Non-Abelian BIonic brane intersections

Paul L. H. Cook,[1,*] Robert de Mello Koch,[1,3,†] and Jeff Murugan[2,3,‡]
[1]*Department of Physics and Center for Theoretical Physics, University of the Witwatersrand, Wits, 2050, South Africa*
[2]*Department of Mathematics and Applied Mathematics, University of Cape Town, South Africa*
[3]*Stellenbosch Institute for Advanced Studies, Stellenbosch, South Africa*
(Received 29 July 2003; published 23 December 2003)

We study ''fuzzy funnel'' solutions to the non-Abelian equations of motion of the D string. Our funnel describes $n^6/360$ coincident D-strings ending on $n^3/6$ D7-branes, in terms of a fuzzy six-sphere which expands along the string. We also provide a dual description of this configuration in terms of the world volume theory of the D7-branes. Our work makes use of an interesting nonlinear higher dimensional generalization of the instanton equations.

DOI: 10.1103/PhysRevD.68.126007 PACS number(s): 11.25.Uv

I. INTRODUCTION

Many new results in string theory have been obtained by studying the low energy world volume theory of D-branes [1]. A fascinating example is the appearance of noncommutative geometry. In particular, an interesting class of solutions has been obtained by studying a set of D1-branes that end on an orthogonal D3-brane [2] or on an orthogonal D5-brane [3]. These fuzzy funnel solutions consist of a fuzzy sphere geometry which expands along the length of the string.

Fuzzy spheres themselves are a fascinating example of noncommutative geometry. They arise as solutions to matrix brane actions [4,2,3] and may also play a role in a spacetime explanation of the stringy exclusion principle [5]. The geometry of even dimensional fuzzy spheres has been investigated in [6] and the detailed SO(m) decomposition of the matrix algebras of the fuzzy spheres has been given in [7]. For fuzzy spheres S^m with $m>2$, it turns out that the matrix algebras contain more representations than is needed to describe functions on the sphere. In fact, in the classical limit (limit of large matrices), the matrix algebras related to even dimensional fuzzy spheres approach the algebra of functions of the higher dimensional space $\mathrm{SO}(2k+1)/\mathrm{U}(k)$. It has been argued that the appearance of these extra dimensions is a consequence of the Myers effect [8].

In this paper we study ''fuzzy funnel'' solutions to the non-Abelian equations of motion of the D-string. Our funnel describes $n^6/360$ coincident D-strings ending on $n^3/6$ D7-branes. The geometry of our solution is that of a fuzzy S^6 which expands along the string. This connection between the number of D-strings and the number of D7-branes has also been obtained directly from the noncommutative geometry of the S^6. This solution is a natural generalization of the D1⊥D3 [2] and the D1⊥D5 [3] solutions which made use of the fuzzy S^2 and fuzzy S^4, respectively. We also provide a dual description of this configuration in terms of the world volume theory of the D7-branes. The D7-brane theory gauge field configurations have nonvanishing third Chern character on the six-sphere surrounding the end points of the D-strings. The energy, charge, and radial profile of our solution computed in the two descriptions agree exactly.

Our paper is organized as follows. Since our solution makes use of the fuzzy S^6, we review the relevant matrix algebra in Sec. II. In Sec. III we develop the description of our system using the low energy D-string theory. In Sec. IV we recover the same results using the low energy D7-brane theory. In Sec. V we consider the simplest fluctuations on the fuzzy funnel solution. Finally in Sec. VI we make some comments on the domains of validity of both the D-string and the D7-brane theories.

II. FUZZY SIX-SPHERE

In this section we review the construction of the fuzzy six-sphere. This is done to establish notation and to derive a number of identities that will be used in later sections. In preparing this section we found [9] helpful.

To construct the fuzzy six-sphere, we need to construct solutions to the equation

$$\sum_{i=1}^{7} X^i X^i = c\mathbf{1}, \tag{2.1}$$

with X^i a matrix, $\mathbf{1}$ the identity matrix, and c a constant. Schur's lemma can be used to obtain a simple construction of the matrices X^i. Toward this end, consider the Clifford algebra

$$\{\Gamma^i,\Gamma^j\}=2\delta^{ij}, \quad i,j=1,2,...,7.$$

Denote the space on which the Γ^i matrices act by V. The n-fold tensor product of V is written as $V^{\otimes n}$. The X^i are now obtained by taking

$$X^i=(\Gamma^i\otimes 1\otimes\cdots\otimes 1+1\otimes\Gamma^i\otimes\cdots\otimes 1+\cdots$$
$$+1\otimes 1\otimes\cdots\otimes\Gamma^i)_{st}.$$

The subscript st is to indicate that the above X^i are to be restricted to the completely symmetric and traceless tensor

*Email address: pcook@neo.phys.wits.ac.za
†Email address: robert@neo.phys.wits.ac.za
‡Email address: jeff@hbar.mth.uct.ac.za

product space.[1] To prove that the above X^i do indeed provide coordinates for the fuzzy six-sphere, one shows that $\Sigma_{i=1}^7 X^i X^i$ commutes with the generators of SO(7)

$$X^{kl}=\frac{1}{2}([\Gamma^k,\Gamma^l]\otimes 1\otimes\cdots\otimes 1+1\otimes[\Gamma^k,\Gamma^l]\otimes\cdots\otimes 1+\cdots +1\otimes 1\otimes\cdots\otimes[\Gamma^k,\Gamma^l])_{st}.$$

The result (2.1) now follows from Schur's lemma. Further, using the Clifford algebra we easily find $c=n(n+6)$. The X^{ij} matrices generate the SO(7) Lie algebra. The matrix algebra associated with the fuzzy S^6 includes both the X^i and the X^{ij}. Together these matrices generate the SO(7,1) Lie algebra.

The symmetric traceless representation we work with has dimension

$$N=\frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5),$$

which identifies the representation generated by the X^{ij} as the $\mathbf{r}=(n/2,n/2,n/2)$ irreducible representation of SO(7).

Using the above definitions and the Clifford algebra, it is straightforward to derive the following identities (as usual, repeated indices are summed):

$$[X^{ij},X^{kl}]=2\delta^{ik}X^{il}-2\delta^{ik}X^{jl}+2\delta^{jl}X^{ki}-2\delta^{il}X^{kj},$$

$$[X^{ij},X^k]=2(\delta^{jk}X^i-\delta^{ik}X^j),$$

$$X^iX^i=c\mathbf{1},$$

$$X^{ij}X^j=6X^i=X^jX^{ji},$$

$$X^jX^{jk}X^{kl}X^l=6^2c\mathbf{1},$$

$$X^jX^{jk}X^{kl}X^{lm}X^{mn}X^n=6^4c\mathbf{1},$$

$$X^jX^{jk}X^{kl}X^{lm}X^{mn}X^{np}X^{pq}X^q=6^6c\mathbf{1},$$

$$X^{ij}X^{jl}=6X^{il}-X^iX^l+c\,\delta^{il}\mathbf{1},$$

$$X^{ij}X^{ji}=6c\mathbf{1},$$

$$X^{ij}X^{jk}X^{kl}X^{li}=6c^2\mathbf{1},$$

$$X^{ij}X^{jk}X^{kl}X^{lm}X^{mn}X^{ni}=6c^3\mathbf{1},$$

$$\epsilon^{ijklmnq}X^iX^jX^kX^lX^mX^n=i(384+288n+48n^2)X^q.$$

The geometry of the fuzzy six-sphere has been studied in detail in [6]. These authors argue that the fuzzy six-sphere is a bundle over the sphere S^6. In the classical limit the fiber over the sphere is the symmetric space SO(6)/U(3). The result of relevance to us, following from this geometrical analysis, is that one can identify points in the base, and as a consequence it is possible to read off the 6-brane charge as $\frac{1}{6}(n+1)(n+2)(n+3)$. We will see that it is possible to reproduce this purely noncommutative geometric derivation of the charge using either a dynamical analysis based on the non-Abelian Born-Infeld description of N coincident D-strings in the large N limit, or by using the non-Abelian Born-Infeld description of $n^3/6$ D7-branes, in the large n limit. The D6 brane charge will correspond to a D7-brane charge in our T-dual description.

In the remainder of this paper we work in the large n limit. Consequently we use

$$N=\frac{n^6}{360},\quad c=n^2$$

and take the 6-brane charge to be $n^3/6$.

III. DESCRIPTION OF THE D1⊥D7 SYSTEM IN TERMS OF N D1 BRANES

In this section we study the fuzzy geometry of the D7⊥D1 system, using the non-Abelian theory describing N coincident D-strings. Our construction employs the fuzzy six-sphere to construct a fuzzy funnel in which the D-strings expand into orthogonal D7-branes. We use an approach based on minimizing the energy [10], which generalizes the results obtained in [2] for D1-branes expanding into orthogonal D3-branes and the results in [3] for D1-branes expanding into orthogonal D5-branes.

The low energy effective action for N D-strings is given by the non-Abelian Born-Infeld action [11,12]

$$S=-T_1\int d^2\sigma\,\mathrm{STr}\sqrt{-\det\begin{bmatrix}\eta_{ab} & \lambda\partial_a\Phi^j\\ -\lambda\partial_b\Phi^i & Q^{ij}\end{bmatrix}}\equiv -T_1\int d^2\sigma\,\mathrm{STr}\sqrt{-\det M}, \tag{3.1}$$

where

$$Q^{ij}=\delta^{ij}+i\lambda[\Phi^i,\Phi^j],\quad \lambda=2\pi l_s^2.$$

The symmetrized trace prescription [13] (indicated by STr in the above action) instructs us to symmetrize over all permutations of $\partial_a\Phi^i$ and $[\Phi^i,\Phi^j]$ within the trace over the gauge group indices, after expanding the square root. We are using static gauge so that the worldsheet coordinates are identified with spacetime coordinates as $\tau=x^0$ and $\sigma=x^9$. The transverse coordinates are now the non-Abelian scalars Φ^i, $i=1,...,8$. These scalars are $N\times N$ matrices transforming in the adjoint representation of the U(N) gauge symmetry present on the worldsheet of the D1s.

[1]This restriction is important if one is to obtain an irreducible representation, which is assumed in the application of Schur's lemma.

We seek static solutions with seven of the scalars excited. It is a tedious but straightforward exercise to show that it is consistent with the equations of motion to make use of a static ansatz that involves seven of the scalars, at the level of the action. With this ansatz and a rather lengthly calculation we obtain

$$
\begin{aligned}
-\det(M)=&1+\frac{\lambda^2}{2}\Phi^{ij}\Phi^{ji}+\frac{\lambda^4}{8}(\Phi^{ij}\Phi^{ji})^2-\frac{\lambda^4}{4}\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{li}\\
&+\lambda^6\left(\frac{(\Phi^{ij}\Phi^{ji})^3}{48}-\frac{\Phi^{mn}\Phi^{nm}\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{li}}{8}+\frac{\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{lm}\Phi^{mn}\Phi^{ni}}{6}\right)+\lambda^2\partial_\sigma\Phi^i\partial_\sigma\Phi^i\\
&+\lambda^4\left(\frac{\partial_\sigma\Phi^k\partial_\sigma\Phi^k\Phi^{ij}\Phi^{ji}}{2}-\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k\right)-\lambda^6\left(\frac{\partial_\sigma\Phi^m\partial_\sigma\Phi^m\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{li}}{4}-\frac{\partial_\sigma\Phi^i\partial_\sigma\Phi^i(\Phi^{ij}\Phi^{ji})^2}{8}\right.\\
&\left.+\frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k\Phi^{ml}\Phi^{lm}}{2}-\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{lm}\partial_\sigma\Phi^m\right)\\
&-\lambda^8\left(-\frac{\partial_\sigma\Phi^k\partial_\sigma\Phi^k(\Phi^{ij}\Phi^{ji})^3}{48}+\frac{\partial_\sigma\Phi^p\partial_\sigma\Phi^p\Phi^{ij}\Phi^{ji}\Phi^{kl}\Phi^{lm}\Phi^{mn}\Phi^{nk}}{8}-\frac{\partial_\sigma\Phi^p\partial_\sigma\Phi^p\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{lm}\Phi^{mn}\Phi^{ni}}{6}\right.\\
&+\frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k(\Phi^{ml}\Phi^{lm})^2}{8}-\frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\partial_\sigma\Phi^k\Phi^{ml}\Phi^{ln}\Phi^{np}\Phi^{pm}}{4}-\frac{\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{ln}\partial_\sigma\Phi^n\Phi^{mp}\Phi^{pm}}{2}\\
&\left.+\partial_\sigma\Phi^i\Phi^{ij}\Phi^{jk}\Phi^{kl}\Phi^{ln}\Phi^{np}\Phi^{pm}\partial_\sigma\Phi^m\right),
\end{aligned}
$$

where

$$\Phi^{ij}=[\Phi^i,\Phi^j].$$

Our ansatz for the funnel solution is given by

$$\Phi^i=R(\sigma)X^i.$$

We have checked that the equation determining $R(\sigma)$ obtained by substituting this ansatz into the equations of motion [following from Eq. (3.1)] agree with the equations obtained by inserting this ansatz into the action (3.1) and varying with respect to $R(\sigma)$. Following this second procedure, inserting the above ansatz into Eq. (3.1) we obtain

$$
\begin{aligned}
S&=-T_1\int d^2\sigma\,\mathrm{STr}\sqrt{\left(1+\left(\frac{d\bar{R}}{d\sigma}\right)^2\right)(1+f(\bar{R}))},\\
f(\bar{R})&=12\frac{\bar{R}^4}{c\lambda^2}+48\frac{\bar{R}^8}{c^2\lambda^4}+64\frac{\bar{R}^{12}}{c^3\lambda^6},
\end{aligned}
\tag{3.2}
$$

where we have introduced the physical radius

$$\bar{R}=\sqrt{c}\lambda R.$$

In obtaining this result, use has been made of the identities listed in Sec. II. The formula (3.2) is not exact—it catches only the leading large N contribution. If we expand the square root in Eq. (3.1) and implement the symmetrization of the trace for each term in the expansion, we find corrections to Eq. (3.2) of order $1/c$ relative to the leading term. Thus our results are only valid for large N.

Since this is a static configuration, it is easy to obtain the following expression for the energy of our solution:

$$
\begin{aligned}
E&=NT_1\int d\sigma\sqrt{\left(1+\left(\frac{d\bar{R}}{d\sigma}\right)^2\right)(1+f(\bar{R}))}\\
&=NT_1\int d\sigma\sqrt{\left(\frac{d\bar{R}}{d\sigma}\pm\sqrt{f(\bar{R})}\right)^2+\left(1\mp\frac{d\bar{R}}{d\sigma}\sqrt{f(\bar{R})}\right)^2}\\
&\geqslant NT_1\int d\sigma\left(1\mp\frac{d\bar{R}}{d\sigma}\sqrt{f(\bar{R})}\right).
\end{aligned}
$$

The above inequality is saturated when

$$0=\frac{d\bar{R}}{d\sigma}\pm\sqrt{\frac{12\bar{R}^4}{c\lambda^2}+\frac{48\bar{R}^8}{c^2\lambda^4}+\frac{64\bar{R}^{12}}{c^3\lambda^6}}.$$

For small $\bar{R}$ it is simple to obtain

$$\frac{d\bar{R}}{d\sigma}=\mp\frac{2\sqrt{3}\bar{R}^2}{\sqrt{c}\lambda}\Rightarrow\bar{R}=\pm\frac{\sqrt{c}\lambda}{2\sqrt{3}(\sigma-\sigma_0)}.$$

This is the same behavior as was found in both the D3-brane funnel [2] and the D5-brane funnel [3]. We have reproduced the expected behavior for any D-string funnel in the region where the funnel is well approximated by the D-string. Consider now the large $\bar{R}$ region. If our funnel is to expand into an orthogonal D7-brane at large $\bar{R}$, the expansion must be given by a harmonic function in seven spatial dimensions. At large $\bar{R}$ we find

$$\frac{d\bar{R}}{d\sigma}=\mp\frac{8\bar{R}^6}{c^{3/2}\lambda^3}\Rightarrow\sigma-\sigma_0=\pm\frac{c^{3/2}\lambda^3}{40\bar{R}^5},$$

which is indeed the correct harmonic behavior needed for a D7-brane to appear at $\sigma=\sigma_0$.

Further evidence that we have a funnel expanding into coincident D7-branes is provided by computing the RR charge and energy of this solution. The energy of our solution is

$$E=NT_1\int d\sigma\left(1+\frac{d\bar{R}}{d\sigma}\sqrt{\frac{12\bar{R}^4}{c\lambda^2}+\frac{48\bar{R}^8}{c^2\lambda^4}+\frac{64\bar{R}^{12}}{c^3\lambda^6}}\right)$$

$$=NT_1\int_0^\infty d\sigma+NT_1\int_0^\infty d\bar{R}\sqrt{\frac{12\bar{R}^4}{c\lambda^2}+\frac{48\bar{R}^8}{c^2\lambda^4}+\frac{64\bar{R}^{12}}{c^3\lambda^6}}.$$

The first term is easily identified as the energy of N semi-infinite D-strings stretching from $\sigma=0$ to $\sigma=\infty$. Now consider the second term. We compute this term for large $\bar{R}$, where we expect that the funnel is expanding into a number of coincident D7-branes. Using the identities

$$N=\frac{n^6}{360},\quad c=n^2,$$

which are valid for large n, as well as the known relation between the tension of the D-string and the D7-brane and of the D-string and the D3-brane

$$T_7=\frac{T_1}{(2\pi l_s)^6},\quad T_3=\frac{T_1}{(2\pi l_s)^2},$$

it is straightforward to obtain the following result for the energy:

$$E=NT_1\int_0^\infty d\sigma+\frac{n^3}{6}T_7\left(\frac{16\pi^3}{15}\int d\bar{R}\,\bar{R}^6\right)+\frac{n^5}{240}T_3\int d\bar{R}4\,\pi\bar{R}^2+\Delta E, \tag{3.3}$$

where

$$\Delta E=NT_1c^{1/4}\sqrt{\frac{\lambda}{2}}\int_0^\infty\left[\sqrt{u^{12}+3u^8+3u^4}-u^6-\frac{3}{2}u^2\right]du$$

$$\approx(0.2629...)NT_1c^{1/4}\sqrt{\lambda}.$$

The second term in Eq. (3.3) is precisely the energy of $n^3/6$ D7-branes, so that we have reproduced the noncommutative geometric derivation of the charge given in [6]. The two terms given provide the analogue of the two terms providing the total energy of the supersymmetric D3⊥D1 system [2]. The fact that there are further contributions to the energy matches what one finds in the analysis of the D5⊥D1 system [3]. In the D5⊥D1 context, this was interpreted as a consequence of the fact that the system is not supersymmetric. The third term in Eq. (3.3) is apparently the energy of $n^5/240$ D3 branes. Recall that the zero scale size limit of an instanton in a Dp-brane corresponds to a D$(p-4)$ brane bound to the Dp-brane [14]. Thus this term is naturally interpreted as an instanton contribution in the D7-brane theory. It is interesting to note that the corresponding term in the D5⊥D1 system arises from a D1 contribution, which can be interpreted as an instanton contribution in the D5-brane theory. It would be interesting to understand the physical origin of this term, perhaps as a consequence of the Myers effect. The last term represents a finite binding energy.

We have evidence that our solution describes a funnel expanding into a number of coincident D7-branes located at $\sigma=0$. The D7 branes expand to fill the X^i, $i=1,2,...,7$ directions. If this is indeed the case, this configuration should be a source for the eight-form RR-potential $C^{(8)}_{012345678}$. We check this, providing a further check of the D7-brane charge computed by studying the energy of our configuration. The relevant source term comes from the following contribution to the non-Abelian Wess-Zumino action:

$$S_{\rm WZ}=-i\frac{\lambda^3}{6}\mu_1\int \mathrm{STr}\,P[(i_\Phi i_\Phi)^3C^{(8)}].$$

Evaluating the value of this term for our solution

$$S_{\rm WZ}=-i\frac{\lambda^4}{6}\mu_1\int d\sigma d\tau C^{(8)}_{01234567}\times\mathrm{STr}(\epsilon^{ijklmnp}\Phi^i\Phi^j\Phi^k\Phi^l\Phi^m\Phi^n\partial_\sigma\Phi^p)$$

$$=-i\frac{\lambda^4\mu_1}{6\lambda^7c^{7/2}}\int d\sigma d\tau C^{(8)}_{01234567}\times\mathrm{STr}(\epsilon^{ijklmnp}G^iG^jG^kG^lG^mG^nG^p)\bar{R}^6\frac{d\bar{R}}{d\sigma},$$

using the identities given in Sec. II, the relation between D7 and D1 charges

$$\mu_7=\frac{\mu_1}{(2\pi l_s)^6},$$

and working in the large n limit, we obtain

$$S_{\rm WZ}=\frac{n^3}{6}\mu_7\left(\frac{16\pi^3}{15}\int d\bar{R}\,C^{(8)}_{01234567}\bar{R}^6\right).$$

This is exactly the seven-brane source term we would expect to get if we have $n^3/6$ D7-branes in complete agreement with our energy computation.

Up to now, we have obtained solutions by employing a method which minimizes the energy. We end this section with a direct analysis of the equations of motion. Requiring that Eq. (3.2) is stationary with respect to variations of $\bar{R}$, we obtain the following equation of motion:

$$\sqrt{1+\left(\frac{d\bar{R}}{d\sigma}\right)^2}\,\frac{d\sqrt{1+f(R)}}{d\bar{R}} = \frac{d}{d\sigma}\left(\sqrt{\frac{1+f(\bar{R})}{1+\left(\frac{d\bar{R}}{d\sigma}\right)^2}}\,\frac{d\bar{R}}{d\sigma}\right).$$

After some straightforward manipulations, this equation of motion can be written as

$$\left(\frac{d\bar{R}}{d\sigma}\right)^{-1}\frac{d}{d\sigma}\left(\sqrt{\frac{1+f(\bar{R})}{1+\left(\frac{d\bar{R}}{d\sigma}\right)^2}}\right)=0,$$

which is easily integrated to give

$$\frac{d\bar{R}}{d\sigma}=\pm\sqrt{kf(\bar{R})-1}, \tag{3.4}$$

where k is a non-negative dimensionless constant of integration. For $k=1$, we reproduce the energy we obtained above by minimizing the energy. For $0\leqslant k\leqslant 1$, the solution reaches $\bar{R}=0$ at a finite value of σ so that the funnel "pinches" off. As explained in [2] this solution can naturally be continued past $\bar{R}=0$ by matching to a second pinched off funnel. This configuration provides the description of two parallel sets of coincident D7-branes, joined by N finite length D-strings. If $k>1$, the solution reaches $d\bar{R}/d\sigma=0$ at finite σ and terminates. Again [2], this solution is naturally continued by matching to a second funnel. In this case, the double funnel describes N finite D-strings joining a set of coincident anti-D7 branes with a set of parallel coincident D7-branes.

This concludes our discussion of the D-string theory. In the next section we turn to a dual description of the same configuration, which employs the non-Abelian world volume theory of the coincident D7-branes.

IV. D1⊥D7 CONFIGURATION USING A D7 WORLD VOLUME DESCRIPTION

In the previous section we have argued that our funnel describes $N=n^6/360$ D-strings expanding into $n^3/6$ D7-branes. Consequently the D7-brane world volume theory is a $7+1$ dimensional non-Abelian Born-Infeld theory with gauge group $U(n^3/6)$. Further, to describe the D-strings, we will also have to excite one of the transverse scalars. This scalar has to reside in the overall U(1) component of the $U(n^3/6)$ gauge group, since it describes a deformation of the geometry of all of the D7-branes. Consequently, we consider the action

$$S=-T_7\int d^8\sigma\,\mathrm{STr}\sqrt{-\det(G_{ab}+\lambda^2\partial_a\phi\partial_b\phi+\lambda F_{ab})}.$$

We employ spherical coordinates on the D7 world volume

$$ds^2=G_{ab}d\sigma^a d\sigma^b=-dt^2+dr^2+r^2 g_{ij}d\alpha^i d\alpha^j,$$

with g_{ij} the metric on the six-sphere of unit radius, r is the radial coordinate, and α^i the angles. In analogy to the D5⊥D1 system [3], we make the following ansatz for the scalar and gauge fields:

$$\phi=\phi(r),\quad A_r=0,\quad A_{\alpha^i}=A_{\alpha^i}(\alpha^j).$$

Once again we have examined the full equations of motion and have verified that this is indeed a consistent ansatz. Inserting this ansatz into the above action, we obtain

$$\begin{aligned} S_7&=-T_7\int d^8\sigma\sqrt{\left(1+\lambda^2\left(\frac{d\phi}{dr}\right)^2\right)g}\;\mathrm{STr}\sqrt{h(r)},\\ g&=\det g_{ij}=-T_7\int d^8\sigma L_7,\\ h(r)&=r^{12}+\frac{1}{2}r^8\lambda^2F^{ij}F_{ij}\\ &\quad+\frac{1}{128}r^4\lambda^4\epsilon_{ijklmn}\epsilon^{ijopqr}F^{kl}F^{mn}F_{op}F_{qr}\\ &\quad+\frac{1}{2304}\lambda^6(\epsilon_{ijklmn}F^{ij}F^{kl}F^{mn})^2. \end{aligned}\tag{4.1}$$

In the above expression, $F_{ij}\equiv F_{\alpha^i\alpha^j}$, indices on the field strength are raised and lowered with the metric g_{ij}, and $\epsilon_{123456}=g$. The equation of motion for the scalar is

$$\frac{d}{dr}\left(\frac{\partial L_7}{\partial\phi'}\right)=0,\quad \phi'=\frac{d\phi}{dr}.$$

This is easily integrated to obtain

$$\frac{\lambda^2\phi'}{\sqrt{1+\lambda^2(\partial_r\phi)^2}}=\frac{f(\alpha^i)}{\sqrt{g}\,\mathrm{STr}\sqrt{h(r)}},$$

where $f(\alpha^i)$ is an arbitrary function of integration depending only on the angles α^i. The left-hand side of the above equation is independent of the α^i, so we must have $\mathrm{STr}\sqrt{h(r)}$ independent of the angles and further,

$$f(\alpha^i)=\frac{\sqrt{g}\lambda^4}{b},$$

with b a dimensionless constant. With this choice we obtain

$$\lambda\phi'=\pm\frac{1}{\sqrt{\frac{b^2[\mathrm{STr}(\sqrt{h(r)})]^2}{\lambda^6}-1}}. \tag{4.2}$$

After identifying $\sigma=\lambda\phi$ we have

$$\lambda\frac{d\phi}{dr}=\frac{d\sigma}{dr}.$$

With this identification and $r=\overline{R}$, the radial profile (4.2) can be matched to the result we obtained from the D-string world volume theory (3.4) by setting

$$kf(r)=\frac{(\mathrm{STr}\sqrt{h(r)})^2b^2}{\lambda^6}.$$

This last condition can be satisfied by choosing

$$F^{ij}F_{ij}=\frac{3c}{2}\mathbf{1},$$

$$\epsilon_{ijklmn}\epsilon^{ijopqr}F^{kl}F^{mn}F_{op}F_{qr}=24c^2\mathbf{1}, \tag{4.3}$$

$$(\epsilon_{ijklmn}F^{ij}F^{kl}F^{mn})^2=36c^3\mathbf{1},$$

where $\mathbf{1}$ is the $n^3/6\times n^3/6$ unit matrix. It is interesting to note that these last three identities reduce to a single independent equation if one chooses

$$8\sqrt{c}F^{ij}=\epsilon^{ijklmn}F_{kl}F_{mn}. \tag{4.4}$$

This last equation provides an interesting nonlinear higher dimensional generalization of the instanton equation. This relation is also suggested by the D-string description [6]. In matrix theory, the commutator $X^{\mu\nu}=i[X^\mu,X^\nu]$ of the matrix valued coordinates is naturally interpreted as a field strength. The state for which

$$X^7|s\rangle=-n|s\rangle,\quad X^i|s\rangle=0,\quad i<7$$

corresponds to a point at the north pole of the sphere. Locally at the north pole, directions i with $i<7$ correspond to the α^i directions. Acting on this state, we find that the only nonzero "field strengths" are

$$i[X^1,X^2]|s\rangle=2n|s\rangle,\quad i[X^3,X^4]|s\rangle=-2n|s\rangle,$$

$$i[X^5,X^6]|s\rangle=-2n|s\rangle.$$

Since $\sqrt{c}=n$, we see that the field strengths at the north pole do indeed satisfy Eq. (4.4). In the remainder of this section we will assume that our field strengths satisfy Eqs. (4.3) and (4.4). We will not address the issue of obtaining an actual gauge field solution from which we can compute these field strengths. Note that the above field strengths satisfy

$$\frac{1}{48\pi^3}\int\mathrm{Tr}\left(\frac{\epsilon_{ijklmn}F^{ij}F^{kl}F^{mn}}{8}\right)\sqrt{g}d^6\alpha=\frac{n^6}{360}=N,$$

exactly as one would expect for any six-sphere surrounding the D-string end points.

We now turn to a computation of the energy of this solution. To compare to the energy of the configuration that saturated the energy bound, we now set $k=1$. The energy is

$$E=T_7\int\sqrt{g}d^6\alpha dr\sqrt{1+\lambda^2\left(\frac{d\phi}{dr}\right)^2}\frac{n^3}{6}\times\sqrt{r^{12}+\frac{3r^8\lambda^2c}{4}+\frac{3r^4\lambda^4c^2}{16}+\frac{\lambda^6c^3}{64}}.$$

After using Eq. (4.2) this becomes

$$\begin{aligned}E&=T_1\frac{n^6}{360}\int_0^\infty dr\lambda\frac{d\phi}{dr}\\&\quad+T_7\frac{16\pi^3}{15}\frac{n^3}{6}\int_0^\infty dr\sqrt{r^{12}+\frac{3r^8\lambda^2c}{4}+\frac{3r^4\lambda^4c^2}{16}}\\&=NT_1\int_0^\infty d\sigma+\frac{n^3}{6}T_7\left(\frac{16\pi^3}{15}\int d\overline{R}\,\overline{R}^6\right)\\&\quad+\frac{n^5}{240}T_3\int d\overline{R}\,4\pi\overline{R}^2+\Delta E,\end{aligned}$$

where again

$$\begin{aligned}\Delta E&=NT_1c^{1/4}\sqrt{\frac{\lambda}{2}}\int_0^\infty\left[\sqrt{u^{12}+3u^8+3u^4}-u^6-\frac{3}{2}u^2\right]du\\&\approx(0.2629...)NT_1c^{1/4}\sqrt{\lambda}.\end{aligned}$$

This exactly matches the energy computed using the D-string description. Thus the energy, radial profile of the funnel, and charge computed using the D7 world volume theory is in exact agreement with the calculations performed using the D-string world volume theory.

V. FLUCTUATIONS

In this section we study the propagation of fluctuations on the fuzzy funnel solution obtained in Sec. III. For a similar analysis of fluctuations for the D3⊥D1 and the D5⊥D1 systems see [2] and [3], respectively.

Since our funnel has the topology $R\times S^6$, the fluctuations of this geometry are naturally decomposed in terms of the spherical harmonics on the S^6. Of course, we have a fuzzy S^6, so it is natural to expand the fluctuations in terms of traceless symmetric products of the X^i, which provide the deformation of the usual algebra of functions on S^6. One consequence of the fact that we use a fuzzy sphere is simply that there is a highest angular momentum $l\leq l_{\max}=n$. Concretely, we consider fluctuations of the form

$$\delta\Phi^8=\mathcal{C}_{i_1i_2\cdots i_n}(\tau,\sigma)X^{i_1}X^{i_2}\cdots X^{i_n},\quad\delta\Phi^i=0,\quad i<8,$$

where $\mathcal{C}_{i_1i_2\cdots i_n}(\tau,\sigma)$ is required to be a traceless symmetric tensor. Our goal in this section is simply to show that these modes, which correspond to partial waves of angular momentum n, see the correct angular momentum barrier.

The lowest order equation of motion is

$$(-\partial_\tau^2+\partial_\sigma^2)\Phi^i=[\Phi^j,[\Phi^j,\Phi^i]].$$

This equation of motion is valid for small Φ^j and hence corresponds to the region of small $R(\sigma)$. The linearized equation for the fluctuation following from this lowest order equation of motion is

$$(-\partial_\tau^2+\partial_\sigma^2)\,\delta\Phi^8=[\delta\Phi^j,[\Phi^j,\Phi^8]]+[\Phi^j,[\delta\Phi^j,\Phi^8]]+[\Phi^j,[\Phi^j,\delta\Phi^8]].$$

Since $\Phi^8=0$ and $\delta\Phi^j=0$ for $j<8$, this simplifies to

$$(-\partial_\tau^2+\partial_\sigma^2)\,\delta\Phi^8=[\Phi^j,[\Phi^j,\delta\Phi^8]]. \tag{5.1}$$

To evaluate the right-hand side, we need to use the result

$$\begin{aligned}[\Phi^j,[\Phi^j,\delta\Phi^8]]&=R^2(\sigma)C^{i_1i_2\cdots i_n}[G^j,[G^j,G^{i_1}G^{i_2}\cdots G^{i_n}]]\\&=4n(n+4)R^2(\sigma)C^{i_1i_2\cdots i_n}G^{i_1}G^{i_2}\cdots G^{i_n}.\end{aligned}$$

Identifying $R^2(\sigma)=1/12\sigma^2$ which is valid when $R(\sigma)$ is small, we obtain

$$\left(\partial_\tau^2-\partial_\sigma^2+\frac{n(n+4)}{3\sigma^2}\right)C^{i_1i_2\cdots i_n}(\tau,\sigma)=0.$$

Thus the double commutator on the right-hand side of Eq. (5.1) has indeed reproduced the correct angular momentum barrier.

VI. FINAL COMMENTS

We have obtained a description of the D1⊥D7 system in terms of a fuzzy six-sphere which expands along the string. We have studied the energy, charge, and radial profile of this configuration using the non-Abelian equations of motion of the D-string and also by using the dual description provided by the world volume theory of the D7-branes. Our analysis is limited to the low energy world volume theory in each case. The agreement between descriptions is perfect. Further, we have found that the configuration describes $n^6/360$ coincident D-strings ending on $n^3/6$ D7-branes. This relation between the number of D-strings and D7-branes has also been obtained from a direct study of the noncommutative geometry of the fuzzy S^6.

This precise agreement between the two descriptions is also a feature of the D1⊥D3 and D1⊥D5 systems. For the system we have studied in this paper, we would expect the D7-brane world volume theory to provide a reliable description for those regions of the funnel that have opened up to fill out a seven-dimensional spatial volume and are hence well approximated as a D7-brane. The D-string world volume theory should provide a reliable description of the funnel in the regions where the funnel is very thin and hence well approximated by a D-string. Thus we have two complementary descriptions of the D1⊥D7 system. How are we to understand the agreement between the two descriptions of the D1⊥D7 system?

There are two potential sources of corrections to both descriptions. There are both higher derivative corrections and higher order commutator corrections. Following [3] we assume that we can ignore higher derivative corrections when $l_s|\partial^2\Phi|\ll|\partial\Phi|$. For the D-string theory, we easily find that this condition implies that $r\ll(n^3\pi^3/12)^{1/5}l_s$. For the D7-brane theory this condition implies that $r\gg 2l_s$. Thus for large n there is a significant region $[2l_s\ll r\ll(n^3\pi^3/12)^{1/5}l_s]$ where both descriptions do not receive higher derivative (α') corrections. A conservative bound for the region in which higher commutator terms are avoided is obtained by requiring that the Taylor expansion of the square root in the D-string action should converge very rapidly. This implies that $r\ll\sqrt{n}l_s$. For large n we have $\sqrt{n}\gg(n^3\pi^3/12)^{1/5}$, so that this is less restrictive than what we obtained above. We have not established the analogous region in which higher commutator terms are avoided in the D7-brane theory.

Since the D1⊥D7 system is supersymmetric, it is natural to ask if our solution for a D1 ending on a D7 preserves some supersymmetry. We will now argue that it is possible to *lower* the energy of our solution, something which is *not* possible for the D1⊥D5 system. The fact that we can lower the energy of our solution suggests that our configuration is not supersymmetric or stable. To reproduce the correct D1 charge, we need to consider a field strength in the D7 brane world volume theory which satisfies

$$\int_{S^6}F\wedge F\wedge F\propto N.$$

If we consider a field strength, which is nonzero only in a volume V on the S^6, we can estimate the magnitude of this field strength as

$$F\propto\left(\frac{N}{V}\right)^{1/3}.$$

After expanding the Born-Infeld action and keeping only the term quadratic in the field strength, we obtain the following formula for the energy per unit length:

$$\int_{S^6}F^2\propto V^{1/3}N^{2/3}.$$

Minimizing the volume V on which the field strength is nonzero would clearly lower this energy per unit length. Thus, under the assumptions stated, the configuration with a homogeneous field strength over the full S^6 must be unstable. This is in contrast to the D1⊥D5 system as we now explain. For the D1⊥D5 system, the D1 charge is reproduced by a field strength in the D5 world volume which satisfies

$$\int_{S^4}F\wedge F\propto N,$$

so that we would estimate

$$F \propto \sqrt{\frac{N}{V}}.$$

Hence the energy per unit length of the configuration

$$\int_{S^4} F^2 \propto N$$

is independent of V.

ACKNOWLEDGMENTS

We would like to thank Sanjaye Ramgoolam for extremely helpful discussions and for comments on the manuscript. We would also like to thank Rocco Duvenhage, Phil Ferrer, and Joao Rodrigues for pleasant discussions. The work of R.d.M.K. and P.C. is supported by NRF grant number Gun 2047219. The work of J.M. is supported by the Lindbury trust. R.d.M.K. and J.M. would like to thank Fritz Hahne, Bernard Lategan, and Maria Mouton for their warm hospitatility at the Stellenbosch Institute for Adavanced Studies, where this work was completed.

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