Damping of tensor modes in cosmology

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An analytic formula is given for the traceless transverse part of the anisotropic stress tensor due to free streaming neutrinos, and used to derive an integro-differential equation for the propagation of cosmological gravitational waves. The solution shows that anisotropic stress reduces the squared amplitude by 35.6% for wavelengths that enter the horizon during the radiation-dominated phase, independent of any cosmological parameters. This decreases the tensor temperature and polarization correlation functions for these wavelengths by the same amount. The effect is less for wavelengths that enter the horizon at later times. At the longest wavelengths the decrease in the tensor correlation functions due to neutrino free streaming ranges from 10.7% for $\Omega_M h^2 = 0.1$ to 9.0% for $\Omega_M h^2 = 0.15$. An appendix gives a general proof that tensor as well as scalar modes satisfy a conservation law for perturbations outside the horizon, even when the anisotropic stress tensor is not negligible.

I. INTRODUCTION

It is widely expected that the observation of cosmological tensor fluctuations through measurements of the polarization of the microwave background may provide a uniquely valuable check on the validity of simple inflationary cosmologies. For instance, for a large class of inflationary theories with single scalar fields satisfying the "slow roll" approximation, the wave-number dependence $\mathcal{P}_S \propto k^{n_S-1}$ and $\mathcal{P}_T \propto k^{n_T}$ of the scalar and tensor power spectral functions and the ratio of these spectral functions after horizon exit during inflation are related by [1]

$$\mathcal{P}_T / \mathcal{P}_S = -n_T/2. \tag{1}$$

But in order to use observations to check such relations, we need to know what happens to the fluctuations between the time of inflation and the present. There is a very large literature on the scalar modes, but ever since the first calculations [2] of the production of tensor modes in inflation, with only one exception [3] known to me, the interaction of these modes with matter and radiation has simply been assumed to be negligible in studies of the cosmic microwave background [4]. It is not included in the widely used computer program of Seljak and Zaldarriaga [5]. As we shall see, the effect is not negligible even at the relatively low values of ℓ where the B-type polarization multipole coefficients $C_{B\ell}$ are likely to be first measured, and becomes quite significant for larger values of ℓ.

II. DAMPING EFFECTS IN THE WAVE EQUATION

The interaction of tensor modes with matter and radiation vanishes in the case of perfect fluids, but not in the presence of traceless transverse terms in the anisotropic stress tensor. In general, the tensor fluctuation satisfies

$$\ddot{h}_{ij} + \left(\frac{3\dot{a}}{a}\right)\dot{h}_{ij} - \left(\frac{\nabla^2}{a^2}\right)h_{ij} = 16\pi G\pi_{ij}, \tag{2}$$

where dots indicate ordinary time derivatives. Here the components of the perturbed metric are

$$g_{00} = -1, \quad g_{i0} = 0, \quad g_{ij}(\mathbf{x},t) = a^2(t)[\delta_{ij} + h_{ij}(\mathbf{x},t)] \tag{3}$$

where $h_{ij}(\mathbf{x},t)$ is treated as a small perturbation; and $\pi_{ij}(\mathbf{x},t)$ is the anisotropic part of the stress tensor, defined by writing the spatial part of the perturbed energy-momentum tensor as $T_{ij} = \bar{p}g_{ij} + a^2\pi_{ij}$, or equivalently

$$T^i{}_j = \bar{p}\,\delta_{ij} + \pi_{ij}, \tag{4}$$

where $\bar{p}$ is the unperturbed pressure. In these formulas we are considering only tensor perturbations, so that

$$h_{ii} = 0, \quad \partial_i h_{ij} = 0, \quad \pi_{ii} = 0, \quad \partial_i \pi_{ij} = 0. \tag{5}$$

For a perfect fluid $\pi_{ij} = 0$, but this is not true in general. For instance, in any imperfect fluid with shear viscosity η, we have [6] $\pi_{ij} = -\eta\dot{h}_{ij}$. Nevertheless, as we shall show in the Appendix, even where hydrodynamic approximations are inapplicable, h_{ij} becomes time-independent as the wavelength of a mode leaves the horizon, and remains time-independent until horizon re-entry. All modes of cosmological interest are still far outside the horizon at the temperature $\approx 10^{10}$ K, where neutrinos are going out of equilibrium with electrons and photons, so h_{ij} *can be effected by anisotropic inertia only later, when neutrinos are freely streaming.*

We can calculate the contribution of freely streaming neutrinos to π_{ij} exactly [7]. We define a density $n(\mathbf{x},\mathbf{p},t)$ as

$$n(\mathbf{x},\mathbf{p},t) \equiv \sum_r \left(\prod_{i=1}^{3} \delta[(x^i - x_r^i(t)]\right)\left(\prod_{i=1}^{3} \delta[p_i - p_{ri}(t)]\right), \tag{6}$$

with r labeling individual neutrino and antineutrino trajectories. The relativistic equations of motion in phase space for any metric with $g_{00} = -1$ and $g_{i0} = 0$ are

$$\dot{x}_r^i = \frac{p_r^i}{p_r^0}, \quad \dot{p}_{ri} = \frac{p_r^j p_r^k}{2p_r^0} \left(\frac{\partial g_{jk}}{\partial x^i} \right)_{x=x_r}. \tag{7}$$

It then follows that n satisfies a Boltzmann equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x^i} \frac{p^i}{p^0} + \frac{\partial n}{\partial p_i} \frac{p^j p^k}{2p^0} \frac{\partial g_{jk}}{\partial x^i} = 0, \tag{8}$$

it being understood that p^i and p^0 are expressed in terms of the independent variable p_i by $p^i = g^{ij} p_j$ and $p^0 = (g^{ij} p_i p_j)^{1/2}$. At a time t_1 soon after neutrinos started free streaming, n had the ideal gas form (assuming zero chemical potentials)

$$n(\mathbf{x}, \mathbf{p}, t_1) = \frac{N}{(2\pi)^3} \{ \exp[\sqrt{g^{ij}(\mathbf{x}, t_1) p_i p_j}/k_B T_1] + 1 \}^{-1}$$

$$\equiv n_1(\mathbf{x}, \mathbf{p}), \tag{9}$$

where N is the number of types of neutrinos, counting antineutrinos separately, and k_B is Boltzmann's constant. We therefore write

$$n(\mathbf{x}, \mathbf{p}, t) = n_1(\mathbf{x}, \mathbf{p}) + \delta n(\mathbf{x}, \mathbf{p}, t) \tag{10}$$

so that δn vanishes for $t = t_1$.

In the absence of metric perturbations, Eq. (8) and the initial condition (9) have the solution $n(\mathbf{p}) = \bar{n}(p)$, where $\bar{n}(p)$ is the zeroth-order part of n_1:

$$\bar{n}(p) = \frac{N}{(2\pi)^3} [\exp(p/k_B T_1 a_1) + 1]^{-1}, \tag{11}$$

and $p \equiv \sqrt{p_i p_i}$. To first order in metric perturbations, Eq. (8) gives

$$\frac{\partial \delta n(\mathbf{x}, \mathbf{p}, t)}{\partial t} + \frac{\hat{p}_i}{a(t)} \frac{\partial \delta n(\mathbf{x}, \mathbf{p}, t)}{\partial x^i}$$

$$= -\frac{p}{2a(t)} \bar{n}'(p) \hat{p}_i \hat{p}_j \hat{p}_k \frac{\partial}{\partial x^k} [h_{ij}(\mathbf{x}, t) - h_{ij}(\mathbf{x}, t_1)], \tag{12}$$

where hats denote unit vectors. [In putting the Boltzmann equation in this form, we use that fact that n_1 depends on $\mathbf{x}$ and p_i only through the combination $g^{ij}(\mathbf{x}, t_1) p_i p_j$, so that to first order $\partial_k n_1(\mathbf{x}, \mathbf{p}) = -p \bar{n}'(p) \hat{p}_i \hat{p}_j \partial_k h_{ij}(\mathbf{x}, t_1)$.]

We now suppose that the $\mathbf{x}$ dependence of $h_{ij}(\mathbf{x}, t)$ is contained in a factor $\exp(i\mathbf{k} \cdot \mathbf{x})$, where $\mathbf{k}$ is a co-moving wave number.[1] Equation (12) and the initial condition that $\delta n = 0$ at $t = t_1$ then have the solution

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[1]Conventionally the comoving coordinate $\mathbf{x}$ and wave number $\mathbf{k}$ are normalized by defining $a(t)$ so that $a = 1$ at present. Here we will leave this normalization arbitrary.

$$\delta n(\mathbf{p}, u) = -\frac{i}{2} p \bar{n}'(p) \hat{p} \cdot \hat{k} \hat{p}_i \hat{p}_j \int_0^u du' \, e^{i\hat{p} \cdot \hat{k}(u' - u)} [h_{ij}(u')$$

$$- h_{ij}(0)], \tag{13}$$

where we now drop the position argument, and write δn and h_{ij} as functions of a variable u instead of t, with u defined as the wave number times the conformal time

$$u \equiv k \int_{t_1}^t \frac{dt'}{a(t')}. \tag{14}$$

The space part of the neutrino energy-momentum tensor is given by

$$T_{\nu j}^i = \frac{1}{\sqrt{\text{Det}g}} \sum_r \frac{p_r^i p_{rj}}{p_r^0} \delta^3(\mathbf{x} - \mathbf{x}_r)$$

$$= \frac{1}{\sqrt{\text{Det}g}} \int \left(\prod_{k=1}^3 dp_k \right) \frac{n p^i p_j}{p^0}. \tag{15}$$

This yields terms of first order in $h_{ij}(u)$ from $p^i = g^{ij} p_j$ and $p^0 = \sqrt{g^{ij} p_i p_j}$, a term of first order in $h_{ij}(0)$ from the term n_1 in n, and a term of first order in $h_{ij}(u) - h_{ij}(0)$ from δn. Collecting all these terms and using Eq. (5) yields a surprisingly simple formula for π_{ij}:

$$\pi_{ij}(u) = -4\bar{\rho}_\nu(u) \int_0^u K(u - U) h_{ij}'(U) dU, \tag{16}$$

where primes now indicate derivatives with respect to U or u; K is the kernel

$$K(s) \equiv \frac{1}{16} \int_{-1}^{+1} dx (1 - x^2)^2 e^{ixs} = -\frac{\sin s}{s^3} - \frac{3\cos s}{s^4} + \frac{3\sin s}{s^5}, \tag{17}$$

and $\bar{\rho}_\nu = a^{-4} \int d^3 p \, p \bar{n}(p)$ is the unperturbed neutrino energy density.

To continue, we use Eq. (16) in Eq. (2) and express time derivatives in terms of u-derivatives. This gives an integro-differential equation for $h_{ij}(u)$ [8]:

$$h_{ij}''(u) + \frac{2a'(u)}{a(u)} h_{ij}'(u) + h_{ij}(u)$$

$$= -24 f_\nu(u) \left(\frac{a'(u)}{a(u)} \right)^2 \int_0^u K(u - U) h_{ij}'(U) dU, \tag{18}$$

where $f_\nu \equiv \bar{\rho}_\nu / \bar{\rho}$.

We took the initial time t_1 to be soon after neutrinos started free streaming, so interesting perturbations are outside the horizon then, and for some time after. As we show in the Appendix, h_{ij} rapidly became time independent after horizon exit, and remained so until horizon re-entry. In terms of u, we then have the initial condition

$$h'_{ij}(0) = 0. \tag{19}$$

The solution of Eq. (18) can therefore be put in the general form

$$h_{ij}(u) = h_{ij}(0)\chi(u), \tag{20}$$

where $\chi(u)$ satisfies the same integro-differential equation as $h_{ij}(u)$

$$\chi''(u) + \frac{2a'(u)}{a(u)}\chi'(u) + \chi(u)$$

$$= -24f_\nu(u)\left(\frac{a'(u)}{a(u)}\right)^2 \int_0^u K(u-U)\chi'(U)dU,$$

$$\tag{21}$$

and the initial conditions

$$\chi(0) = 1, \quad \chi'(0) = 0. \tag{22}$$

III. SHORT WAVELENGTHS

We will first consider wavelengths short enough to have re-entered the horizon during the radiation-dominated era (though long after neutrino decoupling), and then turn to the general case in Sec. IV. We can take the initial time t_1 to be early enough so that it can be approximated as $t_1 \simeq 0$, with the zero of time defined so that during the radiation-dominated era we have $a \propto \sqrt{t}$. Then in Eq. (21) we can set $a'/a = 1/u$, while for 3 neutrino flavors f_ν takes the constant value $f_\nu(0) = 0.40523$. Then Eq. (21) becomes

$$\chi''(u) + \frac{2}{u}\chi'(u) + \chi(u) = -\frac{24f_\nu(0)}{u^2}\int_0^u K(u-U)\chi'(U)dU. \tag{23}$$

Because of the decrease of the factor $1/u^2$, the right-hand of Eq. (23) becomes negligible for $u \gg 1$, so deep inside the horizon the solution of Eqs. (22) and (23) approaches a homogeneous solution

$$\chi(u) \rightarrow A\sin(u+\delta)/u \tag{24}$$

as compared with the solution $\sin(u)/u$ for $f_\nu = 0$. A numerical solution of Eqs. (22) and (23) shows that $\chi(u)$ follows the $f_\nu = 0$ solution pretty accurately until $u \approx 1$, when the perturbation enters the horizon, and thereafter rapidly approaches the asymptotic form (24), with $A = 0.8026$ and δ very small. This asymptotic form provides the initial condition for the later period when the matter energy density becomes first comparable to and then greater than that of radiation, so the effect of neutrino damping at these later times is still only to reduce the tensor amplitude by the same factor $A = 0.8026$. Hence, for wavelengths that enter the horizon after electron–positron annihilation and well before radiation-matter equality, all quadratic effects of the tensor modes in the cosmic microwave background, such as the tensor contribution to the temperature multipole coefficients C_ℓ and the whole of the "B-B" polarization multipole coef-

ficients $C_{\ell B}$, are 35.6% less than they would be without the damping due to free-streaming neutrinos. (Photons also contribute to π_{ij}, but this effect is much smaller because at last scattering photons contribute much less than 40% of the total energy.)

IV. GENERAL WAVELENGTHS

To deal with perturbations that may enter the horizon after the matter energy density has become important, let us switch the independent variable from u to $y \equiv a(t)/a_{EQ}$, where a_{EQ} is $a(t)$ at the time t_{EQ} of radiation-matter equality. To see how they are related, note that

$$\frac{dy}{du} = \frac{\dot{a}}{a_{EQ}k/a} = \frac{a^2}{a_{EQ}k}H_0\sqrt{\Omega_M\left(\frac{a_0}{a}\right)^3 + (\Omega_\gamma + \Omega_\nu)\left(\frac{a_0}{a}\right)^4}. \tag{25}$$

The redshift of matter-radiation equality is given by $1 + z_{EQ} = a_0/a_{EQ} = \Omega_M/(\Omega_\gamma + \Omega_\nu)$, so Eq. (25) can be simplified to read

$$\frac{du}{dy} = \frac{Q}{\sqrt{1+y}}, \tag{26}$$

where

$$Q \equiv \frac{k}{a_0 H_0\sqrt{\Omega_M(1+z_{EQ})}}. \tag{27}$$

Since $u \rightarrow 0$ for $y \rightarrow 0$, the solution of Eq. (26) is

$$u = 2Q(\sqrt{1+y} - 1). \tag{28}$$

The Hubble constant at matter-radiation equality has the value $H_{EQ} = H_0\sqrt{2\Omega_M(1+z_{EQ})^3}$, so Eq. (28) can be written

$$Q = \sqrt{2}k/k_{EQ}, \tag{29}$$

where $k_{EQ} \equiv a_{EQ}H_{EQ}$ is the wave number of perturbations that just enter the horizon at the time of radiation-matter equality. (Hence in particular the results of the previous section apply for $Q \gg 1$.)

The fraction of the total energy density in neutrinos is well known,

$$f_\nu(y) = \frac{\Omega_\nu(a_0/a)^4}{\Omega_M(a_0/a)^3 + (\Omega_\gamma + \Omega_\nu)(a_0/a)^4} = \frac{f_\nu(0)}{1+y}, \tag{30}$$

where

$$f_\nu(0) = \frac{\Omega_\nu}{\Omega_\nu + \Omega_\gamma} = 0.40523. \tag{31}$$

A little algebra then lets us put Eq. (21) in the form

$$(1+y)\frac{d^2\chi(y)}{dy^2}+\left(\frac{2(1+y)}{y}+\frac{1}{2}\right)\frac{d\chi(y)}{dy}+Q^2\chi(y)$$

$$=-\frac{24f_\nu(0)}{y^2}\int_0^y K(y,y')\frac{d\chi(y')}{dy'}dy',\tag{32}$$

where $K(y,y')$ is the same as the $K(s)$ given by Eq. (17), but with s now given by

$$s\equiv u-u'=2Q(\sqrt{1+y}-\sqrt{1+y'}).\tag{33}$$

The initial conditions (22) now read

$$\chi(0)=\left.\frac{d\chi(y)}{dy}\right|_{y=0}=0.\tag{34}$$

We now have to face the complication that for general Q the value of y at last scattering is not in an asymptotic region where the effect of anisotropic inertia is simply to damp $\chi(t)$ by some constant factor. We therefore now have to consider what feature of $\chi(t)$ is related to observations of the cosmic microwave background. It is $\dot\chi$ that enters into the Boltzmann equation for perturbations to the temperature and Stokes parameters [9], so in the approximation of a sudden transition from opacity to transparency, we expect all tensor multipole coefficients to depend on $\chi(y)$ only through a factor $|\chi'(y_L)|^2$, where $y_L=(1+z_{EQ})/(1+z_L)$ is the value of y at last scattering. Hence we will be primarily interested in calculating the value of $|\chi'(y_L)|^2$ for various values of Q, and comparing these values with what they would be in the absence of anisotropic inertia.

For $T_{\gamma0}=2.725\,$K, we have $\Omega_\gamma+\Omega_\nu=4.15\times10^{-5}h^{-2}$, so, taking $1+z_L=1090$, the parameter y_L is

$$y_L=22.1\Omega_M h^2.$$

It will be useful also to have an idea of the value of ℓ for which the multipole coefficients in various correlation functions are dominated by perturbations with a given Q. The dominant contribution to a multipole coefficient of order ℓ comes from wave numbers $k\simeq a_L\ell/d_L$, where a_L is $a(t)$ at the time of last scattering, and d_L is the angular diameter distance of the surface of last scattering, which for flat geometries is

$$d_L=\frac{1}{H_0(1+z_L)}\int_{1/(1+z_L)}^1\frac{dx}{\sqrt{\Omega_M x+(1-\Omega_M)x^4}},$$

where z_L is the redshift of last scattering. Thus the multipole order that receives its main contribution from wavelengths that are just coming into the horizon at matter-radiation equality is

$$\ell_{EQ}\equiv\frac{d_L k_{EQ}}{a_L}$$

$$=\sqrt{2\Omega_M(1+z_{EQ})}\int_{1/(1+z_L)}^1\frac{dx}{\sqrt{\Omega_M x+(1-\Omega_M)x^4}},\tag{35}$$

where z_{EQ} is the redshift of matter-radiation equality. For present radiation temperature $T_{\gamma0}=2.725\,$K and $\Omega_M h^2=0.15$ this redshift is $z_{EQ}=3613$. If also $\Omega_M=0.3$ and $1+z_L=1090$ then the integral in Eq. (35) has the value 3.195, and so Eq. (35) gives $\ell_{EQ}=149$. Hence for these cosmological parameters, Eq. (29) gives

$$Q=\sqrt{2}\frac{\ell}{\ell_{EQ}}\simeq\frac{\ell}{105}.$$

When referring below to specific values of ℓ, it will always be understood that the conversion from Q to ℓ has been made using these cosmological parameters, but it should be kept in mind that the dependence of the function $\chi(y)$ on y and Q is independent of cosmological parameters, and that the value of y at last scattering depends only on $T_{\gamma0}$, $1+z_L$, and $\Omega_M h^2$, not on Ω_M or Ω_{vac}.

Let us first consider the case $Q\ll1$, which for the above cosmological parameters corresponds to $\ell\ll100$. Here the kernel $K(y,y')$ has the constant value 1/15, and Eqs. (32) and (34) have a solution of the form

$$\chi(y)\rightarrow1-Q^2g(y)\quad\text{for }Q\rightarrow0\tag{36}$$

where $g(y)$ is independent of Q, and satisfies the inhomogeneous differential equation

$$(1+y)\frac{d^2g(y)}{dy^2}+\left(\frac{2(1+y)}{y}+\frac{1}{2}\right)\frac{dg(y)}{dy}+\frac{8f_\nu(0)}{5y^2}g(y)=1\tag{37}$$

and the initial conditions

$$g(0)=g'(0)=0.\tag{38}$$

According to the above discussion, the streaming of free neutrinos damps the various tensor correlation functions of the cosmic microwave background by a factor $|\chi'(y_L)/\chi_0'(y_L)|^2$, which for $Q\ll1$ becomes $|g'(y_L)/g_0'(y_L)|^2$, the subscript 0 denoting quantities calculated ignoring this damping, i.e., for $f_\nu=0$, and y_L again equal to the ratio of $a(t)$ at last scattering to that at matter-radiation equality. Numerical solutions of Eqs. (37) and (38) for $f_\nu(0)=0.40523$ and for $f_\nu=0$ show that the damping factor $|g'(y_L)/g_0'(y_L)|^2$ is very close to a linear function of y_L and hence of $\Omega_M h^2$ for observationally favored values of $\Omega_M h^2$, increasing from 0.893 at $\Omega_M h^2=0.10$ to 0.910 for $\Omega_M h^2=0.15$.

This damping is relatively insensitive to Q for small Q. For instance, numerical integration of Eqs. (32) and (34)

shows that for $\Omega_M h^2 = 0.15$, the damping has only decreased from 9% to 8% for $Q = 0.55$ ($\ell \simeq 58$), and to 7% for $Q = 0.8$ ($\ell \simeq 84$). Matters are more complicated for larger values of Q and ℓ, because the damping factor $|\chi'(y_L)/\chi_0'(y_L)|^2$ is the ratio of two oscillating functions with slightly different phases, so that the plot of $|\chi'(y_L)/\chi_0'(y_L)|^2$ vs Q shows narrow spikes: this ratio becomes infinite at values of Q for which $\chi_0'(y_L)$ vanishes and then almost immediately drops to zero at the slightly larger value of Q for which $\chi'(y_L)$ vanishes. [Even if we average over the small range of y values over which last scattering occurs, the plot of $\langle |\chi'(y_L)/\chi_0'(y_L)|^2 \rangle$ vs Q still shows finite though high narrow spikes at the zeroes of $\chi_0'(y_L)$.] These spikes are not particularly interesting, because they occur at values of Q where $\chi'(y_L)$ is particularly small, so that the multipole coefficients in the various tensor temperature and polarization correlation functions will be very difficult to measure for the corresponding values of ℓ. The values of $|\chi'(y_L)/\chi_0'(y_L)|^2$ in the relatively flat regions between the spikes steadily decreases from the value $\simeq 0.9$ for $Q \ll 1$ to a value close to the result 0.644 found in the previous section for $Q \simeq 10$.

The effects considered in this paper will doubtless eventually be taken into account in the computer programs used to analyze data from PLANCK and other future facilities. In the meanwhile, the planning of future observations should take into account that the damping of cosmological gravitational waves is not negligible.

Notes added in proof. After the preprint of this paper was circulated, the damping of tensor modes was included in the computer program described in Ref. [5]. Because the anisotropic part of the stress tensor is gauge invariant, it vanishes outside the horizon in the scalar and tensor modes given, respectively, by Eqs. (A3) and (A4). It follows that for the scalar mode given by Eq. (A3), the general solution of Eq. (A5) is $\epsilon = (\mathcal{R}/a) \int a \, dt$.

APPENDIX: SUPERHORIZON CONSERVATION LAWS

This appendix will prove a result quoted in Sec. II, that in all cases there is a tensor mode whose amplitude remains constant outside the horizon, even where some particles may have mean free times comparable to the Hubble time. The argument is similar to one used previously to show the existence under very general conditions of two scalar modes for which a quantity $\mathcal{R}$ is constant outside the horizon [10]. It is based on the observation that for zero wave number the

Newtonian gauge field equations and the dynamical equations for matter and radiation as well as the condition $k = 0$ are invariant under coordinate transformations that are *not* symmetries of the unperturbed metric.[2] The most general such transformations are

$$x^0 \to x^0 + \epsilon(t), \quad x^i \to \left(\delta_{ij} - \frac{1}{2} \omega_{ij} \right) x^j, \quad \text{(A1)}$$

where $H \equiv \dot{a}/a$, $\epsilon(t)$ is an arbitrary function of time, and $\omega_{ij} = \omega_{ji}$ is an arbitrary constant matrix. Under these conditions we have something like a Goldstone theorem: since the metric satisfies the field equations both before and after the transformation, the change in the metric under these transformations must also satisfy the field equations. This change is simply

$$\delta g_{00} = \dot{\epsilon}(t), \quad \delta g_{i0} = 0, \quad \delta g_{ij} = a^2(t)[-H(t)\epsilon(t)\delta_{ij} + \omega_{ij}]. \quad \text{(A2)}$$

This means that for zero wave number we always have a solution with scalar modes

$$\Psi = H\epsilon - \omega_{ii}/3, \quad \Phi = -\dot{\epsilon} \quad \text{(A3)}$$

and a tensor mode

$$h_{ij} = \omega_{ij} - \frac{1}{3} \delta_{ij} \omega_{kk}. \quad \text{(A4)}$$

(The notation for Φ and Ψ is standard, and the same as in Ref. [10].) These are just gauge modes for zero wave number, but if they can be extended to non-zero wave number they become physical modes, since transformations (A1) are not symmetries of the field equations except for zero wave number. For the scalar modes there are field equations that disappear in the limit of zero wave number, so that conditions $\Phi = \Psi - 8\pi G \pi_S$ and $\delta u = \epsilon$ (where π_S is the scalar part of the anisotropic inertia, called σ in Ref. [10]) and δu is the perturbation to the velocity potential) must be imposed on solutions (A3) for them to have an extension to non-zero wave number. It follows then that the zero wave number scalar modes that become physical for non-zero wave number satisfy

$$\dot{\epsilon} = -H\epsilon + \omega_{kk}/3 - 8\pi G \pi_S, \quad \delta u = \epsilon. \quad \text{(A5)}$$

Then for zero wave number the quantity $\mathcal{R} \equiv -\Psi + h\,\delta u$ has the time-independent value

$$\mathcal{R} = \omega_{kk}/3. \quad \text{(A6)}$$

For tensor modes there are no field equations that disappear for zero wave number, so the solution $h_{ij} = \text{constant}$ automatically has an extension to a physical mode for a nonzero wave number.

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[2] In this respect, the theorem proved here is similar to the Goldstone theorem [11] of quantum field theory. The modes for which $\mathcal{R}$ or h_{ij} are constant outside the horizon take the place here of the Goldstone bosons that become free particles for long wavelengths.

As examples, we note that both the anisotropic stress tensor $\pi_{ij} = -\eta \dot{h}_{ij}$ for an imperfect fluid with shear viscosity η and tensor (16) for freely streaming neutrinos vanish for $\dot{h}_{ij} = 0$, so in the limit of zero wave numbers Eq. (2) has a solution with $\dot{h}_{ij} = 0$. The above theorem shows that this result applies even when some particle's mean free time is comparable with the Hubble time, in which case neither the hydrodynamic nor the free-streaming approximations are applicable.

The solution with $\dot{h}_{ij} = 0$ for zero wave number is not the only solution, but the other solutions decay rapidly after horizon exit. There is no anisotropic inertia in scalar field theories, and in the absence of anisotropic inertia, Eq. (2) for zero wave number has two solutions, one with h_{ij} a constant, and the other with $\dot{h}_{ij} \propto a^{-3}$, for which h_{ij} rapidly becomes a constant. The energy-momentum tensor of the universe departs from the perfect fluid form later, during neutrino decoupling, and perhaps also during reheating or periods of baryon or lepton nonconservation, but during all these epochs cosmologically interesting tensor fluctuations are far outside the horizon, and hence remain constant.

[1] A. Starobinsky, Sov. Astron. Lett. **11**, 133 (1985); E.D. Stewart and D.H. Lyth, Phys. Lett. B **302**, 171 (1993).

[2] V.A. Rubakov, M. Sazhin, and A. Veryaskin, Phys. Lett. **115B**, 189 (1982); R. Fabbri and M.D. Pollock, *ibid.* **125B**, 445 (1983); L.F. Abbott and M.B. Wise, Nucl. Phys. **B244**, 541 (1984); A.A. Starobinskii, Sov. Astron. Lett. **11**, 133 (1985).

[3] The effects of anisotropic inertia due to both neutrinos and photons were included in a large program of numerical calculations reported by J. R. Bond, in *Cosmology and Large Scale Structure*, *Les Houches Session LX*, edited by R. Schaeffer, J. Silk, and J. Zinn-Justin (Elsevier, Amsterdam, 1996). Bond concluded from the numerical results that there is an average "~20%" reduction of the squared tensor amplitude for multipole order ℓ larger than about 100, and that this would not be observable in measurements of C_ℓ because according to Eq. (1) tensor modes already make a much smaller contribution to C_ℓ than scalar modes. It is the prospect of cosmic microwave background polarization measurements that makes the effect of anisotropic inertia on the tensor amplitude important.

[4] See, e.g., V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger, Phys. Rep. **215**, 203 (1992); M.S. Turner, M. White, and J.E. Lidsey, Phys. Rev. D **48**, 4613 (1993); M.S. Turner, *ibid.* **55**, 435 (1997); D.J. Schwarz, Ann. Phys. (Leipzig) **12**, 220 (2003).

[5] U. Seljak and M. Zaldarriaga, Astrophys. J. **469**, 437 (1996).

[6] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Eq. (15.10.39). (It should be noted that h_{ij} as defined in this reference is a^2 times the h_{ij} used in the present work.) For $k/a \gg \dot{a}/a$, this formula for π gives the damping of gravitational waves that had been calculated by S.W. Hawking, Astrophys. J. **145**, 544 (1966).

[7] Differential equations for both the scalar and tensor parts of the anisotropic stress tensor were given by J.R. Bond and A.S. Szalay, Astrophys. J. **274**, 443 (1983), using an orthonormal basis instead of the coordinate basis used here, but the result was applied only for the scalar modes.

[8] For perturbations outside the horizon, where $z \ll 1$, we can replace $K(z-y)$ with $K(0) = 1/15$, and the integral in Eq. (8) becomes just $[h_{ij}(z) - h_{ij}(0)]/15$. Aside from the term $h_{ij}(0)$, this equation in the radiation-dominated case is then equivalent to Eq. (4.3) of C.W. Misner, Astrophys. J. **151**, 431 (1968), which was derived to study a phenomenon different from that considered here: the approach to isotropy of a homogeneous anisotropic cosmology. [This equation was generalized to the case of finite mean free times by C. Misner and R. Matzner, *ibid.* **171**, 415 (1972).] Misner took $h_{ij}(0) = 0$ [but $h'_{ij}(0) \neq 0$], on the ground that a constant term in h_{ij} could be made to vanish by a coordinate transformation, and found a decaying solution. But a constant term in h_{ij} is only a gauge mode when k is strictly zero. As remarked in the Appendix, the existence of this gauge mode means that there is a *physical* mode with $k \neq 0$ for which h_{ij} becomes constant outside the horizon, where k is negligible, but which becomes time-dependent when the wavelength re-enters the horizon. [After the preprint of this work was circulated, I learned of an article by A.K. Rebhan and D.J. Schwarz, Phys. Rev. D **50**, 2541 (1994), which obtained an integro-differential equation like Eq. (18), but with extra terms representing more general initial conditions. No attempt was made to identify the initial conditions that would actually apply cosmologically, or to calculate the damping effect relevant to the cosmic microwave background.]

[9] See, e.g., M. Zaldarriaga and U. Seljak, Phys. Rev. D **55**, 1830 (1997).

[10] S. Weinberg, Phys. Rev. D **67**, 123504 (2003).

[11] J. Goldstone, Nuovo Cimento **9**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).