Nonperturbative Faddeev-Popov formula and the infrared limit of QCD

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We show that an exact nonperturbative quantization of continuum gauge theory is provided by the Faddeev-Popov formula in the Landau gauge, $\delta(\partial \cdot A)\det[-\partial \cdot D(A)]\exp[-S_{YM}(A)]$, restricted to the region where the Faddeev-Popov operator is positive $-\partial \cdot D(A) > 0$ (Gribov region). Although there are Gribov copies inside this region, they have no influence on expectation values. The starting point of the derivation is stochastic quantization which determines the Euclidean probability distribution $P(A)$ by a method that is free of the Gribov critique. In the Landau-gauge limit the support of $P(A)$ shrinks down to the Gribov region with Faddeev-Popov weight. The cutoff of the resulting functional integral on the boundary of the Gribov region does not change the *form* of the Dyson-Schwinger (DS) equations because $\det[-\partial \cdot D(A)]$ vanishes on the boundary, so there is no boundary contribution. However this cutoff does provide *supplementary conditions* that govern the choice of solution of the DS equations. In particular the "horizon condition," though consistent with the perturbative renormalization group, puts QCD into a nonperturbative phase. The infrared asymptotic limit of the DS equations of QCD is obtained by neglecting the Yang-Mills action S_{YM}. We sketch the extension to a BRST-invariant formulation. In the infrared asymptotic limit, the BRST-invariant action becomes BRST exact, and defines a topological quantum field theory with an infinite mass gap. Confinement of quarks is discussed briefly.

I. INTRODUCTION

Since the work of Gribov [1], a nonperturbative formulation of continuum gauge theory has appeared problematical due to the existence of Gribov copies. These are distinct but gauge-equivalent configurations $A^{(2)} = {}^g A^{(1)}$ that both satisfy the gauge condition, $\partial \cdot A^{(1)} = \partial \cdot A^{(2)} = 0$, where ${}^g A_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g$ is a local gauge transformation. The difficulty arises when one wishes to quantize by *gauge fixing* namely by taking a single representative configuration on each gauge orbit. It has been proven that this cannot be done in a continuous way when space-time is compactified [2]. Geometrically this reflects the intricacy of *gauge orbit space*, the space of configurations A modulo local gauge transformations g.

There is, however, an approach that bypasses the difficulties of Gribov copies by operating directly in A space. This approach is stochastic quantization. For our purposes it is most conveniently expressed by the time-independent Fokker-Planck equation (given below) that determines the Euclidean probability distribution $P(A)$. The geometric structure of the equation assures that $P(A)$ is correctly weighted. Although one cannot solve the Fokker-Planck equation exactly for finite values of the gauge parameter a, one can transform it into a system of Dyson-Schwinger (DS) equations for the correlation functions that may be solved nonperturbatively, as has been done recently [3]. However, these equations are more cumbersome than the DS equations in an action formalism.

In Secs. II–IV, we find the exact solution of the time-independent Fokker-Planck equation in the Landau-gauge limit $a \to 0$. The solution is remarkably simple. It is the fa-

miliar Faddeev-Popov weight, but restricted to the Gribov region Ω,[1]

$$P(A) = N\delta_\Omega(\partial \cdot A)\det[-\partial \cdot D(A)]\exp[-S_{YM}(A)]. \tag{1.1}$$

The Gribov region Ω is, by definition, the region in A space where A is transverse, and the Faddeev-Popov operator $M(A) \equiv -\partial \cdot D(A)$ is positive,

$$\Omega \equiv \{A: \partial \cdot A = 0; -\partial \cdot D(A) > 0\}. \tag{1.2}$$

The first factor $\delta_\Omega(\partial \cdot A)$ in Eq. (1.1) is the restriction of $\delta(\partial \cdot A)$ to the region where $M(A)$ is positive. Observables $\mathcal{O}(A)$ are required to be gauge-invariant, $\mathcal{O}({}^g A) = \mathcal{O}(A)$ and, by Eq. (1.1), expectation-values are calculated from

$$\langle \mathcal{O}(A) \rangle = \int dA\, \mathcal{O}(A)P(A)$$

$$= N\int_\Omega dA^{tr}\mathcal{O}(A^{tr})\exp[-S_{YM}(A^{tr})]$$

$$\times \det[-\partial \cdot D(A^{tr})], \tag{1.3}$$

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[1] The Yang-Mills action is given by $S_{YM}(A) = (1/4)\int d^4x F_{\mu\nu}^2$ where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_0 f^{abc} A_\mu^b A_\nu^c$, and the gauge-covariant derivative by $[D_\mu(A)\omega]^a \equiv \partial_\mu \omega^a + g_0 f^{abc} A_\mu^b \omega^c$. The Faddeev-Popov operator $M(A) \equiv -\partial \cdot D(A)$ is symmetric when A is transverse, $M(A) = -\partial \cdot D(A) = -D(A) \cdot \partial = M^\dagger(A)$. Positivity of $M(A)$ means all its nontrivial eigenvalues $\lambda_n(A)$ are positive. There is a trivial null eigenvalue with constant eigenvectors $\partial_\mu \omega = 0$ that are generators of global gauge transformations. In Appendix C we establish three simple properties of the Gribov region.

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where A^{tr} is the transverse part of A. Two comments are in order.

(i) Gribov region Ω vs fundamental modular region Λ. Formula (1.3) is paradoxical because the Gribov region Ω is *not* free of Gribov copies [4]. The history of this formula is amusing. It was originally proposed by Gribov who conjectured in his seminal work [1] that there are no Gribov copies in Ω. The same formula was also derived from stochastic quantization [5] by a method similar to the one presented in the present paper (but using globally defined coordinates instead of coordinates defined only on a coordinate patch), and was interpreted to mean that the Gribov region Ω is free of Gribov copies. However, it was then proven [4], with details provided in [6], that there are Gribov copies inside Ω. Moreover, numerical studies [7–11] revealed that in general there are many Gribov copies of a given configuration inside Ω. Consequently Eq. (1.3) was generally abandoned as an exact formula in favor of an integration over a region free of Gribov copies, known as the fundamental modular region Λ,

$$\langle \mathcal{O}(A) \rangle = N \int_{\Lambda} DA^{\mathrm{tr}} \det M(A^{\mathrm{tr}}) \mathcal{O}(A^{\mathrm{tr}}) \exp[-S_{\mathrm{YM}}(A^{\mathrm{tr}})]. \quad (1.4)$$

The last formula is certainly correct and appears to contradict Eq. (1.3). It was subsequently argued nevertheless [12] that the functional integral (1.3) is in fact dominated by configurations on the common boundary of Ω and Λ. The derivation given in Secs. II–IV shows that Eq. (1.3) is indeed correct. This is most fortunate because it is difficult to give an explicit description of Λ. In Appendix A we examine concretely how the paradox is resolved. The lesson is that the *normalized* pobability distributions over Λ and Ω are equal in the sense that their moments of finite order n are equal. These are the correlation functions $\langle A(x_1)A(x_2)\cdots A(x_n)\rangle$. This is possible in an infinite-dimensional space, where the probability distribution may sit on a lower dimensional subspace such as a boundary or part of a boundary. This conclusion is consistent with the numerical investigation of "Gribov noise," namely the effect on measured quantities of taking different Gribov copies. Indeed for the gluon propagator in Landau gauge on reasonably large lattices, Gribov noise is quite small, of the same magnitude as the numerical accuracy [13–15]. The situation is quite different for a finite-dimensional integral, and the analogous problem for a finite lattice is also discussed in Appendix A. Formula (1.3) is also supported by a recent calculation in which the DS equation for the gluon propagator was derived from the time-independent Fokker-Planck equation at finite gauge parameter a. It was found to agree with the DS equation for the gluon propagator in Faddeev-Popov theory in the Landau gauge limit, $a \rightarrow 0$, see particularly Eqs. (9.4), (10.13), (10.14), and (10.17) of [3].

(ii) The form of the DS equations is unchanged by the cutoff on the boundary of Ω. The DS equations are a set of equations for the correlation functions $\langle A(x_1)A(x_2)\cdots A(x_n)\rangle$. We shall derive them for the distribution (1.3) in Secs. V and VI. They are compactly ex-

pressed as a single functional differential equation for the partition function or generating functional of correlation functions,

$$Z(J) = N \int_{\Omega} dA^{\mathrm{tr}} \det[-\partial \cdot D(A^{\mathrm{tr}})] \exp[-S_{\mathrm{YM}}(A^{\mathrm{tr}}) + (J, A^{\mathrm{tr}})]. \quad (1.5)$$

The functional DS equation for $Z(J)$ follows from the identity

$$0 = N \int_{\Omega} dA^{\mathrm{tr}} \frac{\delta}{\delta A^{\mathrm{tr}}} (\det[-\partial \cdot D(A^{\mathrm{tr}})] \exp[-S_{\mathrm{YM}}(A^{\mathrm{tr}})$$
$$+ (J, A^{\mathrm{tr}})]), \quad (1.6)$$

which states that the integral of a derivative vanishes when there is no boundary contribution. There is in fact no boundary contribution, despite the cutoff on the boundary $\partial\Omega$, defined by the equation $\lambda_1(A^{\mathrm{tr}}) = 0$, because the Faddeev-Popov determinent $\det[-\partial \cdot D(A^{\mathrm{tr}})] = \Pi_n \lambda_n(A^{\mathrm{tr}})$ *vanishes* on $\partial\Omega$. Thus the *form* of the DS equation is the same as if the integral were extended to infinity [16]. Again this is most fortunate because it means that implementing the restriction to the Gribov region causes no complication at all in the DS equations.

Although the restriction to the interior of the Gribov horizon does not change the form of the DS equations, it does provide *supplementary conditions* that govern the choice of solution. In fact the properties that result from the restriction to Ω, in particular the positivity of the weight $P(A)$ and of the Faddeev-Popov operator $M(A)$, dictate the natural choice of solution of the DS equation that has been implemented previously, without necessarily invoking explicitly the cutoff at $\partial\Omega$, [3,17–23], and reviewed in [24]. Another property is the *horizon condition* [25]. This is an enhancement,[2] compared to $1/k^2$, of the ghost propagator $\widetilde{G}(k)$ in the infrared, $\lim_{k \to 0}[k^2 \widetilde{G}(k)]^{-1} = 0$.[3] In Sec. VII we show that the horizon condition is most conveniently expressed as a formula for the ghost-propagator renormalization constant $\widetilde{Z}_3$. Although this formula flagrantly contra-

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[2]*Entropy* favors population near the boundary, in a configuration space with a high number N of dimensions, because of the volume element $r^{N-1} dr$. The boundary $\partial\Omega$ of the Gribov region Ω occurs where the lowest nontrivial eigenvalue of the Faddeev-Popov operator $M(B)$ vanishes, so, for typical configurations B on a large Euclidean volume V, $M(B)$ has a very small eigenvalue. More precisely, compared to the Laplacian operator, $M(B)$ has a high density per unit volume of eigenvalues $\rho(\lambda, B)$ at $\lambda = 0$ [25]. This enhances the ghost propagator $G(x-y) = \langle M_{xy}^{-1}(A) \rangle$ in the infrared.

[3]The confinement criterion of Kugo and Ojima [26–28] yields the same condition in the Minkowskian theory. However, for gauge-noninvariant quantities, the relation of the present approach, with a cutoff at the Euclidean Gribov horizon, to the Minkowskian theory remains to be clarified, perhaps along the lines of Appendix B. The relation of numerical gauge fixing by minimization in (Euclidean) lattice gauge theory to the Minkowskian theory is also not clear.

dicts perturbation theory, it is nevertheless consistent with the perturbative renormalization group. The horizon condition puts QCD into a nonperturbative phase.

In Sec. VIII we deduce the asymptotic infrared limit of QCD by neglecting the terms in the DS equations that are subdominant in the infrared. It is found that the subdominant terms and only the subdominant terms come from the Yang-Mills action $S_{\mathrm{YM}}(A)$, so the infrared asymptotic limit of QCD is obtained by setting $S_{\mathrm{YM}}(A)=0$. This is a continuum analog of the strong coupling limit of lattice gauge theory. The functional integral with $\exp[-S_{\mathrm{YM}}(A)]$ replaced by 1 converges because it is cutoff at the Gribov horizon.

In Appendix B we outline the local BRST-invariant formulation of the present nonperturbative formulation. This assures that the Slavnov-Taylor identities hold at the nonperturbative level. In the infrared asymptotic limit, obtained by setting $S_{\mathrm{YM}}(A)=0$, the BRST-invariant action becomes BRST exact, and defines a topological quantum field theory. As shown in Sec. IX, this theory possess an infinite mass gap in the physical sector. In Sec. X the extension to quarks is sketched out.

The starting point of our derivation will be stochastic quantization of gauge fields. In the remainder of the Introduction we give a brief review of this subject so the reader may judge the well-foundedness of this approach at the nonperturbative level.

A. Review of stochastic quantization of gauge fields

Historically, stochastic quantization originated [29] with the observation that the formal, unnormalizable Euclidean probability distribution $P_0(A)=N\exp[-S_{\mathrm{YM}}(A)]$, with four-dimensional Euclidean Yang-Mills action $S_{\mathrm{YM}}(A)$, is the equilibrium distribution of the stochastic process defined by the equation

$$\frac{\partial P}{\partial t}=\int d^4x\,\frac{\delta}{\delta A_\mu^a(x)}\left(\frac{\delta P}{\delta A_\mu^a(x)}+\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)}P\right) \quad (1.7)$$

for the time-dependent probability distribution $P(A,t)$. This equation is a continuum analogue of the diffusion equation in the presence of the drift force K_i,

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial A^i}\left(\frac{\partial P}{\partial A^i}-K_iP\right)=0, \quad (1.8)$$

that is known as the Fokker-Planck equation. If the drift force is conservative, $K_i=-\partial S_{\mathrm{YM}}/\partial A^i$, then $\exp[-S_{\mathrm{YM}}(A)]$ is a time-independent solution. In Euclidean quantum field theory, t is an artificial fifth time that corresponds to the number of sweeps in a Monte Carlo simulation, and that will be eliminated shortly. The same stochastic process may equivalently be represented by the Langevin equation

$$\frac{\partial A_\mu^a}{\partial t}=-\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a}+\eta_\mu^a, \quad (1.9)$$

where $A_\mu^a=A_\mu^a(x,t)$ depends on the artificial fifth time. Here $\eta_\mu^a=\eta_\mu^a(x,t)$ is Gaussian white noise defined by $\langle\eta_\mu^a(x,t)\rangle=0$ and $\langle\eta_\nu^b(x,t)\,\eta_\mu^a(x,t)\rangle=2\,\delta(x-y)\,\delta_{\mu\nu}\delta^{ab}\delta(t-t')\rangle$. If $N\exp[-S_{\mathrm{YM}}(A)]$ were a normalizable probability distribution, which it is not, every normalized solution to Eq. (1.7) would relax to it as equilibrium distribution. However, the process defined by Eq. (1.7) or Eq. (1.9) does not provide a restoring force in gauge orbit directions, so probability escapes to infinity along the gauge orbits, and as a result $P(A,t)$ does not relax to a well-defined limiting distribution $\lim_{t\to\infty}P(A,t)\neq N\exp[-S_{\mathrm{YM}}(A)]$ (although expectation-values of gauge-invariant observables formally do relax to an equilibrium value).

A remedy is provided by the observation [30] that the Langevin equation may be modified by the addition of an infinitesimal gauge transformation, $D_\mu^{ac}v^c$,

$$\frac{\partial A_\mu^a}{\partial t}=-\frac{\delta S}{\delta A_\mu^a}+D_\mu^{ac}v^c+\eta_\mu^a, \quad (1.10)$$

where v^c is at our disposal. This cannot alter the expectation value of gauge-invariant quantities, for only a harmless infinitesimal gauge transformation $K_{\mathrm{gt},\mu}=D_\mu v$ has been introduced. In the language of the diffusion equation, we may say that the additional drift force $K_{\mathrm{gt},\mu}$ is tangent to the gauge orbit. The modified Langevin equation is equivalent to the modified Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\int d^4x\,\frac{\delta}{\delta A_\mu^a(x)}\left(\frac{\delta P}{\delta A_\mu^a(x)}-K_\mu^a(x)P\right),$$

$$K_\mu^a(x)\equiv-\frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(x)}+D_\mu^{ac}v^c(x). \quad (1.11)$$

We will choose $v^c(x)$ to make $D_\mu^{ac}v^c(x)$ globally restoring along gauge orbit directions, so every normalized solution $P(A,t)$ relaxes to a unique equilibrium distribution $\lim_{t\to\infty}P(A,t)=P(A)$.

Stochastic quantization in the time-dependent formulation has been developed by a number of authors who have expressed the solution as a functional integral [31] and demonstrated the renormalizability of this approach [32,33]. A systematic development is presented in [34–39], and reviewed in [40], that includes the four- and five-dimensional Dyson-Schwinger equation for the quantum effective action, an extension of the method to gravity, and gauge-invariant regularization by smoothing in the fifth time. Renormalizability has also been established by an elaboration of BRST techniques [41,42]. Stochastic quantization may be and has been exactly simulated numerically including on rather large lattices, of volume $(48)^4$, [43–47].

B. Time-independent stochastic quantization

When the drift force is globally restoring, $P(A)$ may be calculated directly without reference to the artificial fifth time as the positive normalized solution of the time-independent Fokker-Planck equation

$$HP \equiv \int d^4x \frac{\delta}{\delta A_\mu^a(x)} \left(-\frac{\delta P}{\delta A_\mu^a(x)} + K_\mu^a P \right) = 0,$$

$$K_\mu^a(x) \equiv -\frac{\delta S_{\text{YM}}}{\delta A_\mu^a(x)} + D_\mu^{ac} v^c(x), \tag{1.12}$$

and Euclidean expectation values are calculated from $\langle \mathcal{O} \rangle = \int dA\, \mathcal{O}(A) P(A)$. We call H the "Fokker-Planck Hamiltonian." (It is *not* the quantum mechanical Hamiltonian.) It has been proven directly [3], without reference to the artificial time, that the expectation value $\langle \mathcal{O} \rangle_v$ of a gauge-invariant observable $\mathcal{O}(^g A) = \mathcal{O}(A)$, is independent of v. Equation (1.12) determines a probability distribution $P(A)$ directly in A space that is correctly weighted at the nonperturbative level. The Gribov problem of globally correct gauge fixing by identifying gauge orbits is bypassed. By contrast, in the Hamiltonian formulation of gauge theory, Gauss's law states that the wave functional $\Psi(\mathbf{A})$ is gauge-invariant and is thus a functional defined on the space of gauge orbits [48].

To ensure that $K_{\text{gt},\mu} = D_\mu v$ is globally restoring, we introduce a minimizing functional [49], [50], and [4], and choose $K_{\text{gt},\mu}$ to be in the gauge-orbit direction of steepest descent. A convenient choice of minimizing functional[4] is the Hilbert norm $\|A\|^2 = \int d^4x |A|^2$. For an infinitesimal variation in the gauge-orbit direction $\delta A_\mu = \epsilon D_\mu v$, we have

$$\delta\|A\|^2 = 2(A_\mu, \delta A_\mu) = 2\epsilon(A_\mu, D_\mu v) = 2\epsilon(A_\mu, \partial_\mu v)$$
$$= -2\epsilon(\partial_\mu A_\mu, v), \tag{1.13}$$

so steepest descent among gauge orbit directions of the minimizing functional is provided by $v = a^{-1}\partial \cdot A$ with $a > 0$, and the time-independent Fokker-Planck equation is now specified to within a single gauge parameter,

$$HP = \int d^4x \frac{\delta}{\delta A_\mu^a(x)} \left(-\frac{\delta P}{\delta A_\mu^a(x)} + K_\mu^a P \right) = 0,$$

$$K_\mu^a(x) \equiv -\frac{\delta S_{\text{YM}}}{\delta A_\mu^a(x)} + a^{-1} D_\mu^{ac} \partial \cdot A^c(x), \tag{1.14}$$

(Symmetry and power-counting arguments also determine $v^a = a^{-1}\partial_\lambda A_\lambda^a = a^{-1}\partial \cdot A^a$.)

Having introduced the minimizing functional, we note that the Gribov region Ω may be characterized as the set of

relative minima[5] with respect to local gauge transformations $g(x)$ of the minimizing functional $F_A(g) \equiv \|^g A\|^2$, whereas the fundamental modular region Λ may be characterized as the set of *absolute* minima. The set of absolute minima is free of Gribov copies, apart from the identification of gauge-equivalent points on the boundary $\partial\Lambda$, and may be identified with the gauge orbit space. In a lattice discretization the minimization problem is of spin-glass type, and one expects many nearly degenerate local minima on a typical gauge orbit, as is verified by numerical studies. Thus Λ is a proper subset of Ω, $\Lambda \subset \Omega$, but $\Lambda \neq \Omega$.

C. Region of stable equilibrium of K_{gt}

The gauge transformation "force" K_{gt} is not conservative, and cannot be written, like the first term, as the gradient of some four-dimensional gauge-fixing action, $K_{\text{gt},\mu} = a^{-1}D_\mu^{ac}\partial \cdot A^c(x) \neq -\delta S_{\text{gf}}/\delta A_\mu^a(x)$, so we cannot write the solution $P(A)$ explicitly in general. However, we shall solve Eq. (1.14) for $P(A)$ exactly in the limit $a \to 0$. In this limit $P(A)$ gets concentrated in the region of *stable equilibrium* of the force $K_{\text{gt},\mu} = a^{-1}D_\mu\partial \cdot A$.

Assertion: The region of stable equilibrium under the gauge transformation force $K_{\text{gt},\mu} = D_\mu\partial \cdot A$ is the Gribov region Ω. *Proof*: Transversality is a sufficient condition for equilibrium because $\partial \cdot A = 0$ implies $K_{\text{gt},\mu} = 0$. It is also necessary. Consider the flow under this force, $\dot{A}_\mu = D_\mu\partial \cdot A$. We have $\partial\|A\|^2/\partial t = 2(A_\mu, \dot{A}_\mu) = 2(A_\mu, D_\mu\partial \cdot A) = 2(A_\mu, \partial_\mu\partial \cdot A) = -2\|\partial \cdot A\|^2 \leq 0$, which is negative unless $\partial \cdot A = 0$. We conclude that the region of equilibrium under K_{gt}, which may be stable or unstable, is the set of transverse configurations. To find the region of stable equilibrium, observe that under this flow, we have $\partial/\partial t\partial \cdot A = \partial \cdot \dot{A} = \partial \cdot D(A)\partial \cdot A$. We linearize this equation to first order in $\partial \cdot A$, which means taking $\partial \cdot D(A) \to \partial \cdot D(A^{\text{tr}}) \equiv -M(A^{\text{tr}})$, and we have $\partial/\partial t\partial \cdot A = -M(A^{\text{tr}})\partial \cdot A$. Thus the equilibrium is stable when all eigenvalues of $M(A^{\text{tr}})$ are positive, and it is unstable otherwise. QED.

II. A WELL-DEFINED CHANGE OF VARIABLE

In order to solve the time-independent Fokker-Planck equation (1.14) in the limit $a \to 0$, we only need the solution for small a in a coordinate patch $\mathcal{U}$ in A space that includes the Gribov region Ω. In $\mathcal{U}$, we make the change of variable $A \to (B, g)$, defined by the gauge transformation

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[4]More generally, we may take for the minimizing function $\int d^4x A_\mu^a(x)\alpha_{\mu\nu}A_\nu^a(x)$, where $\alpha_{\mu\nu}$ is a constant positive symmetric matrix. This defines a set of Lorentz-noncovariant but normalizable gauges that includes the Coulomb gauge as a limiting case [51]. To include different instanton sectors, one may choose as a minimizing functional $\|A - A_n\|^2$, where A_n is a fixed configuration of the given instanton number. An alternative minimizing functional suitable for the Higgs phase was proposed in [42].

[5]At any minimum, this functional is (a) stationary, and (b) the matrix of second derivatives is non-negative. These two conditions fix the properties that define the Gribov region: (a) transversality, $\partial \cdot A = 0$, and (b) positivity of the Faddeev-Popov operator $-D(A) \cdot \partial$. Property (a) follows from Eq. (1.13), which states that the first variation of the minimizing functional is $\delta\|A\|^2 = -2(\omega, \partial \cdot A)$. Property (b) follows because the second variation is $\delta^2\|A\|^2 = -2[\omega, \partial \cdot D(A)\omega]$.

$$A_\mu = A_\mu(B,g) = {}^g B_\mu = -g^{-1}\partial_\mu g + g^{-1}B_\mu g,$$

$$\text{with} \quad \partial \cdot B = 0 \quad \text{and} \quad M(B) > 0, \qquad (2.1)$$

where $B \in \Omega$. Local gauge transformations are parametrized by $g(x) = \exp[t^a \theta^a(x)]$ where, for each x, the $\theta^a(x)$ are coordinates for the SU(N) group.[6] The notation $A = A(B,g)$ is understood to stand for $A = A(B,\theta)$, and we have $B = A(B,0)$.

Gribov's critique of the Faddeev-Popov method is that this change of variable is not well-defined for all transverse B and g. We shall show, however, that it is well-defined in a coordinate patch $\mathcal{U}$ that includes Ω. This is true, even though there are Gribov copies within Ω, because the gauge orbits intersect Ω transversely. The coordinate patch $\mathcal{U}$ must be small enough in the θ directions that the gauge transformations $g(\theta)$ that relate these Gribov copies are not in $\mathcal{U}$.

To verify that the gauge-orbits intersect Ω transversely, it is sufficient to show that the change of variables (2.1) is invertible for infinitesimal angles $\theta^a(x) = \epsilon^a(x)$ for all $B \in \Omega$. It follows that it is also invertible, and thus well-defined, on some finite coordinate patch $\mathcal{U}$ that includes Ω.

To first order in ϵ, the change of variable (2.1) is given by $A_\mu = B_\mu + D_\mu(B)\epsilon$. The divergence of this equation reads $\partial \cdot A = \partial \cdot D(B)\epsilon = -M(B)\epsilon$, which shows that $\partial \cdot A$ depends linearly on ϵ. Note that $\partial \cdot A$ is orthogonal to the trivial null space of $M(B)$, consisting of constant functions, and we specify that ϵ is also orthogonal to this null space.[7] Since $B \in \Omega$ by assumption, $M(B)$ is a strictly positive operator on the orthogonal space, and thus invertible, and we have $\epsilon = -M^{-1}(B)\partial \cdot A$. We solve for B in the form $B_\mu = A_\mu + D_\mu(B)M^{-1}(B)\partial \cdot A$. To zeroth order in ϵ we have $B = A = A^{\mathrm{tr}}$, where $A_\mu^{\mathrm{tr}} \equiv A_\mu - \partial_\mu(\partial^2)^{-1}\partial \cdot A$ is the transverse part of A. This gives the inversion formulas $B_\mu = A_\mu + D_\mu(A^{\mathrm{tr}})M^{-1}(A^{\mathrm{tr}})\partial \cdot A$ and $\epsilon = -M^{-1}(A^{\mathrm{tr}})\partial \cdot A$, valid to first order in ϵ or $\partial \cdot A$. Thus for each $A^{\mathrm{tr}} \in \Omega$, the change of variable (2.1) is invertible to first order in the small quantity $\partial \cdot A$. QED

Concerning the shape of the coordinate patch $\mathcal{U}$, note that as the configuration $B \in \Omega$ approaches the boundary $\partial\Omega$ of the Gribov region, the lowest nontrivial eigenvalue $\lambda_1(B)$ of the Faddeev-Popov operator $M(B)$ approaches 0. Consequently the width in longitudinal or θ directions of the coordinate patch $\mathcal{U}$ shrinks to zero as the boundary $\partial\Omega$ is approached. We may picture $\mathcal{U}$ as a very high-dimensional clam, shown in Fig. 1.

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[6]Here and below we use the notation $A_\mu \equiv t^a A_\mu^a$ and $B_\mu \equiv t^a B_\mu^a$. The t^a are a set of anti-Hermitian traceless matrices that form the fundamental representation of the Lie algebra of SU(N), $[t^a, t^b] = f^{abc}t^c$, where the structure constants f^{abc} are completely antisymmetric.

[7]The constant angles $\partial_\mu \theta^a = 0$ parametrize global SU(N) transformations. These act *within* Ω. However, we may safely ignore them because they have finite volume that we normalize to unity. The spectrum of $M(B)$ is discrete by quantization in a finite Euclidean volume.
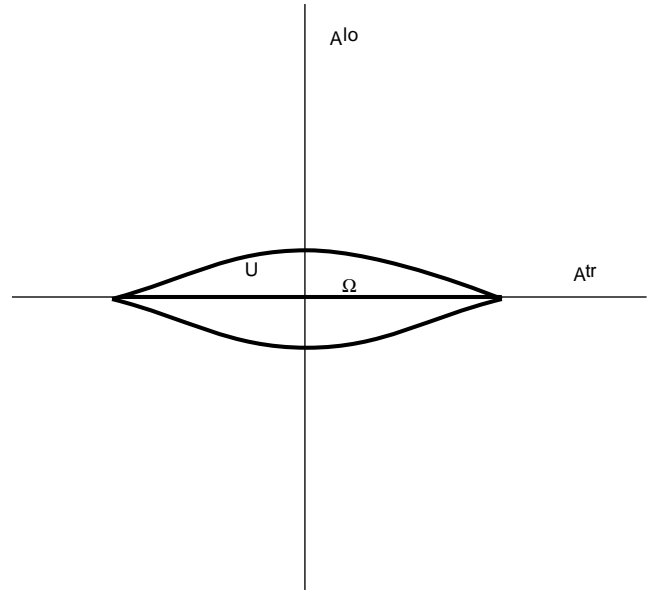

FIG. 1. The coordinate patch $\mathcal{U}$ in A space is the clam-shaped region viewed edge on. The Gribov region Ω is represented by the thick horizontal line.

III. CHANGE OF VARIABLE IN FOKKER-PLANCK EQUATION

To change variables in the Fokker-Planck equation, one takes over to functional variables the standard formulas of differential geometry. The mechanics of the calculation are similar to the computation of the Coulomb Hamiltonian by Christ and Lee [52], but there the change of variable was done globally whereas here it is done only in a coordinate patch. We freely go back and forth from continuum to discrete notation by the replacements $A_\mu^a(x) \leftrightarrow A^i$ and $[B_\mu^a(x), \theta^a(x)] \leftrightarrow u^\alpha$. In terms of A^i, the Fokker-Planck equation reads

$$-HP \equiv \frac{\partial}{\partial A^i}\delta^{ij}\left(\frac{\partial P}{\partial A^j} - K_j P\right) = 0, \qquad (3.1)$$

and expectation values are given by $\langle F \rangle = \int \prod_i dA^i F(A)P(A)$. The coordinates A^i are Cartesian, but the coordinate transformation $A = A(B,\theta) = A(u)$ is nonlinear, and the $u = (B,\theta)$ are curvilinear coordinates. In terms of these, the Fokker-Planck equation reads

$$-HP = \frac{1}{\sqrt{G}}\frac{\partial}{\partial u^\alpha}\left[\sqrt{G}G^{\alpha\beta}\left(\frac{\partial P}{\partial u^\beta} - K_\beta^{(u)}P\right)\right] = 0, \quad (3.2)$$

and expectation-values are given by $\langle F \rangle = \int \prod_\alpha du^\alpha \sqrt{G}F(u)P(u)$. The metric tensor is given by $dA^i dA^i = du^\alpha(\partial A^i/\partial u^\alpha)(\partial A^i/\partial u^\beta)du^\beta = du^\alpha G_{\alpha\beta}du^\beta$, with volume element $\sqrt{G} = \det \partial u/\partial A$. The covariant and contravariant components of any Cartesian vector field K_i are given by $K_\alpha = (\partial A^i/\partial u^\alpha)K_i$ and $K^\alpha = (\partial u^a/\partial A^i)K_i$.

We now calculate these quantities explicitly in functional form. From $A_\mu = g^{-1}Bg + g^{-1}\partial_\mu g$, we obtain

$$\delta A_\mu = g^{-1}(\delta B_\mu g + \partial_\mu(\delta g g^{-1}) + [B, \delta g g^{-1}])g, \quad (3.3)$$

where

$$\omega \equiv dg g^{-1} = \frac{\partial g}{\partial \theta^\beta} g^{-1} d\theta^\beta = \omega_\beta d\theta^\beta = t^a \omega_\beta^a d\theta^\beta \quad (3.4)$$

is the Maurer-Cartan form. It satisfies $d\omega = dg g^{-1} \wedge dg g^{-1} = \omega \wedge \omega$ or, in terms of components,

$$\frac{\partial \omega_\beta^c}{\partial \theta^\alpha} - \frac{\partial \omega_\alpha^c}{\partial \theta^\beta} = f^{cab}[\omega_\alpha^a, \omega_\beta^b]. \quad (3.5)$$

We also have $g^{-1} t^a g = R_{ab} t^b$, where the real orthogonal matrices $R_{ab} = R_{ab}(\theta) = R_{ba}^{-1}$ are in the adjoint representation of the gauge group. From $A_\mu = t^a A_\mu$, and $B_\mu = t^a B_\mu$, we obtain

$$\delta A_\mu^a = R_{ab}^{-1}[\delta B_\mu^b + D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha)], \quad (3.6)$$

where δB_μ is purely transverse, $\partial_\mu \delta B_\mu = 0$, and $D_\mu^{ac} \equiv D_\mu^{ac}(B)$ is the gauge-covariant derivative with the connection B_μ^a as argument. The last expression is the functional form of $\delta A^i = (\partial A^i / \partial u^\alpha) \delta u^\alpha$. It gives the functional operator that corresponds to $\partial A^i / \partial u^\alpha$, and we have for the metric tensor

$$ds^2 = \int d^4x \, \delta A_\mu^a \delta A_\mu^a$$

$$= \int d^4x [\delta B_\mu^b + D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha)][\delta B_\mu^b + D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha)]. \quad (3.7)$$

To calculate $\sqrt{G} = \det \partial A / \partial u$, we start by writing the linear transformation (3.6) as the product of two transformations, $\delta A_\mu^a = R_{ab}^{-1} \delta C_\mu^a$, and

$$\delta C_\mu^a = \delta B_\mu^b + D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha). \quad (3.8)$$

The matrix R_{ab} is orthogonal, so $\det R = 1$, and it is sufficient to calculate the determinent of the linear transformation (3.8). We do this in two steps. We first transform from δC_μ^a to its transverse part $(\delta C)_\lambda^{\text{tr},a} \equiv P_{\lambda \mu}^{\text{tr}}(\delta C)_\mu^a$, and its divergence $\delta L^a \equiv \partial_\mu \delta C_\mu^a$, where $P_{\lambda \mu}^{\text{tr}} = \delta_{\lambda \mu} - \partial_\lambda (\partial^2)^{-1} \partial_\mu$ is the projector onto transverse vector fields. This linear transformation is independent of the variables $u = (B, \theta)$, so its determinent is a constant, and will be ignored. The linear transformation from δB and $\delta \theta$ to δC^{tr} and δL is given by

$$(\delta C)_\lambda^{\text{tr},a} = \delta B_\mu^b + P_{\lambda \mu}^{\text{tr}} D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha)$$

$$\delta L^a = \partial_\mu D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha), \quad (3.9)$$

where we have used the transversality of δB_μ^b. This linear transformation is a triangular matrix, and its determinent is the product of the determinents of its diagonal submatrices. This gives

$$\sqrt{G} = \det I \det[-\partial_\mu D_\mu(B)\omega(\theta)]$$

$$= \det[-\partial_\mu D_\mu(B)]\text{Det}\omega(\theta)$$

$$= \det M(B)\prod_x \det \omega[\theta(x)], \quad (3.10)$$

which contains the Faddeev-Popov determinant $\det M(B)$. It has been obtained by a purely local calculation at a fixed point $A = {}^g B$, without integrating globally over the gauge group. The volume element $\sqrt{G}$ is the product of $\det M(B)$ that depends only on B, and the functional determinant $\text{Det}\,\omega(\theta) \equiv \Pi_x \det \omega[\theta(x)]$ that depends only on θ. Here $\{\det \omega[\theta(x)]\Pi_\alpha d\theta^\alpha(x)\}$ is the Haar measure of the SU(N) gauge group at x. It is common to write $\int Dg = \int D\theta \, \text{Det}[\omega(\theta)]$.

We next find the inverse matrix $\partial A^i / \partial u^\alpha$ by solving for δB_μ^b and $\delta \theta^\alpha$. From Eq. (3.6) we obtain

$$R_{ba} \delta A_\mu^a = [\delta B_\mu^b + D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha)]. \quad (3.11)$$

We take the divergence of this equation and use $\partial_\mu \delta B_\mu = 0$ to obtain

$$\partial_\mu(R_{ba} \delta A_\mu^a) = \partial_\mu D_\mu^{bc}(\omega_\alpha^c \delta \theta^\alpha), \quad (3.12)$$

which gives the first inverse formula

$$\delta \theta^\alpha = J_c^\alpha[(\partial \cdot D)^{-1}]^{cb} \partial_\mu(R_{ba} \delta A_\mu^a), \quad (3.13)$$

where $J_c^\alpha(\theta) \equiv (\omega^{-1})_c^\alpha(\theta)$. The Faddeev-Popov operator $M(B) \equiv -\partial \cdot D(B) = -D(B) \cdot \partial$ is symmetric and positive, so its inverse is well defined. To avoid a proliferation of indices, we write the last and similar equations in operator notation,

$$\delta \theta = J(\partial \cdot D)^{-1} \partial \cdot (R \delta A). \quad (3.14)$$

Inserting this into Eq. (3.11), we obtain the second inverse formula

$$\delta B_\lambda = [\delta_{\lambda \mu} - D_\lambda(\partial \cdot D)^{-1}\partial_\mu](R \delta A_\mu). \quad (3.15)$$

One sees that δB_λ is transverse, $\partial_\lambda \delta B_\lambda = 0$. The last two equations give the operators corresponding to the matrices $\partial u^\alpha / \partial A^i$. From them we read off the continuum version of $\partial / \partial A^i = \partial u^\alpha / \partial A_i \partial / \partial u^\alpha$ namely,

$$\frac{\delta}{\delta A_\mu} = \widetilde{R}\bigg([\delta_{\mu \lambda} - \partial_\mu(D \cdot \partial)^{-1}D_\lambda]$$

$$\times \frac{\delta}{\delta B_\lambda} - \partial_\mu(D \cdot \partial)^{-1}J(\theta)\frac{\delta}{\delta \theta}\bigg), \quad (3.16)$$

where $\widetilde{R}$ is the transpose of R. The $(J\delta/\delta\theta)_b = J_b^\alpha(\theta)(\delta/\delta\theta^\alpha) \equiv (\omega^{-1})_b^\alpha(\theta)(\delta/\delta\theta^\alpha)$ are the angular momentum or Lie differential operators of the gauge group. They satisfy the Lie algebra commutation relations of the local gauge group

$$\left[J_a^\alpha[\theta(x)]\frac{\delta}{\delta\theta(x)^\alpha},J_b^\beta[\theta(y)]\frac{\delta}{\delta\theta(y)^\beta}\right]$$

$$=-\delta(x-y)f^{abc}J_c^\gamma[\theta(x)]\frac{\delta}{\delta\theta(x)^\gamma},\quad(3.17)$$

that follow from Eq. (3.5).

We need the curvilinear components of the drift force $K_\mu=K_{\mathrm{YM},\mu}+a^{-1}K_{\mathrm{gt},\mu}$ where $K_{\mathrm{YM},\mu}(A)=-\delta S/\delta A_\mu=D_\lambda F_{\lambda\mu}(A)$ and $K_{\mathrm{gt},\mu}=D_\mu\partial\cdot A$. We shall see that the one-form or covariant θ-component of K_{YM} vanishes [because the action $S_{\mathrm{YM}}(^gB)=S_{\mathrm{YM}}(B)$ is gauge invariant], while the tangent-vector or contravariant B-component of K_{gt} vanishes (because K_{gt} is tangent to the gauge orbit). Thus the Fokker-Planck equation (3.2) in curvilinear coordinates $u=(B,\theta)$, reads $HP=0$, where

$$H=H_{BB}+H_{B\theta}+H_{\theta G}+H_{\theta\theta},\quad(3.18)$$

$$-H_{BB}\equiv\frac{1}{\sqrt{G}}\frac{\partial}{\partial B^\alpha}\sqrt{G}G_{(BB)}^{\alpha\beta}\left(\frac{\partial}{\partial B^\beta}-K_{\mathrm{YM},\beta}^{(B)}\right),$$

$$-H_{B\theta}\equiv\frac{1}{\sqrt{G}}\frac{\partial}{\partial B^\alpha}\sqrt{G}G_{(B\theta)}^{\alpha\beta}\frac{\partial}{\partial\theta^\beta},$$

$$(3.19)$$

$$-H_{\theta B}\equiv\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta^\alpha}\sqrt{G}G_{(\theta B)}^{\alpha\beta}\left(\frac{\partial}{\partial B^\beta}-K_{\mathrm{YM},\beta}^{(B)}\right),$$

$$-H_{\theta\theta}\equiv\frac{1}{\sqrt{G}}\frac{\partial}{\partial\theta^\alpha}\sqrt{G}\left(G_{(\theta\theta)}^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}-K_{\mathrm{gt},(\theta)}^\alpha\right).$$

We use the continuum version of the formula $K_{\mathrm{YM},i}\delta A^i=K_{\mathrm{YM},\alpha}^{(B)}\delta B^\alpha+K_{\mathrm{YM},\alpha}^{(\theta)}\delta\theta^\alpha$ to obtain the one-form components of K_{YM}. We have

$$\int d^4x K_{\mathrm{YM},\mu}^a(A)\delta A_\mu^a=\int d^4x D_\lambda F_{\lambda\mu}^a(^gB)\delta A_\mu^a$$

$$=\int d^4x R_{ab}^{-1}D_\lambda F_{\lambda\mu}^b(B)\delta A_\mu^a$$

$$=\int d^4x D_\lambda F_{\lambda\mu}^b(B)[\delta B_\mu^b$$

$$+D_\mu^{bc}(\omega_\alpha^c\delta\theta^a)]$$

$$=\int d^4x D_\lambda F_{\lambda\mu}^b(B)\delta B_\mu^b,\quad(3.20)$$

by Eq. (3.6), where we have performed an integration by parts, and used $(D_\mu D_\lambda F_{\lambda\mu})^a=(1/2)g_0 f^{abc}F_{\mu\lambda}^b F_{\lambda\mu}^c=0$. Thus the one-form components of K_{YM} are given by

$$K_{\mathrm{YM},\alpha}=(K_{\mathrm{YM},\alpha}^{(B)},K_{\mathrm{YM},\alpha}^{(\theta)})=[D_\lambda F_{\lambda\mu}^b(B),0].\quad(3.21)$$

We use the continuum version of $K_{\mathrm{gt},i}(\partial/\partial A^i)=K_{\mathrm{gt},\alpha}^{(B)}(\partial/\partial B^\alpha)+K_{\mathrm{gt},\alpha}^{(\theta)}(\partial/\partial\theta^\alpha)$ to obtain the contravariant or tangent-vector components of $K_{\mathrm{gt},\mu}=D_\mu\partial\cdot A$. We have

$$\partial_\lambda A_\lambda=\partial_\lambda(g^{-1}B_\lambda g+g^{-1}\partial_\lambda g)$$

$$=g^{-1}(\partial_\lambda(\partial_\lambda gg^{-1})+[B,\partial_\lambda gg^{-1}])g$$

$$=g^{-1}D_\lambda(B)(\partial_\lambda gg^{-1})g$$

$$=g^{-1}D_\lambda(B)(\omega_\alpha\partial_\lambda\theta^\alpha)g,\quad(3.22)$$

where we have used $\partial_\lambda gg^{-1}=(\partial g/\partial\theta_\alpha)g^{-1}\partial_\lambda\theta^a=\omega_\alpha\partial_\lambda\theta^\alpha$, and ω is again the Maurer-Cartan form. In index and operator notation this reads

$$\partial_\lambda A_\lambda^a=\tilde{R}_{ab}D_\lambda^{bc}(\omega_\alpha^c\partial_\lambda\theta^\alpha)\leftrightarrow\partial_\lambda A_\lambda=\tilde{R}D_\lambda(\omega\partial_\lambda\theta),$$

$$(3.23)$$

where $D_\lambda\equiv D_\lambda(B)$. By the gauge transformation property of the gauge covariant derivative $D(A)=D(^gB)$, this gives

$$D_\mu(A)\partial_\lambda A_\lambda=\tilde{R}D_\mu(B)D_\lambda(B)(\omega\partial_\lambda\theta),\quad(3.24)$$

By Eq. (3.16) we obtain

$$\int d^4x K_{\mathrm{gt},\mu}^a\frac{\delta}{\delta A_\mu^a}=\int d^4x D_\mu(B)D_\lambda(B)(\omega\partial_\lambda\theta)$$

$$\times\left(\left[\delta_{\mu\nu}-\partial_\mu(D\cdot\partial)^{-1}D_\nu\right]\right.$$

$$\left.\times\frac{\delta}{\delta B_\nu}-\partial_\mu(D\cdot\partial)^{-1}J(\theta)\frac{\delta}{\delta\theta}\right).$$

$$(3.25)$$

We perform an integration by parts and use $D_\mu[\delta_{\mu\nu}-\partial_\mu(D\cdot\partial)^{-1}D_\nu]\delta/\delta B_\nu=0$ to obtain

$$\int d^4x K_{\mathrm{gt},\mu}^a\frac{\delta}{\delta A_\mu^a}=\int d^4x[D_\lambda(B)(\omega\partial_\lambda\theta)]^a\left[J(\theta)\frac{\delta}{\delta\theta}\right]_a.$$

$$(3.26)$$

Thus the tangent-vector components of K_{gt} are given by

$$K_{\mathrm{gt}}^\alpha=(K_{\mathrm{gt}}^{(B),\alpha},K_{\mathrm{gt}}^{(\theta),\alpha})=\{0,J_b^\beta(\theta)[D_\lambda(B)(\omega\partial_\lambda\theta)]^b\}.$$

$$(3.27)$$

From Eq. (3.16) we obtain the Laplacian operator $1/\sqrt{G}\partial/\partial u^\alpha\sqrt{G}\partial u^\alpha/\partial A^i\partial u^\beta/\partial A^i\partial/\partial u^\beta$ in curvilinear coordinates,

$$\int d^4x\frac{1}{\sqrt{G}}\left(\frac{\delta}{\delta B_\lambda}[\delta_{\lambda\mu}-D_\lambda(\partial\cdot D)^{-1}\partial_\mu]\right.$$

$$+\frac{\delta}{\delta\theta}\tilde{J}(\theta)(\partial\cdot D)^{-1}\partial_\mu\Bigg)$$

$$\times\sqrt{G}\Bigg([\delta_{\mu\nu}-\partial_\mu(D\cdot\partial)^{-1}D_\nu]$$

$$\times\frac{\delta}{\delta B_\nu}-\partial_\mu(D\cdot\partial)^{-1}J(\theta)\frac{\delta}{\delta\theta}\Bigg).\quad(3.28)$$

Putting all terms together, the explicit expressions for the terms in Eq. (3.19) are

$$-H_{BB} = \frac{1}{\det M(B)} \int d^4x \frac{\delta}{\delta B_\lambda} \det M(B)[\delta_{\lambda\mu} - D_\lambda$$

$$\times (\partial \cdot D)^{-1}\partial_\mu] \times [\delta_{\mu\nu} - \partial_\mu(D\cdot\partial)^{-1}D_\nu]$$

$$\times \left[\frac{\delta}{\delta B_\nu} - D_\kappa F_{\kappa\nu}(B)\right], \qquad (3.29)$$

$$-H_{B\theta} = \frac{1}{\det M(B)} \int d^4x \frac{\delta}{\delta B_\lambda} \det M(B)$$

$$\times [-\partial_\lambda + D_\lambda(\partial\cdot D)^{-1}\partial^2](D\cdot\partial)^{-1}J(\theta)\frac{\delta}{\delta\theta}, \qquad (3.30)$$

$$-H_{\theta B} = \frac{1}{\mathrm{Det}\,\omega(\theta)} \int d^4x \frac{\delta}{\delta\theta} \mathrm{Det}\,\omega(\theta)\tilde{J}(\theta)$$

$$\times (\partial\cdot D)^{-1}[\partial_\nu - \partial^2(D\cdot\partial)^{-1}D_\nu]$$

$$\times \left[\frac{\delta}{\delta B_\nu} - D_\lambda F_{\lambda\mu}(B)\right], \qquad (3.31)$$

$$-H_{\theta\theta} = \frac{1}{\mathrm{Det}\,\omega(\theta)} \int d^4x \frac{\delta}{\delta\theta} \mathrm{Det}\,\omega(\theta)\tilde{J}(\theta)$$

$$\times \left((\partial\cdot D)^{-1}(-\partial^2)(D\cdot\partial)^{-1}J(\theta)\frac{\delta}{\delta\theta} - \frac{1}{a}D_\lambda \right.$$

$$\left. \times [\omega(\theta)\partial_\lambda\theta] \right). \qquad (3.32)$$

IV. SOLUTION IN LANDAU-GAUGE LIMIT

We shall solve the Fokker-Planck equation $HP=0$ in the limit $a\to 0$. In this limit the drift force in the gauge-orbit or θ-direction is dominant. This situation is reminiscent of the Born-Oppenheimer method in molecular physics. The θ variables equilibrate rapidly, like the electron positions in a molecular wave function, and the dependence on the B variable is determined by an average over the θ variable, like the nuclear variables.

We expect that the solution gets concentrated close to $\theta = 0$. We rescale variable according to $\theta = a^{1/2}\Theta$, and find that H_{BB} is independent of a and unchanged, whereas

$$-H_{B\theta} = \frac{1}{a^{1/2}} \frac{1}{\det M(B)} \int d^4x \frac{\delta}{\delta B_\lambda} \det M(B)$$

$$\times [-\partial_\lambda + D_\lambda(\partial\cdot D)^{-1}\partial^2](D\cdot\partial)^{-1} \times J(a^{1/2}\Theta)\frac{\delta}{\delta\Theta}$$

$$\qquad (4.1)$$

and

$$-H_{\theta B} = \frac{1}{a^{1/2}} \frac{1}{\mathrm{Det}\,\omega(a^{1/2}\Theta)} \int d^4x \frac{\delta}{\delta\Theta}$$

$$\times \mathrm{Det}\,\omega(a^{1/2}\Theta)\tilde{J}(a^{1/2}\Theta)(\partial\cdot D)^{-1}$$

$$\times [\partial_\nu - \partial^2(D\cdot\partial)^{-1}D_\nu] \left[\frac{\delta}{\delta B_\nu} - D_\lambda F_{\lambda\mu}(B)\right]$$

$$\qquad (4.2)$$

are of leading order $1/a^{1/2}$, while

$$-H_{\theta\theta} = \frac{1}{a} \frac{1}{\mathrm{Det}\,\omega(a^{1/2}\Theta)} \int d^4x \frac{\delta}{\delta\Theta} \mathrm{Det}\,\omega(a^{1/2}\Theta)\tilde{J}(a^{1/2}\Theta)$$

$$\times \left((\partial\cdot D)^{-1}(-\partial^2)(D\cdot\partial)^{-1}J(a^{1/2}\Theta) \right.$$

$$\left. \times \frac{\delta}{\delta\Theta} - D_\lambda[\omega(a^{1/2}\Theta)\partial_\lambda\Theta] \right) \qquad (4.3)$$

is of leading order $1/a$.

The Fokker-Planck Hamiltonian has an expansion in a given by $H = a^{-1}H_0 + a^{-1/2}H_1 + H_2 + O(a^{1/2})$. We seek a solution of the form $P = P_0 + a^{1/2}P_1 + aP_2 + \cdots$, which gives

$$(a^{-1}H_0 + a^{-1/2}H_1 + H_2 + \cdots)(P_0 + a^{1/2}P_1 + aP_2 + \cdots) = 0. \qquad (4.4)$$

To leading order we obtain

$$-H_0P_0 = \int d^4x \frac{\delta}{\delta\Theta} \left((\partial\cdot D)^{-1}(-\partial^2)(D\cdot\partial)^{-1} \right.$$

$$\left. \times \frac{\delta}{\delta\Theta} - D\cdot\partial\Theta \right) P_0 = 0 \qquad (4.5)$$

or

$$\int d^4x \frac{\delta}{\delta\Theta} \left(V\frac{\delta}{\delta\Theta} + M\Theta \right) P_0 = 0, \qquad (4.6)$$

where $D \equiv D(B)$, $M \equiv M(B)$. The operator $V = V(B)$ is defined by $V \equiv M^{-1}(-\partial^2)M^{-1}$. It is symmetric and positive.

The last equation is solved by a Gaussian in Θ,

$$P_0(B,\Theta) = Q(B)N(\det X)^{1/2} \exp[-(\Theta, X\Theta)/2]$$

$$= Q(B)N(\det X)^{1/2} \exp[-(\theta, X\theta)/(2a)], \qquad (4.7)$$

where $(\theta, X\theta) \equiv \int d^4x\, \theta^a(x)(X\theta)^a(x)$. Here $X = X(B)$ is a symmetric operator to be determined, and N is fixed by

$$\int D\theta\ N(\det X)^{1/2} \exp[-(\theta, X\theta)/(2a)] = 1. \qquad (4.8)$$

The upper limit on the θ integration is actually finite, but this gives a correction of order $\exp(-1/a)$ that we neglect. The solution (4.7) decreases rapidly as $|\theta|$ increases away from 0,

as expected, with a Gaussian width $|\theta| \sim a^{1/2}$. In the limit $a \to 0$, the support of the solution $P(B, \theta)$ shrinks to $\theta = 0$, and is given by

$$P(B, \theta) = \delta(\theta) Q(B). \tag{4.9}$$

We now check that Eq. (4.7) is actually the solution. Equation (4.6) yields two equations for X,

$$(\Theta, XVX\Theta) - (\Theta, XM\Theta) = 0,$$
$$\text{tr}(VX - M) = 0, \tag{4.10}$$

that hold identically for all Θ. The first equation yields $2XVX = XM + MX$, or $MY + YM = 2V$ for $Y \equiv X^{-1}$. Moreover, when this equation is satisfied, the second equation is automatically satisfied. To solve for Y, we take matrix elements in the basis provided by the eigenfunctions of the Faddeev-Popov operator $Mu_n = \lambda_n u_n$, and obtain $2(u_m, Vu_n) = (\lambda_m + \lambda_n)(u_m, Yu_n)$ or

$$(u_m, X^{-1}u_n) = (u_m, Yu_n) = 2(\lambda_m + \lambda_n)^{-1}(u_m, Vu_n)$$
$$= 2\int_0^\infty dt(u_m, \exp(-Mt)V\exp(-Mt)u_n). \tag{4.11}$$

This gives

$$X^{-1} = Y = 2\int_0^\infty dt \exp(-Mt)V\exp(-Mt)$$
$$= 2M^{-1}\int_0^\infty dt \exp(-Mt)(-\partial^2)\exp(-Mt)M^{-1}, \tag{4.12}$$

and $X = X(B)$ is indeed a positive operator, as is necessary for the normalizability of the Gaussian (4.7).

The coefficient function $Q(B)$ in Eq. (4.7) is left undetermined by the equation $H_0 P_0 = 0$. Since the leading term in the Hamiltonian $H = 1/a H_0 + \cdots$ leaves the solution indeterminate, we are in the case of degenerate perturbation theory, and the lowest order solution is determined by a higher order perturbation. To obtain an equation for $Q(B)$, we integrate the exact equation $HP = 0$ over Θ,

$$\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta)HP = 0, \tag{4.13}$$

where, we recall, $H = H_{BB} + H_{B\theta} + H_{\theta B} + H_{\theta\theta}$. This kills the $H_{\theta\theta}$ term that is of order $1/a$, for, by Eq. (4.3), it is the integral of an exact derivative, and thus vanishes identically, $\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta) \, H_{\theta\theta} P = \int D\Theta \, \delta/\delta\Theta \cdots = 0$. For the same reason it kills the $H_{\theta B}$ term that is of order $1/a^{1/2}$, $\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta) \, H_{\theta B} P = 0$. It also kills the $H_{B\theta}$ term that is of order $1/a^{1/2}$ because, by Eq. (4.1), the integral $\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta)H_{B\theta}P$ is of the form

$$\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta)J(a^{1/2}\Theta)\frac{\delta}{\delta\Theta}F$$
$$= -\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta)FJ(a^{1/2}\Theta)\frac{\delta}{\delta\Theta}1 = 0, \tag{4.14}$$

where the explicit form of F is not needed.[8] The first equality holds by the Lie group property that makes $J(a^{1/2}\Theta)\delta/\delta\Theta$ anti-Hermitian with respect to Haar measure $\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta)$.

[It is easy to verify that the equation

$$\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta) \, H_{B\theta}P = 0$$

holds in the small-a limit. This is the same as the small angle approximation, and we have, to the order required, $g(\theta) = \exp(\theta) = 1 + t^\alpha \theta^\alpha + (1/2)(t^\alpha \theta^\alpha)^2$. For the Maurer-Cartan form $(\partial g/\partial\theta^\beta)g^{-1} = t^a \omega_\beta^a$ we obtain, to the order required, $\omega_\beta^a = \delta^{a\beta} + (1/2)f^{a\gamma\beta}\theta^\gamma = \delta^{a\beta} + (a^{1/2}/2)f^{a\gamma\beta}\Theta^\gamma$. The second term is an antisymmetric matrix so for the Haar measure we get $\det \omega(a^{1/2}\Theta) = 1 + O(a)$, and for the matrix J_a^β, defined by $J_a^\beta \omega_\beta^c = \delta_a^c$, we get $J_a^\beta(a^{1/2}\Theta) = \delta^{a\beta} + (a^{1/2}/2)f^{a\gamma\beta}\Theta^\gamma + O(a)$. This gives

$$\int D\Theta \, \text{Det } \omega(a^{1/2}\Theta)J_a^\beta(a^{1/2}\Theta)\frac{\delta}{\delta\Theta^\beta}F$$
$$= \int D\Theta\left[\left(\delta^{\alpha\beta} + \frac{a^{1/2}}{2}f^{a\gamma\beta}\Theta^\gamma\right)\frac{\delta}{\delta\Theta^\beta} + O(a)\right]F.$$

The term in $\delta/\delta\Theta^\beta$ is an exact derivative because $f^{a\gamma\beta}$ is antisymmetric, and gives vanishing contribution. The leading term in F is of order $1/a^{1/2}$, so the remainder is of order $a^{1/2}$ and vanishes in the small a limit.]

We conclude that in Eq. (4.13), the only surviving term is H_{BB}, given in Eq. (3.29). It is independent of a and Θ, and Eq. (4.13) simplifies to

$$H_{BB}Q = 0. \tag{4.15}$$

From Eq. (3.29) we see that this equation is of the form

$$\cdots[\delta_{\mu\nu} - \partial_\mu(D \cdot \partial)^{-1}D_\nu]\left[\frac{\delta}{\delta B_\nu} - D_\kappa F_{\kappa\nu}(B)\right]Q = 0.$$

The left factor is orthogonal on ν to longitudinal fields, so it may be written

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[8]The fact that the integral on $D\Theta$ surgically kills the $H_{\theta B}$ and $H_{B\theta}$ terms is the payoff for using the curvilinear coordinates (B, θ). In a previous calculation by the author [3], the time-independent Fokker-Planck equation was solved using Cartesian coordinates A^{tr} and A^{lo} instead of (B, θ). This gave an additional contribution, not surgically killed by the corresponding integration over DA^{lo}, that was mistakenly neglected, and that was needed to cancel a spurious term, called K_2, in the effective drift force. Fortunately K_2 was neglected in [3], so what was thought to be an approximate formula there is in fact exact, and the calculation reported there is correct.

$$\cdots [\delta_{\mu\nu} - \partial_\mu (D\cdot\partial)^{-1} D_\nu] P_{\nu\lambda}^{\mathrm{tr}} \left[\frac{\delta}{\delta B_\lambda} + \frac{\delta S_{\mathrm{YM}}(B)}{\delta B_\lambda} \right] Q = 0,$$

where we have used the fact that functional differentiation with respect to a transverse field is ordinary functional differentiation with a transverse projector that comes from

$$\frac{\delta B_\mu^b(y)}{\delta B_\lambda^a(x)} = P_{\lambda\mu}^{\mathrm{tr}}(x-y)\,\delta^{ab}. \tag{4.16}$$

Thus the equation, $H_{BB} Q(B) = 0$, has the simple solution

$$Q(B) = N \exp[-S_{\mathrm{YM}}(B)]. \tag{4.17}$$

In continuum gauge theory, the Gribov region Ω is convex, as shown in Appendix C, and therefore it is connected, so the normalization of the solution (4.17) is unique. We have obtained the solution in the coordinate patch $\mathcal{U}$, in the limit $a \to 0$,

$$P(B,\theta) = N\delta(\theta) \exp[-S_{\mathrm{YM}}(B)]. \tag{4.18}$$

We express the solution $P(B,\theta)$ in terms of the original Cartesian coordinates A. The volume element is of course $\int dA$. To first order in θ we have $A = B + D(B)\theta$, and $\partial\cdot A = \partial\cdot D(B)\theta$, so

$$\delta(\theta) = \delta(\partial\cdot A)\det[-\partial\cdot D(A)]. \tag{4.19}$$

Inside the coordinate patch $\mathcal{U}$, the solution reads

$$P(A) = N\delta(\partial\cdot A)\det[-\partial\cdot D(A)]\exp[-S_{\mathrm{YM}}(A)]. \tag{4.20}$$

Its support lies on $\partial\cdot A = 0$, and it vanishes with $\det[-\partial\cdot D(A)]$ on the boundary $\partial\Omega$ of the Gribov region. We extend it to all of A space by stipulating that it vanishes outside $\mathcal{U}$. For the diffusion equation with a drift force, the equilibrium distribution is unique [53].

V. DYSON-SCHWINGER EQUATION FOR PARTITION FUNCTION

To be of use, the nonperturbative Faddeev-Popov formula (1.3) must be supplemented with a prescription for how the functional integral, restricted to the Gribov region Ω, is to be evaluated nonperturbatively. An earlier approach [54] is to insert a θ function $\theta[\lambda_1(B)]$ that effects a cutoff at the Gribov horizon. The θ function is given a suitable representation as an integral over auxiliary fields with a local effective action, and one integrates over all B without restriction and over the auxiliary fields. A far simpler approach [16] rests on the observation that the Gribov horizon $\partial\Omega$ is *a nodal surface* of the integrand because the Faddeev-Popov determinent $\det M(B) = \Pi_{n=1}^\infty \lambda_n(B)$ vanishes with $\lambda_1(B)$, that is to say, on $\partial\Omega$. The DS equations, which are derived by a partial integration, do not pick up a boundary term, and would have the same form if the integral were extended to infinity. In this approach we never have to know where the Gribov horizon actually is.

The partition function for the distribution (1.3) is given by

$$Z(J) = N \int_\Omega dB \det M(B) \exp[-S_{\mathrm{YM}}(B) + (J,B)], \tag{5.1}$$

where we have written $B \equiv A^{\mathrm{tr}}$, and $(J,B) \equiv \int d^4x\, J_\mu^a(x) B_\mu^a(x)$. Only the transverse part of J contributes, and we also take J to be identically transverse, $J = J^{\mathrm{tr}}$. (The extension of the present nonperturbative approach with a cutoff at the Gribov horizon to an off-shell gauge condition with a local and BRST-invariant action is sketched in Appendix B.) The Faddeev-Popov determinent $\det M(B)$ vanishes on the boundary $\partial\Omega$, so the identity

$$0 = \int_\Omega dB \frac{\delta}{\delta B_\mu^b(x)} (\det M(B) \exp[-S_{\mathrm{YM}}(B) + (J,B)]) \tag{5.2}$$

holds, *without any contribution from boundary terms even though the integral is cut off at the Gribov horizon $\partial\Omega$*. It is shown in Appendix C that the Gribov horizon surrounds the origin at a finite distance in all directions.

To derive the functional DS equation for $Z(J)$, we write $\det M(B) = \exp[\mathrm{Tr}\ln M(B)]$, and define the total action

$$\Sigma(B) \equiv S_{\mathrm{YM}}(B) - \mathrm{Tr}\ln M(B), \tag{5.3}$$

so Eq. (5.2) reads

$$0 = \int_\Omega dB \left(J_\mu^b(x) - \frac{\delta\Sigma(B)}{\delta B_\mu^b(x)} \right)(\det M(B) \times \exp[-S_{\mathrm{YM}}(B) + (J,B)]). \tag{5.4}$$

Although $\Sigma(B)$ is not local in B, we shall derive the same DS equations as one gets from the usual local action of gluons and ghosts. We have

$$\frac{\delta\Sigma(B)}{\delta B_\mu^b(x)} = -[D_\lambda F_{\lambda\mu}(B)]^{b,\mathrm{tr}}(x) - \mathcal{I}_{\mathrm{gh},\mu}^b(x;B) \tag{5.5}$$

by Eq. (4.16), where "tr" means transverse part, $[X_\mu]^{\mathrm{tr}} \equiv X_\mu - \partial_\mu(\partial^2)^{-1}\partial_\nu X_\nu$, and the ghost current is given by

$$\mathcal{I}_{\mathrm{gh},\mu}^b(x;B) \equiv \frac{\delta[\mathrm{Tr}\ln M(B)]}{\delta B_\mu^b(x)} = \mathrm{Tr}\left(\frac{\delta M(B)}{\delta B_\mu^b(x)} M^{-1}(B) \right)$$

$$= -\int d^4y\, \frac{\delta[\partial^2\delta^{ac} + g_0 f^{adc} B_\lambda^d(y)\partial_\lambda]}{\delta B_\mu^b(x)}$$

$$\times (M^{-1})^{ca}(y,z;B)\bigg|_{z=y}$$

$$= -g_0 f^{abc} \int d^4y\, P_{\mu\lambda}^{\mathrm{tr}}(x-y)\partial_\lambda(M^{-1})^{ca}$$

$$\times (y,z;B)\bigg|_{z=y}. \tag{5.6}$$

Here and below, derivatives act on the left argument of propagators. The identity (5.4) reads

$$0 = \int_\Omega dB (J^b_\mu(x) + [D_\lambda F^b_{\lambda\mu}(B)]^{\mathrm{tr}}(x) + \mathcal{I}^b_{\mathrm{gh},\mu}(x;B))$$

$$\times (\det M(B) \exp[-S_{\mathrm{YM}}(B) + (J,B)]) \qquad (5.7)$$

and yields the functional DS equation for the partition function $Z(J)$,

$$-\left(D_\lambda F^b_{\lambda\mu}\left(\frac{\delta}{\delta J}\right)\right)^{\mathrm{tr}}(x) Z(J) = [\mathcal{I}^b_{\mathrm{qu,gh},\mu}(x;J) + J^b_\mu(x)] Z(J),$$
$$(5.8)$$

where $D_\lambda F^b_{\lambda\mu}(\delta/\delta J)$ is a cubic polynomial in $\delta/\delta J$. The quantum ghost current in the presence of the source J is, by Eq. (5.6),

$$\mathcal{I}^b_{\mathrm{qu,gh},\mu}(x;J) \equiv \langle \mathcal{I}^b_{\mathrm{gh},\mu}(x;B) \rangle_J$$

$$= -g_0 f^{abc} \int d^4 y P^{\mathrm{tr}}_{\mu\lambda}(x-y) \partial_\lambda \mathcal{G}^{ca}(y,z;J) \Big|_{z=y}.$$
$$(5.9)$$

Here we have introduced the ghost propagator in the presence of the source J,

$$\mathcal{G}^{ca}(x,y;J) \equiv \langle (M^{-1})^{ca}(x,y;B) \rangle_J, \qquad (5.10)$$

where $\langle \mathcal{O} \rangle_J$ denotes the mean value of $\mathcal{O}(B)$ in the presence of the source J,

$$\langle \mathcal{O} \rangle_J = Z^{-1}(J) N \int_\Omega dB \det M(B) \mathcal{O}(B)$$

$$\times \exp[-S_{\mathrm{YM}}(B) + (J,B)]. \qquad (5.11)$$

To obtain a closed system of equations, we need a DS equation for the ghost propagator $\mathcal{G}^{ab}(x,y;J)$. It contains a term proportional to $\lambda_1^{-1}(B)$, so *we must avoid integrating by parts on B or introducing ghost sources* (see Appendix B). Fortunately the functional DS equation for $\mathcal{G}^{ab}(x,y;J)$ follows from the trivial identity $I = M(B) M^{-1}(B)$, that we average with $P(B) \exp[(J,B)]$,

$$\delta(x-y) \delta^{ab} Z(J) = \int_\Omega DB M^{ac}(B)$$

$$\times (M^{-1})^{cb}_{xy}(B) P(B) \exp[(J,B)]$$

$$= M^{ac}\left(\frac{\delta}{\delta J}\right) \int_\Omega DB (M^{-1})^{cb}_{xy}(B) P(B)$$

$$\times \exp[(J,B)], \qquad (5.12)$$

where $M^{ac}(\delta/\delta J) = -\partial^2 \delta^{ac} - g_0 f^{abc}(\delta/\delta J^b_\mu)\partial_\mu$. Here $P(B) = \det M(B) \exp[-S_{\mathrm{YM}}(B)]$ is the probability distribution, although the form of the DS equation for the ghost propgator is independent of $P(B)$. This gives the DS equation for the ghost propagator

$$M^{ac}\left(\frac{\delta}{\delta J}\right)[\mathcal{G}^{cb}(x,y;J) Z(J)] = \delta(x-y) \delta^{ab} Z(J).$$
$$(5.13)$$

Equations (5.8) and (5.13) and formula (5.9) provide a complete system of functional DS equations for the partition function $Z(J)$ and the ghost propagator $\mathcal{G}^{cb}(x,y;J)$.

VI. FUNCTIONAL DS EQUATION FOR GLUON AND GHOST PROPAGATORS

We change the variable from $Z(J) = \exp W(J)$ to the "free energy" $W(J)$. For the ghost propagator we obtain

$$M^{ac}\left(\frac{\delta W}{\delta J} + \frac{\delta}{\delta J}\right) \mathcal{G}^{cb}(x,y;J) = \delta(x-y) \delta^{ab}. \qquad (6.1)$$

We again change variables by Legendre transformation from the free energy $W(J)$ to the quantum effective action

$$\Gamma(B_{\mathrm{cl}}) = J_x B_{\mathrm{cl},x} - W(J), \qquad (6.2)$$

where the new variable $B^a_{\mathrm{cl},\mu}(x)$ is defined by

$$B^a_{\mathrm{cl},\mu}(x;J) \equiv \frac{\delta W(J)}{\delta J^a_\mu(x)} = \frac{1}{Z} \frac{\delta Z(J)}{\delta J^a_\mu(x)} = \langle B^a_\mu(x) \rangle_J. \qquad (6.3)$$

It is identically transverse, $B_{\mathrm{cl},\mu} = B^{\mathrm{tr}}_{\mathrm{cl},\mu}$, and takes values in Ω because $B_{\mathrm{cl}}(J) = \langle B \rangle_J$ is an average with a positive probability, $N \det M(B) \exp(B,J)$, over the convex region Ω. Inversion of $B_{\mathrm{cl}} = B_{\mathrm{cl}}(J)$ to obtain $J = J(B_{\mathrm{cl}})$ is possible because the gluon propagator in the presence of the source J,

$$\mathcal{D}_{xy}(J) \equiv \langle (B_x - \langle B_x \rangle_J)(B_y - \langle B_y \rangle_J) \rangle_J = \frac{\partial^2 W}{\partial J_x \partial J_y} = \frac{\partial B_y(J)}{\partial J_x},$$
$$(6.4)$$

is a positive matrix. The gluon propagator is expressed in terms of the Legendre-transformed variables B and $\Gamma(B)$ by

$$\mathcal{D}^{-1}_{xy}(B) = \frac{\partial^2 \Gamma(B)}{\partial B_x \partial B_y}. \qquad (6.5)$$

Here and below, we write B instead of B_{cl}. The gluon propagator and its inverse are identically transverse, $\partial_\lambda \mathcal{D}_{\lambda\mu}(x,y;B) = 0$.

Under the Legendre transformation, derivatives transform according to

$$\frac{\delta}{\delta J^a_\lambda(x)} = \left(\mathcal{D}\frac{\delta}{\delta B}\right)^a_\lambda(x) \equiv \int d^4 y \mathcal{D}^{ab}_{\lambda\mu}(x,y;B) \frac{\delta}{\delta B^b_\mu(y)}, \qquad (6.6)$$

as one sees from Eq. (6.4). In terms of the Legendre transformed variables, the DS equation (6.1) for the ghost propagator reads
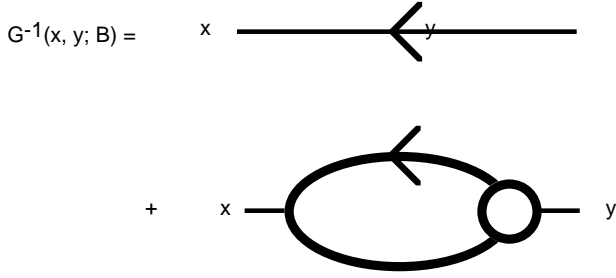
$$G^{-1}(x,y;B) =$$

FIG. 2. The functional DS equation (6.10) for the complete ghost propagator $\mathcal{G}(x,y;B)$ in the presence of the source B. The thin line is the tree-level term. The heavy line with (without) the arrow is the complete ghost (gluon) propagator $\mathcal{G}(x,y;B)$ ($\mathcal{D}(x,y;B)$) in the presence of the source B. The circle is the complete ghost-ghost-gluon vertex in the presence of the source B.

$$\delta(x-y)\delta^{ab} = M^{ac}\left(B + \mathcal{D}\frac{\delta}{\delta B}\right)\mathcal{G}^{cb}(x,y;B)$$

$$= M^{ac}(B)\mathcal{G}^{cb}(x,y;B)$$

$$- g_0 f^{adc}\int dz \mathcal{D}^{de}_{\mu\nu}(x,z;B)\frac{\delta}{\delta B^e_\nu(z)}\partial_\mu$$

$$\times \mathcal{G}^{cb}(x,y;B),\qquad (6.7)$$

where $\mathcal{G}^{cb}(x,y;B) \equiv \mathcal{G}^{cb}[x,y;J(B)]$ is the ghost propagator expressed in terms of the source B. Finally, instead of $\mathcal{G}^{ab}(x,y;B)$, we take as the new unknown variable the inverse ghost propagator $\Gamma^{ab}_{\text{gh}}(x,y;B)$ defined by

$$\Gamma_{\text{gh},xy}(B) \equiv \mathcal{G}^{-1}_{xy}(B)$$

$$\leftrightarrow \int dy\,\Gamma^{ab}_{\text{gh}}(x,y;B)\mathcal{G}^{bc}(y,z;B)$$

$$= \delta(x-z)\delta^{ac}.\qquad (6.8)$$

We substitute

$$\frac{\partial}{\partial B_z}\mathcal{G}_{xy}(B) = -\mathcal{G}_{xu}(B)\frac{\partial\Gamma_{\text{gh},uv}(B)}{\partial B_z}\mathcal{G}_{vy}(B)\qquad (6.9)$$

into the previous DS equation, and multiply on the right by the matrix $\Gamma_{\text{gh},yw}$ to obtain the functional DS equation for the inverse ghost propagator

$$\Gamma^{ab}_{\text{gh}}(x,y;B) = M^{ab}(B)\delta(x-y)$$

$$+ g_0 f^{adc}\int dz\,du\,\mathcal{D}^{de}_{\mu\nu}(x,z;B)\partial_\mu$$

$$\times \mathcal{G}^{cf}(x,u;B)\frac{\delta\Gamma^{fb}_{\text{gh}}(u,y;B)}{\delta B^e_\nu(z)},\qquad (6.10)$$

that is represented diagrammatically in Fig. 2. Here

$\delta\Gamma^{fb}_{\text{gh}}(u,y;B)/\delta B^e_\nu(z)$ is the complete ghost-ghost-gluon vertex in the presence of the source.

We make the same changes of variable in the functional DS equation (5.8) for $Z(J) = \exp[W(J)]$. We evaluate[9]

$$\nabla_\lambda F^a_{\lambda\kappa}\left(\frac{\delta}{\delta J}\right)(x)\exp[W(J)]$$

$$= \exp[W(J)]\Big[\partial_\lambda(\partial_\lambda B^a_\kappa - \partial_\kappa B^a_\lambda + g_0 f^{abc}B^b_\lambda B^c_\kappa$$

$$+ \mathcal{D}^{bc}_{\lambda\kappa}(x,x,B)) + \ldots + g_0 f^{abc}\left(B^b_\lambda + \frac{\delta}{\delta J^b_\lambda}\right)$$

$$\times g_0 f^{cde}[B^d_\lambda B^e_\kappa + \mathcal{D}^{de}_{\lambda\kappa}(x,x,B)]\Big]$$

$$= \exp[W(J)][\nabla_\lambda F^a_{\lambda\kappa}(B) + \mathcal{I}^b_{\text{qu.gl},\mu}(x;B)],\qquad (6.11)$$

where $B_\mu = \delta W(J)/\delta J_\mu$ and $\mathcal{D}_{\mu\nu}(x,y;B) = \delta B_\nu(x)/\delta J_\mu(y)$. The quantum gluon current in the presence of the source B is defined by

$$\mathcal{I}^b_{\text{qu.gl},\kappa}(x;B) \equiv \Big(g_0 f^{abc}(\delta_{\lambda\mu}\delta_{\nu\kappa} + \delta_{\lambda\kappa}\delta_{\mu\nu} - 2\delta_{\lambda\nu}\delta_{\kappa\mu})$$

$$\times \nabla^{bd}_\lambda \mathcal{D}^{dc}_{\mu\nu}(x,z;B)\Big|_{z=x} - g_0^2 f^{abc}f^{cde}$$

$$\times \int dy\,dz\,dw\,\mathcal{D}^{bf}_{\lambda\rho}(x,y;B)\mathcal{D}^{dg}_{\lambda\sigma}(x,z;B)$$

$$\times \mathcal{D}^{eh}_{\kappa\tau}(x,w;B)\Gamma^{fgh}_{\rho\sigma\tau}(y,z,w;B)\Big)^{\text{tr}}.\quad (6.12)$$

Here

$$\Gamma^{fgh}_{\rho\sigma\tau}(y,z,w;B) \equiv \frac{\delta^3\Gamma(B)}{\delta B^f_\rho(y)\delta B^g_\sigma(z)\delta B^h_\tau(w)}\qquad (6.13)$$

is the complete triple-gluon vertex in the presence of the source B. This gives the functional DS equation for $\Gamma(B)$,

$$\frac{\delta\Gamma(B)}{\delta B^a_\mu(x)} = -\nabla_\lambda F^a_{\lambda\mu}(B)(x) - \mathcal{I}^b_{\text{qu.gh},\mu}(x;B) - \mathcal{I}^b_{\text{qu.gl},\mu}(x;B),$$
$$(6.14)$$

where, by Eq. (5.9),

$$\mathcal{I}^b_{\text{qu.gh},\mu}(x;B) \equiv \mathcal{I}^b_{\text{qu.gh},\mu}[x;J(B)]$$

$$= -g_0 f^{abc}\int d^4 y\,P^{\text{tr}}_{\mu\lambda}(x-y)\partial_\lambda \mathcal{G}^{ca}(y,z;B)\Big|_{z=y}$$
$$(6.15)$$

is the quantum ghost current in the presence of the source B.

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[9]In this section we write $\nabla^{ac}_\mu(A) = \partial_\mu \delta^{ac} + g_0 f^{abc}A^b_\mu$ for the gauge-covariant derivative instead of $D^{ac}_\mu(A)$ to avoid confusion with the gluon propagator $\mathcal{D}$.

A more explicit form of this equation is obtained by differentiating with respect to $B_\tau^g(u)$, which yields a functional DS equation for the inverse gluon propagator,

$$\frac{\delta^2 \Gamma(B)}{\delta B_\kappa^a(x) \delta B_\tau^g(u)} = (-\delta_{\kappa\tau}(\nabla_\lambda \nabla_\lambda)^{ag} + (\nabla_\kappa \nabla_\tau)^{ag} - 2g_0 f^{acg} F_{\kappa\tau}^c)^{\mathrm{tr}}$$

$$\times \delta(x-u) + (\text{ghost loop})$$

$$+ (1 \text{ gluon loop}) + (\text{tadpole})$$

$$+ (2 \text{ gluon loops}), \qquad (6.16)$$

where

$$(\text{ghost loop}) \equiv -g_0 f^{abc} \left(\int dy dz \partial_\kappa \mathcal{G}^{bd}(x,y;B) \right.$$

$$\left. \times \mathcal{G}^{ce}(x,z;B) \frac{\delta \Gamma_{\mathrm{gh}}^{de}(y,z;B)}{\delta b_\tau^g(u)} \right)^{\mathrm{tr}}, \quad (6.17)$$

$$(1 \text{ gluon loop}) \equiv \int dy dz$$

$$\times (g_0 f^{abc} (\delta_{\lambda\mu}\delta_{\kappa\nu} + \delta_{\kappa\lambda}\delta_{\mu\nu} - 2\delta_{\lambda\nu}\delta_{\kappa\mu})$$

$$+ \nabla_\lambda^{bd} \mathcal{D}_{\mu\rho}^{de}(x,y;B) \mathcal{D}_{\nu\sigma}^{cf}(x,z;B))^{\mathrm{tr}}$$

$$\times \Gamma_{\rho\sigma\tau}^{efg}(y,z,u;B), \qquad (6.18)$$

$$(\text{tadpole}) \equiv g_0^2 f^{abc} f^{bgd} ((\delta_{\lambda\mu}\delta_{\kappa\nu} + \delta_{\kappa\lambda}\delta_{\mu\nu} - 2\delta_{\lambda\nu}\delta_{\kappa\mu})$$

$$\times \delta(x-u) \mathcal{D}_{\mu\nu}^{dc}(x,x;B))^{\mathrm{tr}}, \qquad (6.19)$$

where superscript "tr" means projection onto transverse parts on (x,κ) and (u,τ). The complete ghost-ghost-gluon vertex in the presence of the source B, $\delta \Gamma_{\mathrm{gh}}^{de}(y,z;B)/\delta B_\tau^g(u)$, reappears in Eq. (6.17), and the complete triple-gluon vertex $\Gamma_{\rho\sigma\tau}^{efg}(y,z,u;B)$ in the presence of the source B is defined in Eq. (6.13). We do not write out explicitly the two-loop term, but all terms are expressed graphically in Fig. 3.

The pair of equations (6.10) and (6.16) are a complete system of functional DS equations for the quantum effective action $\Gamma(B)$, and for the inverse ghost propagator $\Gamma_{\mathrm{gh}}^{ab}(x,y;B)$. These functional equations are converted to equations for the coefficient functions by differentiating an arbitrary number of times with respect to B, and then setting $B=0$.

VII. HORIZON CONDITION AND RENORMALIZATION

Solutions are subject to the *supplementary conditions* that both the gluon and ghost inverse propagators $\delta^2 \Gamma(B)/\delta B_\kappa^a(x) \delta B_\lambda^b(y)$ and $\Gamma_{\mathrm{gh}}^{ab}(x,y;B)$ be positive matrices. Another supplementary condition results from the fact, discussed in Appendix A, that in a space of high-dimension, entropy favors a high concentration of population very near the boundary $\partial\Omega$ of the bounded region Ω. The boundary
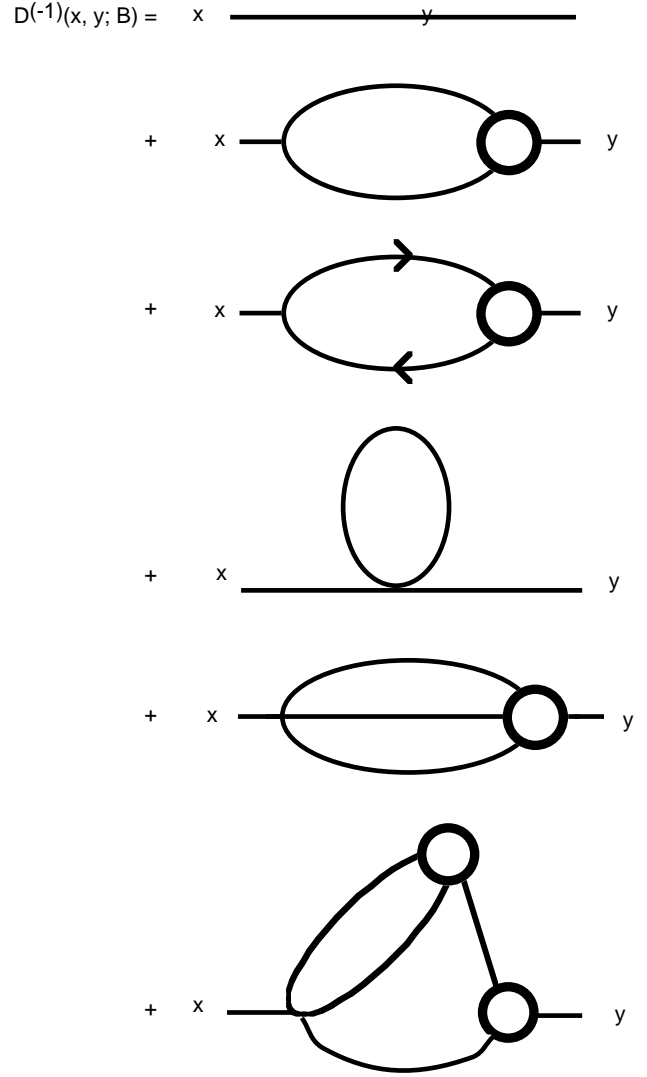

FIG. 3. The functional DS equation (6.16) for the complete gluon propagator $\mathcal{D}(x,y;B)$ in the presence of the source B. The thin line is the tree-level term. The heavy line with (without) the arrow is the complete ghost (gluon) propagator $\mathcal{G}(x,y;B)$ ($\mathcal{D}(x,y;B)$) in the presence of the source B. The circles are complete 3- and 4-vertices in the presence of the source B.

occurs where the lowest nontrivial eigenvalue of the Faddeev-Popov operator $M(B)$ vanishes. Thus, for typical configurations B, the positive operator $M(B)$ has a very small eigenvalue and, in fact, it has a high density of eigenvalues $\rho(\lambda,B)$ at $\lambda=0$, per unit Euclidean volume V, as compared to the Laplacian operator [25]. This makes the ghost propagator, $G(x-y)\delta^{ab} = \langle (M^{-1})^{ab}(x,y;B) \rangle$, long range, so in momentum space it is enhanced at $p=0$ compared to $1/p^2$, $\lim_{p\to 0}[p^2 \tilde{G}(p)]^{-1} = 0$ [1,25]. This property will provide a nonperturbative formula for the ghost-propagator renormalization constant $\tilde{Z}_3$ that moreover is consistent with the perturbative renormalization group.

The gluon and ghost propagators, with source $B=0$, are given in momentum space by

$$D_{\mu\nu}(x) = (2\pi)^{-4} \int d^4k \widetilde{D}_{\mu\nu}(k) \exp(ik \cdot x),$$

$$G(x) = (2\pi)^{-4} \int d^4p \widetilde{G}(p) \exp(ip \cdot x), \qquad (7.1)$$

and the ghost-gluon vertex by

$$f^{abc} \Gamma_\mu(x-y, y-z) \equiv \left. \frac{\delta \Gamma_{gh}^{ac}(x, z; B)}{\delta B_\mu^b(y)} \right|_{B=0}$$

$$= f^{abc} (2\pi)^{-8} \int d^4p d^4q \, \widetilde{\Gamma}_\mu(p, q)$$

$$\times \exp[ip \cdot (x-y) + iq \cdot (y-z)]. \qquad (7.2)$$

The DS equation for the ghost propagator $\widetilde{G}(p)$, obtained from Eq. (6.10) by setting $B=0$, reads

$$\widetilde{G}^{-1}(p) = p^2 - Ng_0 ip_\mu (2\pi)^{-4} \int d^4k \widetilde{D}_{\mu\nu}(k)$$

$$\times \widetilde{G}(p-k) \widetilde{\Gamma}_\nu(p-k, p). \qquad (7.3)$$

All quantities are unrenormalized, and we have used $f^{abc} f^{cde} = N\delta^{ae}$ for SU(N).

Factorization of the external ghost momentum is a well-known special property of the Landau gauge that makes it less divergent than other gauges. To make it explicit, we note that the ghost-ghost-gluon vertex $\widetilde{\Gamma}_\mu(p, q)$ is a function of two linearly independent 4-vectors. It is also transverse, $(p - q)_\mu \widetilde{\Gamma}_\mu(p, q) = 0$, because the transversality condition is imposed on-shell, so it may be written

$$\widetilde{\Gamma}_\mu(p, q) = -ig_0 P_{\mu\nu}^{tr}(k) p_\nu V(p^2, k^2, q^2), \qquad (7.4)$$

where $k \equiv q - p$. The scalar vertex function $V(p^2, k^2, q^2)$ is symmetric $V(p^2, k^2, q^2) = V(q^2, k^2, p^2)$ in consequence of the symmetry $\mathcal{G}^{ac}(x, z; B) = \mathcal{G}^{ca}(z, x; B)$. The DS equation for the ghost propagator reads

$$\widetilde{G}^{-1}(p) = p^2 - Ng_0^2 p_\mu p_\nu (2\pi)^{-4} \int d^4k \widetilde{D}_{\mu\nu}(k) \widetilde{G}(p-k)$$

$$\times V[(p-k)^2, k^2, p^2], \qquad (7.5)$$

where the factorization of the two external ghost momenta p_μ and p_ν is now explicit.

This equation is divergent and must be renormalized. In perturbative renormalization theory, quantities renormalize according to

$$D_{\mu\nu} = Z_3 D_{R,\mu\nu}; \quad G = \widetilde{Z}_3 G_R; \quad V = \widetilde{Z}_1^{-1} V_R;$$

$$g_0 = \widetilde{Z}_1 (\widetilde{Z}_3 Z_3^{1/2})^{-1} g_R, \qquad (7.6)$$

and in Landau gauge the additional special property

$$\widetilde{Z}_1 = 1; \quad V = V_R; \quad g_0 = (\widetilde{Z}_3 Z_3^{1/2})^{-1} g_R \qquad (7.7)$$

holds. In terms of renormalized quantities, the DS equation for the ghost propagator reads

$$\widetilde{G}_R^{-1}(p) = p^2 \widetilde{Z}_3 - Ng_R^2 p_\mu p_\nu (2\pi)^{-4} \int d^4k \widetilde{D}_{R,\mu\nu}(k)$$

$$\times \widetilde{G}_R(p-k) V_R[(p-k)^2, k^2, p^2]. \qquad (7.8)$$

To avoid infrared difficulties, the ghost propagator is usually renormalized at some finite renormalization mass μ. However, the horizon condition, $\lim_{p^2 \to 0}[p^2 G(p)]^{-1} = 0$, allows us to renormalize at $p = 0$. It tells us that in the DS equation (7.8), the first term, $p^2 \widetilde{Z}_3$, must be cancelled by the term of order p^2 in the second term. This gives a renormalization condition at $p = 0$, in the form of an equation for $\widetilde{Z}_3$,

$$\widetilde{Z}_3 = Ng_R^2 (2\pi)^{-4} \int_{|k| < \Lambda} d^4k \hat{p}_\mu \hat{p}_\nu \widetilde{D}_{R,\mu\nu}(k)$$

$$\times \widetilde{G}_R(k) V_R(k^2 k^2, 0), \qquad (7.9)$$

where Λ is an ultraviolet cutoff. We have set $p = 0$ in the integrand, and the integral is independent of the direction $\hat{p}$. This statement of the horizon condition shows that it is flagrantly nonperturbative because, in perturbation theory, the left-hand side is of order 1, but the right-hand side is of leading order g_R^2.

The last equation gives the renormalization-group flow,

$$\Lambda \frac{\partial \widetilde{Z}_3}{\partial \Lambda} = Ng_R^2 (p_\mu p_\nu / p^2)(2\pi)^{-4} \Lambda^4 \widetilde{D}_R(\Lambda) \widetilde{G}_R(\Lambda)$$

$$\times V_R(\Lambda^2, \Lambda^2, 0) \int d^3\hat{k} (\delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\nu)$$

$$= Ng_R^2 (4\pi)^{-2}(3/2)\Lambda^4 \widetilde{D}_R(\Lambda) \widetilde{G}_R(\Lambda) V_R(\Lambda^2, \Lambda^2, 0). \qquad (7.10)$$

As a check, we note that if we take the tree values $\widetilde{D}_R(\Lambda) = \widetilde{G}_R(\Lambda) = 1/\Lambda^2$, and $V(p^2, k^2, q^2) = 1$, we obtain

$$\Lambda \frac{\partial \widetilde{Z}_3}{\partial \Lambda} = (4\pi)^{-2}(3/2)Ng_0^2 + O(g_0^4). \qquad (7.11)$$

The term of order g_0^2 is scheme-independent, and agrees with the standard one-loop expression in Landau gauge. Thus the horizon condition provides a normalization condition for the ghost propagator at $p = 0$ that is in flagrant disagreement with perturbation theory, but nevertheless satisfies the perturbative renormalization-group flow equation.

We substitute Eq. (7.9) into the DS equation (7.8) for the ghost propagator, and obtain

$$\widetilde{G}_R^{-1}(p) = N g_R^2 p_\mu p_\nu (2\pi)^{-4} \int d^4k \widetilde{D}_{R,\mu\nu}(k)$$

$$\times \{ \widetilde{G}_R(k) V_R(k^2, k^2, 0)$$

$$- \widetilde{G}_R(p-k) V_R[(p-k)^2, k^2, p^2] \}. \quad (7.12)$$

This is a finite, renormalized DS equation for the ghost propagator. It is invariant under the renormalization group in the sense that it is form-invariant under the transformations (7.6) and (7.7) of perturbative renormalization theory in Landau gauge. This equation, from which the tree term k^2 has been eliminated by the horizon condition, gives the ghost propagator an infrared anomalous dimension a_G, so it behaves likes $G(k) \sim (\mu^2)^{a_G}/(k^2)^{1+a_G}$ in the infrared. This puts QCD into a nonperturbative phase.

VIII. EXACT INFRARED ASYMPTOTIC LIMIT OF QCD

Recent solutions of the truncated coupled DS equations for the gluon and ghost propagators yield ghost propagators that are enhanced in the infrared, and gluon propagators that are infrared suppressed [3,17–23]. Typical values for the infrared asymptotic form of the gluon and ghost propagators [20] and [21] are

$$D^{\mathrm{as}}(k) = \mu^{2a_D}/(k^2)^{1+a_D} \approx (k^2)^{0.187}/(\mu^2)^{1.187},$$

$$G^{\mathrm{as}}(k) = \mu^{2a_D}/(k^2)^{1+a_G} \approx (\mu^2)^{0.595}/(k^2)^{1.595},$$

$$(8.1)$$

$a_G = (93 - \sqrt{1201})/98 \approx 0.595$, $a_D = -2a_G$, where a_D and a_G are the infrared critical exponents of the ghost and gluon. The gluon propagator $\widetilde{D}(k)$ is so strongly suppressed at $k = 0$ that it vanishes $\widetilde{D}(0) = 0$. With $D(x-y) = \langle A(x)A(y) \rangle$, this corresponds to suppression of the low-frequency modes of $A(x)$ in the functional integral. The actual values of the infrared critical exponents do not depend too strongly on the truncation scheme [20]. The salient infrared features are easily understood. The cutoff of the functional integral at the Gribov horizon is implemented in the DS equations by the horizon condition. It states that the ghost propagator $G(k)$ is enhanced in the infrared or, equivalently, that the infrared critical exponent of the ghost is positive, $a_G > 0$. The DS equations yield $a_D = -2a_G$, so enhancement of the ghost causes suppression of the gluon in the infrared. This is the expression in the DS equations of the proximity of the Gribov horizon in infrared directions.

The results of calculation with the DS equations are in at least qualitative agreement with numerical evaluations of gluon and ghost propagators [13,55–59], which, on sufficiently large lattices, yield a gluon propagator $D(k)$ that turns over and decreases as k decreases [60–67] [possibly extrapolating to $D(0) = 0$ at infinite lattice volume], with a turnover point $k_{\max}$ that scales like a physical mass [68]. The only explanation for this counterintuitive turnover is the strong suppression of infrared components by the proximity of the Gribov horizon in infrared directions. The agreement of DS and numerical calculations gives us confidence that we

have a reliable picture of the gluon and ghost propagators including, in particular, the infrared region.

One may use the above expressions for the asymptotic propagators to estimate the convergence and magnitude of the various terms on the right-hand side of the DS equations, simply by counting powers of momentum. The dominant terms in the infrared region are the ones that contain the most ghost propators $G(k)$ in the loop integrals. The infrared limit of the truncated DS equations are found to have the following remarkable properties.

(i) The infrared limit of the DS equations decouples from the degrees of freedom associated with finite momentum and is free of ultraviolet divergences. Technically, what is found is that when the external momenta k_e are small compared to Λ_{QCD}, then the internal loop momenta k_i scale like the k_e, and the contribution when the k_i are large compared to k_e may be neglected. As a result, when the k_e are small, one may replace the propagators and vertices in internal loops by their infrared asymptotic forms $G^{\mathrm{as}}(k)$ and $D^{\mathrm{as}}(k)$, etc. The loop contributions that are dominant in the infrared are the ones that contain the most ghost propagators $\widetilde{G}(k)$.[10] The asymptotic infrared limit of the DS equations is highly convergent in the ultraviolet because $G^{\mathrm{as}}(k)$ is strongly suppressed there. In fact the asymptotic gluon equation, given below, is finite without renormalization, and the asymptotic ghost equation is finite with the renormalization (7.12). We conclude that the DS equations possess an infrared asymptotic limit that is well-defined, and decoupled from propagators and vertices at finite momentum.

(ii) The terms that are dominant in the infrared limit come from the action $-\mathrm{Tr}\ln M(B)$, whereas the subdominant terms come from Yang-Mills action $S_{\mathrm{YM}}(B)$. It is instructive to classify terms that are dominant or subdominant on the right-hand side of the DS equations accordingly as they originate with the action, $-\mathrm{Tr}\ln M(B)$, or with the Yang-Mills action, $S_{\mathrm{YM}}(B)$. Because the ghost propagator is enhanced in the infrared while the gluon propagator is suppressed, one finds that all subdominant terms and only the

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[10]For the ghost-propagator equation (7.3) or (7.12), both terms on the right-hand side are dominant, and both originate from the action $-\mathrm{Tr}\ln M(B)$. The gluon-propagator equation (6.16), with source $B = 0$, reads

$$\widetilde{D}_{\mu\nu}^{-1}(k) = (\delta_{\mu\nu}k^2 - k_\mu k_\nu) + (\text{gluon loops})$$

$$+ N g^2 (2\pi)^{-d} \int d^d p \widetilde{G}(p) + \widetilde{D}_{\mu\nu}^{-1}(k)$$

$$= (\delta_{\mu\nu}k^2 - k_\mu k_\nu) + (\text{gluon loops}) + N g^2 (2\pi)^{-d}$$

$$\times \int d^d p \widetilde{G}(p+k)(p+k)_\mu \widetilde{G}(p) \Gamma_\nu(p, p+k). \quad (8.2)$$

The tree term, of order k^2, is subdominanant in the infrared compared to $(D^{\mathrm{as}})^{-1}(k) \sim (k^2)^{-0.187}$. The dominant term on the right-hand side is the ghost loop that originates from the action $-\mathrm{Tr}\ln M(B)$, whereas the subdominant terms—namely the tree term, the gluon loop, and the two-loop term—all originate from the Yang-Mills action $S_{\mathrm{YM}}(B)$.
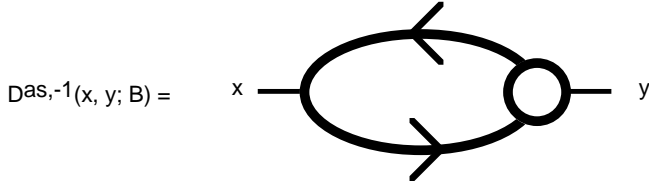
FIG. 4. The functional DS equation (8.4) for the complete infrared asymptotic gluon propagator $\hat{\mathcal{D}}(x,y;B)$ in the presence of the source B. There is no tree term nor any gluon loop, but only the ghost loop. The heavy line with the arrow is the complete infrared asymptotic ghost propagator $\hat{\mathcal{G}}(x,y;B)$ in the presence of the source B. The functional DS equation (8.6) for the complete infrared asymptotic ghost propagator $\hat{\mathcal{G}}(x,y;B)$ in the presence of the source B is as in Fig. 2.

subdominant terms disappear if one sets $S_{YM}(B)=0$ in the derivation of the DS equations given in Secs. V and VI.

Because the solutions of the truncated DS equations are consistent with numerical evaluations of the gluon and ghost propagators, the effects of truncation should not be too drastic. We therefore expect that properties (i) and (ii) of the truncated DS equations hold also for the solutions of the exact, untruncated, DS equations, that is, that there exists an exact infrared asymptotic limit of the DS equations that is obtained by setting $S_{YM}(B)=0$. This implies that the cutoff at the Gribov horizon suffices to make the functional integral over B converge, even though $\exp[-S_{YM}(B)]$ is replaced by 1.

We now write in functional form the exact infrared asymptotic DS equations (without truncation) that are obtained by setting $S_{YM}(B)=0$. We designate the generating functionals where the coefficient functions are given their asymptotic forms by $\hat{\Gamma}(B)$ and $\hat{\Gamma}_{gh}(x,y;B)$. The infrared asymptotic gluon and ghost inverse propagators are designated $(\hat{\mathcal{D}})_{xy}^{-1}(B)=\partial^2\hat{\Gamma}(B)/\partial B_x\partial B_y$ and $(\hat{\mathcal{G}})_{xy}^{-1}(B)=\hat{\Gamma}_{gh,xy}(B)$. The functional DS equation (6.10) for the ghost propagator is unchanged in form, as represented in Fig. 2,

$$\hat{\Gamma}_{gh}^{ab}(x,y;B)=(-\partial^2\delta^{ab}-g_0 f^{acb}B_\mu^c\partial_\mu)\delta(x-y)$$

$$+g_0 f^{adc}\int dzdu\,\hat{\mathcal{D}}_{\mu\nu}^{de}(x,z;B)\partial_\mu\hat{\mathcal{G}}^{cf}(x,u;B)$$

$$\times\frac{\delta\hat{\Gamma}_{gh}^{fb}(u,y;B)}{\delta B_\nu^e(z)}. \qquad (8.3)$$

In the infrared asymptotic limit, only the ghost loop contributes to the functional DS equation for the gluon propagator (6.16), which reads

$$(\hat{\mathcal{D}}^{-1})_{\mu\nu}^{ag}(x,y;B)=-g_0 f^{abc}\left(\int dzdu\,\partial_\mu\hat{\mathcal{G}}^{bd}(x,u;B)\right.$$

$$\left.\times\hat{\mathcal{G}}^{ce}(x,z;B)\frac{\delta\hat{\Gamma}_{gh}^{de}(u,z;B)}{\delta B_\nu^g(y)}\right)^{\mathrm{tr}}, \qquad (8.4)$$

and is diagrammed in Fig. 4.

An enormous simplification is apparent here, because the last equation allows an exact elimination of the asymptotic functional gluon propagator $\hat{\mathcal{D}}_{\mu\nu}^{ab}(x,y;B)$. The one remaining unknown is the inverse ghost propagator $\hat{\Gamma}_{gh,xy}(B)$.

When supplemented by the horizon condition (7.9), Eqs. (8.3) and (8.4) are a finite system when expressed in terms of renormalized quantities. Indeed, with $\tilde{Z}_1=1$, renormalization of the exact functional asymptotic equations is accomplished by writing

$$B=Z_3^{1/2}B_R; \quad \hat{\Gamma}(B)=\hat{\Gamma}_R(B_R);$$

$$\hat{\Gamma}_{gh}(x,y;B)=\tilde{Z}_3^{-1}\hat{\Gamma}_{gh,R}(x,y;B_R);$$

$$g_0=(\tilde{Z}_3 Z_3^{1/2})^{-1}g_R;$$

$$\hat{\mathcal{G}}(B)=\tilde{Z}_3\hat{\mathcal{G}}_R(B_R);$$

$$\hat{\mathcal{D}}_{\mu\nu}(x,y;B)=Z_3\hat{\mathcal{D}}_{R,\mu\nu}(x,y;B_R). \qquad (8.5)$$

Upon making these substitutions, the functional equation for the ghost propagator reads

$$\hat{\Gamma}_{gh,R}^{ab}(x,y;B_R)=(-\partial^2\delta^{ab}\tilde{Z}_3-g_R f^{abc}B_{R,\mu}^c\partial_\mu)\delta(x-y)$$

$$+g_R f^{adc}\int dzdu\,\hat{\mathcal{D}}_{R,\mu\nu}^{de}(x,z;B_R)\partial_\mu$$

$$\times\hat{\mathcal{G}}_R^{cf}(x,u;B_R)\frac{\delta\hat{\Gamma}_{gh,R}^{fb}(u,y;B_R)}{\delta B_{R,\nu}^e(z)}, \qquad (8.6)$$

where $\tilde{Z}_3$ is given in Eq. (7.9), and the renormalized infrared asymptotic functional gluon propagator is given by

$$(\hat{\mathcal{D}}_R^{-1})_{\mu\nu}^{ag}(x,y;B_R)=-g_R f^{abc}\left(\int dzdu\,\partial_\mu\hat{\mathcal{G}}_R^{bd}(x,u;B_R)\right.$$

$$\left.\times\hat{\mathcal{G}}_R^{ce}(x,z;B_R)\frac{\delta\hat{\Gamma}_{R,gh}^{de}(u,z;B_R)}{\delta B_{R,\nu}^g(y)}\right)^{\mathrm{tr}}. \qquad (8.7)$$

When Eq. (8.6) is expanded in a functional power series in B, $\tilde{Z}_3$ appears only in the equation of order $(B)^0$ that determines the ghost propagator with source $B=0$. This equation is finite as in the preceding section. All the higher order equations are independent of $\tilde{Z}_3$ and finite. Equations (8.6) and (8.7) are a complete system of functional DS equations, diagrammed in Figs. 4 and 2, that are free of divergences, and that define the asymptotic infrared theory. The gluon propagator may be eliminated exactly from Eq. (8.7), and the asymptotic infrared theory is defined by the functional inverse ghost propagator $\hat{\Gamma}_{gh,R}(x,y;B_R)$.

With $\tilde{Z}_3$ given in Eq. (7.9), these equations are invariant under the finite renormalization-group transformations

$$B_R = z_3^{1/2} B_R'; \quad \hat{\Gamma}_R(B_R) = \hat{\Gamma}_R'(B_R');$$

$$\hat{\Gamma}_{\text{gh},R}(x,y;B_R) = \hat{z}_3^{-1} \hat{\Gamma}_{\text{gh},R}'(x,y;B_R);$$

$$g_R = (\tilde{z}_3 z_3^{1/2})^{-1} g_R';$$

$$\hat{\mathcal{G}}_R(B_R) = \tilde{z}_3 \hat{\mathcal{G}}_R'(B_R');$$

$$\hat{\mathcal{D}}_{\mu\nu}(x,y;B_R) = z_3 \hat{\mathcal{D}}_{R,\mu\nu}'(x,y;B_R'). \quad (8.8)$$

The quantity $g_R^2 D_R(k) G_R^2(k) = g_0^2 D(k) G^2(k)$ is invariant under the renormalizations (7.6) and (7.7). Consequently a scheme-independent running coupling constant, characteristic of the Landau gauge, may be defined [17] by $\alpha_{\text{land}}(k) \equiv (4\pi)^{-1} g_0^2 D(k) G^2(k) (k^2)^3$. The asymptotic infrared theory is characterized, in addition to the infrared critical exponents a_G and a_D, by $\alpha_{\text{land}}(0) \approx 8.915/N$, for color SU($N$) [20].[11]

The limit, in which the Yang-Mills action $S_{\text{YM}}(B)$ is systematically neglected, is a continuum analogue of the lattice strong-coupling limit. Indeed if one rescales the gauge connection by the change of variable $A' \equiv g_0 A$, the effective action, from which the DS equations were derived, reads

$$\Sigma(A) = -\text{Tr} \ln M(A) + S_{\text{YM}}(A)$$

$$= -\text{Tr} \ln M'(A') + (g_0^2)^{-1} S_{\text{YM}}'(A'), \quad (8.9)$$

where $M'(A')$ and $S_{\text{YM}}'(A')$ are independent of g_0. Neglect of $S_{\text{YM}}(A)$ is the same as setting $g_0^{-2} = 0$ or, after renormalization, $g_R^{-2} = 0$. The asymptotic infrared limit is described by the effective action

$$\hat{\Sigma} = -\text{Tr} \ln M(A). \quad (8.10)$$

If one extends the nonperturbative formulation to a BRST-invariant theory, as outlined in Appendix B, the BRST-invariant local action reads

$$S = \int d^4x [s(\partial_\mu \bar{c} A_\mu) + S_{\text{YM}}(A)], \quad (8.11)$$

where the BRST operator acts according to

$$s A_\mu = D_\mu c; \quad sc = -c^2; \quad s\bar{c} = \lambda; \quad s\lambda = 0. \quad (8.12)$$

The asymptotic infrared limit is described by the local BRST-invariant action

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[11] A scheme-independent running coupling constant may be defined in the Coulomb gauge [69] by $\alpha_{\text{Coul}}(k) = (4\pi)^{-1} [12N/(11N - 2N_f)] k^2 \tilde{V}(k)$, with N_f quarks in the fundamental representation, where $\tilde{V}(|\mathbf{k}|) \equiv g_0^2 \lim_{k_4 \to \infty} D_{44}(\mathbf{k}, k_4)$. By contrast with $\alpha_{\text{land}}(k)$ that is finite at $k = 0$, it appears that $\alpha_{\text{Coul}}(k)$ diverges like $1/k^2$ at small k, in a realization of infrared slavery that features a string tension, $V(r) \sim \sigma_{\text{Coul}} r$ at large r [70] and [71].

$$\hat{S} \equiv \int d^4x s(\partial_\mu \bar{c} A_\mu) = \int d^4x (-\partial_\mu \bar{c} D_\mu c + \partial_\mu \lambda A_\mu), \quad (8.13)$$

and the infrared asymptotic correlators satisfy the Slavnov-Taylor identities.

IX. MASS GAP

The action $\hat{S}$ that describes the infrared asymptotic theory is not only BRST invariant, it is BRST exact, $\hat{S} = sX$, and defines a topological quantum field theory. To see what its properties may be, recall that $\hat{S}$ describes the asymptotic infrared limit, in which external momenta k were small compared to Λ_{QCD}, so it is the limit $\Lambda_{\text{QCD}} \to \infty$. If QCD is a theory with a mass gap of order Λ_{QCD}, then physical correlation lengths should vanish in the asymptotic theory, $R \sim \Lambda_{\text{QCD}}^{-1} \to 0$.

To show this, consider a gauge-invariant correlator, for example

$$C(x) = \langle F_x^2(A) F_0^2(A) \rangle$$

$$= N \int_\Omega dA\, dc\, d\bar{c}\, d\lambda F_x^2(A) F_0^2(A) \exp(-\hat{S}), \quad (9.1)$$

with $x \neq 0$, where Lorentz indices are suppressed $F^2(x) \to F_{\kappa\lambda}^a(x) F_{\mu\nu}^a(x)$, and the connected part is understood. Since the action is topological, we may make any transformation that commutes with s, without changing expectation values. As an example, consider the change of variable corresponding to a coordinate transformation $x_\mu' = x_\mu'(x)$ of A and c, leaving $\bar{c}$ and λ unchanged,

$$A_\mu'(x') = \frac{\partial x^\lambda}{\partial x'^\mu} A_\lambda(x); \quad c'(x') = c(x);$$

$$\bar{c}'(x) = \bar{c}(x); \quad \lambda'(x) = \lambda(x). \quad (9.2)$$

(The result is the same if $\bar{c}$ and λ are also transformed.) The infinitesimal form of this change of variable, with $x'^\mu = x^\mu - \xi^\mu(x)$, is given by

$$A_\mu(x) \to A_\mu'(x) = A_\mu(x) + \delta A_\mu(x)$$

$$= A_\mu(x) + \xi^\lambda \partial_\lambda A_\mu(x) + \partial_\mu \xi^\lambda A_\lambda(x),$$

$$c(x) \to c'(x) = c(x) + \delta c(x) = c + c(x) + \xi^\lambda \partial_\lambda c(x), \quad (9.3)$$

$$\bar{c}(x) \to \bar{c}'(x) = \bar{c}(x), \quad \lambda(x) \to \lambda'(x) = \lambda(x).$$

Upon making this change of variable in the functional integral, we obtain

$$C(x) = N \int_{\Omega'} dA\, dc\, d\bar{c}\, d\lambda F_x^2(A') F_0^2(A')$$

$$\times \exp[-\hat{S}(A', c', \bar{c}, \lambda)], \quad (9.4)$$

where $A' \equiv A + \delta A$, and

$$\hat{S}(A',c',\bar{c},\lambda) = \int d^4x[-\partial_\mu\bar{c}D_\mu(A')c' + \partial_\mu\lambda A'_\mu].$$
(9.5)

The integration in A space is cut off at the Gribov horizon $\partial\Omega'$ corresponding to $M(A')$. Integration over the ghost fields gives $\det M(A')$ which vanishes on the boundary $\partial\Omega'$. One may change the cutoff to the Gribov horizon $\partial\Omega$ corresponding to $M(A)$ because the error is only of order ξ^2. Moreover, $F_x^2(A')F_0^2(A')$ is the coordinate transform of $F_x^2(A)F_0^2(A)$, which we write as

$$F_x^2(A')F_0^2(A') = [1 + \xi^\lambda\partial_\lambda + L(\partial\xi)]F_x^2(A)F_0^2(A),$$
(9.6)

where $L(\partial\xi)$ is a numerical matrix that is linear in $\partial_\lambda\xi_\mu$ and acts on the tensorial indices of $F_x^2(A)$ $F_0^2(A)$, and we have

$$C(x) = [1 + \xi^\lambda\partial_\lambda + L(\partial\xi)]N\int_\Omega dAdcd\bar{c}d\lambda F_x^2(A)F_0^2(A)$$

$$\times \exp[-\hat{S}(A',c',\bar{c},\lambda)].$$
(9.7)

One may verify that the s operator commutes with the coordinate transformation, $sA' = D(A')c'$, so

$$\hat{S}(A',c',\bar{c},\lambda) = \int d^4x s(\partial_\mu\bar{c}A'_\mu) = \hat{S}(A,c,\bar{c},\lambda) + s\delta X,$$
(9.8)

where $\delta X = \int d^4x\partial_\mu\bar{c}\delta A_\mu$. Thus the variation of $\hat{S}$ is also s exact, and we have

$$C(x) = [1 + \xi^\lambda\partial_\lambda + L(\partial\xi)]N\int_\Omega dAdcd\bar{c}d\lambda F_x^2(A)F_0^2(A)$$

$$\times(1 - s\delta X)\exp[-\hat{S}(A,c,\bar{c},\lambda)]$$

$$= [1 + \xi^\lambda\partial_\lambda + L(\partial\xi)]\langle F_x^2(A)F_0^2(A)(1 - s\delta X)\rangle. \quad (9.9)$$

Gauge-invariant operators are s invariant,

$$\langle F_x^2(A)F_0^2(A)s\delta X\rangle = \langle s[F_x^2(A)F_0^2(A)\delta X]\rangle = 0,$$
(9.10)

which vanishes because it is the expectation value of an s-exact observable. This gives $C(x) = [1 + \xi^\lambda\partial_\lambda + L(\partial\xi)]C(x)$, so $C(x)$ is invariant under arbitrary coordinate transformation. Thus it is a number independent of x. It vanishes for $x = \infty$, so $C(x) = 0$ for $x \neq 0$. The argument holds for a generic gauge-invariant correlator.

We have shown that the correlation length R of gauge-invariant observables vanishes in the gauge-invariant, physical sector of asymptotic theory defined by $\hat{S}$. In other words, the mass gap is infinite, $M = 1/R = \infty$, in the physical sector of the asymptotic theory. It is tempting to conclude from this that there is a finite mass gap in the physical sector of the exact nonasymptotic theory, otherwise we would have obtained nonzero correlators in the infrared limit. However, local gauge-invariant observables like $F^2(x)$ are composite

operators, and so far we have discussed only the correlators of elementary fields. To establish that the mass gap in the nonasymptotic theory is finite, one should check that the correlators of local gauge-invariant operators in the limit of large separation are also given by the infrared asymptotic theory defined by $\hat{S}$.

X. QUARKS

So far we have neglected quarks, but they may be included in the time-independent Fokker-Planck equation [3]. The derivation of the nonperturbative Faddeev-Popov formula, including quarks, proceeds as in Secs. II–IV, by changing quark variables according to $\psi = g^{-1}\Psi$ and $\bar{\psi} = \bar{\Psi}_g$. The result is that the quark action $S_{qu} = \int d^4x\bar{\psi}(\gamma_\mu D_\mu + M)\psi$ gets added to the gluon action Σ or S. According to the latest DS calculations that include $N_f = 3$ flavors of dynamical quarks, the quark-loop term in the DS equation for gluons is subdominant in the infrared [72]. Provided that the effects of truncation are not too drastic, the quark contribution will also be subdominant in the infrared limit of the exact functional DS equation for the gluon propagator. In this case the inclusion of quarks does not disturb the simplicity of the gluon sector described by $\hat{S}$.

If the intrinsic mass of the quarks is finite, then the quark sector does not appear in the asymptotic infrared limit. If the instrinsic mass of the quarks is zero, the pion is a massless Goldstone boson associated with spontaneous breaking of chiral symmetry. However, even in this case, in the (truncated) DS equation for the quark propagator given in [72], the infrared limit of the quark propagator does *not* decouple from the degrees of freedom associated with finite momentum (in contrast to the gluon). This is to be expected because the parameters that characterize the dynamics of massless quarks, $\langle\bar{\psi}\psi\rangle$ and f_π, are finite multiples of Λ_{QCD}, but the infrared asymptotic limit corresponds to $\Lambda_{QCD}\to\infty$. Nevertheless one may ask if chiral symmetry is broken in the asymptotic infrared theory. The chiral-symmetry breaking parameter is given by $\langle\bar{\psi}\psi\rangle = \pi\langle\rho(0,A)\rangle$, where $\rho(\lambda,A)$ is density of eigenvalues λ, per unit volume, of the Dirac operator $i\gamma\cdot D(A)$ in the configuration A. In the infrared asymptotic limit, the expectation value $\langle\rho(\lambda,A)\rangle$ is evaluated in the theory defined by the action $\hat{S}$. One would expect that it gives $\langle\bar{\psi}\psi\rangle = \infty$, since this corresponds to $\Lambda_{QCD} = \infty$. Thus in the theory defined by $\hat{S}$, the average density of levels per unit volume $\langle\rho(0,A)\rangle$ of the Dirac operator $i\gamma\cdot D(A)$ should be infinite at $\lambda = 0$.

The infrared asymptotic theory is far simpler than full QCD and provides a valuable model in which the characteristic features of the confining phase, as described in the Landau gauge, are revealed. To understand confinement in the asymptotic theory, note that while the infrared components of $A(x)$ are severely suppressed by the cutoff at the Gribov horizon, its short-wavelength components fluctuate widely because the factor $\exp[-S_{YM}(A)]$ is replaced by 1. Indeed, the infrared asymptotic gluon propagator $D^{as}(k)$, Eq. (8.1), is strongly enhanced in the ultraviolet. This suggests a pic-

ture of confinement in the infrared asymptotic theory in which the short-wavelength fluctuations of $A^a(x)$ in color directions cause the decoherence of any field that carries a color charge. Indeed transport of a color vector $q(\tau)$ along a path $z_\mu(\tau)$ is described by $P \exp(g_0 \int A_\mu \dot{z}^\mu dt)$. In a highly random field $A_\mu^b(x)$, superposition of different paths is incoherent, so a field that bears a color charge does not propagate. In full QCD in Landau gauge, the dominant fluctuations of $A(x)$ responsible for confinement should be on the length scale Λ_{QCD}^{-1}. This picture of confinement is quite different from the scenario in Coulomb gauge, where confinement of color charge is attributed to a realization of infrared slavery by an instantaneous, long-range color-Coulomb potential [73,70,71].

XI. CONCLUSION

We briefly review the salient features of the nonperturbative continuum Euclidean formulation of QCD developed here.

(i) In Landau gauge one may integrate the Faddeev-Popov weight over the Gribov region Ω instead of over the fundamental modular region Λ.

(ii) The *form* of the Dyson-Schwinger equations is unchanged by the cutoff of the functional integral on the boundary $\partial\Omega$ of the Gribov region because the Faddeev-Popov determinant vanishes there. This simplicity makes the DS equations the method of choice for nonperturbative calculations in QCD.

(iii) The restriction to the Gribov region provides *supplementary conditions* that govern the choice of solution of the DS equations. Two conditions are the positivity of the gluon and ghost propagators. Another is the *horizon condition* which is the statement that the ghost propagator $G(k)$ is more singular than $1/k^2$ in the infrared, $\lim_{k\to 0}[k^2 G(k)]^{-1} = 0$. This fixes the ghost-propagator renormalization constant $\tilde{Z}_3$ to the value (7.9). Although Eq. (7.9) is in flagrant disagreement with the perturbative expression for $\tilde{Z}_3$, nevertheless it is consistent with the perturbative renormalization group.

(iv) Implementation of the horizon condition in the DS equations puts QCD into a nonperturbative phase.

(v) Recent solutions of the truncated DS equations possess an asymptotic infrared limit that is obtained by systematically neglecting the terms in the DS equations that come from the Yang-Mills action $S_{YM}(A)$, but keeping the Faddeev-Popov determinant and the cutoff at the Gribov horizon. If the effects of truncation are not too drastic, this also gives an exact asymptotic infrared limit of QCD that is a continuum analogue of the strong-coupling limit in lattice gauge theory. This is possible because convergence of the A integration without the Yang-Mills factor $\exp[-S_{YM}(A)]$ may be assured by the cutoff at the Gribov horizon.

(vi) The asymptotic infrared limit of QCD is defined by the functional DS equations (8.6) and (8.7). The gluon propagator may be eliminated exactly from Eq. (8.7), and the asymptotic infrared theory is completely characterized by the functional inverse ghost propagator $\hat{\Gamma}_{gh}(x,y;B)$.

(vii) There exists a local BRST-invariant extension of the present nonperturbative formulation, sketched out in Appendix B. This ensures that the Slavnov-Taylor identities hold in the nonperturbative theory. The asymptotic infrared limit of QCD, valid at distances large compared to $1/\Lambda_{QCD}$, is described by the BRST-exact action, $\hat{S} = \int d^4x s(\partial_\mu \bar{c} A_\mu)$, that defines a topological quantum field theory with an infinite mass gap.

(viii) The extension of the nonperturbative formulation to include the quark action $\int d^4x \bar{\psi}(\gamma_\mu D_\mu + m)\psi$ is immediate. The presence of quarks does not disturb the asymptotic infrared limit of the gluon sector.

(ix) The asymptotic infrared theory provides a simple model in the Landau gauge in which the characteristic features of confinement may be understood. A picture of confinement of color charge emerges, in which the highly random fluctuations of the gluon field A cause the superposition from the transport of color charge along different paths to interfere incoherently, so the fields that bear a color charge do not propagate.

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APPENDIX A: RESOLUTION OF PARADOX

At first sight it is surprising that expectation-values taken over the fundamental modular region Λ and the Gribov region Ω are equal. In this appendix we show how this paradox is resolved.

1. Argument of Semenov-Tyan-Shanskii and Franke

The proof by Semenov-Tyan-Shanskii and Franke [4] that the Gribov region Ω and the fundamental modular region Λ are different, substantiated by instances given in [6], was long considered to disprove Eq. (1.3). We review the argument of [4]. Let $g(t) = \exp(t\omega)$ be a one-parameter subgroup of the local gauge group with generator $\omega = \omega(x)$. To be definite, we normalize ω to $(\omega, \omega) = V$, where V is the Euclidean volume. Let $A_\mu(t, \omega, B) \equiv g(t)^{-1} B_\mu g(t) + g(t)^{-1} \partial_\mu g(t)$ be the gauge-transform of B_μ under $g(t) = \exp(t\omega)$, so $A(0, \omega, B) = B$, and let $F_B(t, \omega)$ be the Hilbert square norm of $A(t, \omega, B)$, regarded as a function of t and ω for fixed B,

$$F_B(t, \omega) = \|A(t, \omega, B)\|^2 = \int d^4x |A_\mu(t, \omega, B)|^2. \quad \text{(A1)}$$

The fundamental modular region Λ is the set of B such that $F_B(0, \omega)$ is an absolute minimum, $F_B(0, \omega) \leq F_B(t, \omega)$ for all ω and t. The Gribov region Ω is the set of B for which $F_B(0, \omega)$ is a relative minimum $F_B(0, \omega) \leq F_B(t, \omega)$ for all ω and sufficiently small t.

We differentiate $F_B(t,\omega)$ with respect to t, and use $A'_\mu = D_\mu(A)\omega \equiv D_\mu\omega$,

$$F'_B(t,\omega) = 2(D_\mu\omega, A_\mu) = 2(\partial_\mu\omega, A_\mu) = -2(\omega, \partial_\mu A_\mu),$$

$$F''_B(t,\omega) = 2(\partial_\mu\omega, D_\mu\omega) = -2(\omega, \partial_\mu D_\mu\omega),$$

$$\tag{A2}$$

$$F'''_B(t,\omega) = 2(\partial_\mu\omega, D_\mu\omega \times \omega),$$

$$F''''_B(t,\omega) = 2[\partial_\mu\omega, (D_\mu\omega \times \omega) \times \omega],$$

where $X \times Y = [X, Y]$ is the commutator in the Lie algebra. These formulas show that the interior of Ω consists of all transverse configurations B, $\partial \cdot B = 0$, such that all nontrivial eigenvalues of $M(B) = -\partial_\mu D_\mu(B)$ are strictly positive, $\lambda_n(B) > 0$. Moreover, for B on the boundary $\partial\Omega$, $M(B)$ has at least one nontrivial eigenvalue that vanishes, $\lambda_1(B) = 0$.

We specialize to the SU(2) group, so the commutator $X \times Y$ is the ordinary 3-vector cross product. The vector triple product gives

$$F''''_B(t,\omega) = 2[\partial_\mu\omega, (\omega \cdot D_\mu\omega\omega - \omega^2 D_\mu\omega)]$$

$$= 2(\partial_\mu\omega, \omega \cdot \partial_\mu\omega\omega) + 2[\omega, \partial_\mu(\omega^2 D_\mu\omega)]$$

$$= 2(\omega \cdot \partial_\mu\omega, \omega \cdot \partial_\mu\omega) + 2[\omega, \partial_\mu(\omega^2)D_\mu\omega]$$

$$+ 2(\omega, \omega^2 \partial_\mu D_\mu\omega)$$

$$= (1/2)[\partial_\mu(\omega^2), \partial_\mu(\omega^2)] + 2[\omega \cdot \partial_\mu\omega, \partial_\mu(\omega^2)]$$

$$+ 2(\omega^2, \omega \cdot \partial_\mu D_\mu\omega)$$

$$= (3/2)[\partial_\mu(\omega^2), \partial_\mu(\omega^2)] + 2(\omega^2, \omega \cdot \partial_\mu D_\mu\omega),$$

$$\tag{A3}$$

where the dot is contraction on color indices.

Let B be a point on the Gribov horizon $\partial\Omega$, so B is transverse $\partial_\mu B_\mu = 0$, and the Faddeev-Popov operator $-\partial_\mu D_\mu(B)$ is non-negative, but with at least one nontrivial null eigenvalue, $\partial_\mu D_\mu(B)\omega_0 = 0$, for some ω_0. By Eq. (A2), we have $F'_B(0,\omega_0) = F''_B(0,\omega_0) = 0$. For B on $\partial\Omega$, it follows that in general $F_B(0,\omega)$ is *not* a local minimum on the gauge orbit through B because, in general, $F'''_B(0,\omega_0) \neq 0$, so $F_B(t,\omega_0) - F_B(0,\omega_0)$ changes sign at $t = 0$. By continuity this implies that nearby points inside the Gribov region Ω cannot be absolute minima, even though they are relative minima. They are Gribov copies inside Ω. This is the argument of [4], and examples for which $F'''_B(0,\omega_0) \neq 0$ are given in [6].

But let us evaluate the fourth derivative at $t = 0$, in the direction ω_0. With $\partial_\mu D_\mu(B)\omega_0 = 0$, we have from Eq. (A3),

$$F''''_B(0,\omega_0) = (3/2) \int d^4x [\partial_\mu(\omega_0^2)]^2. \tag{A4}$$

This is the integral of a positive density, and we expect that $F''''_B(0,\omega_0)$ is *large and positive*.

The relevant question for comparing the expectation values over Ω and over Λ is not whether these regions coincide—they do not—but whether the normalized aver-

ages over these sets are equal in the thermodynamic limit. Here we implicitly suppose a lattice discretization and configurations that are sampled from the Wilson ensemble. In the thermodynamic limit, the probability may get concentrated on a subset that consists of a boundary or part of a boundary. The boundaries of Λ and Ω may approach each other in the thermodynamic limit for *typical* configurations on the boundary. If $F''''_B(0,\omega_0)$ is large, and $F'''_B(0,\omega_0)$ is small, then there is a local minimum near B, which could be the absolute minimum on the gauge orbit. If the distance to the absolute minimum vanishes in the thermodynamic limit for a typical configuration, then the argument of [4] does not disprove Eq. (1.3).

We normalize ω_0 to $(\omega_0, \omega_0) = V$, where V is the volume of Euclidean space. We estimate quantities using this normalization, and we shall verify that the conclusions do not depend on the normalization of ω_0. With this normalization, we estimate that $\omega_0(x) = O(1)$. Since $F''''_B(0,\omega_0)$ is the integral of a positive local density over a volume V, we estimate that $F''''_B(0,\omega_0) = O(V)$ for a typical configuration B on the Gribov horizon. On the other hand, the density that appears in $F'''_B(0,\omega_0)$ has no definite sign. For a typical configuration, sampled from the Wilson ensemble, we make the crudest statistical estimate namely random density, so $F'''_B(0,\omega_0) = O(V^{1/2})$. This is small compared to $F''''_B(0,\omega_0)$. We seek a nearby minimum on the gauge orbit through B. For simplicity we assume that all nontrivial eigenvalues of $M(B)$ are strictly positive, apart from the zero eigenvalue belonging to ω_0, which is the only dangerous direction. We write $F(t) \equiv F_B(t,\omega_0)$, and we have

$$F(t) = F(0) + (1/3!)F'''(0)t^3 + (1/4!)F''''(0)t^4, \tag{A5}$$

with neglect of higher order terms. The minimum is found at $F'(t_{cr}) = 0$, which gives $t_{cr} = -3F'''(0)/F''''(0)$, and one has

$$F(t_{cr}) = F(0) - (9/8)\frac{[F'''(0)]^4}{[F''''(0)]^3}. \tag{A6}$$

This is lower than $F(0)$, in agreement with the argument of [4]. This expression is independent of the normalization of ω_0, as one sees from Eq. (A2), so our estimate for this quantity is independent of the normalization of ω_0. By the above estimates, the second term is of order $(V^{1/2})^4/(V)^3 = V^{-1}$. It is small compared to the first term, $F(0) = \|B\|^2$, which is of order V. The configuration at the nearby minimum is

$$B_\mu(x, t_{cr}) = B_\mu(x) + t_{cr}[D_\mu(B)\omega_0](x)$$

$$= B_\mu(x) - \frac{3F'''(0)}{F''''(0)}[D_\mu(B)\omega_0](x), \tag{A7}$$

which is again independent of the normalization of ω_0. According to the above estimates, the second term is of order $V^{-1/2}$. Thus in the thermodynamic limit of lattice gauge theory, $V \to \infty$, the nearby minimum approaches the point B on the Gribov horizon. In actuality, the problem of minimizing the functional $F_A(g) = \|{}^gA\|$ on the lattice is a problem of

spin-glass type, so one expects many, nearly degenerate, relative minima, and the one found here is not necessarily the absolute minimum. Nevertheless the point remains that [4] does not disprove the equality of expectation values on Λ and Ω in the thermodynamic limit.

2. Many Gribov copies inside the Gribov region from numerical simulations

We now consider the fact that in numerical gauge-fixing to Landau gauge in lattice gauge theory, there are many local minima (i.e., Gribov copies inside the Gribov region, Ω) on a typical gauge orbit [7–11]. Their number grows with the lattice size as is characteristic of a spin-glass. In this sense Ω is very large compared to Λ. However, the number of dimensions of configuration space is high, and our geometrical intuition from 3-space may be misleading. Indeed, on a lattice of Euclidean volume V, the dimension D of configuration space is $D = fV$, where f is the number of degrees of freedom per lattice site, and the dimension D of configuration space diverges with the Euclidean volume V.

In continuum gauge theory Λ and Ω are both convex and bounded in every direction [4]. By simple entropy considerations, the population in a bounded region of a high-dimensional space gets concentrated on the boundary. For example, inside a sphere of radius R in a D-dimensional space, the radial density is given by $r^{D-1} dr$, and for $r \lesssim R$ is highly concentrated near the boundary $r = R$. To take the simplest example, consider two spheres (in configuration space), the first of radius R, and the second of radius $R + cV^{-1/2}$. In the spirit of the previous estimates, these would be the radii of Λ and Ω. The ratio of the radii $(R + cV^{-1/2})/R$ approaches unity, in the limit $V \to \infty$, so all nth moments, $\langle r^n \rangle$ for finite n, of the two spheres become equal. On the other hand the ratio of their volumes is given by $[(R + cV^{-1/2})/R]^D = [(R + cV^{-1/2})/R]^{fV}$, where $D = fV$ is the dimension of configuration space. For large V the ratio of the volumes of the two spheres is thus $\exp(afV^{1/2}/R)$, which diverges exponentially like $V^{1/2}$. In this example the ratio of the volumes of the two spheres diverges with V, but all *finite* moments of the two spheres become equal. In field theory the nth moments of the distribution are the n-point functions $\langle A(x_1) \cdots A(x_n) \rangle$. So again, the fact that there are many Gribov copies inside Ω does not disprove that averages calculated over Λ or Ω are equal.

3. Gauge theory on a finite lattice

For a finite lattice the paradox becomes acute. Stochastic quantization may also be defined in lattice gauge theory [10]. As in the continuum theory, a drift force $a^{-1}K_{\text{gt}}$ tangent to the gauge orbit may be chosen in the direction of steepest descent of a suitable minimizing function, and is globally resoring. It appears that one may solve the lattice Fokker-Planck equation in the limit $a \to 0$ on a finite lattice, by the method used in Secs. II–IV, for it depends only on general geometrical properties that are common to lattice and continuum gauge theories. If so, one would again be led to the conclusion that the weight inside the Gribov region is given by the lattice analogue of Eq. (4.17), namely

$N \exp[-S_W(U)]$, where $S_W(U)$ is the Wilson action, and U is a configuration in the lattice Gribov region Ω. However, on a finite lattice the distinction between the fundamental modular region Λ and the Gribov region Ω can surely not be ignored. The resolution of this paradox would appear to be that in lattice gauge theory the Gribov region Ω is made of disconnected pieces Ω_i. In each piece, the solution is indeed given by $Q_i(U) = N_i \exp[-S_W(U)]$, for $U \in \Omega_i$, where the normalizations N_i are left indeterminate by the method of Secs. II–IV. Presumably, the average with the lattice Faddeev-Popov weight over all the disconnected pieces Ω_i of the Gribov region, with the correct normalization N_i in each piece, will agree with same integral over the fundamental modular region Λ.

APPENDIX B: BRST-INVARIANT FORMULATION

New issues arise when the nonperturbative approach is extended to a theory with a local BRST-invariant action.

1. Off-shell transversality condition

To obtain a local action, on must take the transversality condition "off-shell." The off-shell partition function is given by

$$Z(J,L) \equiv \int_\Omega DA D\lambda \, \det M(A)$$
$$\times \exp[-S_{\text{YM}}(A) + i(\lambda, \partial \cdot A) + i(J,A) + i(L,\lambda)],$$
(B1)

where λ is the Nakanishi-Lautrup Lagrange multiplier field that enforces the gauge condition $\partial \cdot A = 0$, and L is its source. This reduces to Eq. (5.1) for $L = 0$. It is not immediately obvious what region Ω to integrate over because A is not transverse for $L \neq 0$, so $M(A) = -\partial \cdot D(A)$ is not a symmetric operator. One must also take the Gribov horizon $\partial \Omega$ off-shell when the gauge condition is off-shell. If we effect the λ integration, the last integral becomes

$$Z(J,L) = \int_\Omega DA \, \det M(A) \, \delta(\partial \cdot A + L)$$
$$\times \exp[-S_{\text{YM}}(A) + i(J,A)].$$
(B2)

Only configurations A of the form $A = B - \partial(\partial^2)^{-1}L$ are relevant, where B is transverse. We regard the partition function $Z(J,L)$ as a formal power series in the source L. Both the lowest nontrivial eigenvalue $\lambda_1[B - \partial(\partial^2)^{-1}L]$ of the Faddeev-Popov operator, $M[B - \partial(\partial^2)^{-1}L]$, and the points $B_0(L)$ where it vanishes, may be calculated by formal perturbation theory as a power series in L. Here $B_0(0)$ is a point on the on-shell horizon. In this way we may take the Gribov horizon $\partial \Omega$ off-shell.

2. Faddeev-Popov ghosts

One may make the action local by writing

$$\det M(B) = \int DcD\bar{c} \exp[\bar{c}, M(B)c], \qquad \text{(B3)}$$

where c and $\bar{c}$ are anticommuting ghost and antighost fields. Grassmannian sources, η and $\bar{\eta}$, are then introduced, so this gets replaced by

$$\int DcD\bar{c} \exp\{[\bar{c}, M(B)c] + (\bar{\eta}, c) + (\bar{c}, \eta)\}$$

$$= \det M(B) \exp[\bar{\eta}, M^{-1}(B)\eta]. \qquad \text{(B4)}$$

This expression does not vanish on the boundary $\partial\Omega$. For by an eigenfunction expansion of $M^{-1}(B)$, we obtain for the last expression

$$\prod_n \lambda_n \exp\left[\sum_n \frac{1}{\lambda_n} \bar{\eta}_n \eta_n\right] = \prod_n \lambda_n \prod_n \left(1 + \frac{1}{\lambda_n} \bar{\eta}_n \eta_n\right)$$

$$= \prod_n (\lambda_n + \bar{\eta}_n \eta_n). \qquad \text{(B5)}$$

It does not contain as a factor $\lambda_1(B)$ that vanishes on $\partial\Omega$. For this reason, we did not use Faddeev-Popov ghost fields and their sources in the derivation of the DS equations in Secs. V and VI. Nevertheless *we obtained the same DS equations, including the ghost propagators, that we would have obtained if we had introduced the ghost fields and their sources*. For this reason, and by use of the off-shell Gribov horizon, it should be possible to extend the nonperturbative approach to the theory defined by the familiar BRST-invariant local action (8.11), integrated over the off-shell Gribov region.

APPENDIX C: PROPERTIES OF THE GRIBOV REGION

We note three properties of the Gribov region Ω defined in Eq. (1.2). (i) Ω contains the origin $A = 0$. (ii) It is bounded in every direction. (iii) It is convex. We give the one-line proofs of these properties [74]. They follow from the expression $M(A) = M_0 + M_1(A)$, where $M_0^{ac}(A) = -\partial^2 \delta^{ac}$, and $M_1^{ac}(A) = -g_0 f^{abc} A_\mu^b \partial_\mu$, where A is transverse. Property (i) is obvious since $M_0 = -\partial^2 \delta^{ac}$ is strictly positive. To establish (ii), note that $M_1(A)$ has zero trace, since it is traceless on color indices $f^{aba} = 0$. Thus, for any given A, there exists a state ω for which the expectation value of $M_1(A)$ is negative, $E \equiv (\omega, M_1(A)\omega) < 0$. Moreover, $M_1(A)$ is linear in A, $M_1(\lambda A) = \lambda M_1(A)$, so upon replacing A by λA, where λ is a positive number, we have $(\omega, M(\lambda A)\omega) = (\omega, M_0\omega) + \lambda[\omega, M_1(A)\omega] = (\omega, M_0\omega) + \lambda E$. By taking λ sufficiently large and positive, the expectation value is negative $[\omega, M(\lambda A)\omega] < 0$. This establishes (ii). To establish convexity, we must show that $M(\alpha A_1 + \beta A_2)$ is a strictly positive operator when $M(A_1)$ and $M(A_2)$ are both strictly positive operators, for all positive α and β, with $\alpha + \beta = 1$. This is immediate because $M_1(A)$ depends linearly on A, and we have $M(\alpha A_1 + \beta A_2) = \alpha M(A_1) + \beta M(A_2)$. QED

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