Self-similarity and singularity formation in a coupled system of Yang-Mills-dilaton evolution equations

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We study both analytically and numerically a coupled system of spherically symmetric SU(2) Yang-Mills-dilaton equations in 3+1 Minkowski space-time. It has been found that the system admits a hidden scale invariance which becomes transparent if a special ansatz for the dilaton field is used. This choice corresponds to a transition to a frame rotated in the $\ln r$-t plane at a definite angle. We find an infinite countable family of self-similar solutions which can be parametrized by the N—the number of zeros of the relevant Yang-Mills (YM) function. According to the performed linear perturbation analysis, the lowest solution with $N=0$ only occurred to be stable. The Cauchy problem has been solved numerically for a wide range of smooth finite-energy initial data. It has been found that if the initial data exceed some threshold, the resulting solutions in a compact region shrinking to the origin attain the lowest $N=0$ stable self-similar profile, which can pretend to be a global stable attractor in the Cauchy problem. The solutions reside a finite time in a self-similar regime and then the unbounded growth of the second derivative of the YM function at the origin indicates a singularity formation, which is in agreement with the general expectations for the supercritical systems.

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I. INTRODUCTION

Singularity formation in nonlinear evolution equations is a general problem which arises in various branches of mathematical physics. The term singularity means that the solutions to the evolutional differential equations cease to be differentiated at some point or region. However, the strong-field regime of the approach to the singularity can provide some universality and additional symmetries in the solutions behavior that is of particular interest in the corresponding theoretical field models, including gravity.

It is well known that the space-time singularities are the most generic features of Einstein's equations [1] and it is believed that the mixmaster-type singularity can pretend to be a generic one. However, the nature of the space-time singularity is model dependent and various matter fields provide various scenarios of singularity formation. In the typical case of gravitational collapse the late-time dynamics of the space-time singularity development is hidden under the event horizon formed. However in the last decade studies of the massless fields collapse initiated in the pioneering work of Choptuik [2] opened a new direction for understanding the singularity formation dynamics.

It was realized [2] that black holes with arbitrary small masses would be obtained and the mass scale low was discovered. This was called type-II behavior which is characterized by a mass gap absence in the black hole spectrum. The limiting case of a vanishing black hole mass is of particular interest. It has been found these critical solutions observed at the threshold of the black hole formation are discretely self-similar.

Later on, numerical studies of the gravitational collapse of self-interacting massless fields were performed [3,4] and the type-I behavior was observed as well. The type-I behavior is characterized by a finite mass gap in the black hole spectrum. This type-I behavior takes place if the considered system of Einstein-matter equations admits static finite-energy asymptotically flat solutions. In this case the smallest black hole has a finite mass which is equal to the mass of the lowest static solution. In the Einstein-Yang-Mills (EYM) system of equations the $N=1$ Bartnick-McKinnon (BK) static solution [5] is the lowest-mass static solution which determines a minimal mass of the formed black hole. Moreover, this $N=1$ BK solution is occurred to be an intermediate attractor which the collapsing solution should attain in order to turn to the type-I black hole formation scenario [3,6].

On the other hand, the analysis of the blowup in the nonlinear wave equations in various field models without gravity showed that singularity formation in gravity and blowup in the nonlinear wave equations share many common features [7–11]. Based on these observations, Bizon and Tabor put forward the conjecture [8] that all basic properties of the gravitational collapse of massless fields such as universality, self-similarity, and mass scaling, originally observed for Einstein's equations, are just the basic properties of a wide class of a supercritical evolution partial differential equations (PDEs). This class includes Einstein equations, the Yang-Mills equation in 5+1 Minkovski space, and many others.

Following this conjecture, we consider a coupled system of Yang-Mills-dilaton (YM-dilaton) equations in 3+1 Minkovski space which is of interest for several reasons. First of all, this system is a truncated version of a theoretical field model inspired by the heterotic string. Then the dilaton field itself which is also called a scalar graviton provides many key features characteristic of a gravity. For example the static system of spherically symmetric SU(2) Yang-Mills-dilaton equations has a countable infinite set of regular finite-mass solutions [12] which are similar to the BK solutions in the Einstein-Yang-Mills system of equations. Note that a similar set of the regular solutions exists in the Einstein-Yang-Mills-

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dilaton system [13,14]. This fact stresses the dominant role of the Yang-Mills field for the YM-dilaton, EYM, and EYM-dilaton systems dynamics. One can expect that the YM-dilaton model should exhibit all main features relevant for the EYM and EYM-dilaton models. Hence a study of the singularity formation in the YM-dilaton system should shed new light on a singularity development hidden under the formed event horizon in the EYM and EYM-dilaton models.

We have found that the considered system of spherically symmetric SU(2) YM-dilaton equations admits a hidden scale invariance. The corresponding criticality index is equal to $+1$, similar to the Einstein's equations in $3+1$ dimensions and Yang-Mills equation in $5+1$ Minkovski space-time. This means that the YM dilaton is supercritical and singularity should arise if the initial data exceed some threshold. Note that a pure Yang-Mills system of equations in $3+1$ Minkovski space-time is subcritical and the solutions should remain everywhere smooth during the evolution [15]. We have found the late-time asymptotics of the YM-dilaton solutions prior to the singularity formation is universal and is described by the lowest $N=0$ self-similar profile. This $N=0$ profile for the YM field is similar to those found in [8,9] for a pure YM field in $5+1$ Minkovski space. We have shown the whole family of self-similar solutions exists labeled by the $N=0,1,2,3,\ldots,\infty$—the number of the nodes of the relevant YM function.

The paper is organized as follows. In the next section we introduce the main definitions and discuss the scale properties of the obtained YM-dilaton system of spherically symmetric equations. In Sec. III the self-similar solutions of the YM-dilaton system and their linear stability are considered. And in Sec. IV the results of the numerical simulations of the evolutional Cauchy problem are discussed. We conclude with the main results and discuss briefly some open questions in the last section.

II. MAIN EQUATIONS AND DEFINITIONS

We consider a coupled system of Yang-Mills-dilaton fields, which is given by the action

$$S=\frac{1}{4\pi}\int\left[\frac{1}{2}(\partial\Phi)^2-\frac{\exp(k\Phi)}{4g^2}F^{a\mu\nu}F^{a}_{\mu\nu}\right]d^3x\,dt, \tag{1}$$

where Φ is the dilaton field and $F^{a\mu\nu}$ the Yang-Mills field. Note that this action is a truncated version of the bosonic part of the heterotic string effective action in four dimensions in Einstein frame [16].

In the case of spherical symmetry, the dilaton field and SU(2) Yang-Mills potential can be parametrized in terms of two independent functions $\Phi(t,r)$ and $f(t,r)$ as follows:

$$\Phi=\Phi(r,t),\quad A_t^a=0,\quad A_i^a=\epsilon_{aik}\frac{x^k}{r^2}[f(r,t)-1]. \tag{2}$$

After substitution of this ansatz into the action and field equations, the rescaling $\Phi\to\Phi/k$, $r\to(k/g)r$, $t\to(k/g)t$, and $S\to g^2 S$ removes the dependence on k and g and therefore we can put $k=1$, $g=1$ in what follows without restrictions. After integrating out the angular variables, the resulting action and the field equations become

$$S=-\int\left\{\frac{1}{2}r^2\Phi'^2-\frac{1}{2}r^2\dot{\Phi}^2+e^{\Phi}\left[f'^2-\dot{f}^2+\frac{(f^2-1)^2}{2r^2}\right]\right\}dr\,dt, \tag{3}$$

$$\ddot{f}+\dot{f}\dot{\Phi}-f''-f'\Phi'=\frac{f(1-f^2)}{r^2}, \tag{4}$$

$$\ddot{\Phi}-\Phi''-\frac{2\Phi'}{r}=-\frac{e^{\Phi}}{r^2}\left[f'^2-\dot{f}^2+\frac{(f^2-1)^2}{2r^2}\right]. \tag{5}$$

Here and below [except Eq. (26)] an over dot stands for a partial derivative with respect to t, while a prime is a partial derivative with respect to r.

It is also useful to write down the expression for the total energy

$$E=\int_0^\infty\left\{\frac{1}{2}r^2\Phi'^2+\frac{1}{2}r^2\dot{\Phi}^2+e^{\Phi}\left[f'^2+\dot{f}^2+\frac{(f^2-1)^2}{2r^2}\right]\right\}dr, \tag{6}$$

which is, of course, conserved *on shell*.

For further analysis it is important to note that the system of equations (4), (5) admits a hidden scale-invariant form. Indeed, Eq. (4) is scale invariant in sense that if $f(t,r)$, $\Phi(t,r)$ is a solution to Eq. (4), then

$$\tilde{f}(t,r)=f\left(\frac{t}{\lambda},\frac{r}{\lambda}\right),\quad \tilde{\Phi}(t,r)=\Phi\left(\frac{t}{\lambda},\frac{r}{\lambda}\right)$$

is also a solution. The same is not true for Eq. (5) because of the factor e^{Φ}/r^2 on the right-hand side of Eq. (5), which breaks the scale invariance. However, it is possible to extract a scale-invariant part $\phi(t,r)$ from the dilaton function $\Phi(t,r)$ that makes transparent the hidden scale invariance of the system. Indeed, if we consider the ansatz

$$\Phi(r,t)=\phi(r,t)+2\ln r, \tag{7}$$

the system of equations (4), (5), rewritten in terms of $f(t,r)$, $\phi(t,r)$, becomes scale invariant. The energy, expressed in terms of the functions $f(t,r)$, $\phi(t,r)$, becomes

$$E=\int_0^\infty dr\left\{\frac{1}{2}r^2\phi'^2+2r\phi'+2+\frac{1}{2}r^2\dot{\phi}^2+r^2e^{\phi}\left[f'^2+\dot{f}^2+\frac{(f^2-1)^2}{2r^2}\right]\right\}, \tag{8}$$

providing the corresponding homogeneous scale low for the energy as

$$E\left[f\left(\frac{t}{\lambda}\right),\phi\left(\frac{r}{\lambda}\right)\right]=\lambda E[f(t,r),\phi(t,r)].$$

According to the PDE general theory [17] the degree α of the scale parameter λ as it enters in the energy homogeneous scale low defines the criticality class of the PDE system. The criticality class of the system indicates the possibility of a singularity formation in the corresponding well-posed Cauchy problem as follows. If $\alpha<0$, the system is subcritical: then, all initially regular solutions should remain globally regular during the evolution. If $\alpha>0$, the system is supercritical and one should expect singularity formations for a finite time for all initial data if it exceeds a some threshold values. If $\alpha=0$, the system is critical and there are no definite expectations on singularity formation. Since in our case we have $\alpha=+1$, the system is supercritical and one should expect singularity formations, which will be confirmed in the last section of this paper.

The revealed invariance under the scale dilations

$$f(t,r)\to f\left(\frac{t}{\lambda},\frac{r}{\lambda}\right),\quad \phi(t,r)\to\phi\left(\frac{t}{\lambda},\frac{r}{\lambda}\right)$$

allows one to search for solutions depending on r and t variables in a combination r/t. Because of the time translation invariance, it is useful to introduce some positive constant T, which transforms similarly under the dilations, and search for general scale-invariant solutions in a self-similar form:

$$f(r,t)=f(x),\quad \phi(r,t)=\phi(x),\quad x=\frac{T-t}{r}. \tag{9}$$

The constant T has the meaning of a blowup time—the absolute value of the time in the evolution Cauchy problem, when the expected singularity starts development and the solution ceases to be smooth. We also will use alternatively the inverse independent variable $\eta=1/x=r/(T-t)$, which is more natural for the study of the Cauchy problem. The coordinate η covers half of the complete Minkowski spacetime only, corresponding to the past region of the blowup point $t=T$, $r=0$. The coordinate x covers complete Minkowski space and we will use it mainly for the analysis of the self-similar solutions in the next section.

III. SELF-SIMILAR SOLUTIONS AND THEIR LINEAR STABILITY ANALYSIS

In order to bring the system of the scale-invariant equations to the form, suitable for further analysis, it is convenient to introduce a new function $s(x)$ as follows:

$$\phi(x)=\ln[x^2 s(x)].$$

As we will see below, $s(x)$ represents the regular part of the function $\phi(x)$ on the semiaxis $x\in[1,\infty)$.

In terms of the functions $f(x)$ and $s(x)$ the system of PDEs (4), (5) transforms to a system of ordinary differential equations (ODEs):

$$f_{,x,x}+\frac{2f_{,x}}{x}+\frac{f_{,x}s_{,x}}{s}=\frac{f(1-f^2)}{1-x^2}, \tag{10}$$

$$\frac{s_{,x,x}}{s}-\frac{s_{,x}^2}{s^2}-x^2 s f_{,x}^2-\frac{2}{x^2}=\frac{2}{1-x^2}-\frac{x^2 s(1-f^2)^2}{2(1-x^2)}. \tag{11}$$

This system has four singular points $x=-\infty$, -1, $+1$, $+\infty$. At the first step we restrict ourselves to the interval $x\in[1,+\infty)$ which covers the interior of the past light cone of the point $t=T$, $r=0$.

The natural requirement of the regularity for the YM function $f(x)$ on the past light cone of the point $t=T$, $r=0$ provides the following local solution of the system (10), (11) near $x=1$:

$$\begin{aligned}f(x)_{x\to1}&=f_1(x-1)-\frac{f_1}{8}(10+s_1)(x-1)^2+f_1\left(\frac{31}{48}-\frac{2}{3}f_1^2\right.\\&\quad\left.+\frac{1}{96}s_1^2+\frac{5}{32}s_1\right)(x-1)^3+O((x-1)^4),\\ s(x)_{x\to1}&=4+s_1(x-1)+\left(8+8f_1^2+\frac{1}{8}s_1^2+\frac{1}{2}s_1\right)(x-1)^2\\&\quad+\left(-8f_1^2+\frac{4}{3}s_1f_1^2+\frac{7}{3}s_1+\frac{1}{96}s_1^3+\frac{7}{48}s_1^2-\frac{4}{3}\right)\\&\quad\times(x-1)^3+O((x-1)^4),\end{aligned} \tag{12}$$

where f_1 and s_1 are free parameters. The regularity requirement of the YM function f at the origin on each slice $t<T$ (until the blowup time $t=T$) leads to the following series expansion of the solutions near $x=+\infty$, written by making use of the inverse $\eta=1/x$ self-similar variable:

$$\begin{aligned}f(\eta)_{\eta\to0}&=\pm1+f_2\eta^2+\frac{1}{10}f_2\left(-4s_0f_2^2+3f_2+\frac{10}{3}\right)\\&\quad\times\eta^4+O(\eta^6),\\ s(\eta)_{\eta\to0}&=s_0+s_0\left(s_0f_2^2-\frac{1}{3}\right)\eta^2\\&\quad+\frac{1}{20}s_0\left(8s_0^2f_2^4+8s_0f_2^3-\frac{8}{9}\right)\eta^4+O(\eta^6).\end{aligned} \tag{13}$$

Here f_2 and s_0 are also free parameters.

These local solutions at the singular points satisfy appropriate boundary conditions

$$\begin{aligned}&f(1)=0,\quad s(1)=4,\quad f(\infty)=\pm1,\\&f_{,x}(x)\to_{x\to\infty}0,\quad s_{,x}(x)\to_{x\to\infty}0.\end{aligned} \tag{14}$$

As a result we get a suitable desingularised boundary value problem (BVP) for the system (10), (11) with the boundary conditions of Eq. (14). Alternative strategy consists of solving of the system (10), (11) as an initial value problem start-

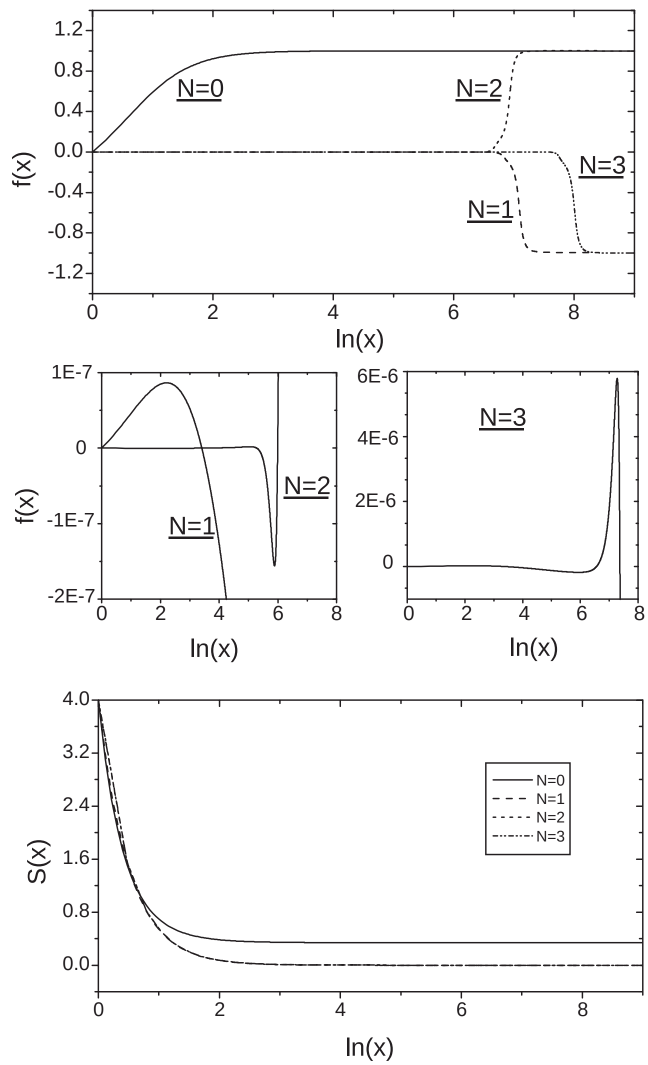


FIG. 1. Self-similar solutions for $N=0,1,2,3$: YM functions $f_N(x)$ (top), dilaton functions $s_N(x)$ (bottom). Note that as N increases the solution $s_N(x)$ tends uniformly to the limiting curve $4/x^2$.

ing with the series expansion (12) near the $x=1$. In this case the problem transforms to the shooting method search for some discrete sets of free parameters f_1, s_1, so that they provide the meeting of the local solutions to Eq. (13) at infinity. The method of shooting technique also gives a powerful tool to prove global existence, developed by Breitenlohner, Forgach, and Maison [18]. We are planning to consider the proof of the existence in a separate paper.

In order to find the above solutions numerically, we consider the above BVP using the method, described in detail in [19]. In accordance with the general expectations, we have found that the BVP (10), (11), (14) admits infinitely many solutions and all of then can be labeled by the N—the total number of zeros of the YM function $f(x)$ on the open semiaxis $x\in(1,\infty)$. The solutions with the lowest N, $N=0,1,2,3$ are given in Fig. 1. One can also extract the values of the corresponding free parameters f_1, s_1 or f_2, s_0 as they enter the local series expansions, Eq. (12) and Eq. (13). The values of f_1 and s_1 are displayed in Table I.

TABLE I. The values of free parameters f_1, s_1 for the self-similar solutions with various N.

N	f_1	s_1
0	0.498934096775465	-8.92179247
1	1.13571×10^{-12}	-8.00124
2	8.98718×10^{-13}	-8.00095
3	8.06871×10^{-13}	-8.00068
...	...	...
∞	0.0	-8.0

Note that the solution which corresponds to $N=\infty$ is simply the trivial solution $f(x)=0$, $s(x)=4/x^2$, which in terms of the primary function reads $\phi(x)=\ln 4$.

If N is finite and $N\gg 1$, the solution of the system of equations (10), (11) can be easily found by applying the linear perturbations theory around the trivial solution in the oscillation region. Indeed, after the substitution of $f(x)=0+\epsilon\tilde{f}(x)$, $\phi(x)=\ln 4+\epsilon\tilde{\phi}(x)$ into the system of equations (10), (11), the linearized system for the perturbations $\tilde{f}(x),\tilde{\phi}(x)$ which holds at the interval $1\ll x_0\leq x\leq x_N$ [$x_N\sim\exp(2\pi/\sqrt{3}N)$], becomes

$$\frac{d^2\tilde{f}}{dx^2}=-\frac{1}{x^2}\tilde{f},\quad \frac{d^2\tilde{\phi}}{dx^2}=\frac{2}{x^2}\tilde{\phi}. \tag{15}$$

The solution, bounded at infinity, can be found in terms of elementary functions as follows:

$$\tilde{f}(x)=C_1\sqrt{x}\sin\left(\frac{\sqrt{3}}{2}\ln x+\delta\right),\quad \tilde{\phi}=\frac{C_2}{x}, \tag{16}$$

where C_1, C_2, and δ are some integration constants. This oscillating solution gives a strong argument in favor of the existence of solutions with arbitrary $N>0$ zeros of YM function.

We found the solutions, which are defined on the interval $x\in[1,+\infty)$. Now one can analytically continue them in terms of the functions $f(x)$, $\phi(x)$ to the left of the point $x=1$. It has been done numerically. The continued functions behave monotonically at the interval $x\in[0,1]$ and they meet their corresponding values at the regular point $x=0$: $f(x=0)=f_\infty$, $\phi(x=0)=\phi_\infty$ which are also their asymptotic values at the spatial infinity $r\to+\infty$ on each slice $t=\text{const}$, $t<T$ in the self-similar regime. This can be seen from the series expansion of the solutions in terms of the η variable at $\eta\to+\infty$ (f_∞, ϕ_∞, $\overline{f_1}$, $\overline{\phi_1}$ are free parameters):

$$f(\eta)_{\eta\to\infty}=f_\infty+\overline{f_1}\eta^{-1}+\frac{1}{2}(-\overline{f_1}\,\overline{\phi_1}+f_\infty-f_\infty^3)\eta^{-2}+O(\eta^{-3}),$$

$$\phi(\eta)_{\eta\to\infty}=\phi_\infty+\overline{\phi_1}\eta^{-1}+\frac{1}{2}\left[2+e^{\phi_\infty}\left(\overline{f_1^2}-\frac{1}{2}+f_\infty^2-\frac{1}{2}f_\infty^4\right)\right]\times\eta^{-2}+O(\eta^{-3}). \tag{17}$$

For example, the values $f(x=0)=f_\infty=-0.5072593\ldots$, $\phi(x=0)=\phi_\infty=2.1214115\ldots$ correspond to the $N=0$ solution (see Fig. 5 below).

Further continuation of the solutions to the region of the negative x meets an obstacle at the next singular point $x=-1$. It was found that all self-similar solutions (except the trivial one), labeled by the $N=0,1,2,3,\ldots\infty$, ceased to be differentiable in the vicinity of the singular point $x=-1$ and therefore they cannot be continued smoothly to the complete manifold, covered by the x coordinate.

The stability analysis of the obtained self-similar solutions can be done in terms of the linear perturbation theory. The negative energy states in the spectrum of the linearized system are usually related to the instability of the background solutions.

Now we consider the problem on the semiaxis $x\in[1,+\infty)$. Let us introduce in addition to the variable x a second independent variable τ so that the set of the lines $x=$const is orthogonal to the set of the lines $\tau=$const:

$$x=\frac{T-t}{r},\quad \tau=-\ln\sqrt{(T-t)^2-r^2}. \tag{18}$$

The action (1) in the x,τ variables after the substitution of the anzatz (7) for the dilaton field becomes

$$S=\int dx d\tau e^{-2\tau}\Bigg\{\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{1}{2(x^2-1)^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 +\frac{2}{(x^2-1)^2}\left(\frac{\partial\phi}{\partial\tau}\right)-\frac{2x}{x^2-1}\left(\frac{\partial\phi}{\partial x}\right)+\frac{2}{x^2-1} +e^{\phi}\left[\left(\frac{\partial f}{\partial x}\right)^2-\frac{1}{(x^2-1)^2}\left(\frac{\partial f}{\partial\tau}\right)^2+\frac{(f^2-1)^2}{2(x^2-1)}\right]\Bigg\}. \tag{19}$$

We look for spherically symmetric perturbed solutions of the following form:

$$f(x,\tau)=f_N(x)+\epsilon\sqrt{2(x^2-1)}v(x)e^{i\omega\tau},$$
$$\phi(x,\tau)=\phi_N(x)+\epsilon e^{-\phi_N(x)/2}\sqrt{x^2-1}u(x)e^{i\omega\tau}, \tag{20}$$

where the background $f_N(x),\phi_N(x)$ are some solutions of the BVP (10), (11), (14), parametrized by N—the number of zeros of the YM function $f(x)$.

It is helpful to use the variable ρ instead of the x:

$$\rho=\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right),\quad x=\frac{1+e^{2\rho}}{1-e^{2\rho}}. \tag{21}$$

Then, following the lines of the Ref. [12], one can bring the effective action for the perturbations to the standard form of two-channel Schrödinger equation

$$\tilde{S}=-\int d\tau d\rho e^{2\tau}[(\chi)^+_{,\rho}(\chi)_{,\rho}+\chi^+U\chi-(\omega^2-1)\chi^+\chi], \tag{22}$$

where the column χ and the function $A(\rho)$ are introduced as follows (σ_2 is Pauli matrix):

$$\chi=e^{-i\sigma_2 A(\rho)}\begin{pmatrix}v(\rho)\\ u(\rho)\end{pmatrix},$$

$$A(\rho)=\int_0^\rho\frac{\exp[\phi_N(\xi)/2]f_N(\xi)_{,\xi}}{\sqrt{2}}d\xi. \tag{23}$$

Finally we get a matrix equation, which describes the spectrum of the linear perturbations,

$$-\chi_{,\rho,\rho}+U\chi=\Omega^2\chi,\quad \Omega^2=\omega^2-1, \tag{24}$$

where

$$U=\begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}=e^{-i\sigma_2 A}Ve^{i\sigma_2 A}, \tag{25}$$

and the elements of the matrix V are

$$V_{11}=e^{\varphi_N}\left[f_N'^2+\frac{(f_N^2-1)^2}{2\sinh^2\rho}\right],$$

$$V_{12}=V_{21}=\frac{1}{\sqrt{2}\sinh^2\rho}(\sinh^2\rho e^{\varphi_N/2}f_N')',$$

$$V_{22}=\frac{1}{2}\varphi_N''+\frac{1}{4}\varphi_N'^2+\varphi_N'\coth\rho+\frac{3f_N^2-1}{\sinh^2\rho}. \tag{26}$$

Note that the prime in formula (26) stands for a derivative with respect to the ρ variable.

For the background solution $f_0(\rho)$, $\phi_0(\rho)$, which corresponds to the $N=0$, the matrix elements $U_{11}(\rho)$, $U_{12}(\rho)$, $U_{22}(\rho)$ are shown in Fig. 2. Background solutions with other $N>0$ provide the matrix elements U_{ij} of a similar shape.

Using the method of the phase functions shift, introduced by Calogero [20] and developed by Degasperis [21], we have found that the self-similar solutions $f_N(\rho)$, $\phi_N(\rho)$ with N zeros of the YM function f have N unstable modes in the spectrum of the linear perturbations. Hence the only self-similar solution with $N=0$ is linearly stable.

IV. CAUCHY PROBLEM

In this section we consider the Cauchy problem for the system of equations (4), (5) starting with regular initial data. The goal is to study the behavior of the solutions near the origin, their possible attaining to some self-similar solutions, and their late-time asymptotics prior to the expected blowup.

The system of equations (4), (5) is a coupled system of nonlinear wave equations. The boundary conditions at the origin and at the radial infinity as well as the initial conditions on the slice $t=0$ have to be defined in order to get a

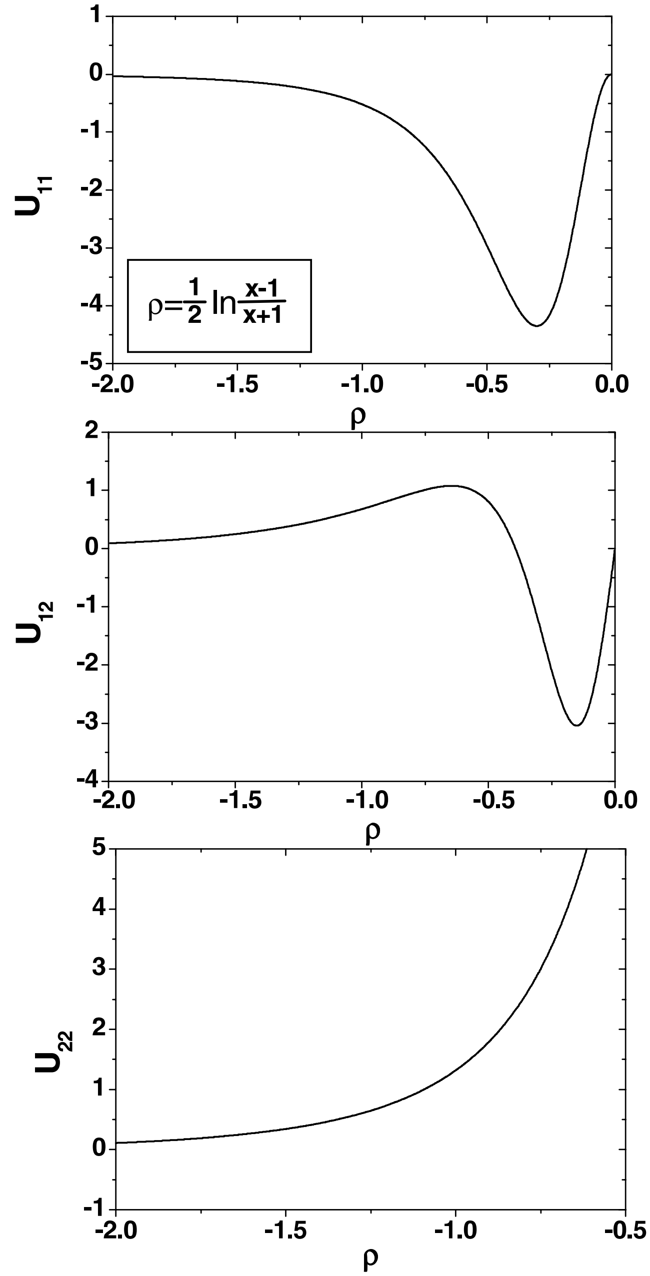


FIG. 2. Matrix elements $U_{ij}(\rho)$ for the background self-similar solution with $N=0$: U_{11} (top), $U_{12}=U_{21}$ (middle), U_{22} (bottom).

well-posed Cauchy problem. In fact, such a problem with the initial and boundary conditions imposed is called a mixed-type Cauchy problem. However, we will call it a Cauchy problem, for short.

The system of equations (4), (5), similarly to its static version, has two singular boundary points $r=0$ and $r=+\infty$. The regularity requirement provides the only possible series expansion near the origin:

$$f(t,r)_{r\to 0}=1-b(t)r^2+O(r^4),$$

$$\Phi(t,r)_{r\to 0}=\Phi_0(t)+\Phi_2(t)r^2+O(r^4), \tag{27}$$

where $b(t)$, $\Phi_0(t)$, $\Phi_2(t)$ are finite smooth functions. Hence the corresponding boundary conditions at the origin, which must hold during the evolution, are as follows:

$$f(t,r=0)=1, \quad f'(t,r=0)=0,$$

$$\Phi(t,r=0)=\Phi_0(t), \quad \Phi'(t,r=0)=0. \tag{28}$$

It is important to note that the value of the dilaton function at the origin $\Phi(t,r=0)$ is a free parameter and there are no reasons to keep it equal to its initial value. Indeed, in our numerical algorithm we have put only the condition $\Phi(t=0,r=0)=0$ while $\Phi(t,r=0)$ has been calculated on each slice $t>0$ according to the evolution equations (4), (5). This means we have put free boundary condition for $\Phi(t,r=0)$.

The asymptotic behavior of the solutions regular at infinity is given by the series expansion

$$f(t,r)_{r\to\infty}=\pm\left(1-\frac{c}{r}+O(r^{-2})\right),$$

$$\Phi(t,r)_{r\to\infty}=\Phi_\infty-\frac{d}{r}+O(r^{-4}), \tag{29}$$

where c, d, and Φ_∞ are constants. This gives the boundary conditions at the spatial infinity as follows:

$$f(t,r=\infty)=\pm 1, \quad f'(t,r=\infty)=0,$$

$$\Phi(t,r=\infty)=\Phi_\infty, \quad \Phi'(t,r=\infty)=0. \tag{30}$$

In order to impose the initial conditions in the Cauchy problem for the system of equations (4), (5), it is enough to set the initial distribution and its derivatives for the YM function $f(t=0,r)$ only. Then the dilaton distribution $\Phi(t=0,r)$ can be obtained from the field equation (5), in a way similar to the Einstein equations, where the dilaton plays a role of the relevant metric function now.

So we always consider smooth regular initial distributions $f(t=0,r)=h(r)$, where $h(r)$ provides nonvacuum values for the YM field function in some compact region for r outside the origin. It is convenient to use two different types of the initial profiles for $h(r)$:

$$h(r)=1-Ar^2\exp[-\sigma(r-R)^2], \tag{31}$$

which is a Gauss-type (A, σ, and R constants) initial profile that connects the same YM vacua $f=+1$, and

$$h(r)=\frac{1-ar^2}{1+ar^2}, \tag{32}$$

which is a kink-type ($a=\text{const}$), which connects two topologically distinct YM vacua $f=\pm 1$.

After the initial distribution $h(r)$ is fixed, one can define a YM radial wave, propagating towards the origin (ingoing wave), as follows:

$$f(0,r)=h(r), \quad \dot{f}(0,r)=f'(0,r)=h'(r). \tag{33}$$

We also put

$$\dot{\Phi}(0,r)=0. \tag{34}$$

The dilaton function at $t=0$ can be obtained now from Eq. (5) with the initial data at the origin imposed:

$$-\Phi''-\frac{2\Phi'}{r}=-\frac{e^{\Phi}}{r^2}\left[\frac{(h^2-1)^2}{2r^2}\right],$$

$$\Phi(0,r=0)=\Phi'(0,r=0)=0. \tag{35}$$

The symmetric initial conditions that lead to two YM radial waves, propagating towards (ingoing) and outwards (outgoing) the origin, are determined in a similar way:

$$f(0,r)=h(r),\quad \dot{f}(0,r)=0,\quad \dot{\Phi}(0,r)=0,$$

$$-\Phi''-\frac{2\Phi'}{r}=-\frac{e^{\Phi}}{r^2}\left[h'^2+\frac{(h^2-1)^2}{2r^2}\right],$$

$$\Phi(0,r=0)=\Phi'(0,r=0)=0. \tag{36}$$

The Cauchy problem has been studied numerically with the help of a finite-difference scheme that preserves the total energy during the evolution. An adaptive mesh refinement algorithm was also installed in order to get a better resolution on a small scales prior to the singularity formation.

The system has been solved for various initial profiles $h(r)$ and initial data (35), (36). The results can be summarized as follows. If we start with small initial data, controlled by the values of the parameters in Eqs. (31), (32), the ingoing wave solution gets smoothly bounced near the origin and after which is radiated away qualitatively similarly to the typical solution of a linear wave equation. If we gradually change the relevant parameters in the initial conditions towards more strength initial data, after it exceeds some threshold value, the ingoing YM wave solution demonstrates universal behavior at the small scales. In fact, the YM function $f(t,r)$ attains the self-similar $N=0$ solution $f_0(r/(T-t))$; see Figs. 3 and 4. As was expected, in this regime the corresponding solution for the dilaton function $\Phi(t,r)$ develops in a non-self-similar way. However, as it can be easily seen

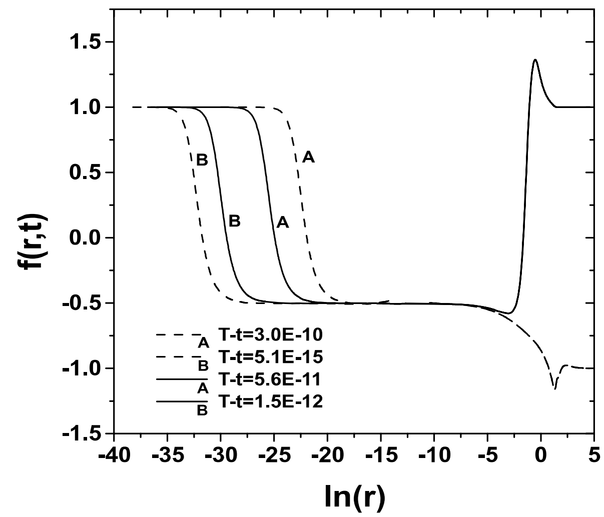
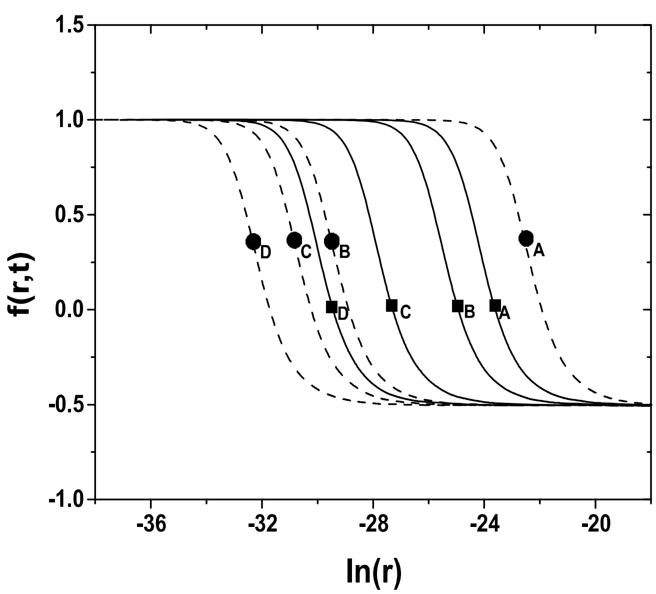


FIG. 3. The set of YM profiles $f(t,r)$ at the various times prior to the singularity formation. The ingoing waves at the small radial scales exhibit universal self-similar behavior whereas the outgoing waves at the right side of the figure are almost frozen at the given time scale. The solutions correspond to the following initial data: (I) $f(0,r)=(1-ar^2)/(1+ar^2)$, kink type, $a=0.281$ (dashed line); (II) $f(0,r)=1-Ar^2\exp[-\sigma(r-r_0)^2]$, Gauss type, $A=0.2$, $\sigma=10$, $r_0=2$ (solid line).



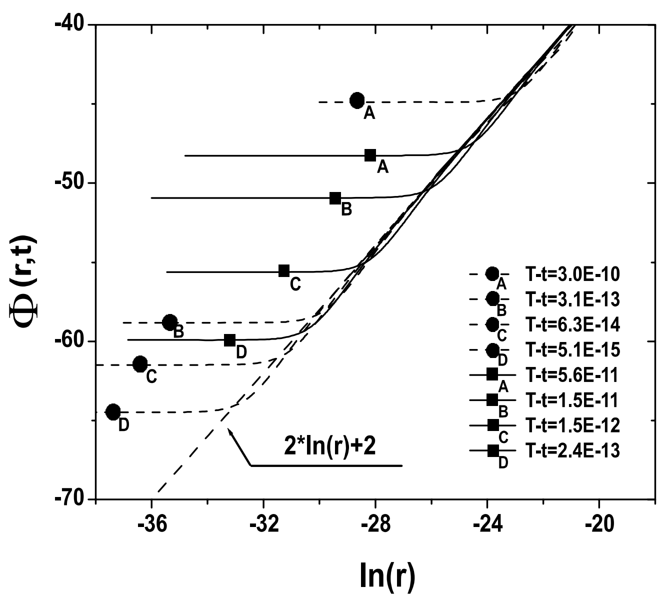


FIG. 4. Self-similar regime of the solutions behavior: YM function $f(t,r)$ (left), dilaton function $\Phi(t,r)$ (right) at various t prior to the blowup. The solutions effectively depends on $r/(T-t)$, where T is the blowup time. The initial data are the same as in Fig. 3: kink-type initial profile (circles), Gauss-type initial profile (squares).

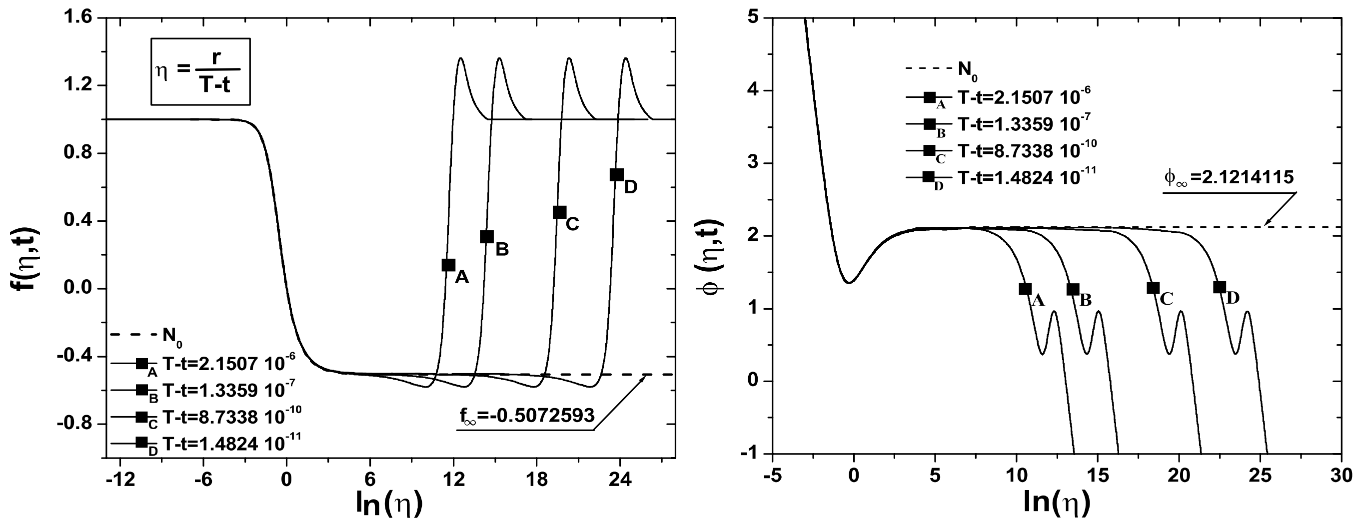


FIG. 5. The late-time evolution of a some typical Cauchy problem solution prior to the blowup. YM profiles $f(\eta,t)$ (left) and the scale-invariant part of the dilaton profiles $\phi(\eta,t)=\Phi(r,t)-2\ln(r)$ (right) attain the self-similar solution $N=0$ (dashed line). The values f_∞, ϕ_∞ are the asymptotical values of the self-similar $N=0$ solution at $\eta\to\infty$. The initial conditions are the same as in Fig. 3 for Gauss-type initial profile.

from Fig. 4, the whole $\Phi(t,r)$ profile propagates along the straight line $2\ln r+2.1214$. This means that in the appropriately shifted and rotated at angle $\beta=\arctan(-2)$ back clockwise frame the obtained dilaton function $\phi(t,r)$,

$$\tilde{\phi}(t,r)=\Phi(t,r)\cos\beta+\sin\beta\ln r,$$

exhibits the self-similar behavior and attains the self-similar $N=0$ solution $\phi_0(t,r)$. The dilaton function $\Phi(t,r)$ can be expressed now in terms of its self-similar part $\Phi(t,r)=\tilde{\phi}(t,r)/\cos\beta+2\ln r=\phi(t,r)+2\ln r$, which reproduces the ansatz (7). Figure 5 illustrates the late-time evolution of some typical solution to the Cauchy problem which attains the self-similar solution $f_0(\eta)$, $\phi_0(\eta)$.

The solutions evolve in a self-similar regime during a finite time and at some blowup time T the second derivative of the YM function f exhibits unbounded growth at the origin. This is an indication for a singularity formation. The blowup time T is just the total time in the Cauchy problem and it depends on the initial data. However, as the solutions attain the self-similar profile $f_0(\eta)$, $\phi_0(\eta)$ their dependence on time enters effectively only in the form $T-t$. Hence, the late-time asymptotics becomes universal for an arbitrary solution with the initial data, which leads to the blowup. According to our studies, the self-similar solution $f_0(\eta)$, $\phi_0(\eta)$ is linearly stable and, as a result, it can pretend to be a global stable attractor in the Cauchy problem. The singularity formation at the origin is not accompanied by an energy concentration which is typical for the supercritical systems. Indeed, the total energy, Eq. (8), at time $t\leqslant T$ inside the past light cone $\int_0^{T-t}dr(\cdots)$ of the point $(T,0)$ can be calculated using the $N=0$ self-similar solution $f_0(x)$, $\phi_0(x)$ as follows:

$$M=E=(T-t)\int_1^\infty dx\Bigg\{\frac{x^2+1}{2x^2}\phi_0(x)_{,x}^2-\frac{2}{x}\phi_0(x)_{,x}+\frac{2}{x^2}+e^{\phi_0(x)}\Bigg[\frac{x^2+1}{x^2}f_0(x)_{,x}^2+\frac{(f_0(x)^2-1)^2}{2x^2}\Bigg]\Bigg\}\sim 4(T-t),$$

and vanishes as $t\to T$. Note that numerically the integral is equal to 4 with accuracy 10^{-9}. The corresponding Schwarzschild radius is equal to $r_S\sim 8(T-t)$. It means that if gravity is included, the obtained late-time self-similar attractor (if exists) has to be hidden under the formed event horizon.

V. CONCLUSIONS AND DISCUSSION

We conclude with the following summary. Using the special ansatz for the dilaton field, we brought the system of the SU(2) spherically symmetric Yang-Mills-dilaton equations to a scale-invariant form and found an infinite countable family of self-similar solutions, labeled by the $N\geqslant 0$—number of zeros of the relevant YM function. Among them the only lowest solution, which corresponds to $N=0$, is stable in the framework of the linear perturbation theory. Being a scale invariant, the considered system has a criticality index equal to $+1$ which means that the system is a supercritical one in PDE terminology and should exhibit a singularity formation if the initial data in the evolutional Cauchy problem exceed some threshold value. The Cauchy problem has been solved numerically for a wide range of smooth finite-energy initial data and the results we found are in agreement with this general expectation.

It has been found that if the initial data exceed some threshold, the resulting solutions in a compact region, shrinking to the origin, attain the lowest $N=0$ stable self-similar profile, which plays the role of an attractor in the Cauchy problem. If the solutions attain the $N=0$ self-similar attractor, they evolve in a universal way during a finite time until the second derivative of the YM function at the origin starts growing infinitely.

The problem of the threshold that separates the disperse and blowup behavior of the Cauchy problem solutions requires more detailed studies. This threshold is usually related to intermediate attractors, which are some local minima or saddle points of the effective action. In our system both the static $N=1$ [12] and self-similar solution $f_1(\eta)$, $\phi_1(\eta)$ are saddle points of the effective action and can play the role of such intermediate attractors. The study of the decay of the static and self-similar solutions with $N\geqslant 1$ along their unstable modes is of particular interest for the threshold understanding.

As was noted earlier, the system of Yang-Mills-dilaton fields is the bosonic part of the appropriately truncated supersymmetric field theory which also contains spin-1/2 dilatino and gaugino fields in the fermionic sector. The study of the fermionic fields in the obtained self-similar bosonic backgrounds seems to be a very interesting task itself and also can shed new light on singularity formation thresholds of the bosonic fields.

All these tasks are under considerations and will be reported elsewhere.

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