d}{d\mu}W_R=0, \tag{2.14}$$

where now the functional determinant appearing in Eq. (1.3) is regularized according to Eq. (2.10). In this way we get

$$\begin{aligned} W_R&=S-\frac{1}{4}\omega''(0|L/\mu^2)\\ &=S-\frac{1}{4}[\omega''(0|L)+2\ln\mu^2\omega'(0|L)+(\ln\mu^2)^2\omega(0|L)]. \end{aligned} \tag{2.15}$$

Making use of Eqs. (2.13)–(2.15), we finally get (at the one-loop level)

$$\begin{aligned} \mu\frac{d}{d\mu}S&=\omega'(0|L)+\ln\mu^2\omega(0|L)\\ &=B_2(L)-(\gamma+\ln\mu^2)P_2(L). \end{aligned} \tag{2.16}$$

If the theory is renormalizable, the action and the heat coefficients have the same structure in terms of the fields. More precisely, if the action has the form

$$S=\int dV\sum_\alpha \lambda_\alpha(\mu)F_\alpha, \tag{2.17}$$

then

$$B_2(L)=\int dV\sum_\alpha k_\alpha(\mu)F_\alpha,$$

$$P_2(L)=\int dV\sum_\alpha h_\alpha(\mu)F_\alpha, \tag{2.18}$$

where $F_\alpha\equiv(1,\phi^2/2,\phi^4/24,\dots)$ are the independent building blocks, $\lambda_\alpha\equiv(\Lambda,m^2,\lambda,\dots)$ the collection of all coupling constants, including the ones concerning the gravitational action, while k_α and h_α are constants, which can be directly read off from the form of the heat coefficients. From Eqs. (2.16)–(2.18) one obtains the differential equations from the beta functions in the form

$$\beta_\alpha\equiv\mu\frac{d\lambda_\alpha}{d\mu}=k_\alpha-(\gamma+\ln\mu^2)h_\alpha, \tag{2.19}$$

which of course give the usual result when $P_2=0$.

III. SCALAR FIELDS IN A FRIEDMANN-ROBERTSON-WALKER SPACE-TIME

Now we will provide an application of the formalism previously developed. We shall study a scalar field defined on a spacetime of the kind $\mathbb{R}\times\Sigma_3$, where Σ_3 is the constant curvature spatial section. The physical motivations are due to the fact that, in a suitable coordinates system, the metric of the FRW space-time is conformally related to the metric of $R\times\Sigma_3$ and moreover, as we shall see, in usual cases, the renormalization properties do not depend on the conformal transformation. This statement will be clarified later on.

We start with a four-dimensional FRW space-time with the standard metric

$$ds^2=-dT^2+a^2(T)d\sigma_3^2, \tag{3.1}$$

$d\sigma_3^2$ being the metric associated with a three-dimensional manifold with constant curvature. Then we introduce the related conformal time η by

$$\eta=\int\frac{dT}{a(T)}. \tag{3.2}$$

In this way the metric assumes the form

$$ds^2=a^2(\eta)(-d\eta^2+d\sigma_3^2). \tag{3.3}$$

This means that (locally) the space-time is conformally related to a constant curvature manifold $M^4=R\times\Sigma_3$, possibly equipped with a nontrivial topology. In particular we shall investigate in detail the case of a noncompact and nonsmooth hyperbolic spatial section $\Sigma_3=H^3/\Gamma$, with finite volume, H^3 being the three-dimensional hyperbolic manifold and Γ a discrete group of isometries containing hyperbolic and parabolic elements [15]. We denote by $\tilde{M}^4$ the original space-time with the metric $\tilde{g}_{ij}$ conformally related to the metric g_{ij} of the constant curvature manifold M^4, that is

$$\tilde{g}_{ij}=e^{2\sigma}g_{ij},\quad \sigma=\ln a(\eta), \tag{3.4}$$

$\sigma:=\sigma(x)$ being a scalar function.

In general, by conformal transformations, the partition function is not invariant (see Appendix A), but one has

$$\tilde{W}=W[\tilde{\phi},\tilde{g}]=W[\phi,g]-\ln J[g,\tilde{g}], \tag{3.5}$$

where $J[g,\tilde{g}]$ is the Jacobian of the conformal transformation. Such a Jacobian (also called a cocycle function or an induced effective action) can be computed for any infinitesimal conformal transformation (see, for example, Refs. [18–21] and references cited therein). Its expression in four-dimensions reads (see Appendix A)

$$\ln J[g,\tilde{g}]=\frac{1}{(4\pi)^2}\int_0^1 dq\int d^4x\sqrt{g^q}[b_2(x|L_q)-(\gamma+\ln\mu^2)p_2(x|L_q)], \tag{3.6}$$

where $b_2(x|L_q)$ and $p_2(x|L_q)$ are the local quantities related to the coefficients $B_2(L_q)$ and $P_2(L_q)$ respectively, while L_q is the field operator in the metric $g^q_{ij}=\exp(2q\sigma)g_{ij}$. Thus, in principle, the knowledge of the partition function $Z[\phi,g]$ in the manifold M^4 and the heat coefficients b_2 and p_2, are sufficient in order to get the partition function in the original manifold $\tilde{M}^4$. For such a reason here we shall study the heat-kernel asymptotics and the one-loop effective action for scalar fields in $M^4=\mathbb{R}\times H^3/\Gamma$. If Γ contains parabolic elements, the heat-kernel asymptotics for the Laplacian contains also logarithmic terms [15].

As can be trivially seen, in the standard case the relevant heat-kernel coefficient $a_2(x|L/\mu^2)$ does not depend on μ and the Jacobian $J[g,\tilde{g}]$ is finite. This means that W and $\tilde{W}$ have the same one-loop divergences and give rise to the same renormalization group equations. The situation completely change in the "nonstandard case" we are going to consider, since the Jacobian factor explicitly depends on μ, as one can see by looking at Eq. (3.6).

A. Heat-kernel expansion for scalar fields on $\mathbb{R}\times H^3/\Gamma$

We start with the classical Euclidean action for a massive, self-interacting scalar field in $M^4\equiv\mathbb{R}\times H^3/\Gamma$, H^3 being the three-dimensional hyperbolic manifold and Γ a group containing the identity, hyperbolic and parabolic elements. H^3/Γ is a rank-1 symmetric space with constant curvature R. This latter is also the scalar curvature of M^4. (For more details concerning the geometry of this spatial section, see Ref. [15].)

The action for the matter field has the form

$$S[\phi,g]=\int\left[-\frac{1}{2}\phi\Delta\phi+V_c(\phi)\right]\sqrt{g}d^4x, \tag{3.7}$$

where the classical potential reads

$$V_c(\phi)=\frac{\lambda\phi^4}{24}+\frac{m^2\phi^2}{2}+\frac{\xi R\phi^2}{2}, \tag{3.8}$$

m being the mass of the field, λ the self-interacting coupling constant and ξ a coupling constant which takes into account of a possible nonminimal coupling between matter and gravitation (see, for example, Refs. [22,12]).

Within the one-loop approximation, one splits the field as $\phi=\phi_c+\Phi$, ϕ_c being the background (classical) field and Φ the quantum fluctuation. In this way, the relevant operator associated with the quantum fluctuation is given by

$$L=-\Delta+M^2,\quad M^2=m^2+\xi R+\frac{\lambda\phi_c^2}{2}. \tag{3.9}$$

Since we are dealing with an ultrastatic space-time, we have

$$L=-\partial_\eta^2-\Delta_3+M^2=-\partial_\eta^2+L_3,$$
$$L_3=-\Delta_3+M^2, \tag{3.10}$$

Δ_3 being the Laplace-Beltrami operator acting on functions in H^3/Γ.

In the regularization scheme we have proposed in previous sections, the one-loop effective action is given by

$$W=S+\frac{1}{2}\ln\det\frac{L}{\mu^2}=S-\frac{1}{4}\omega''(0|L/\mu^2). \tag{3.11}$$

For the case under consideration, the heat-kernel trace has the form

$$\mathrm{Tr}\, e^{-tL}=\frac{\ell}{\sqrt{4\pi t}}\mathrm{Tr}\, e^{-tL_3}, \tag{3.12}$$

ℓ being the "infinite volume" of $\mathbb{R}$. For the rank-1 symmetric space H^3/Γ, the trace of the heat kernel can be computed by using the Selberg trace formula. In our case (the group Γ contains identity, hyperbolic and parabolic elements) we get (see Ref. [15] for details)

$$\mathrm{Tr}\, e^{-tL_3}\sim\sum_{j=0}^{\infty}[B_j(L_3)+P_j(L_3)\ln t]t^{j-3/2}, \tag{3.13}$$

where

$$B_0(L_3)=\frac{v_F}{(4\pi)^{3/2}},$$

$$B_1(L_3)=-\frac{v_F\delta^2}{(4\pi)^{3/2}}+\frac{C}{\sqrt{4\pi}}, \tag{3.14}$$

$$B_2(L_3)=\frac{v_F\delta^4}{2(4\pi)^{3/2}}+\frac{1}{6\sqrt{\pi}}-\frac{C\delta^2}{\sqrt{4\pi}},$$

$$P_0(L_3)=0,\quad P_1(L_3)=\frac{1}{8\sqrt{\pi}},$$

$$P_2(L_3)=-\frac{\delta^2}{8\sqrt{\pi}}, \tag{3.15}$$

C being a known constant, v_F the (dimensionless) fundamental volume and $\delta^2=|\kappa|+M^2$, where $\kappa=R/6$ is the negative, constant (Gaussian) curvature of the manifold. We have written only the coefficients which we need in the paper, but in principle all coefficients can be computed. We have also used units in which $\kappa=-1$.

From Eqs. (3.12) and (3.13) we immediately obtain the expansion we are interested in, that is

$$\mathrm{Tr}\, e^{-tL}\sim\sum_{j=0}^{\infty}[B_j(L)+P_j(L)\ln t]t^{j-2}, \tag{3.16}$$

where trivially

$$B_j(L)=\frac{\ell B_j(L_3)}{\sqrt{4\pi}},\quad P_j(L)=\frac{\ell P_j(L_3)}{\sqrt{4\pi}}. \tag{3.17}$$

Then, in the standard units, we finally obtain

$$B_0(L)=\frac{V}{16\pi^2},$$

$$B_1(L)=-\frac{V}{16\pi^2}\left(\delta^2+\frac{2\pi CR}{3v_F}\right), \tag{3.18}$$

$$B_2(L)=\frac{V}{16\pi^2}\left(\frac{\delta^4}{2}+\frac{\pi R^2}{27v_F}+\frac{2\pi C\delta^2 R}{3v_F}\right),$$

$$P_0(L)=0,\quad P_1(L)=-\frac{V}{16\pi^2}\frac{\pi R}{6v_F},$$

$$P_2(L)=\frac{V}{16\pi^2}\frac{\pi R\delta^2}{6v_F}. \tag{3.19}$$

Expanding the previous quantity we have in particular

$$\begin{aligned}B_2(L)=\frac{V}{16\pi^2}\Bigg\{&\frac{m^4}{2}+\lambda m^2\frac{\phi_c^2}{2}+3\lambda^2\frac{\phi_c^4}{24}\\&+\left[\lambda\left(\xi-\frac{1}{6}\right)+\frac{2\pi\lambda C}{3v_F}\right]\frac{R\phi_c^2}{2}\\&+\left[m^2\left(\xi-\frac{1}{6}\right)+\frac{2\pi m^2 C}{3v_F}\right]R\\&+\left[\frac{\pi}{27v_F}+\frac{1}{2}\left(\xi-\frac{1}{6}\right)^2+\frac{2\pi C}{3v_F}\left(\xi-\frac{1}{6}\right)\right]R^2\Bigg\},\end{aligned} \tag{3.20}$$

$$P_2(L)=\frac{V}{16\pi^2}\left\{\frac{\pi\lambda}{6v_F}\frac{R\phi_c^2}{2}+\frac{\pi m^2}{6v_F}R+\frac{\pi}{6v_F}\left(\xi-\frac{1}{6}\right)R^2\right\}. \tag{3.21}$$

B. The renormalized one-loop effective action and the one-loop renormalization group equations

Here we would like to explicitly compute the beta functions for the model considered above. In order to have the renormalisation of the model, to the matter action (3.7) we have to add the action for the (classical) gravitational field, then

$$S[\phi,g]=\int\left[\Lambda-\frac{1}{2}\phi\Delta\phi+V_c(\phi)+g_1R+g_2R^2\right]\sqrt{g}d^4x. \tag{3.22}$$

Λ being the bare cosmological constant, In this way we have the independent building blocks

$$F_\alpha\equiv\left(1,\frac{\phi^2}{2},\frac{\phi^4}{24},\frac{R\phi^2}{2},R,R^2\right) \tag{3.23}$$

and the corresponding coupling constants $\lambda_\alpha(\mu)\equiv(\Lambda,m^2,\lambda,\xi,g_1,g_2)$.

Now, from Eqs. (2.19), (3.20), and (3.21), we directly get

$$\mu\frac{d\Lambda}{d\mu}=\frac{m^4}{32\pi^2},\quad \mu\frac{dm^2}{d\mu}=\frac{\lambda m^2}{16\pi^2},\quad \mu\frac{d\lambda}{d\mu}=\frac{3\lambda^2}{16\pi^2}, \tag{3.24}$$

$$\mu\frac{d\xi}{d\mu}=\frac{\lambda(\xi-1/6)}{16\pi^2}+\frac{\lambda C}{24\pi v_F}-\frac{\lambda(\gamma+\ln\mu^2)}{96\pi v_F},$$

$$\mu\frac{dg_1}{d\mu}=\frac{m^2(\xi-1/6)}{(16\pi^2}+\frac{m^2C}{24\pi v_F}-\frac{m^2(\gamma+\ln\mu^2)}{96\pi v_F},$$

$$\mu\frac{dg_2}{d\mu}=\frac{1}{432\pi v_F}+\frac{(\xi-1/6)^2}{32\pi^2}+\frac{C(\xi-1/6)}{24\pi v_F}-\frac{(\xi-1/6)(\gamma+\ln\mu^2)}{96\pi v_F}. \tag{3.25}$$

Some remarks are in order. First, as in the 4-dimensional smooth case, there is no renormalization of the wave function. Second, the RGEs concerning the coupling constants Λ, m^2 and λ [see Eq. (3.24)] are exactly the same which one has in the smooth case (see, for example, Refs. [22,23]), while the RGEs related to ξ, g_1 and g_2 are modified by the presence of the parabolic elements of the group Γ.

IV. CONCLUSION

In this paper we have started an investigation concerning the one-loop effective action for a scalar field in a four-dimensional FRW space-time. The one-loop effective action may be considered as the sum of two contributions. The first one can be computed by considering the field in a constant curvature space-time, conformally related to the original manifold, while the second one takes its origin in the Jacobian of the conformal transformation. This latter contribution is what is usually called the conformal induced effective action.

Here we have studied in detail the first contribution in the particular spacetime $R\times H^3/\Gamma$. Due to the nontrivial topology of the manifold we have considered, new ultraviolet divergences appear in the one-loop effective action. At the one-loop level, we have shown that all divergences may be reabsorbed by suitable counterterms in the classical action. The new divergences depend on the coefficient

$$P_2(L)=\frac{V}{16\pi^2}\left[\frac{\pi\lambda R\phi_c^2}{12v_F}+\frac{\pi m^2R}{6v_F}+\frac{\pi(\xi-1/6)R^2}{6v_F}\right]. \tag{4.1}$$

In the massless and free case with the conformal coupling $\xi=1/6$, one has $P_2=0$ and thus, in this particular situation the usual evaluation of the effective action by means of zeta-function methods works without any modification. It has to be noted that the choices $m=0$, $\lambda=0$, and $\xi=1/6$ are consistent with RGGs. If the P_2 coefficient is not vanishing, a generalization of the zeta-function techniques has been used and the RGEs have been derived in a consistent way, checking that the model is indeed renormalizable at the one-loop level.

As far as the anomaly induced effective action is concerned, in the case $P_2=0$ it can be computed making use of the general expressions reported in Sec. I and in Appendix A [24,25,21]. The evaluation of it when $P_2\neq 0$ is not an easy task and we will investigate it in the near future.

APPENDIX A: CONFORMAL TRANSFORMATIONS

In this appendix we shall consider the conformal properties of the first nontrivial Seeley-DeWitt coefficients (also see Ref. [26]). For more generality, here we work in a D-dimensional Euclidean manifold and for convenience we use the scalar density $\varphi=g^{1/4}\phi$ in place of the scalar field ϕ. The classical action then assumes the form

$$S=\int d^Dx\sqrt{g}\phi L\phi=\int d^Dx\varphi L\varphi, \tag{A1}$$

where $L=-\Delta_g+m^2+\xi R$ is a Laplacian-like operator ($\Delta_g=g^{ij}\nabla_i\nabla_j$, ∇_i being the covariant derivative).

A conformal transformation is defined by

$$\tilde{g}_{ij}=e^{2\sigma}g_{ij},\quad \tilde{g}=|\det\tilde{g}_{ij}|=e^{D\sigma}g,\quad i,j=0,1,\ldots,D-1, \tag{A2}$$

$$\tilde{\varphi}=e^{\sigma}\varphi,\quad \tilde{\phi}=e^{(1-D/2)\sigma}\phi, \tag{A3}$$

$\sigma\equiv\sigma(x)$ being a generic scalar function.

By a straightforword computation, for the connection coefficients, the Riemann and Ricci tensors and scalar curvature one obtains, respectively,

$$\tilde{\Gamma}^k_{ij}=\Gamma^k_{ij}+\Sigma^k_{ij},$$

$$\tilde{R}^i_{jrs}=R^i_{jrs}+\Sigma^i_{jrs},$$

$$\tilde{R}_{ij}=R_{ij}+\Sigma_{ij}, \tag{A4}$$

$$\tilde{R}=e^{-2\sigma}(R+\Sigma),$$

where

$$\Sigma^k_{ij}=\sigma_i\delta^k_j+\sigma_j\delta^k_i-\sigma^k g_{ij}, \quad \sigma_k=\partial_k\sigma. \tag{A5}$$

Furthermore, for any scalar function f,

$$\Delta_{\tilde{g}}f=e^{-2\sigma}[\Delta_g+(D-2)\sigma^k\partial_k]f. \tag{A6}$$

In order to make explicit computations, we now introduce the useful notation

$$\sigma_{ij}=\nabla_i\nabla_j\sigma, \quad \sigma^k_k=\Delta\sigma,$$

$$B_{ij}=B_{ji}=\sigma_{ij}-\sigma_i\sigma_j+\frac{g_{ij}}{2}\sigma^k\sigma_k,$$

$$B=B^k_k=\Delta\sigma+\frac{D-2}{2}\sigma^k\sigma_k,$$

$$B^{ij}B_{ij}=\sigma^{ij}\sigma_{ij}-2\sigma^{ij}\sigma_i\sigma_j-\Delta\sigma\sigma^k\sigma_k+\frac{D(\sigma^k\sigma_k)^2}{2}. \tag{A7}$$

In this way

$$\begin{aligned}\Sigma^i_{jrs}&=\nabla_r\Sigma^i_{sj}-\nabla_s\Sigma^i_{rj}+\Sigma^i_{rl}\Sigma^l_{sj}-\Sigma^i_{sl}\Sigma^l_{rj}\\&=-[g_{ir}B_{js}-g_{is}B_{jr}+g_{js}B_{ir}-g_{jr}B_{is}],\\ \Sigma_{ij}&=\Sigma^k_{ikj}=-[(D-2)B_{ij}+B^k_k g_{ij}],\\ \Sigma&=g^{ij}\Sigma_{ij}\\&=-2(D-1)\Delta_g\sigma-(D-1)(D-2)\sigma^k\sigma_k\\&=-2(D-1)B\Sigma^{ijrs}\Sigma_{ijrs}4(D-2)B^{ij}B_{ij}+4B^2,\end{aligned} \tag{A8}$$

$$\Sigma^{ij}\Sigma_{ij}=(D-2)^2B^{ij}B_{ij}+(3D-4)B^2.$$

Now it is easy to verify that the Weyl tensor

$$C_{ijrs}=R_{ijrs}+\frac{1}{D-2}(g_{ir}R_{js}-g_{is}R_{jr}+g_{js}R_{ir}-g_{jr}R_{is})-\frac{1}{(D-1)(D-2)}R(g_{ir}g_{js}-g_{is}g_{jr}) \tag{A9}$$

is conformally invariant, while the Gauss-Bonnet tensor

$$G=R^{ijrs}R_{ijrs}-4R^{ij}R_{ij}+R^2 \tag{A10}$$

transforms according to

$$\tilde{G}=G+8(D-3)R^{ij}B_{ij}-4(D-3)RB-4(D-2)(D-3)(B^{ij}B_{ij}-B^2). \tag{A11}$$

Recall that in four-dimensions, G is a total divergence.

By definition, $\tilde{S}=S$ (the action is a number), and so

$$\tilde{\varphi}\tilde{L}\tilde{\varphi}=\varphi L\varphi=\tilde{\varphi}e^{-\sigma}Le^{-\sigma}\tilde{\varphi}. \tag{A12}$$

As a consequence

$$\tilde{\varphi}\tilde{L}\tilde{\varphi}=\tilde{\varphi}\left\{-\Delta_{\tilde{g}}+\frac{\lambda}{2}\tilde{\phi}_c^2+\xi_D\tilde{R}+e^{-2\sigma}[m^2+(\xi-\xi_D)R]\right\}\tilde{\varphi}, \tag{A13}$$

where $\tilde{\phi}_c$ is the classical solution (background field) and $\xi_D=(D-1)/4(D-2)$. As a result, for a conformally coupled ($\xi=\xi_D$) massless scalar field, the action in invariant in form.

The first heat-kernel coefficients related to a generic operator of the form $-\Delta-2W^k\nabla_k+M^2$ on a Riemannian, smooth manifold without boundary read

$$a_0=1, \quad a_1=\frac{R}{6}-M^2-W^2-\hat{\nabla}_kW^k,$$

$$a_2=\frac{a_1^2}{2}+\frac{\hat{\Delta}a_1}{6}+\frac{1}{180}(\Delta R+R_{ijrs}R^{ijrs}-R_{ij}R^{ij}). \tag{A14}$$

Since in general W^k could be a matrix, we have introduced the connection $\hat{\nabla}_kf=\nabla_k+[W_k,f]$.

For the operator L we are dealing with, $W_k=0$ and $M^2=\lambda\phi_0^2/2+m^2+\xi R$, while, for $\tilde{L}$, $\tilde{M}^2=\lambda\tilde{\phi}_0^2/2+\xi_D\tilde{R}+e^{-2\sigma}[m^2+(\xi-\xi_D)R]$. Then one has

$$\tilde{a}_1=\frac{\tilde{R}}{6}-\tilde{M}^2=e^{-2\sigma}\left[a_1-\left(\xi_D-\frac{1}{6}\right)\Sigma\right],$$

$$\tilde{a}_2=\frac{\tilde{a}_1^2}{2}+\frac{\Delta_{\tilde{g}}\tilde{a}_1}{6}++\frac{1}{180}(\Delta_{\tilde{g}}\tilde{R}+\tilde{R}_{ijrs}\tilde{R}^{ijrs}-\tilde{R}_{ij}\tilde{R}^{ij}). \tag{A15}$$

The relation between $\tilde{a}_2$ and a_2 in general is very complicated, but in the case of conformally coupled fields in four-dimensions. In such a case we have in fact

$$\tilde{a}_1=e^{-2\sigma}a_1, \quad \tilde{a}_2=e^{-4\sigma}\left(a_2-\frac{1}{3}\Delta\sigma+\nabla_kV^k\right), \tag{A16}$$

where ∇_kV^k is the total divergence due to the geometric part of a_2. From the latter equation, the well known result

$$\tilde{A}_2 = \frac{1}{(4\pi)^2}\int d^4x\sqrt{\tilde{g}}\tilde{a}_2 = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}a_2 = A_2 \tag{A17}$$

follows. For the P_n and Q_n coefficients there are no general expressions in terms of geometrical quantities and so one cannot say anything about their conformal transformation properties, without considering the specific problem.

To conclude this section, we present a simple derivation of the conformal transformation properties of the one-loop effective action, which, according to generalized zeta-function regularization, is given by

$$W = -\log Z = S - \frac{1}{4}\omega''(0|L/\mu^2). \tag{A18}$$

In general, this is not invariant even for conformally coupled fields due to the presence of the functional measure, which breaks the symmetry. This phenomenon is well known and, in the physical literature, it is called a ''conformal or trace anomaly.''

We set

$$\ln\tilde{Z} \equiv \ln Z[\tilde{\phi},\tilde{g}] = J[g,\tilde{g}]\ln Z[\phi,g],$$

$$\ln J[g,\tilde{g}] = -(\tilde{W} - W), \tag{A19}$$

and consider a family of continuous transformations of the forms

$$g^q_{ij} = e^{2q\sigma}g_{ij}, \quad g^0_{ij} = g_{ij}, \quad g^1_{ij} = \tilde{g}_{ij}. \tag{A20}$$

$$\varphi_q = e^{q\sigma}\varphi, \quad L_q = e^{-q\sigma}Le^{-q\sigma},$$

$$W_q = W[\phi_q, g^q]. \tag{A21}$$

For an infinitesimal transformation we get

$$\ln J[g^q, g^{q+\delta q}] = -(W_{q+\delta q} - W_q) = -\delta S + \frac{\delta\omega''(0|L_q/\mu^2)}{4}. \tag{A22}$$

We observe that

$$\delta\omega(s|L_q/\mu^2) = \delta[s\,\mathrm{Tr}\,L_q^{-s}] = 2\,\delta q\int d^Dx\sqrt{g^q}\sigma(x)s\,\omega(s;x|L_q/\mu^2), \tag{A23}$$

where $\omega(s;x|L)$ represents the diagonal kernel of $\omega(s|L)$, that is

$$\omega(s|L) = \int d^Dx\sqrt{g}\,\omega(s;x|L). \tag{A24}$$

In order to go on in arbitrary dimensions, we consider a more general version of Eqs. (2.1) and (2.12), that is

$$\mathrm{Tr}\,e^{-tL} \sim \sum_0^\infty B_j t^{j-D/2} + \sum_0^\infty P_j \ln t\, t^{j-D/2}, \tag{A25}$$

$$\omega(s|L) = -P_{D/2}(L) + [B_{D/2}(L) - \gamma P_{D/2}(L)]s + O(s^2), \tag{A26}$$

and suppose the local version

$$\omega(s;x|L) = -\frac{p_{D/2}(x|L)}{(4\pi)^{D/2}} + \frac{[b_{D/2}(x|L) - \gamma p_{D/2}(x|L)]s}{(4\pi)^{D/2}} + O(s^2), \tag{A27}$$

$$B_n = \frac{1}{(4\pi)^{D/2}}\int d^Dx\sqrt{g}\,b_n(x|L),$$

$$P_n = \frac{1}{(4\pi)^{D/2}}\int d^Dx\sqrt{g}\,p_n(x|L), \tag{A28}$$

to be valid. Then, from Eqs. (A23) and (A23), we get

$$\delta\omega''(s|L_q) = 4\,\delta q\int d^Dx\sqrt{g^q}\sigma(x)[b_{D/2}(x|L_q/\mu^2) - \gamma p_{D/2}(x|L_q/\mu^2)]. \tag{A29}$$

Now, integrating Eq. (A22) with respect to q we obtain the final formula

$$\ln J[g,\tilde{g}] = \frac{1}{(4\pi)^{D/2}}\int_0^1 dq\int d^Dx\sqrt{g^q}[b_{D/2}(x|L_q) - (\gamma + \ln\mu^2)p_{D/2}(x|L_q)]. \tag{A30}$$

Here we have assumed the classical theory to be conformal invariant, that is $\delta S = 0$. In such a case, when $p_{D/2} = 0$, the latter equation gives rise to the well known form of the one-loop effective action.

APPENDIX B: A NONLOCAL INTERACTION EXAMPLE

In this appendix we shall present another explicit example in which logarithmic terms appear in the heat-kernel expansion.

We consider the following toy model which, in a simplified manner, mimics an interacting scalar field defined on a noncompact flat manifold with pairs of noncommuting coordinates. Let the D-dimensional manifold be $M^D = \mathbb{R}^d \times \mathbb{R}^p$. Moreover, let be L_d and L_p Laplacian-like operators on $\mathbb{R}^d$ and $\mathbb{R}^p$, respectively, with $D = p + d$ and $L_D = L_d + L_p$. The model may be defined by the classical Euclidean action

$$S = \int dx\left[\frac{1}{2}\phi\left(L_D + \frac{a^2}{L_p}\right)\phi + V(\phi)\right]. \tag{B1}$$

The nonlocal interaction mimics the ''noncommuting'' manifold $\mathbb{R}^p$ as soon as p is even and the parameter a^2 controls its presence. The self-interacting potential is given by

$$V(\phi)=\frac{\phi^r}{r}. \tag{B2}$$

In renormalizable theories, the power r is nonarbitrary, but it is related to the dimension. In fact one has the possible couples $(D=3,r=6)$, $(D=4,r=4)$, $(D=6,r=3)$, . . . The nonlocal one-loop fluctuation operator reads

$$L=L_D+\frac{a^2}{L_p}+V''(\phi_c)=L_D+\frac{a^2}{L_p}+M^2,$$

$$M^2=\frac{\phi_c^{r-2}}{(r-2)!}. \tag{B3}$$

The heat-kernel trace and zeta function can be exactly evaluated and read, respectively,

$$\mathrm{Tr}\, e^{-tL}=\frac{2V_D a^{p/2}}{(4\pi)^{D/2}\Gamma(p/2)}\frac{e^{-tM^2}K_{p/2}(2at)}{t^d/2}, \tag{B4}$$

$$\zeta(s|L)=\frac{2^{p+1}\sqrt{\pi}V_D a^p}{(4\pi)^{D/2}\Gamma(p/2)(M^2-2a)^{s+(p-d)/2}}\times\frac{\Gamma(s-(p-d)/2)\Gamma(s-D/2)}{\Gamma(s)\Gamma(s+(1-d)/2)}\times F\left(s+\frac{p-d}{2},1+\frac{p}{2};s+\frac{1-d}{2},\frac{M^2-2a}{M^2+2a}\right), \tag{B5}$$

$K_\nu(z)$ being the modified Bessel function and V_D the volume of the whole manifold and $F(\alpha,\beta;\gamma,z)$ the hypergeometric function.

The function $\mathrm{Tr}\, e^{-tL}$ can be written as an exact series of t, which assumes different forms for odd and even p. In fact for odd $p=n+1/2$ we have

$$\mathrm{Tr}\, e^{-tL}=\frac{\sqrt{\pi}V_D e^{-tM^2}e^{-2at}}{(4\pi t)^{D/2}\Gamma(n+1/2)}\sum_{k=0}^{n}\frac{(n+k)!(at)^{n-k}}{k!(n-k)!4^k}, \tag{B6}$$

while, for even $p=2n$,

$$\mathrm{Tr}\, e^{-tL}=\frac{V_D e^{-tM^2}}{(4\pi t)^{D/2}}\Bigg\{\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!(at)^{2k}}{(n-1)!}-(-1)^n\sum_{k=0}^{\infty}\frac{(at)^{2(n+k)}}{(n-1)!k!(n+k)!}\times[2\ln(at)-\psi(k+1)-\psi(n+k+1)]\Bigg\}. \tag{B7}$$

One can see that logarithmic terms in the heat expansion only appear for even p. Moreover, one has a pure logarithmic term only for even d, equal or greater than p. In particular, for $D=4$, $d=p=2$, $M^2=\lambda(\phi_c^2/2)$ one obtains

$$\mathrm{Tr}\, e^{-tL}=\frac{V_4 a e^{-tM^2}}{8\pi^2 t}K_1(2at)\simeq\frac{V_4}{8\pi^2}\left[\frac{1}{2t^2}-\frac{M^2}{2t}+\frac{M^4}{4}+a^2(\gamma+\ln a-1/2)+a^2\ln t+O(t,t\ln t)\right]. \tag{B8}$$

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