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Post-Newtonian quasirigid body

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In this paper, we construct for the first time, in the first post-Newtonian (1PN) approximation, a complete model of a quasirigid body by means of a new constraint on the mass current density and mass density. In our 1PN quasirigid body model most of the relations, such as the spin vector proportional to the angular velocity, the definition of the moment of inertia tensor, the key relation between the mass quadrupole moment and the moment of inertia tensor, the rigid rotating formulas for the mass quadrupole moment, and the moment of inertia tensor, are just an extension of the main relations in the Newtonian rigid body model. When all of the $1/c^2$ terms are neglected, the 1PN quasirigid body model and the corresponding formulas reduce to the Newtonian version. A key relation is obtained in this paper for the first time, which might be very useful in future applications to problems in geodynamics and astronomy.

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I. INTRODUCTION

The idea of a Newtonian rigid body has been used to treat the rotation of astronomical bodies up to now. For example, most Newtonian treatments of the Earth's rotation are based on an accurate rigid body theory (such as SMART 97 [1,2]) plus perturbative arguments from elasticity, the oceans, the atmosphere, the core and so on [3]. The idea of a rigid body in Newtonian theory is very powerful, introducing three principal axes of a body: the spin axis, rotation axis and figure axis without ambiguity. It makes the problem much simpler since there exists a key simple relation between the quadrupole moment and the moment of inertia tensor in a rigid body. But even in Newtonian theory the concept of a rigid body is only an ideal one, because there is no real rigid body in the physical world [4]. Owing to the modern high accuracy requirements, the Newtonian theory has to be replaced by Einstein's general relativity [at least its first post-Newtonian (1PN) approximation]. The problems of the post-Newtonian rigid body have been discussed ever since Born's kinematical rigidity (see Dixon's review [5]). Kinematical rigidity is dependent on the internal velocity distribution within the body while not considering the stress and energy flux contributions to the energy-momentum tensor $T^{\alpha\beta}$. Dixon [5], Thorne and Gürsel [6], Klioner [7–9], and Soffel [10] have a much better way, the so-called dynamical rigidity, in which the $T^{\alpha\beta}$ of the body and the gravitational field caused by the body satisfy a certain interdependency. The interdependency is not the same for different authors. However all of the different interdependencies make the 1PN spin S^i proportional to the angular velocity Ω^a and define a relativistic moment of inertia tensor. Certainly, the interdependency for the Newtonian rigid body just corresponds to a simple formula between mass density and current density [shown in Eq. (1.1)]. But none of them obtains the key simple relation between the 1PN quadrupole moment M_{ab} and the 1PN moment of inertia tensor I_{ab} like the Newtonian one shown in Eq. (1.2). Some even assert that such a key relation is invalid in general relativity [6,7]. Therefore the idea of 1PN rigidity has almost not been directly applied to practical problems up to now. We have a different opinion. We think that because no one has discovered a suitable interdependency inside the energy-momentum tensor and the gravitational field before, the key relation between the 1PN M_{ab} and the 1PN I_{ab} has not been found. In this paper we present a suitable new interdependency to obtain the 1PN I_{ab} and a key relation between the 1PN M_{ab} and the 1PN I_{ab} similar to the Newtonian one. This is the first time the rigidity problem has been solved on the post-Newtonian level. Recently we suggested another interdependency inside the energy-momentum tensor and gravitational field of the quasirigid body on the 1PN level by means of a special gauge condition [11], but the special gauge condition is more or less speculative and not commonly accepted. In this paper we totally discard the special gauge condition, and present a general expression for the rigid spin with or without external field (free precession).

First let us recall the basic aspects of the Newtonian rigid body. We take Σ and $\Sigma^a(=\Sigma V^a)$ as the mass density and the mass current density of a rigid body A respectively. Then the mass multipole moments M_L and spin S^a of body A are defined as $M_L=\int_A d^3X\Sigma\hat{X}^L$ and $S^a=\epsilon_{abc}\int_A d^3XX^b\Sigma^c$,

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where $\hat{X}^L$ is an abbreviation for $X^{\langle i_1}X^{i_2}\cdots X^{i_l\rangle}$, in which i_j ($j=1,2,\ldots,l$) is a spatial index, and the angular brackets mean "symmetrize and take the trace-free part" (STF). In a Newtonian rigid body the rotational angular velocity $\mathbf{\Omega}$ is independent of spatial coordinates; we have

$$\Sigma^a=\epsilon_{abc}\Sigma\Omega^b X^c. \tag{1.1}$$

Inserting Eq. (1.1) into the expression for spin, we have a linear relation between the spin and angular velocity $S^a=I_{ab}\Omega^b$, where the moment of inertia tensor I_{ab} is $I_{ab}=I_{ba}=\int_A d^3X\Sigma(X^2\delta_{ab}-X^aX^b)$.

The mass quadrupole moment and the moment of inertia tensor satisfy the key relation

$$M_{ab}=-I_{ab}+\frac{1}{3}\delta_{ab}I_{cc}. \tag{1.2}$$

By means of the continuity equation $\partial_T\Sigma+\partial_a\Sigma^a=0$, the time derivative of the moment of inertia tensor I_{ab} is proportional to the angular velocity $\mathbf{\Omega}$:

$$\dot{I}_{ab}\equiv\frac{d}{dT}I_{ab}=(\epsilon_{apq}I_{qb}+\epsilon_{bpq}I_{aq})\Omega^p. \tag{1.3}$$

$\dot{M}_{ab}$ satisfies a relation similar to Eq. (1.3). Therefore, I_{ab} and M_{ab}, like constant tensors, will rigidly rotate in space with the angular velocity $\mathbf{\Omega}$.

II. RIGID BLANCHET-DAMOUR (BD) MASS MOMENT AND RIGID PN SPIN

When we discuss 1PN rigidity, we will use the notation, symbols, and conventions following the 1PN theoretical framework presented by Damour, Soffel, and Xu (cited below as the DSX scheme [12–14]), since the DSX scheme is not only rather simple and complete but also describes the 1PN definition of spin in a satisfactory manner. In the DSX scheme a complete 1PN general relativistic celestial mechanics for N arbitrarily composed and shaped, rotating deformable bodies is described. Here we will briefly summarize the notation and definitions in the DSX scheme. In the post-Newtonian expansion we will always abbreviate the order symbol $O(c^{-n})$ simply as $O(n)$. A spatial multi-index containing l indices is simply denoted by L (and K for k indices, etc.), i.e. $L\equiv i_1i_2\cdots i_l$. A multisummation is always understood for repeated multi-indices $S_LT_L=\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_l}S_{i_1i_2\cdots i_l}T_{i_1i_2\cdots i_l}$. Given a spatial vector, n^i, its lth tensorial power is denoted by $n^L\equiv n^{i_1}n^{i_2}\cdots n^{i_l}$. Also, $\partial_L\equiv\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$. In addition to angular brackets the symmetric and trace-free part of a spatial tensor will be denoted by a caret when no ambiguity arises: $\mathrm{STF}_{i_1\cdots i_l}(T_{i_1}\cdots T_{i_l})\equiv T_{\langle i_1\cdots i_l\rangle}=\hat{T}_L$. The spatial indices $i,j=1,2,3$ are freely raised or lowered by means of the Cartesian metric $\delta_{ij}=\delta^{ij}=\mathrm{diag}(+1,+1,+1)$ in Cartesian coordinates. The metric is presented by means of the potential W and vector potential W_a [see Eq. (4.1) of Ref. [12]]. W and W_a can be separated into a self-part (with a "+") and an external part (with an overbar), i.e. $W=W^++\overline{W}$ and $W_a=W_a^++\overline{W}_a$. The self-part W^+ and W_a^+ will be solved from the gravitational mass density Σ and the mass current density Σ^a: $\Sigma\equiv(T^{00}+T^{ss})/c^2$, and $\Sigma^a\equiv T^{0a}/c$, through the 1PN Einstein field equation and the coordinate conditions (gauge conditions) [see Eq. (4.3) of Ref. [12]], where $T^{\alpha\beta}$ is the energy-momentum tensor. W^+ and W_a^+ will be expanded by the STF BD mass moments M_L and STF spin moments S_L [see Eq. (6.11) of Ref. [12]]. The external part $\overline{W}$ and $\overline{W}^a$ can be expanded in terms of the gravito-electric tidal moments G_L and gravito-magnetic tidal moments H_L [see Eq. (4.15) of Ref. [13]]. G_L and H_L are also STF spatial tensors dependent on time only. The BD mass moments [15] are widely accepted as the best 1PN mass moments and have the form

$$M_L^A(T)\equiv\int_A d^3X\hat{X}^L\Sigma+\frac{1}{2(2l+3)c^2}\frac{d^2}{dT^2}\left[\int_A d^3X\hat{X}^LX^2\Sigma\right]-\frac{4(2l+1)}{(l+1)(2l+3)c^2}\frac{d}{dT}\left[\int_A d^3X\hat{X}^{aL}\Sigma^a\right]\quad(l\geqslant0). \tag{2.1}$$

The 1PN spin moment has been discussed for a long time [14,16]. In the DSX scheme, the expression for the 1PN spin of body A [see Eq. (3.9) of [14]] is

$$S_a^{A,\mathrm{PN}}\equiv\epsilon_{abc}\int_A d^3XX^b\left[\Sigma^c\left(1+\frac{4}{c^2}W^A\right)-\frac{4}{c^2}\Sigma\left(W_c^{+A}+\frac{1}{8}\partial_c\partial_TZ_A^+\right)\right]+\frac{1}{c^2}\sum_{l\geqslant0}\frac{1}{l!}\left[\frac{1}{2l+3}H_{aL}^A\hat{N}_L^A-\frac{l}{l+1}M_{aL}^AH_L^A\right]$$
$$-\epsilon_{abc}\frac{1}{2c^2}\sum_{l\geqslant0}\frac{1}{l!(l+2)(2l+5)}[(l+10)\hat{N}_{bL}^A\dot{G}_{cL}^A+8(2l+3)\hat{P}_{bL}^AG_{cL}^A-(l+2)\dot{N}_{\langle bL\rangle}{}^AG_{cL}^A]+O(4), \tag{2.2}$$

where $Z_A^+\equiv G\int_A d^3X'\Sigma(T_A,\mathbf{X}')|\mathbf{X}-\mathbf{X}'|$, $W_c^{+A}\equiv G\int_A d^3X'\Sigma^c(T_A,\mathbf{X}')/|\mathbf{X}-\mathbf{X}'|$, the overdot means the time derivative ∂_T, $\hat{N}_L$ and $\hat{P}_L$ are defined as [see Eq. (2.10) of [14]] $\hat{N}_L\equiv\int_A d^3XX^2\hat{X}^L\Sigma$ and $\hat{P}_L\equiv\int_A d^3X\hat{X}^{aL}\Sigma^a$, respectively. Later we omit the body label, A, on all quantities. In Ref. [14], $\hat{N}_L$ and $\hat{P}_L$ are called "bad moments." With such a definition of the 1PN spin vector, S_a^{PN} satisfies the rotational equation of motion [see Eq. (3.11) of Ref. [14]]. We also have the 1PN continuity equation for (Σ,Σ^a) [see Eq. (5.6b) of Ref. [12]]

$$\partial_T\Sigma+\partial_a\Sigma^a=\frac{1}{c^2}(\partial_T T^{bb}-\Sigma\partial_T W)+O(4). \tag{2.3}$$

Those are the equations which are taken from the DSX scheme and will be used in the following discussion on 1PN rigidity.

The definition of the 1PN quasirigid body has to agree with the Newtonian rigid body when $1/c^2$ terms are neglected. In the 1PN quasirigid body the angular velocity should be independent of the local coordinate X^a of body A. A detail discussion of PN angular velocity will be carried out later after Eq. (3.2). In the DSX scheme we substitute Σ and Σ^c for the energy-momentum tensor $T^{\alpha\beta}$, therefore the interdependency inside the energy-momentum tensor and gravitational field in Refs. [6–8] might be replaced by the interdependency between Σ, Σ^c, and the gravitational field. We expect that the interdependency will produce equations similar to Eqs. (1.2) and (1.3) on the 1PN level. On the 1PN level it is sufficient to replace $T^{\alpha\beta}$ by Σ, Σ^c, and their derivatives [17]. Before a further discussion of the quasirigid body, the rigid BD moments and the rigid 1PN spin vector should be considered. Since $\hat{P}_L$ and $\hat{N}_L$ are 1PN terms in the discussion of the rigid 1PN spin, we can substitute the Newtonian relations [Eq. (1.1) and Newtonian continuity equation] for the definitions of $\hat{N}_L$ and $\hat{P}_L$. It is easy to prove that

$$P_{\langle L\rangle}=-\frac{1}{2l+1}\dot{N}_{\langle L\rangle}. \tag{2.4}$$

Lemma 1. The rigid BD mass moments of rigid body A [Eq. (2.1)] can be simplified to

$$M_L=\int_A d^3X X^{\langle L\rangle}\left[\Sigma+\frac{1}{c^2}\left(\frac{l+9}{2(l+1)(2l+3)}\right)X^2\ddot{\Sigma}\right]+O(4). \tag{2.5}$$

Proof. Beginning with Eq. (2.1) (the definition of the BD mass moment) and replacing the third term in the right-hand side of Eq. (2.1) by Eq. (2.4), the proof will be carried out directly. In fact only the relativistic quadrupole moment is interesting, because in the solar system all of the *relativistic* higher multipole moments are too small to be considered in any modern measurements anticipated these days. Therefore we take only the relativistic quadrupole moments into account:

$$M_{ab}=\int_A d^3X X^{\langle ab\rangle}\left(\Sigma+\frac{11}{42c^2}X^2\ddot{\Sigma}\right). \tag{2.6}$$

We define the 1PN part of the mass density $\Sigma_{\rm PN}\equiv\frac{11}{42}X^2\ddot{\Sigma}$ and the total mass density

$$\bar{\Sigma}\equiv\Sigma+\frac{\Sigma_{\rm PN}}{c^2}. \tag{2.7}$$

Lemma 2. The rigid 1PN spin vector of body A [see Eq. (2.2)] can be reduced to

$$S_a^{\rm PN}=\epsilon_{abc}\int_A d^3X X^b\left\{\Sigma^c+\frac{\Sigma}{c^2}\left(\frac{7}{2}\epsilon_{cde}\Omega^d\partial_e Z^+ +\frac{1}{2}\epsilon_{edf}\Omega^d X^f\partial_{ce}Z^+\right)+\frac{1}{c^2}\sum_{l\geqslant 0}\frac{\Sigma}{l!}\left[4\epsilon_{cde}\Omega^d X^e X^{\langle L\rangle}G_L(T)+\frac{1}{l+2}\epsilon_{ced}X^{\langle dL\rangle}H_{eL}\right.\right.$$
$$\left.\left.-\frac{l+10}{2(l+2)(2l+5)}\hat{X}^L X^2\dot{G}_{cL}+\frac{l+10}{2(l+2)(2l+5)}\partial_T(\ln\Sigma)\hat{X}^L X^2 G_{cL}\right]\right\}. \tag{2.8}$$

Substituting Eqs. (1.1) and (2.4) for the 1PN part of Eq. (2.2), integrating by parts, assuming the surface integration for the whole body A to be zero, and taking some STF formulas, we obtained Eq. (2.8) (the detailed calculations are shown in Appendix A).

We define the 1PN self-part and 1PN external part of the current density as

$$\Sigma^c_{\rm self}\equiv\Sigma\left(\frac{7}{2}\epsilon_{cde}\Omega^d\partial_e Z^+ +\frac{1}{2}\epsilon_{edf}\Omega^d X^f\partial_{ce}Z^+\right), \tag{2.9}$$

$$\Sigma^c_{\rm ext}\equiv\sum_{l\geqslant 0}\frac{\Sigma}{l!}\left[4\epsilon_{cde}\Omega^d X^e X^{\langle L\rangle}G_L(T)+\frac{1}{l+2}\epsilon_{ced}X^{\langle dL\rangle}H_{eL}-\frac{l+10}{2(l+2)(2l+5)}\hat{X}^L X^2\dot{G}_{cL}+\frac{l+10}{2(l+2)(2l+5)}\partial_T(\ln\Sigma)\hat{X}^L X^2 G_{cL}\right]. \tag{2.10}$$

Both $\Sigma^c_{\rm self}$ and $\Sigma^c_{\rm ext}$ as well as Σ^c itself are spatially compact supported. When the tidal moments G_L and H_L are equal to zero, Σ^c and $\Sigma^c_{\rm self}/c^2$ form the 1PN self-part of the spin vector. We can define

$$\overline{\Sigma^c} \equiv \Sigma^c + \frac{\Sigma^c_{\rm self}}{c^2} + \frac{\Sigma^c_{\rm ext}}{c^2}; \tag{2.11}$$

then Eq. (2.8) becomes

$$S^{\rm PN}_a = \epsilon_{abc} \int_A d^3X X^b \overline{\Sigma^c}. \tag{2.12}$$

Comparing the Newtonian definition of spin with Eq. (2.12), $\overline{\Sigma^c}$ is a fully 1PN quantity.

III. MODEL OF A POST-NEWTONIAN QUASIRIGID BODY

We add $(1/c^2)[\partial_T(\Sigma_{\rm PN}) + \partial_a(\Sigma^a_{\rm self} + \Sigma^a_{\rm ext})]$ to both sides of Eq. (2.3) and have

$$\partial_T \overline{\Sigma} + \partial_a \overline{\Sigma^a} = \frac{1}{c^2}[\partial_T T^{bb} - \Sigma \partial_T W + \partial_T \Sigma_{\rm PN} + \partial_a(\Sigma^a_{\rm self} + \Sigma^a_{\rm ext})]. \tag{3.1}$$

Now we construct a model of the 1PN quasirigid body by constraining $\overline{\Sigma^c}$ and $\overline{\Sigma}$ to satisfy

$$\overline{\Sigma^a} + \frac{1}{2c^2} X^a[\partial_T T^{bb} - \Sigma \partial_T W + \partial_T \Sigma_{\rm PN} + \partial_a(\Sigma^a_{\rm self} + \Sigma^a_{\rm ext})] = \epsilon_{abc}\Omega^b X^c \overline{\Sigma} + O(4). \tag{3.2}$$

The relation [Eq. (3.2)] is our most important assumption for the 1PN quasirigid body. In Eq. (3.2) $\mathbf{\Omega}$ is a parameter dependent on time. Because in the Newtonian approximation $\mathbf{\Omega}$ is the angular velocity, we call the parameter $\mathbf{\Omega}$ the angular velocity also. Really, in the 1PN approximation $\mathbf{v} = \mathbf{\Omega} \times \mathbf{X}$ is not valid. The velocity $\mathbf{v}$ has a rather complicated relation with Σ and Σ_a [see Eq. (2.27) of [17]]. Substituting the complicated relation into Eq. (3.2) we obtain a PN relation between the velocity $\mathbf{v}$ and the angular velocity $\mathbf{\Omega}$. Considering that $\overline{\Sigma^c}$ and $\overline{\Sigma}$ are expressed by Σ^c and Σ, which are related to $T^{\alpha\beta}$ in the DSX scheme, then $T^{\alpha\beta}$ is also constrained by Eq. (3.2). When the $1/c^2$ terms are neglected, Eq. (3.2) goes to Eq. (1.1). Later we will see that only in this model do the 1PN mass quadrupole moments and the moment of inertia tensors satisfy similar Newtonian key relations like Eq. (1.2). We were not surprised by the appearance of the time derivative of Σ in Eq. (3.2), since in the DSX scheme $T^{\alpha\beta}$ can be fully represented by Σ and Σ^a and their space and time derivatives without difficulty [17]. In the interdependencies described by Thorne and Gürsel [6] and Klioner [8], they have their own models of a rigid rotating body using another constraint on $T^{\alpha\beta}$. By comparing the constrained equations [see Eq. (A7) in [6] or Eq. (7) in [8] and Eq. (8) in [8]] with Eq. (3.2), we see that Eq. (3.2) is more complicated, but still reasonable.

Substituting Eq. (3.2) for Eq. (2.12), we obtain the linear relation between the 1PN spin vector of the quasirigid body and the angular velocity:

$$S^{\rm PN}_a = I_{ab}\Omega^b + O(4), \tag{3.3}$$

where the moment of inertia tensor is

$$I_{ab} = I_{ba} = \int_A d^3X(\delta_{ab}\mathbf{X}^2 - X^a X^b)\overline{\Sigma} + O(4), \tag{3.4}$$

in which $\overline{\Sigma}$ is defined in Eq. (2.7).

By comparing Eq. (3.4) with Eq. (2.6), we have

$$M_{ab} = -I_{ab} + \frac{1}{3}\delta_{ab}I_{cc} + O(4). \tag{3.5}$$

Equation (3.5) is the key relation between the 1PN mass quadrupole moment (rigid BD moment) and the 1PN moment of inertia tensor. It is just this relation that makes the model of the quasirigid body very useful and applicable on the 1PN level as shown in the Newtonian case. We have obtained in this paper, for the first time, the 1PN key relation. Making use of the extended 1PN continuity equation Eq. (3.1), we immediately have (detailed calculation in Appendix B)

$$\dot{I}_{ab} \equiv \frac{d}{dT}I_{ab} = (\epsilon_{apq}I_{qb} + \epsilon_{bpq}I_{aq})\Omega^p + O(4). \tag{3.6}$$

The 1PN $\dot{M}_{ab}$ satisfies a relation similar to Eq. (3.6). From Eq. (3.6) the behavior of the 1PN I_{ab} (and also 1PN M_{ab}) in our quasirigid model is just like the Newtonian version [Eq. (1.3)], i.e. I_{ab} and M_{ab} rigidly rotate as a whole. Since the 1PN higher mass multiple moments M_L $(L>2)$ do not satisfy a relation similar to Eq. (3.6), therefore our model is called a 1PN quasirigid body, but not a 1PN rigid body.

IV. DISCUSSION

Equation (3.6) means that we can always introduce a rotation matrix $P_{ia}(T)$, which is a time-dependent orthogonal matrix and transforms the PN reference system (RS) to a reference system corotating with the rigid body (RS^+). $P_{ia}(T)$ can be constructed using the rotational angular velocity Ω^a of the rigid body according to the relation $\Omega^a(T) = \frac{1}{2}\epsilon_{abc}P_{ib}(T)\dot{P}_{ic}(T)$ [18]. In the new corotating coordinates we get

$$\frac{d\tilde{I}_{ij}}{dT}=O(4), \tag{4.1}$$

where $\tilde{I}_{ij}=P_{ia}P_{jb}I_{ab}$. P_{ia} satisfies the following relations: $P_{ia}P_{ja}=\delta_{ij}$, $P_{ia}P_{ib}=\delta_{ab}$ and $dP_{ia}/dT=\epsilon_{abc}\Omega^{b}P_{ic}$ (here we use Ω^{b} to substitute for ω^{j} in [18]). The proof is easy by means of Eq. (3.6). Equation (4.1) shows it is possible to introduce the 1PN Tisserand reference system.

Finally, we should emphasize that the calculation of the 1PN moment of inertia tensor Eq. (3.4) is not too difficult, although the constrained relation on $\bar{\Sigma}^{c}$ and $\bar{\Sigma}$ in Eq. (3.2) in the model of the 1PN quasirigid body is complicated. In practical problems, from our 1PN rigid spin and the 1PN moment of inertia tensor [Eqs. (3.3)–(3.5)] it is possible to define the three principal axes of the body, the spin axis, rotation axis and figure axis as described in Newtonian theory, which we will discuss in a separate paper in the future.

In conclusion, the rigid BD (1PN) mass multipole moments Eq. (2.5) and the rigid 1PN spin moment Eq. (2.12) are discussed in this paper. We have successfully constructed a new 1PN model of a quasirigid body in which the constraint on $\bar{\Sigma}^{c}$ and $\bar{\Sigma}$ satisfies Eq. (3.2). Our 1PN quasirigid body model will reduce to the Newtonian one when all of the $1/c^{2}$ terms are neglected. Most of the relations in our 1PN rigid body model, such as the spin vector proportional to the angular velocity $\mathbf{\Omega}$ [Eq. (3.3)], the definition of the moment of inertia tensor [Eq. (3.4)], the key relation between the mass quadrupole moment and the moment of inertia tensor [Eq. (3.5)], the rigidly rotating formulas of I_{ab} and M_{ab} [see Eq. (3.6)] are similar to the Newtonian rigid body model where the corresponding relations are mentioned at the beginning of this paper. In particular, the 1PN key relation between M_{ab} and I_{ab} might be applied to practical problems in geodynamics and astronomy in the future, e.g. the discussion of the relativistic effects of nutation and precession.

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APPENDIX A: THE PROOF OF LEMMA 2

We start from Eq. (2.2):

$$\begin{aligned}S_a^{A,\mathrm{PN}}\equiv\epsilon_{abc}\int_A d^3X X^b\bigg[\Sigma^c\bigg(1+\frac{4}{c^2}W^A\bigg)\\&-\frac{4}{c^2}\Sigma\bigg(W_c^{+A}+\frac{1}{8}\partial_c\partial_T Z_A^+\bigg)\bigg]\\&+\frac{1}{c^2}\sum_{l\geqslant 0}\frac{1}{l!}\bigg[\frac{1}{2l+3}H_{aL}^A\hat{N}_L^A-\frac{l}{l+1}M_{aL}^A H_L^A\bigg]\\&-\epsilon_{abc}\frac{1}{2c^2}\sum_{l\geqslant 0}\frac{1}{l!(l+2)(2l+5)}\\&\times[(l+10)\hat{N}_{bL}^A\dot{G}_{cL}^A+8(2l+3)\hat{P}_{bL}^A G_{cL}^A\\&-(l+2)\dot{N}_{\langle bL\rangle}^A G_{cL}^A]+O(4).\end{aligned} \tag{A1}$$

In the following formulas we omit the body label, A, on all quantities.

Let us first rewrite the second term through the fourth term. Set

$$(E.1)\equiv\frac{1}{c^2}\epsilon_{abc}\int_A d^3X X^b\bigg(4\Sigma^c W-4\Sigma W^{+c}-\frac{1}{2}\Sigma\partial_c\partial_T Z^+\bigg), \tag{A2}$$

where $W=W^{+}+\bar{W}$, and W^{+c} and Z^{+} are defined in Eq. (2.2). Because in Eq. (A2) all the terms are 1PN terms, Eq. (1.1) can be substituted for Σ^{c}. We have

$$\frac{4}{c^2}W^{+c}\Sigma=\frac{4}{c^2}\epsilon_{cde}\Omega^d\Sigma(-\partial_e Z^+ + X^e W^+), \tag{A3}$$

$$\frac{4}{c^2}W\Sigma^c=\frac{4}{c^2}\epsilon_{cde}\Omega^d X^e\Sigma(\bar{W}+W^+), \tag{A4}$$

$$\begin{aligned}-\frac{\Sigma}{2c^2}\epsilon_{abc}\partial_c\partial_T Z^+=-\frac{\Sigma}{2c^2}\epsilon_{abc}(\epsilon_{ced}\Omega^e\partial_d Z^+\\&+\epsilon_{edf}\Omega^d X^f\partial_c\partial_e Z^+).\end{aligned} \tag{A5}$$

By combining Eqs. (A3)–(A5), we have

$$\begin{aligned}(E.1)=\epsilon_{abc}\int_A d^3X X^b\bigg[\frac{\Sigma}{c^2}\bigg(4\epsilon_{cde}\Omega^d X^e\bar{W}+\frac{7}{2}\epsilon_{cde}\Omega^d\partial_e Z^+\\&+\frac{1}{2}\epsilon_{edf}\Omega^d X^f\partial_c\partial_e Z^+\bigg)\bigg],\end{aligned} \tag{A6}$$

where $\bar{W}=\Sigma(1/l!)X^{\langle L\rangle}G_L+O(2)$ [see Eq. (4.15a) of Ref. [13]].

The fifth and sixth terms can be rewritten as

$$\sum_{l\geqslant 0}\frac{1}{l!}\left(\frac{1}{2l+3}\hat{N}_L H_{aL}-\frac{l}{l+1}M_{aL}H_L\right)=-\left\{\sum_{l\geqslant 1}\frac{1}{l!}\frac{l}{l+1}M_{ai_lL-1}H_{i_lL-1}+\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}\left(\frac{1}{(2l+1)(2l+3)}-\frac{l+1}{2l+1}\right)\hat{N}_L H_{aL}\right\}$$
$$=-\left\{\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}\left(M_{abL}H_{bL}+\frac{1}{(2l+1)(2l+3)}\hat{N}_L H_{aL}\right)-\sum_{l\geqslant 0}\frac{(l+1)^2}{(l+2)!(2l+1)}\hat{N}_L H_{aL}\right\}. \quad \text{(A7)}$$

We use the following identity:

$$\delta_{b\langle a}\hat{R}_{L\rangle}\hat{T}_{bL}=\frac{1}{(l+1)(2l+1)}\hat{R}_L\hat{T}_{aL}, \quad \text{(A8)}$$

where $\hat{R}_L$ and $\hat{T}_{aL}$ are arbitrary STF spatial tensors. Hence the second term of Eq. (A7)

$$\frac{1}{(2l+1)(2l+3)}\hat{N}_L H_{aL}=\frac{l+1}{2l+3}\,\delta_{b\langle a}N_{L\rangle}H_{bL}.$$

Therefore the first and second terms of Eq. (A7) can be written as

$$-\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}\left(M_{abL}H_{bL}+\frac{1}{(2l+3)(2l+1)}\hat{N}_L H_{aL}\right)$$
$$=-\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}H_{bL}\int d^3X\Sigma\left(X^{\langle abL\rangle}+\frac{l+1}{2l+3}\,\delta^{b\langle a}X^{L\rangle}X^2\right)$$
$$=-\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}H_{bL}\int d^3X\Sigma X^b X^{\langle aL\rangle}. \quad \text{(A9)}$$

In terms of another identity, $X^2X^{\langle L\rangle}=(2l+1)/(l+1)X^bX^{\langle bL\rangle}$, the third term of Eq. (A7) becomes

$$\sum_{l\geqslant 0}\frac{(l+1)^2}{(l+2)!(2l+1)}H_{aL}\hat{N}_L$$
$$=\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}H_{aL}\int d^3X\Sigma X^b X^{\langle bL\rangle}. \quad \text{(A10)}$$

By combining Eqs. (A9) and (A10), Eq. (A7) has the form

$$-\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}\left[H_{bL}\int_A d^3X\Sigma X^b X^{\langle aL\rangle}-H_{aL}\int_A d^3X\Sigma X^b X^{\langle bL\rangle}\right]$$
$$=-\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}(\delta_{ad}\delta_{be}-\delta_{ae}\delta_{bd})\times\left[H_{eL}\int_A d^3X\Sigma X^b X^{\langle dL\rangle}\right]$$
$$=\sum_{l\geqslant 0}\frac{l+1}{(l+2)!}H_{eL}\boldsymbol{\epsilon}_{abc}\boldsymbol{\epsilon}_{ced}\int_A d^3X\Sigma X^b X^{\langle dL\rangle}$$
$$=\boldsymbol{\epsilon}_{abc}\int_A d^3XX^b\sum_{l\geqslant 0}\frac{1}{l!}\frac{\Sigma}{l+2}\boldsymbol{\epsilon}_{ced}X^{\langle dL\rangle}H_{eL}. \quad \text{(A11)}$$

The seventh term in Eq. (A1) can be rewritten as

$$\frac{1}{2c^2}\boldsymbol{\epsilon}_{abc}\sum_{l\geqslant 0}\frac{l+10}{l!(l+2)(2l+5)}\hat{N}_{bL}\dot{G}_{cL}$$
$$=\frac{1}{2c^2}\boldsymbol{\epsilon}_{abc}\int_A d^3XX^b\Sigma\left(\sum_{l\geqslant 0}\frac{l+10}{l!(l+2)(2l+5)}\dot{G}_{cL}\hat{X}^L X^2\right), \quad \text{(A12)}$$

where in deducing Eq. (A12) we have considered two formulas:

$$X^{\langle bL\rangle}=X^bX^{\langle L\rangle}-\frac{l}{2l+1}X^2\delta^{b\langle a_l}X^{L-1\rangle} \quad \text{(A13)}$$

and

$$\boldsymbol{\epsilon}_{abc}\delta^{b\langle a_l}\hat{X}^{L-1\rangle}\dot{G}_{cL}=\boldsymbol{\epsilon}_{abc}X^{\langle L-1\rangle}\dot{G}_{cbL-1}=0. \quad \text{(A14)}$$

The last two terms of Eq. (A1) can be combined, because of Eq. (2.4),

$$-\frac{1}{2c^2}\epsilon_{abc}\sum_{l\geqslant 0}\frac{1}{l!(l+2)(2l+5)}[8(2l+3)P_{\langle bL\rangle}G_{cL}-(l+2)\dot{N}_{\langle bL\rangle}G_{cL}]$$

$$=\frac{1}{2c^2}\epsilon_{abc}\sum_{l\geqslant 0}\frac{l+10}{l!(l+2)(2l+5)}\dot{N}_{\langle bL\rangle}G_{cL}$$

$$=\frac{1}{2c^2}\epsilon_{abc}\int d^3XX^b\dot{\Sigma}\sum_{l\geqslant 0}\frac{l+10}{l!(l+2)(2l+5)}X^2X^{\langle L\rangle}G_{cL}. \tag{A15}$$

By adding Eqs. (A6), (A11), (A12) and (A15) together, Eq. (A1) for the rigid spin vector becomes

$$S_a^{\rm PN}=\epsilon_{abc}\int_A d^3XX^b\Bigg\{\Sigma^c+\frac{\Sigma}{c^2}\left(\frac{7}{2}\epsilon_{cde}\Omega^d\partial_e Z^+ +\frac{1}{2}\epsilon_{edf}\Omega^d X^f\partial_{ce}Z^+\right)+\frac{1}{c^2}\sum_{l\geqslant 0}\frac{\Sigma}{l!}\Bigg[4\epsilon_{cde}\Omega^d X^e X^{\langle L\rangle}G_L(T)+\frac{1}{l+2}\epsilon_{ced}X^{\langle dL\rangle}H_{eL}$$
$$-\frac{l+10}{2(l+2)(2l+5)}\hat{X}^L X^2\dot{G}_{cL}+\frac{l+10}{2(l+2)(2l+5)}\partial_T(\ln\Sigma)\hat{X}^L X^2 G_{cL}\Bigg]\Bigg\}. \tag{A16}$$

APPENDIX B: THE PROOF OF EQ. (3.6)

From the definition of I_{ab} [see Eq. (3.4)] we have

$$\dot{I}_{ab}=\int_A d^3x\frac{\partial\overline{\Sigma}}{\partial T}(\delta_{ab}X^2-X^aX^b)+O(4). \tag{B1}$$

By means of Eq. (3.1), Eq. (B1) becomes

$$\dot{I}_{ab}=\int_A d^3x\Bigg\{\Bigg[-\partial_d\overline{\Sigma}^d+\frac{1}{c^2}(\partial_T T^{dd}-\Sigma\partial_T W+\partial_T\Sigma_{\rm PN}$$
$$+\partial_d(\Sigma^d_{\rm self}+\Sigma^d_{\rm ext}))\Bigg](\delta_{ab}X^2-X^aX^b)\Bigg\}+O(4). \tag{B2}$$

Because of the surface integration equaling zero, the first part of the integration (B2) reduces to

$$\int_A d^3x\{-\partial_d\overline{\Sigma}^d(\delta_{ab}X^2-X^aX^b)\}$$
$$=\int_A d^3x\overline{\Sigma}^d(2\delta_{ab}X^d-\delta_{ad}X^b-\delta_{bd}X^a). \tag{B3}$$

From Eq. (3.2) we have

$$\overline{\Sigma}^d=\epsilon_{dfc}\Omega^f X^c\overline{\Sigma}-\frac{X^d}{2c^2}h, \tag{B4}$$

where $h=\partial_T T^{ee}-\Sigma\partial_T W+\partial_T\Sigma_{\rm PN}+\partial_e(\Sigma^e_{\rm self}+\Sigma^e_{ext})$.

Substituting Eq. (B4) into Eq. (B3) we get

$$({\rm B3})=\int_A d^3x\Bigg\{\overline{\Sigma}\epsilon_{dec}\Omega^e X^c(-\delta_{ad}X^b-\delta_{db}X^a)$$
$$-\frac{h}{c^2}(\delta_{ab}X^2-X^aX^a)\Bigg\}. \tag{B5}$$

Substituting Eq. (B5) into Eq. (B2) we find the second term in Eq. (B5) just by cancelling with the second part of Eq. (B2). Then Eq. (B2) becomes

$$\dot{I}_{ab}=\int_A d^3x\overline{\Sigma}(-\epsilon_{aec}X^cX^b-\epsilon_{bec}X^cX^a)\Omega^e$$
$$=(\epsilon_{aec}I_{cb}+\epsilon_{bec}I_{ac})\Omega^e+O(4), \tag{B6}$$

where the identity $\epsilon_{aec}\delta_{cb}X^2+\epsilon_{bec}\delta_{ca}X^2=0$ has been used in the last step.

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