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Anomalous specific heat in high-density QED and QCD

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Long-range quasistatic gauge-boson interactions lead to anomalous (non-Fermi-liquid) behavior of the specific heat in the low-temperature limit of an electron or quark gas with a leading $T \ln T^{-1}$ term. We obtain perturbative results beyond the leading log approximation and find that dynamical screening gives rise to a low-temperature series involving also anomalous fractional powers $T^{(3+2n)/3}$. We determine their coefficients in perturbation theory up to and including order $T^{7/3}$ and compare with exact numerical results obtained in the large-N_f limit of QED and QCD.

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It was established long ago [1] in the context of a nonrelativistic electron gas that only weakly screened low-frequency transverse gauge-boson interactions lead to a qualitative deviation from Fermi-liquid behavior. A particular consequence of this is the appearance of an anomalous contribution to the low-temperature limit of entropy and specific heat proportional to $\alpha T \ln T^{-1}$ [1–3], but it was argued that the effect would probably be too small for experimental detection.

More recently, it has been realized that analogous non-Fermi-liquid behavior in ultradegenerate QCD is of central importance to the magnitude of the gap in color superconductivity [4–6], and it was pointed out [7] that the anomalous contributions to the low-temperature specific heat may be of interest in astrophysical systems such as neutron or protoneutron stars, if they involve a normal (nonsuperconducting) degenerate quark matter component.

So far only the coefficient of the $\alpha T \ln T^{-1}$ term in the specific heat was determined (with Ref. [3] correcting the result of Ref. [1] by a factor of 4), but not the complete argument of the leading logarithm. While the existence of the $T \ln T^{-1}$ term implies that there is a temperature range where the entropy or the specific heat *exceeds* the ideal-gas value, without knowledge of the constants "under the log" it is impossible to give numerical values for the required temperatures.

Furthermore, a quantitative understanding of these anomalous contributions is also of interest with regard to the recent progress made in high-order perturbative calculations of the pressure (free energy) of QCD at nonzero temperature and chemical potential [8], where it has been found that dimensional reduction techniques work remarkably well except for a narrow strip in the T-μ plane around the $T=0$ line.

In the present Rapid Communication we report the results of a calculation of the low-temperature entropy and specific heat for ultradegenerate QED and QCD that goes beyond the leading log approximation. In addition to completing the leading logarithm, we find that, for $T/\mu \ll g \ll 1$, where g is either the strong or the electromagnetic coupling constant, the higher terms of the low-temperature series involve also anomalous fractional powers $T^{(3+2n)/3}$, and we give their coefficients through order $T^{7/3}$.

Our starting point is an expression for the thermodynamic potential of QED and QCD that becomes exact in the limit of large flavor number N_f [9,10] and that at finite N_f has an error of order g^4 in the regime $T/\mu \ll g$,

$$P = NN_f\left(\frac{\mu^4}{12\pi^2}+\frac{\mu^2T^2}{6}+\frac{7\pi^2T^4}{180}\right) - N_g\int\frac{d^3q}{(2\pi)^3}\int_0^\infty\frac{dq_0}{\pi} \times\Bigg[2\left([n_b+\tfrac{1}{2}]\,\mathrm{Im}\ln D_T^{-1}-\tfrac{1}{2}\,\mathrm{Im}\ln D_{\rm vac}^{-1}\right) +\left([n_b+\tfrac{1}{2}]\,\mathrm{Im}\ln\frac{D_L^{-1}}{q^2-q_0^2}-\tfrac{1}{2}\,\mathrm{Im}\ln\frac{D_{\rm vac}^{-1}}{q^2-q_0^2}\right)\Bigg] + O(g^4\mu^4), \quad (T/\mu \ll g) \tag{1}$$

where $N=3$, $N_g=8$ for QCD, and both equal 1 for QED. D_T and D_L are the spatially transverse and longitudinal gauge boson propagators at finite temperature T and (electron or quark) chemical potential μ obtained by Dyson-resumming one-loop fermion loops, and $D_{\rm vac}$ is the corresponding quantity at zero temperature and chemical potential.

Nonanalytic terms in the low-temperature expansion arise from the contribution

$$\frac{P_{T,n_b}}{N_g} = -\int\frac{d^3q}{(2\pi)^3}\int_0^\infty\frac{dq_0}{\pi}2n_b\,\mathrm{Im}\ln D_T^{-1}. \tag{2}$$

The bosonic distribution function $n_b=1/[\exp(q_0/T)-1]$ restricts the q_0 integration to $q_0 \lesssim T$ and T is assumed to be the smallest mass scale in the problem. Consistently dropping contributions proportional to T^4 in the pressure (T^3 in the entropy), which for $T/\mu \ll g$ are beyond our perturbative accuracy, it turns out that we only need the $T \to 0$ limit of the inverse propagator D_T^{-1}, and only the lowest orders in q_0/q and q_0/μ:

$$\mathrm{Re}\,D_T^{-1} = q^2[1+O(g_{\rm eff}^2)] + \left(\frac{g_{\rm eff}^2\mu^2}{\pi^2q^2}-1+O(g_{\rm eff}^2q^0)+O(g_{\rm eff}^2q^2/\mu^2)\right)q_0^2 + O(g_{\rm eff}^2q_0^4), \tag{3}$$

$$\mathrm{Im}\, D_T^{-1} = -\frac{g_{\mathrm{eff}}^2 q_0}{48\pi q^3}(q^2-q_0^2)(12\mu^2+3q^2+q_0^2)\,\theta(2\mu-q) \tag{4}$$

where $g_{\mathrm{eff}}^2 = g^2 N_f$ for QED and $g^2 N_f/2$ for QCD.

Keeping only the leading terms in the limit $q_0 \to 0$ gives

$$\mathrm{Im}\ln D_T^{-1} \simeq \arctan\frac{-g_{\mathrm{eff}}^2(4\mu^2+q^2)q_0\,\theta(2\mu-q)}{16\pi q^3}. \tag{5}$$

Inserting this approximation into Eq. (2) leads to the integral

$$\int_0^{2\mu} dq\, q^2 \arctan\frac{q_0(4\mu^2+q^2)}{q^3} \simeq \frac{4\mu^2}{3}q_0\left(\ln\frac{2\mu}{q_0}+\frac{5}{2}\right)+O(q_0^{5/3}). \tag{6}$$

Performing the q_0 integration then gives the following contribution to the entropy $S=\partial P/\partial T$ (per unit volume):

$$\frac{S_{T,n_b}}{N_g} = \frac{g_{\mathrm{eff}}^2\mu^2 T}{36\pi^2}\left(\ln\frac{32\pi\mu}{g_{\mathrm{eff}}^2 T}+1+\gamma_E-\frac{6}{\pi^2}\zeta'(2)\right)+O(T^{5/3}). \tag{7}$$

While this reproduces the coefficient of the anomalous $T\ln T^{-1}$ term reported in Ref. [3], the coefficient under the logarithm as well as the suppressed $O(T^{5/3})$ contribution are still incomplete. To complete the term linear in T, one has to perform an exactly analogous calculation of the longitudinal contribution, which involves

$$\mathrm{Im}\ln D_L^{-1} \simeq \frac{g_{\mathrm{eff}}^2(4\mu^2-q^2)q_0\,\theta(2\mu-q)/(8\pi q)}{q^2+(g_{\mathrm{eff}}^2\mu^2)/\pi^2}, \tag{8}$$

and when inserted into Eq. (1) contributes

$$\frac{S_{L,n_b}}{N_g} = \frac{g_{\mathrm{eff}}^2\mu^2 T}{24\pi^2}\left(\ln\frac{g_{\mathrm{eff}}^2}{4\pi^2}+1\right)+O(g_{\mathrm{eff}}^4)+O(T^3). \tag{9}$$

Finally, the remaining parts of (1), which do not involve the bosonic distribution function, yield

$$\frac{S_{non-n_b}}{N_g} = -\frac{g_{\mathrm{eff}}^2}{8\pi^2}\mu^2 T. \tag{10}$$

The latter contribution matches exactly the one from the standard perturbative result to order g^2 [11], while the contributions (7) and (9) depend on having $T/\mu \ll g$. In this region, all of the contributions listed so far are negligible compared to the zero-temperature contribution $\sim g^4\mu^4$ in the pressure [which is only partially included in Eq. (1)]. However, by considering instead the entropy (and further below the specific heat), the above contributions become the dominant ones.

Adding them all together, we obtain

$$\frac{S-S_0}{N_g} = \frac{g_{\mathrm{eff}}^2\mu^2 T}{36\pi^2}\left(\ln\frac{4g_{\mathrm{eff}}\mu}{\pi^2 T}-2+\gamma_E-\frac{6}{\pi^2}\zeta'(2)\right) + c_{5/3}T^{5/3}+c_{7/3}T^{7/3}+O(T^3), \tag{11}$$

where S_0 is the ideal-gas value of the entropy per unit volume.

To also obtain completely the coefficients of the terms in the low-temperature expansion that involve fractional powers of T we need to include more terms of Eqs. (3) and (4) than those kept in relation (6). A lengthy calculation, whose details will be discussed elsewhere, gives

$$c_{5/3} = -\frac{8\times 2^{2/3}\Gamma\left(\frac{8}{3}\right)\zeta\left(\frac{8}{3}\right)}{9\sqrt{3}\pi^{11/3}}(g_{\mathrm{eff}}\mu)^{4/3}, \tag{12}$$

$$c_{7/3} = \frac{80\times 2^{1/3}\Gamma\left(\frac{10}{3}\right)\zeta\left(\frac{10}{3}\right)}{27\sqrt{3}\pi^{13/3}}(g_{\mathrm{eff}}\mu)^{2/3}. \tag{13}$$

Setting $T/\mu \sim g_{\mathrm{eff}}^{1+\delta}$ with $\delta>0$, one finds that the terms in the expansion (11) correspond to the orders $g_{\mathrm{eff}}^{3+\delta}\ln(c/g_{\mathrm{eff}})$, $g_{\mathrm{eff}}^{3+(5/3)\delta}$, and $g_{\mathrm{eff}}^{3+(7/3)\delta}$, respectively, with a truncation error of the order $g_{\mathrm{eff}}^{3+3\delta}$. Hence, the expansion parameter in this low-temperature series is $T/(g_{\mathrm{eff}}\mu)$, which is also the scaleless parameter appearing in the argument of the leading logarithm (remarkably, however, only after the transverse and the longitudinal contributions have been added together). The combination $g_{\mathrm{eff}}\mu$ is the scale of the Debye mass at high chemical potential, whose leading-order value is $m_D = g_{\mathrm{eff}}\mu/\pi$. In fact, the calculation of the coefficients in Eq. (11) required keeping the leading-order "hard-dense-loop" (HDL) part of the gauge boson propagator [12,13], in particular the dynamic screening in Eq. (4), but also a HDL correction to the real part of the transverse self energy in Eq. (3). The above calculation is therefore in a certain sense another application of HDL resummation [13], which thus turns out to be necessary also for a perturbative treatment of the low-temperature regime $T/\mu \ll g$.

As a check on our result and also as a test of its convergence properties, we compare the anomalous transverse contributions S_{T,n_b} with those of the exactly (albeit only numerically) solvable large-N_f limit [10] in Fig. 1. We find good convergence to the exact result as long as $T/\mu \lesssim g_{\mathrm{eff}}/(2\pi^2)$. This is also the region where the complete large-N_f result for the low-temperature entropy [10] has the anomalous property of exceeding the ideal-gas value.

Our results do not, however, seem to agree with the results of Ref. [7], which recently questioned the presence of a term $\propto \alpha T\ln T^{-1}$. The (renormalization group resummed) result reported therein rather corresponds to a leading nonana-

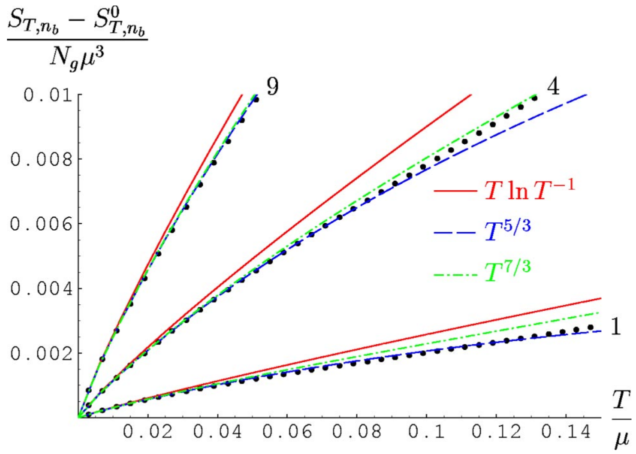

FIG. 1. Transverse n_b contribution to the interaction part of the low-temperature entropy density in the large-N_f limit for the three values $g^2_{\rm eff}=1,4,9$. The large dots give the exact numerical results; the full, dashed, and dash-dotted lines correspond to our perturbative result up to and including the $T\ln T^{-1}$, $T^{5/3}$, and $T^{7/3}$ contributions.

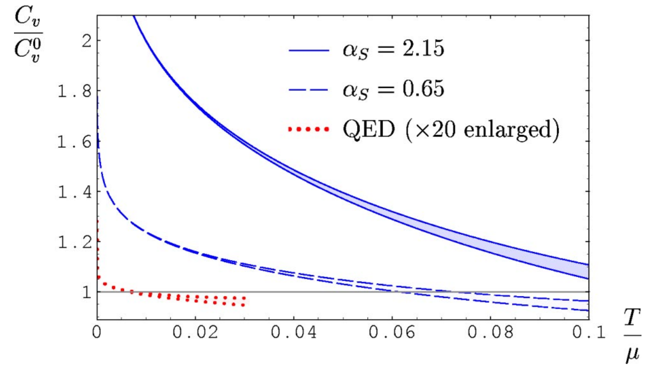

FIG. 2. The perturbative result for the specific heat, normalized to the ideal-gas value, to order $T^{5/3}$ and $T^{7/3}$ (lower and upper curves, respectively) for two particular values of α_s in two-flavor QCD (chosen for comparability to Ref. [7]) and $g_{\rm eff}\approx 0.303$ for QED. The deviation of the QED result from the ideal-gas value is enlarged by a factor of 20, and the plot terminates where the expansion parameter $(\pi^2 T)/(g_{\rm eff}\mu)\approx 1$.

lytic $\alpha T^3 \ln T$ term when expanded out perturbatively, which is in fact the type of nonanalytic terms that appear already in regular Fermi liquids [14].

For potential phenomenological applications in astrophysical systems, the specific heat C_v at constant volume and number density is of more direct interest than the entropy density that we have calculated so far. The former (per unit volume) is given by [15]

$$C_v = T\left\{\left(\frac{\partial S}{\partial T}\right)_\mu - \left(\frac{\partial \mathcal{N}}{\partial T}\right)^2_\mu \left(\frac{\partial \mathcal{N}}{\partial \mu}\right)^{-1}_T\right\}, \tag{14}$$

where $\mathcal{N}$ is the number density, but to the order of accuracy of our expansions, C_v can be simply obtained as the logarithmic derivative of the entropy:

$$C_v = T\left(\frac{\partial S}{\partial T}\right)_\mu + O(T^3). \tag{15}$$

Explicitly, the result is

$$\frac{C_v - C^0_v}{N_g} = \frac{g^2_{\rm eff}\mu^2 T}{36\pi^2}\left(\ln\frac{4 g_{\rm eff}\mu}{\pi^2 T} - 3 + \gamma_E - \frac{6}{\pi^2}\zeta'(2)\right) + \frac{5}{3}c_{5/3}T^{5/3} + \frac{7}{3}c_{7/3}T^{7/3} + O(T^3) \tag{16}$$

with $C^0_v = NN_f\mu^2 T/3 + O(T^3)$, and $c_{5/3}$ and $c_{7/3}$ given by Eqs. (12) and (13).

For illustrative purposes, we evaluate the ratio of C_v as given by Eq. (16) to the ideal-gas value C^0_v for QCD with two massless quark flavors in Fig. 2, using alternatively two values for α_s that have been used also in Ref. [7] and that correspond to one-loop running couplings with renormalization point 0.5 GeV (full line) and 1 GeV (dashed line). The shaded bands shown are limited from below and above by the results to order $T^{5/3}$ and $T^{7/3}$, respectively, and thus indicate the quality of the low-temperature expansion. One may interpret these results as roughly corresponding to QCD with a quark chemical potential of 0.5 GeV and the total variation corresponding to different renormalization schemes with the minimal subtraction scale varied between μ and 2μ. The critical temperature for the color superconducting phase transition may be anywhere between 6 and 60 MeV [16], so the range $T/\mu \geqslant 0.012$ in Fig. 2 might correspond to normal quark matter. While it is certainly questionable to apply perturbative results for $\alpha_s \gtrsim 0.65$, Fig. 2 suggests that the anomalous feature of an excess of the specific heat over its ideal-gas value may possibly come into play in astrophysical situations, in particular in the cooling of (proto-)neutron stars [17,18]. This should be contrasted with the ordinary perturbative estimate for C_v/C^0_v based on the well-known [11] exchange term $\propto g^2$ (which, as we have shown, requires $T/\mu \gg g$). The latter would predict $C_v/C^0_v \lesssim 0.6$ for $\alpha_s \gtrsim 0.65$.

For completeness, we also give the numerical results corresponding to QED, where $g_{\rm eff}\approx 0.303$. Here the range of temperature, where the specific heat exceeds the ideal-gas value, and the deviations from the latter, are much smaller (the deviations from the ideal-gas value have been enlarged by a factor of 20 in Fig. 2 to make them more visible).

To summarize, we have presented a quantitative evaluation of the leading contributions to the entropy and specific heat of high-density QCD and QED in the regime $T/\mu \ll g \ll 1$, which is dominated by non-Fermi-liquid behavior. While the effect remains small in QED, it seems conceivable that the anomalous terms in the specific heat play a noticeable role in the thermodynamics of a normal quark matter component of neutron or protoneutron stars.

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