

## Boulware state and semiclassical thermodynamics of black holes in a cavity

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A black hole, surrounded by a reflecting shell, acts as an effective starlike object with respect to the outer region that leads to vacuum polarization outside, where the quantum fields are in the Boulware state. We find the quantum correction to the Hawking temperature, taking into account this circumstance. It is proportional to the integral of the trace of the total quantum stress-energy tensor over the whole space from the horizon to infinity. For the shell, sufficiently close to the horizon, the leading term comes from the boundary contribution of the Boulware state.

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One of the brightest features of black holes is the fact that a black hole possesses thermal properties such as entropy and temperature. These entities acquire the literal meaning in the state of thermal equilibrium (the Hartle-Hawking state). For the system to achieve this state, two ingredients become essential. First, one should take into account the presence of quantum radiation around the hole. Quantum fields propagating in a black hole background affect the geometry and, in particular, change the surface gravity which determines the value of the Hawking temperature. Second, Hawking radiation should be constrained inside a cavity that prevents quantum fields from escaping to infinity. Thus, some overlap between quantum and boundary effects should exist in black hole thermodynamics. Consider a black hole enclosed inside a perfectly reflecting shell (microcanonical boundary conditions). As the stress-energy tensor of quantum fields in this state is bounded on the horizon, quantum back reaction leads to small corrections to the geometry and Hawking temperature that can be calculated within the perturbative approach. Such a program was realized for different types of fields [1–6]. In doing so, it was usually implied that the region outside the shell represents the usual vacuum, giving no contribution to thermodynamics.

Meanwhile, actually the space outside the shell is not empty. A black hole inside the shell, along with its Hawking radiation, acts as a source of the gravitational field outside and curves spacetime. Therefore, vacuum polarization is inevitable outside and represents the Boulware state (vacuum with respect to the Schwarzschild time) rather than a pure classical vacuum. In this state the average values  $\langle T_{\mu}^{\nu} \rangle \equiv T_{\mu}^{\nu}$  of quantum fields certainly contribute to the mass, measured by a distant observer. Here the tensor  $T_{\mu}^{\nu}$  is supposed to be calculated in the main (one-loop) approximation as usual. The correction to the mass due to the contribution of the vacuum polarization outside the shell does not affect the geometry of the spherically symmetric configuration inside. The outer region does not contribute to the entropy either since the Boulware state does not possess thermal properties. However, by contrast with the mass and entropy, the outer

region should affect the Hawking temperature. Indeed, even for a pure classical Schwarzschild geometry a massive shell between the horizon and infinity changes the Hawking temperature (it can be easily seen if one matches the metric inside and outside the shell). More than that, in the situation under discussion the geometry deviates from the pure Schwarzschild one due to back reaction of quantum fields. Thus, the value of the Hawking temperature should feel the presence of the quantum fields in the outer region. If the radius of the shell is large enough, the region between the shell and infinity does not contribute to physical quantities significantly, and neglecting vacuum polarization is a quite reasonable approximation. However, for a radius of the shell compatible with the horizon, the effect becomes essential.

The aim of the present paper is to find explicitly quantum corrections to the Hawking temperature  $T_H$ , caused by these effects and, thus, elucidate the influence of the Boulware state on black hole thermodynamics.

Consider the metric of a black hole:

$$ds^2 = -U(r)dt^2 + V^{-1}(r)dr^2 + r^2(\sin^2\theta d\phi^2 + d\theta^2). \quad (1)$$

From the Einstein equations it follows

$$V(r) = 1 - \frac{2m(r)}{r}, \quad m(r) = m + m_q(r), \quad (2)$$

$$m_q(r) = 4\pi \int_{2m}^r dr r^2 (-T_0^0), \quad U = V e^{2\psi},$$

$$\psi = 4\pi \int_{\infty}^r dr r \frac{(T_r^r - T_0^0)}{V(r)}.$$

It is assumed that a reflecting shell is placed at  $r=R$ , so the geometry deviates from the Schwarzschildian one due to quantum corrections. For  $r \rightarrow \infty$ ,  $\psi \rightarrow 0$  (provided the quantum stresses decay rapidly enough) and the geometry approaches its Schwarzschildian form. For  $r < R$ , where a black hole and thermal radiation are present at the temperature  $T_H$ , the quantum fields are in the Hartle-Hawking state. For  $r > R$ , the fields are in the Boulware state. As, in general, stresses on both sides of the boundary, calculated in two

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different states, do not coincide, there appears a jump. The total stress energy  $\theta_\mu^\nu$ , including that of the shell, reads

$$\theta_\mu^\nu = T_\mu^{\nu HH} \theta(R-r) + T_\mu^{\nu B} \theta(r-R) + T_\mu^{\nu S}, \quad (3)$$

where  $\theta(r)$  is the step Heaviside function and  $T_\mu^{\nu S}$  describes the contribution from the shell. It is implied that we work in the one-loop approximation, so the stresses (which are responsible for quantum corrections) are calculated with respect to the unperturbed classical background, i.e., the Schwarzschild geometry. Then it follows from the conservation law  $\theta_\mu^\nu{}_{;\nu} = 0$  with  $\mu = r$  in the Schwarzschild background or from the general formalism [7] that nonzero components of  $T_\mu^{\nu S}$ , necessary to maintain equilibrium, are equal to

$$T_\phi^{\phi S} = T_\theta^{\theta S} = -\frac{R}{2} \delta(r-R) [T_r^{r HH}(R) - T_r^{r B}(R)]. \quad (4)$$

In general the Euclidean version of metric (1) possesses a conical singularity at the horizon  $r = r_+ = 2m$ . The only way to avoid it is to set the temperature equal to its Hawking value  $T_H = (4\pi)^{-1} \kappa = (4\pi)^{-1} [U'(r_+)V'(r_+)]^{1/2}$ , where  $\kappa$  is the surface gravity. Then

$$T_H = (8\pi m)^{-1} [1 + 8\pi r_+^2 T_0^0(r_+)] \exp[\psi(r_+)]. \quad (5)$$

Making use of the  $r$  component of the conservation law, one obtains in the Schwarzschild background

$$\frac{1}{4\pi} \frac{\partial \psi}{\partial r} = r \frac{\theta_r' - \theta_0^0}{1 - \frac{2m}{r}} = \frac{1}{m} \left[ r^2 \theta_i^i - \frac{\partial}{\partial r} (r^3 \theta_r^r) \right]. \quad (6)$$

Consider for definiteness the scalar massless field. Then explicit approximate calculations of the stress-energy tensor show [8,9] that in the Boulware state  $T_\mu^\nu \sim r^{-6}$ , when  $r \rightarrow \infty$ , so  $r^3 T_\mu^\nu \rightarrow 0$  as  $r \rightarrow \infty$ . Then it follows from Eqs. (6), (3), and (4) that

$$\psi_+ \equiv \psi(r_+) = \frac{4\pi}{m} \left[ \int_\infty^{r_+} dr r^2 T_i^i - R^3 [T_\mu^{\nu HH}(R) - T_\mu^{\nu B}(R)] - r_+^3 T_r^{r HH}(r_+) \right]. \quad (7)$$

The back reaction strength is governed by the small parameter  $\varepsilon = \hbar/m^2 \ll 1$  which is assumed to cause small corrections to the Schwarzschild metric:  $\psi \ll 1$ . Then, replacing  $e^\psi$  by  $1 + \psi$  and taking into account the regularity condition  $T_0^0(r_+) = T_r^r(r_+)$  which follows from Eq. (6) and the finiteness of  $T_\mu^\nu$  at the horizon in the Hartle-Hawking state, we obtain

$$T_H = (8\pi m)^{-1} (1 + \delta), \quad \delta = \delta_1 + \delta_2 \quad (8)$$

$$\delta_1 = \frac{4\pi}{m} \left[ R^3 T_r^{r HH}(R) - \int_{2m}^R dr r^2 T_i^{i HH} \right],$$

$$\delta_2 = \frac{4\pi}{m} \left[ -R^3 T_r^{r B}(R) - \int_R^\infty dr r^2 T_i^{i B} \right]. \quad (9)$$

It is convenient to rewrite Eq. (8) in terms of the total mass

$$M_{tot} = m + m^{HH}(r_+, R) + m^B(R, \infty), \quad (10)$$

where

$$m^{HH}(r_+, R) = -4\pi \int_{2m}^R dr r^2 T_0^{0 HH},$$

$$m^B(R, \infty) = -4\pi \int_R^\infty dr r^2 T_0^{0 B}. \quad (11)$$

With the same accuracy,

$$T_H = (8\pi M_{tot})^{-1} (1 + \gamma), \quad (12)$$

$$\gamma = \frac{4\pi}{M_{tot}} \left\{ R^3 [T_r^{r HH}(R) - T_r^{r B}(R)] - \int_{2m}^\infty dr r^2 T_\mu^\mu \right\},$$

where we took into account that the conformal anomaly does not depend on the state.

With Eqs. (3) and (4) at hand, it can be rewritten in terms of the total stress-energy tensor as

$$\gamma = -\frac{4\pi}{M_{tot}} \int_{2m}^\infty dr r^2 \theta_\mu^\mu. \quad (13)$$

This expression includes (i) bulk contributions from quantum fields in the Hartle-Hawking state inside the shell, (ii) the boundary term, and (iii) bulk contributions from quantum fields in the Boulware state outside the shell. In Ref. [10] only (i) and (ii) were calculated. Now we generalize that result for the entire system to take vacuum polarization (iii) into account.

The quantity  $\gamma$  in Eq. (13) split into two parts— $\gamma_1$ , depending only on the  $T_\mu^{\nu HH}$ , and  $\gamma_2$ , depending only on  $T_\mu^{\nu B}$ . The latter quantity has, for massless fields, the structure [8,9]

$$T_\mu^{\nu B} = K T_H^4 \left( \frac{r_+}{r} \right)^6 \left[ \frac{A_\mu^\nu}{\left(1 - \frac{r_+}{r}\right)^2} + B_\mu^\nu \right], \quad (14)$$

where tensors  $A_\mu^\nu$  and  $B_\mu^\nu$  are finite everywhere, including the horizon and infinity, and  $K$  is the numerical factor, singled out for convenience.

For large  $R$  the term  $T_r^{r HH}(R)$  tends to the constant, while  $T_\mu^{\nu B}$  and  $T_\mu^{\nu B}$  behave like  $r^{-6}$ , and  $\gamma_2/\gamma_1 \sim (r_+/R)^6$ . Therefore, for  $R \gg r_+$  we return to the situation considered in [1–6]. However, for  $R \sim 2r_+$ , the corrections due to the Boulware state can be significant. When  $R \rightarrow r_+$ ,

$$\delta \simeq -8\pi r_+^2 T_r^{rB}(R) \simeq C T_H^4 r_+^2 \left(1 - \frac{r_+}{R}\right)^{-2},$$

$$C = -8\pi A_1^1(r_+) K. \quad (15)$$

For conformal fields  $K = 2\pi^2/45$ ,  $A_1^1(r_+) = -\frac{1}{4}$ ,  $C = 4\pi^3/45$ . The quantity  $\delta$  can be rewritten as

$$\delta = \varepsilon D \left(1 - \frac{r_+}{R}\right)^{-2}, \quad (16)$$

where the small parameter  $\varepsilon \equiv \hbar/m^2$  governs the back reaction strength and  $D = C/4^5 \pi^4 = 1/11520\pi$ . As  $\delta$  is the product of the small and big quantities, it is possible that  $\delta \ll 1$  and, thus, perturbation theory is still valid. Then the fractional contribution of the Boulware state to the mass contains an additional factor  $(1 - r_+/R)$  and is also small, so with the same accuracy  $\gamma \simeq \delta$ . We can see that, for shells, sufficiently close to the horizon, the Boulware contribution

dominates the correction. In doing so,  $\gamma > 0$ . If the radius of the shell decreases further,  $T_\mu^{\nu B}$  diverges like  $(1 - r_+/R)^{-2}$ , quantum back reaction fails to be small, and the perturbation theory ceases to work.

There are two points, typical of treatment in an infinite space. First, usually the Hartle-Hawking and Boulware states appear in essentially different contexts: the first one is relevant for a black hole metric, while the second one applies to the background, typical of a relativistic star. Second, the relevancy of vacuum polarization in black hole thermodynamics implies, as a rule, the presence of massive fields. We saw, however, that account for the finiteness of the system, containing a black hole, leads to the overlap between both types of states, so the Boulware state does affect black hole thermodynamics even in the case of massless fields.

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